

Integration by Parts

If u and v are function of x , then

$$\int uv \, dx = u \int v \, dx - \int \left(\int v \, dx \right) \cdot u' \, dx$$

Question # 1(i)

$$\text{Let } I = \int x \sin x \, dx \quad \left| \begin{array}{l} u = x \\ v = \sin x \end{array} \right.$$

Integration by parts

$$\begin{aligned} I &= x \cdot (-\cos x) - \int (-\cos x) \cdot (1) \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + c \end{aligned}$$

Question # 1(ii)

$$\text{Let } I = \int \ln x \, dx \quad \left| \begin{array}{l} u = \ln x \\ v = 1 \end{array} \right.$$

$$= \int \ln x \cdot 1 \, dx$$

Integrating by parts

$$\begin{aligned} I &= \ln x \cdot x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + c \end{aligned}$$

Question # 1(iii)

$$\text{Let } I = \int x \ln x \, dx \quad \left| \begin{array}{l} u = \ln x \\ v = x \end{array} \right.$$

Integrating by parts

$$\begin{aligned} I &= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + c \\ &= \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + c \end{aligned}$$

Question # 1(iv)

$$\text{Let } I = \int x^2 \ln x \, dx \quad \left| \begin{array}{l} u = \ln x \\ v = x^2 \end{array} \right.$$

Do yourself

Question # 1(v)

$$\text{Let } I = \int x^3 \ln x \, dx \quad \left| \begin{array}{l} u = \ln x \\ v = x^3 \end{array} \right.$$

Do yourself

Question # 1(vi)

$$\text{Let } I = \int x^4 \ln x \, dx \quad \left| \begin{array}{l} u = \ln x \\ v = x^4 \end{array} \right.$$

Do yourself

Question # 1(vii)

$$\text{Let } I = \int \tan^{-1} x \, dx \quad \left| \begin{array}{l} u = \tan^{-1} x \\ v = 1 \end{array} \right.$$

Integrating by parts

$$I = \tan^{-1} x \cdot x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$\begin{aligned} &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{d}{dx}(1+x^2)}{1+x^2} \, dx \\ &= x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + c \end{aligned}$$

Question # 1(viii)

$$\text{Let } I = \int x^2 \sin x \, dx \quad \left| \begin{array}{l} u = x^2 \\ v = \sin x \end{array} \right.$$

Integrating by parts

$$\begin{aligned} I &= x^2(-\cos x) - \int (-\cos x) \cdot 2x \, dx \\ &= -x^2 \cos x + 2 \int x \cos x \, dx \end{aligned} \quad \left| \begin{array}{l} u = x \\ v = \cos x \end{array} \right.$$

Again integrating by parts

$$\begin{aligned} I &= -x^2 \cos x + 2 \left(x \sin x - \int \sin x \, dx \right) \\ &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c \end{aligned}$$

Question # 1(ix)

$$\text{Let } I = \int x^2 \tan^{-1} x \, dx \quad \left| \begin{array}{l} u = \tan^{-1} x \\ v = x^2 \end{array} \right.$$

Integrating by parts

$$\begin{aligned} I &= \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x \, dx + \frac{1}{3} \int \frac{x}{1+x^2} \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{x^2}{2} + \frac{1}{3} \cdot \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \int \frac{d}{dx}(1+x^2)}{1+x^2} \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln |1+x^2| + c \end{aligned}$$

Question # 1(x)

$$\text{Let } I = \int x \tan^{-1} x \, dx \quad \left| \begin{array}{l} u = \tan^{-1} x \\ v = x \end{array} \right.$$

Integrating by parts

$$\begin{aligned} I &= \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} \, dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \right) dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \end{aligned}$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c \text{ Ans.}$$

Question # 1(xi)

Let $I = \int x^3 \tan^{-1} x \, dx$ $\left| \begin{array}{l} u = \tan^{-1} x \\ v = x^3 \end{array} \right.$

Integrating by parts

$$I = \tan^{-1} x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int x^2 dx + \frac{1}{4} \int dx - \frac{1}{4} \int \frac{1}{1+x^2} dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \frac{x^3}{3} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{x^3}{12} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c$$

Question # 1(xii)

Let $I = \int x^3 \cos x \, dx$ $\left| \begin{array}{l} u = x^3 \\ v = \cos x \end{array} \right.$

Do yourself as Question # 1(viii). Integrate by parts three times.

Question # 1(xiii)

$I = \int \sin^{-1} x \, dx$ $\left| \begin{array}{l} u = \sin^{-1} x \\ v = 1 \end{array} \right.$

$$= \int \sin^{-1} x \cdot 1 \, dx$$

Integrating by parts

$$I = \sin^{-1} x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x - \int (1-x^2)^{-\frac{1}{2}} (x) dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \frac{d}{dx} (1-x^2) dx$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c$$

Question # 1(xiv)

Let $I = \int x \sin^{-1} x \, dx$ $\left| \begin{array}{l} u = \sin^{-1} x \\ v = x \end{array} \right.$

Integrating by parts

$$I = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx$$

$$\Rightarrow I = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} I_1 - \frac{1}{2} \sin^{-1} x \dots (i)$$

Where $I_1 = \int \sqrt{1-x^2} dx$

Put $x = \sin q \Rightarrow dx = \cos q \, dq$

$$\Rightarrow I_1 = \int \sqrt{1-\sin^2 q} \cos q \, dq$$

$$= \int \sqrt{\cos^2 q} \cos q \, dq$$

$$= \int \cos^2 q \, dq = \int \left(\frac{1+\cos 2q}{2} \right) dq$$

$$= \frac{1}{2} \int (1+\cos 2q) \, dq$$

$$= \frac{1}{2} \left[q + \frac{\sin 2q}{2} \right] + c$$

$$= \frac{1}{2} \left[q + \frac{2 \sin q \cos q}{2} \right] + c$$

$$= \frac{1}{2} \left[q + \sin q \sqrt{1-\sin^2 q} \right] + c$$

$$= \frac{1}{2} \left[\sin^{-1} x + x \sqrt{1-x^2} \right] + c$$

Using value of I_1 in (i)

$$I = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{1}{2} \left(\sin^{-1} x + x \sqrt{1-x^2} + c \right) \right]$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + \frac{1}{2} c$$

$$\Rightarrow I = \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + \frac{1}{2} c$$

Question # 1(xv)

Let $I = \int e^x \sin x \cos x \, dx$ $\left| \begin{array}{l} u = e^x \\ v = \sin 2x \end{array} \right.$

$$= \frac{1}{2} \int e^x \cdot 2 \sin x \cos x \, dx$$

$$= \frac{1}{2} \int e^x \sin 2x \, dx \quad \because \sin 2x = 2 \sin x \cos x$$

Integrating by parts

$$I = \frac{1}{2} \left[e^x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^x dx \right]$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \int e^x \cos 2x \, dx$$

Again integrating by parts

$$I = -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} e^x \right)$$

$$\begin{aligned}
 &= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - \frac{1}{2} \int e^x \sin 2x \right) \\
 &= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - I \right) + c \\
 &= -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x - \frac{1}{4} I + c \\
 \Rightarrow I + \frac{1}{4} I &= -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c \\
 \Rightarrow \frac{5}{4} I &= -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c \\
 \Rightarrow I &= -\frac{1}{5} e^x \cos 2x + \frac{1}{10} e^x \sin 2x + \frac{4}{5} c
 \end{aligned}$$

Question # 1(xvi)

Let $I = \int x \sin x \cos x dx$

$$\begin{aligned}
 &= \frac{1}{2} \int x \cdot 2 \sin x \cos x dx \\
 &= \frac{1}{2} \int x \cdot \sin 2x dx \quad \left| \begin{array}{l} u = x \\ v = \sin 2x \end{array} \right.
 \end{aligned}$$

Integrating by parts

$$I = \frac{1}{2} \left[x \left(\frac{-\cos 2x}{2} \right) - \int \left(\frac{-\cos 2x}{2} \right) (1) dx \right]$$

Now do yourself

Question # 1(xvii)

Let $I = \int x \cos^2 x dx$

$$\begin{aligned}
 &= \int x \left(\frac{1 + \cos 2x}{2} \right) dx \\
 &= \frac{1}{2} \int x(1 + \cos 2x) dx \quad \left| \begin{array}{l} u = x \\ v = \cos 2x \end{array} \right. \\
 &= \frac{1}{2} \int x dx + \frac{1}{2} \int x \cos 2x dx \\
 &= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2} \left[x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot (1) dx \right] \\
 &= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x - \frac{1}{4} \int \sin 2x dx \\
 &= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x - \frac{1}{4} \left(\frac{-\cos 2x}{2} \right) + c \\
 &= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x + \frac{1}{8} \cos 2x + c
 \end{aligned}$$

Question # 1(xviii)

Let $I = \int x \sin^2 x dx$

$$\begin{aligned}
 &= \int x \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{1}{2} \int x(1 - \cos 2x) dx \\
 &= \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx \quad \left| \begin{array}{l} u = x \\ v = \cos 2x \end{array} \right.
 \end{aligned}$$

Integrating by parts

$$\begin{aligned}
 I &= \frac{1}{2} \frac{x^2}{2} - \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot (1) dx \right] \\
 &= \frac{x^2}{4} - \frac{1}{4} x \sin 2x + \frac{1}{4} \int \sin 2x dx \\
 &= \frac{x^2}{4} - \frac{1}{4} x \sin 2x + \frac{1}{4} \left(\frac{-\cos 2x}{2} \right) + c \\
 &= \frac{x^2}{4} - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + c
 \end{aligned}$$

Question # 1(xix)

Let $I = \int (\ln x)^2 dx$

$$= \int (\ln x)^2 \cdot 1 dx \quad \left| \begin{array}{l} u = (\ln x)^2 \\ v = 1 \end{array} \right.$$

Integrating by parts

$$\begin{aligned}
 I &= (\ln x)^2 \cdot x - \int x \cdot 2(\ln x) \cdot \frac{1}{x} dx \\
 &= x(\ln x)^2 - 2 \int (\ln x) dx
 \end{aligned}$$

Again integrating by parts

$$\begin{aligned}
 I &= x(\ln x)^2 - 2 \left[\ln x \cdot x - \int x \cdot \frac{1}{x} dx \right] \\
 &= x(\ln x)^2 - 2x \ln x + 2 \int dx \\
 &= x(\ln x)^2 - 2x \ln x + 2x + c
 \end{aligned}$$

Question # 1(xx)

Let $I = \int \ln(\tan x) \sec^2 x dx$

$$\left| \begin{array}{l} u = \ln(\tan x) \\ v = \sec^2 x \end{array} \right.$$

Integrating by parts

$$\begin{aligned}
 I &= \ln(\tan x) \cdot \tan x - \int \tan x \cdot \frac{1}{\tan x} \cdot \sec^2 x dx \\
 &= \tan x \ln(\tan x) - \int \sec^2 x dx \\
 &= \tan x \ln(\tan x) - \tan x + c
 \end{aligned}$$

Question # 1(xxi)

Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

$$\left| \begin{array}{l} u = \sin^{-1} x \\ v = (1-x^2)^{-\frac{1}{2}} (-2x) \end{array} \right.$$

$$\begin{aligned}
 &= \int \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} (x) dx \\
 &= -\frac{1}{2} \int \sin^{-1} x \cdot (1-x^2)^{-\frac{1}{2}} (-2x) dx
 \end{aligned}$$

Integrating by parts

$$\begin{aligned}
 I &= -\frac{1}{2} \left[\sin^{-1} x \cdot \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right. \\
 &\quad \left. - \int \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cdot \frac{1}{\sqrt{1-x^2}} dx \right] \\
 &= -\frac{1}{2} \left[\sin^{-1} x \cdot \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} \right. \\
 &\quad \left. - \int \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{\sqrt{1-x^2}} dx \right] \\
 &= -\frac{1}{2} \left[2(1-x^2)^{\frac{1}{2}} \sin^{-1} x - 2 \int dx \right] \\
 &= -\sqrt{1-x^2} \sin^{-1} x + \int dx \\
 &= -\sqrt{1-x^2} \sin^{-1} x + x + c \\
 &= x - \sqrt{1-x^2} \sin^{-1} x + c
 \end{aligned}$$

Question # 2(i)

Let $I = \int \tan^4 x dx$

$$= \int \tan^2 x \cdot \tan^2 x dx$$

$$\begin{aligned}
 &= \int \tan^2 x (\sec^2 x - 1) dx \\
 &= \int (\tan^2 x \sec^2 x - \tan^2 x) dx \\
 &= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\
 &= \int \tan^2 x \frac{d}{dx}(\tan x) dx - \int (\sec^2 x - 1) dx \\
 &= \frac{\tan^{2+1} x}{2+1} - \int \sec^2 x dx - \int dx \\
 &= \frac{1}{3} \tan^3 x - \tan x - x + c
 \end{aligned}$$

Question # 2(ii)

Let $I = \int \sec^4 x dx$

$$\begin{aligned}
 &= \int (\sec^2 x) \cdot (\sec^2 x) dx \\
 &= \int (1 + \tan^2 x) \cdot (\sec^2 x) dx \\
 &= \int \sec^2 x dx + \int \tan^2 x \sec^2 x dx \\
 &= \tan x + \int (\tan x)^2 \frac{d}{dx}(\tan x) dx \\
 &= \tan x + \frac{\tan^3 x}{3} + c
 \end{aligned}$$

Question # 2(iii)

Let $I = \int e^x \sin 2x \cos x dx$

$$\begin{aligned}
 &= \frac{1}{2} \int e^x (2 \sin 2x \cos x) dx \\
 &= \frac{1}{2} \int e^x (\sin(2x+x) + \sin(2x-x)) dx \\
 &= \frac{1}{2} \int e^x (\sin 3x + \sin x) dx \\
 &= \frac{1}{2} \int e^x \sin 3x dx + \frac{1}{2} \int e^x \sin x dx \\
 &= \frac{1}{2} I_1 + \frac{1}{2} I_2 \dots\dots\dots (i)
 \end{aligned}$$

Where $I_1 = \int e^x \sin 3x dx$ and $I_2 = \int e^x \sin x dx$
 Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).

Question # 2(iv)

$$\begin{aligned}
 I &= \int \tan^3 x \cdot \sec x dx \\
 &= \int \tan^2 x \cdot \tan x \cdot \sec x dx \\
 &= \int (\sec^2 x - 1) \cdot \sec x \tan x dx
 \end{aligned}$$

Put $t = \sec x \Rightarrow dt = \sec x \tan x dx$

So $I = \int (t^2 - 1) dt$

$$\begin{aligned}
 &= \frac{t^3}{3} - t + c \\
 &= \frac{\sec^3 x}{3} - \sec x + c
 \end{aligned}$$

Question # 2(v)

Let $I = \int x^3 e^{5x} dx$ | $u = x^3$
 $v = e^x$

Integrating by parts

$$I = x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 3x^2 dx$$

$$= \frac{1}{5} x^3 e^{5x} - \frac{1}{5} \int x^2 e^{5x} dx \quad \left| \begin{array}{l} u = x^2 \\ v = e^x \end{array} \right.$$

Again integrating by parts

$$\begin{aligned}
 I &= \frac{1}{5} x^3 e^{5x} - \frac{1}{5} \left[x^2 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 2x dx \right] \\
 &= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \int x e^{5x} dx
 \end{aligned}$$

Again integrating by parts

$$\begin{aligned}
 I &= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} \\
 &\quad + \frac{6}{25} \left[x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot (1) dx \right] \\
 &= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \int e^{5x} dx \\
 &= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \cdot \frac{e^{5x}}{5} + c \\
 &= \frac{e^{5x}}{5} \left(x^3 - \frac{3}{5} x^2 + \frac{6}{25} x - \frac{6}{125} \right) + c
 \end{aligned}$$

Question 2(vi)

Let $I = \int e^{-x} \sin 2x dx$ | $u = e^{-x}$
 $v = \sin 2x$

Integrating by parts

$$\begin{aligned}
 I &= e^{-x} \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^{-x} (-1) dx \\
 &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \int e^{-x} \cos 2x dx
 \end{aligned}$$

Again integrating by parts

$$\begin{aligned}
 I &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \left[e^{-x} \cdot \frac{\sin 2x}{2} \right. \\
 &\quad \left. - \int \frac{\sin 2x}{2} \cdot e^{-x} (-1) dx \right] \\
 &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x - \frac{1}{4} \int e^{-x} \sin 2x dx \\
 \Rightarrow I &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x - \frac{1}{4} I + c \\
 \Rightarrow I + \frac{1}{4} I &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x + c \\
 \Rightarrow \frac{5}{4} I &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x + c \\
 \Rightarrow I &= -\frac{2}{5} e^{-x} \cos 2x - \frac{1}{5} e^{-x} \sin 2x + \frac{4}{5} c \\
 &= -\frac{1}{5} e^{-x} (2 \cos 2x + \sin 2x) + \frac{4}{5} c
 \end{aligned}$$

Question # 2(vii)

Let $I = \int e^{2x} \cdot \cos 3x dx$

Do yourself as above

Question # 2(viii)

$I = \int \operatorname{cosec}^3 x dx$ | $u = \operatorname{cosec} x$
 $v = \operatorname{cosec}^2 x$

$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x dx$$

Integrating by parts

$$\begin{aligned}
 I &= \operatorname{csc} x (-\cot x) - \int (-\cot x) (-\operatorname{csc} x \cot x) dx \\
 &= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x dx \\
 &= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) dx \\
 &= -\operatorname{cosec} x \cot x - \int (\operatorname{cosec}^3 x - \operatorname{cosec} x) dx
 \end{aligned}$$

$$\begin{aligned}
 &= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx \\
 &= -\operatorname{cosec} x \cot x - I + \ln |\operatorname{cosec} x - \cot x| + c \\
 \Rightarrow I + I &= -\operatorname{cosec} x \cot x + \ln |\operatorname{cosec} x - \cot x| + c \\
 \Rightarrow 2I &= -\operatorname{cosec} x \cot x + \ln |\operatorname{cosec} x - \cot x| + c \\
 \Rightarrow I &= -\frac{1}{2} \operatorname{csc} x \cot x + \frac{1}{2} \ln |\operatorname{csc} x - \cot x| + \frac{1}{2} c
 \end{aligned}$$

Question # 3

Let $I = \int e^{ax} \sin bx \, dx$
 $\left| \begin{array}{l} u = e^{ax} \\ v = \sin bx \end{array} \right.$

Integrating by parts

$$\begin{aligned}
 I &= e^{ax} \left(-\frac{\cos bx}{b} \right) - \int \left(-\frac{\cos bx}{b} \right) \cdot e^{ax} (a) \, dx \\
 &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx \, dx
 \end{aligned}$$

Again integrating by parts

$$I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \left[e^{ax} \frac{\sin bx}{b} - \int \frac{\sin bx}{b} \cdot e^{ax} a \, dx \right]$$

$$= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx \, dx$$

$$= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I + c_1$$

$$\Rightarrow I + \frac{a^2}{b^2} I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx + c_1$$

$$\Rightarrow \left(\frac{b^2 + a^2}{b^2} \right) I = \frac{e^{ax}}{b^2} (-b \cos bx + a \sin bx) + c_1$$

$$\Rightarrow (b^2 + a^2) I = e^{ax} (a \sin bx - b \cos bx) + b^2 c_1$$

Put $a = r \cos q$ & $b = r \sin q$

Squaring and adding

$$a^2 + b^2 = r^2 (\cos^2 q + \sin^2 q)$$

$$\Rightarrow a^2 + b^2 = r^2 (1) \Rightarrow r = \sqrt{a^2 + b^2}$$

Also

$$\frac{b}{a} = \frac{r \sin q}{r \cos q} \Rightarrow \frac{b}{a} = \tan q$$

$$\Rightarrow q = \tan^{-1} \frac{b}{a}$$

So

$$(b^2 + a^2) I = e^{ax} (r \cos q \sin bx - r \sin q \cos bx) + b^2 c_1$$

$$(b^2 + a^2) I = e^{ax} r (\sin bx \cos q - \cos bx \sin q) + b^2 c_1$$

$$\Rightarrow (a^2 + b^2) I = e^{ax} r \sin (bx - q) + b^2 c_1$$

Putting value of r and q

$$(a^2 + b^2) I = e^{ax} \sqrt{a^2 + b^2} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + b^2 c_1$$

$$\Rightarrow I = \frac{\sqrt{a^2 + b^2}}{(a^2 + b^2)} e^{ax} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + \frac{b^2}{a^2 + b^2} c_1$$

$$\Rightarrow I = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + c$$

Where $c = \frac{b^2}{a^2 + b^2} c_1$

Question # 4(i)

Let $I = \int \sqrt{a^2 - x^2} \, dx$
 $\left| \begin{array}{l} u = \sqrt{a^2 - x^2} \\ v = 1 \end{array} \right.$

 $= \int \sqrt{a^2 - x^2} \cdot 1 \, dx$

Integrating by parts

$$I = \sqrt{a^2 - x^2} \cdot x - \int x \cdot \frac{1}{2} (a^2 - x^2)^{-\frac{1}{2}} \cdot (-2x) \, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{(a^2 - x^2)^{\frac{1}{2}}} \, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{(a^2 - x^2)^{\frac{1}{2}}} \, dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{(a^2 - x^2)^{\frac{1}{2}}} - \frac{a^2}{(a^2 - x^2)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} \, dx + \int \frac{a^2}{\sqrt{a^2 - x^2}} \, dx$$

$$\Rightarrow I = x\sqrt{a^2 - x^2} - I + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} \, dx$$

$$\Rightarrow I + I = x\sqrt{a^2 - x^2} + a^2 \operatorname{Sin}^{-1} \frac{x}{a} + c$$

$$\Rightarrow 2I = x\sqrt{a^2 - x^2} + a^2 \operatorname{Sin}^{-1} \frac{x}{a} + c$$

$$\Rightarrow I = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \operatorname{Sin}^{-1} \frac{x}{a} + \frac{1}{2} c$$

Review

- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c$

- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right| + c$

Question # 4(ii)

Let $I = \int \sqrt{x^2 - a^2} \, dx$
 $\left| \begin{array}{l} u = \sqrt{x^2 - a^2} \\ v = 1 \end{array} \right.$

 $= \int \sqrt{x^2 - a^2} \cdot 1 \, dx$

Integrating by parts

$$I = \sqrt{x^2 - a^2} \cdot x - \int x \cdot \frac{1}{2} (x^2 - a^2)^{-\frac{1}{2}} \cdot (2x) \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{(x^2 - a^2)^{\frac{1}{2}}} \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{(x^2 - a^2)^{\frac{1}{2}}} \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \left(\frac{x^2 - a^2}{(x^2 - a^2)^{\frac{1}{2}}} + \frac{a^2}{(x^2 - a^2)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} \, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} \, dx$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - I - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx$$

$$\Rightarrow I + I = x\sqrt{x^2 - a^2} - a^2 \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$\Rightarrow 2I = x\sqrt{x^2 - a^2} - a^2 \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$\Rightarrow I = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + \frac{1}{2} c$$

Question # 4(iii)

$$\text{Let } I = \int \sqrt{4 - 5x^2} dx$$

$$= \int \sqrt{4 - 5x^2} \cdot 1 dx$$

Integrating by parts

$$I = \sqrt{4 - 5x^2} \cdot x - \int x \cdot \frac{1}{2} (4 - 5x^2)^{-\frac{1}{2}} \cdot (-10x) dx$$

$$= \sqrt{4 - 5x^2} \cdot x - \int \frac{-5x^2}{(4 - 5x^2)^{\frac{1}{2}}} dx$$

$$= \sqrt{4 - 5x^2} \cdot x - \int \frac{4 - 5x^2 - 4}{(4 - 5x^2)^{\frac{1}{2}}} dx$$

$$= \sqrt{4 - 5x^2} \cdot x - \int \left(\frac{4 - 5x^2}{(4 - 5x^2)^{\frac{1}{2}}} - \frac{4}{(4 - 5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4 - 5x^2} \cdot x - \int \left((4 - 5x^2)^{\frac{1}{2}} - \frac{4}{(4 - 5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4 - 5x^2} \cdot x - \int \sqrt{4 - 5x^2} dx + 4 \int \frac{1}{\sqrt{4 - 5x^2}} dx$$

$$\Rightarrow I = \sqrt{4 - 5x^2} \cdot x - I + 4 \int \frac{1}{\sqrt{5 \left(\frac{4}{5} - x^2 \right)}} dx$$

$$\Rightarrow I + I = \sqrt{4 - 5x^2} \cdot x + 4 \int \frac{1}{\sqrt{5} \sqrt{\frac{4}{5} - x^2}} dx$$

$$\Rightarrow 2I = \sqrt{4 - 5x^2} \cdot x + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}} \right)^2 - x^2}} dx$$

$$= \sqrt{4 - 5x^2} \cdot x + \frac{4}{\sqrt{5}} \text{Sin}^{-1} \left(\frac{x}{2/\sqrt{5}} \right) + c_1$$

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Sin}^{-1} \frac{x}{a}$$

$$\Rightarrow I = \frac{x}{2} \sqrt{4 - 5x^2} + \frac{4}{2\sqrt{5}} \text{Sin}^{-1} \left(\frac{\sqrt{5}x}{2} \right) + \frac{1}{2} c_1$$

$$= \frac{x}{2} \sqrt{4 - 5x^2} + \frac{2}{\sqrt{5}} \text{Sin}^{-1} \left(\frac{\sqrt{5}x}{2} \right) + c$$

$$\text{Where } c = \frac{1}{2} c_1$$

Question # 4(iv)

$$\text{Let } I = \int \sqrt{3 - 4x^2} dx$$

Same as above.

Question # 4(v)

Same as Q # 4(ii)

$$\text{Use } \int \frac{dx}{\sqrt{x^2 + 4}} = \ln \left| x + \sqrt{x^2 + 4} \right| + c$$

Question # 4(vi)

$$\text{Let } I = \int x^2 e^{ax} dx$$

Do yourself as Question # 2(v)

Important Formula

$$\text{Since } \frac{d}{dx} (e^{ax} f(x)) = e^{ax} \frac{d}{dx} f(x) + f(x) \frac{d}{dx} e^{ax}$$

$$= e^{ax} f'(x) + f(x) \cdot e^{ax} (a)$$

$$= e^{ax} [a f(x) + f'(x)]$$

On integrating

$$\int \frac{d}{dx} (e^{ax} f(x)) dx = \int e^{ax} [a f(x) + f'(x)] dx$$

$$\Rightarrow e^{ax} f(x) = \int e^{ax} [a f(x) + f'(x)] dx$$

$$\Rightarrow \boxed{\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c}$$

Question # 5(i)

$$\text{Let } I = \int e^x \left(\frac{1}{x} + \ln x \right) dx$$

$$= \int e^x \left(\ln x + \frac{1}{x} \right) dx$$

$$\text{Put } f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$\text{So } I = \int e^x (f(x) + f'(x)) dx$$

$$= e^x f(x) + c = e^x \ln x + c$$

Question # 5(ii)

$$\text{Let } I = \int e^x (\cos x + \sin x) dx$$

$$= \int e^x (\sin x + \cos x) dx$$

$$\text{Put } f(x) = \sin x \Rightarrow f'(x) = \cos x$$

$$\text{So } I = \int e^x (f(x) + f'(x)) dx$$

$$= e^x f(x) + c$$

$$= e^x \sin x + c$$

Question # 5(iii)

$$\text{Let } I = \int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2 - 1}} \right] dx$$

$$\text{Put } f(x) = \sec^{-1} x \Rightarrow f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\text{So } I = \int e^{ax} [a f(x) + f'(x)] dx$$

$$= e^{ax} f(x) + c$$

$$= e^{ax} \sec^{-1} x + c$$

Question # 5(iv)

$$\text{Let } I = \int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x} \right) dx$$

$$= \int e^{3x} \left(\frac{3 \sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int e^{3x} \left(3 \frac{1}{\sin x} - \frac{\cos x}{\sin x \cdot \sin x} \right) dx$$

$$= \int e^{3x} (3 \csc x - \csc x \cot x) dx$$

Put $f(x) = \csc x \Rightarrow f'(x) = \csc x \cot x$

$$\Rightarrow I = \int e^{3x} (3f(x) + f'(x)) dx$$

$$= e^{3x} f(x) + c$$

$$= e^{3x} \csc x + c$$

Question 5(v)

Let $I = \int e^{2x} (-\sin x + 2 \cos x) dx$ *Correction

$$= \int e^{2x} (2 \cos x - \sin x) dx$$

Put $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

So $I = \int e^{2x} (2f(x) + f'(x)) dx$

$$= e^{2x} f(x) + c$$

$$= e^{2x} \cos x + c$$

Question # 5(vi)

Let $I = \int \frac{xe^x}{(1+x)^2} dx$

$$= \int \frac{(1+x-1)e^x}{(1+x)^2} dx$$

$$= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx$$

$$= \int e^x \left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right] dx$$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$\Rightarrow f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$$

So $I = \int e^x (f(x) + f'(x)) dx$

$$= e^x f(x) + c$$

$$= e^x \left(\frac{1}{1+x} \right) + c$$

Question # 5(vii)

Let $I = \int e^{-x} (\cos x - \sin x) dx$

$$= \int e^{-x} ((-1) \sin x + \cos x) dx$$

Put $f(x) = \sin x \Rightarrow f'(x) = \cos x$

So $I = \int e^{-x} ((-1)f(x) + f'(x)) dx$

$$= e^{-x} f(x) + c$$

$$= e^{-x} \sin x + c$$

Question # 5(viii)

Let $I = \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

$$= \int e^{m \tan^{-1} x} \cdot \frac{1}{1+x^2} dx$$

Put $t = \tan^{-1} x \Rightarrow dt = \frac{1}{1+x^2} dx$

So $I = \int e^{mt} dt$

$$= \frac{e^{mt}}{m} + c$$

$$= \frac{1}{m} e^{m \tan^{-1} x} + c$$

Important Integral

Let $I = \int \tan x dx$

$$= \int \frac{\sin x}{\cos x} dx$$

Put $t = \cos x \Rightarrow dt = -\sin x dx$

$$\Rightarrow -dt = \sin x dx$$

So $I = \int \frac{-dt}{t} = -\int \frac{dt}{t}$

$$= -\ln |t| + c$$

$$= -\ln |\cos x| + c$$

$$= \ln |\cos x|^{-1} + c \quad \because m \ln x = \ln x^m$$

$$= \ln \left| \frac{1}{\cos x} \right| + c = \ln |\sec x| + c$$

$$\Rightarrow \boxed{\int \tan x dx = \ln |\sec x| + c}$$

Similarly, we have

$$\boxed{\int \cot x dx = \ln |\sin x| + c}$$

Question # 5(ix)

Let $I = \int \frac{2x}{1-\sin x} dx$

$$= \int \frac{2x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} dx$$

$$= \int \frac{2x(1+\sin x)}{1-\sin^2 x} dx$$

$$= \int \frac{2x+2x \sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{2x}{\cos^2 x} + \frac{2x \sin x}{\cos^2 x} \right) dx$$

$$= \int \frac{2x}{\cos^2 x} dx + \int \frac{2x \sin x}{\cos x \cdot \cos x} dx$$

$$= 2 \int x \sec^2 x dx + 2 \int x \sec x \tan x dx$$

Integrating by parts

$$I = 2 \left[x \cdot \tan x - \int \tan x \cdot 1 dx \right]$$

$$+ 2 \left[x \cdot \sec x - \int \sec x (1) dx \right]$$

$$= 2 \left[x \cdot \tan x - \ln |\sec x| \right]$$

$$+ 2 \left[x \cdot \sec x - \ln |\sec x + \tan x| \right] + c$$

$$= 2x \tan x - 2 \ln |\sec x|$$

$$+ 2x \sec x - 2 \ln |\sec x + \tan x| + c$$

Question # 5(x)

Let $I = \int \frac{e^x(1+x)}{(2+x)^2} dx$

$$= \int \frac{e^x(2+x-1)}{(2+x)^2} dx$$

$$= \int e^x \left(\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right) dx$$

$$= \int e^x \left((2+x)^{-1} - (2+x)^{-2} \right) dx$$

Put $f(x) = (2+x)^{-1} \Rightarrow f'(x) = -(2+x)^{-2}$

So $I = \int e^x (f(x) + f'(x)) dx$

$$= e^x f(x) + c$$

$$= e^x (2+x)^{-1} + c$$

$$= \frac{e^x}{2+x} + c$$

Question # 15(xi)

Let $I = \int \left(\frac{1 - \sin x}{1 - \cos x} \right) e^x dx$

$$= \int \left(\frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) e^x dx$$

$$= \int \left(\frac{1}{2 \sin^2 \frac{x}{2}} - \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) e^x dx$$

$$= \int \left(\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right) e^x dx$$

$$= \int e^x \left(-\cot \frac{x}{2} + \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) dx$$

Put $f(x) = -\cot \frac{x}{2} \Rightarrow f'(x) = \operatorname{cosec}^2 \frac{x}{2} \cdot \frac{1}{2}$

$$\Rightarrow f'(x) = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}$$

So $I = \int e^x (f(x) + f'(x))$

$$= e^x f(x) + c$$

$$= e^x (-\cot x) + c$$

$$= -e^x \cot x + c$$

Found Error - tell us at
<http://www.MathCity.org/error>

Error Analysts	
1-	Kamran Javed - (2006-08) Punjab College of Information Technology, Rawalpindi.

The End

25 Dec. 2008