MULTIVARIABLE CALCULUS

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Preface

Multivariable calculus is a fundamental subject that extends the concepts of single-variable calculus to higher-dimensional spaces. It provides a powerful framework for analyzing and modeling complex phenomena in fields such as physics, engineering, economics, and computer science.

This textbook is designed to provide a comprehensive introduction to multivariable calculus, covering topics such as Vectors, Functions, partial derivatives, multiple integrals, and differential equations, Laplace and Fourier Transformations, Sequence, Series and Complex Integration. Through a combination of theoretical foundations, practical applications, and numerous examples and exercises, we aim to equip students with a deep understanding of the subject matter and its relevance to real-world problems.

Throughout the book, we emphasize the development of problem-solving skills, critical thinking, and mathematical maturity. We also highlight the connections between multivariable calculus and other areas of mathematics, such as linear algebra and differential equations.

Our goal is to make this textbook a valuable resource for students, instructors, and researchers alike, providing a solid foundation for further study and exploration in mathematics, science, and engineering.

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Superior Group of Colleges

Sillanwali, Sargodha

Purpose of this course is to develop the skills to have ground knowledge of multivariate calculus and appreciation for their further computer science courses.

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MULTI VARIABLE FUNCTIONS AND PARTIAL DERIVATIVES

Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

■ verbally (by a description in words)

■ numerically (by a table of values)

■ algebraically (by an explicit formula)

■ visually (by a graph or level curves)

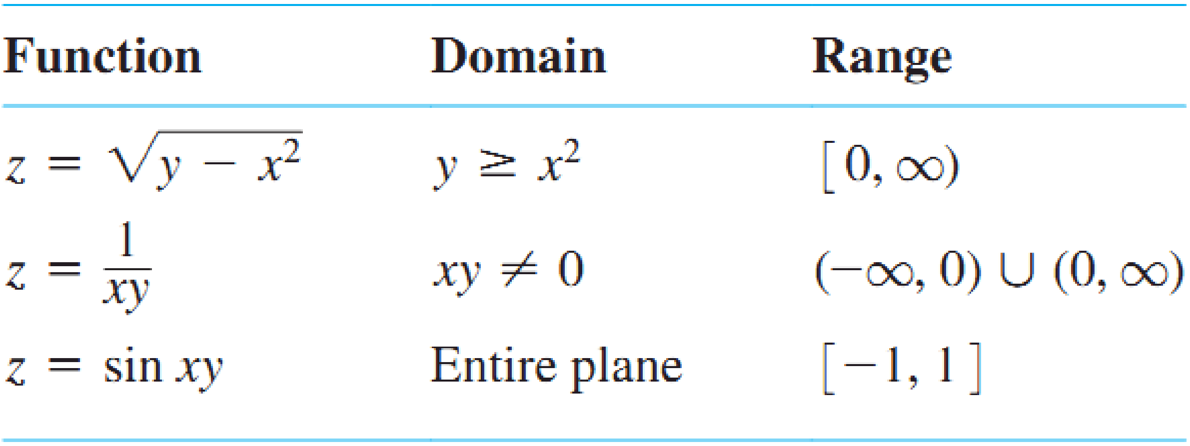
Functions of Two Variables

A function of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by *f(x,y)*. The set D is the domain of *f* and its range is the set of values that *f* takes on, that is, $\{f(x,y) : (x,y) \in D\}$

We often write z = *f(x,y)* to make explicit the value taken on by *f* at the general point *(x,y)*. The variables x and y are independent variables and z is the dependent variable.

Example

These are functions of two variables. Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z.



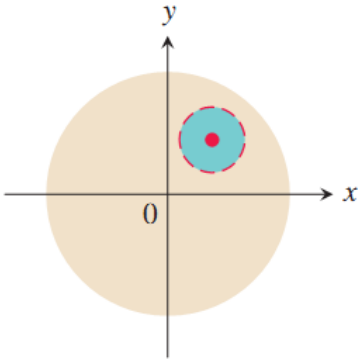
Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

These are functions of three variables with restrictions on some of their domains.

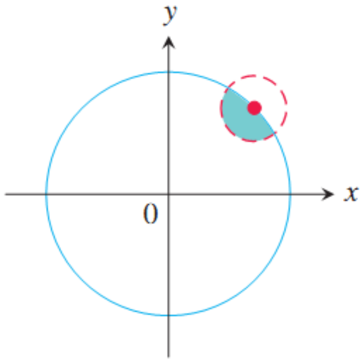
Function	**Domain**	**Range**
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

Definitions

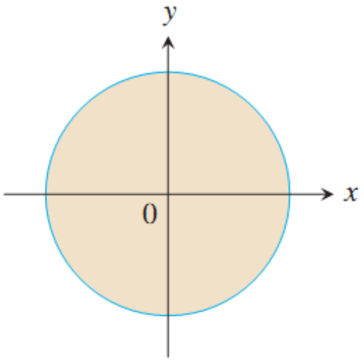
- A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure).
- A point (x_0 , y_0) is a **boundary point** of R if every disk centered at (x_0 , y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)
- The interior points of a region, as a set, make up the **interior** of the region.
- The region's boundary points make up its **boundary**.
- A region is **open** if it consists entirely of interior points.
- A region is **closed** if it contains all its boundary points (Figure).



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.



$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)



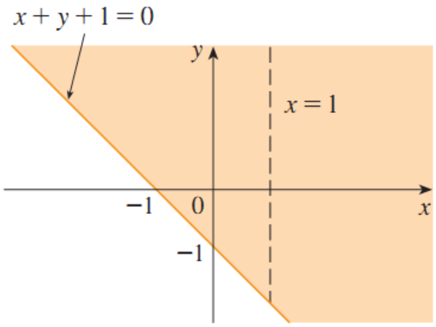
$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all
boundary points.

Example

For the following function, evaluate $f(3,2)$ and find and sketch the domain.

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

Solution



$$f(3, 2) = \frac{\sqrt{3 + 2 + 1}}{3 - 1} = \frac{\sqrt{6}}{2}$$

$$D = \{(x, y) \mid x + y + 1 \geqslant 0,\ x \neq 1\}$$

Example

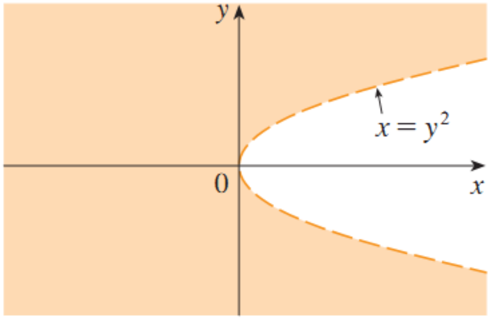
For the following function, evaluate $f(3,2)$ and find and sketch the domain.

$$f(x, y) = x \ln(y^2 - x)$$

Solution

$$f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

$$D = \{(x, y) \mid x < y^2\}$$



Example

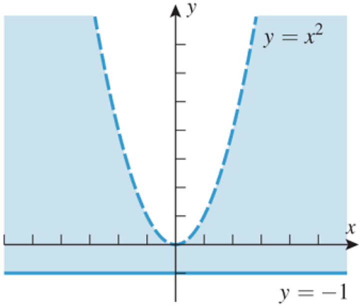
Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural domain of f.

Solution

By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$

To find the natural domain of f, we note that $\sqrt{y+1}$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$



Example

Let

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$ and the natural domain of f.

Solution

By substitution,

$$f\left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

Because of the square root sign, we must have $0 \leq 1 - x^2 - y^2 - z^2$ in order to have a real value for $f(x, y, z)$. Rewriting this inequality in the form

$$x^2 + y^2 + z^2 \leq 1$$

we see that the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1 \blacktriangleleft$$

Bounded and Unbounded Region

A region in the plane is bounded if it lies inside a disk of finite radius. A region is unbounded if it is not bounded.

Examples of bounded sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of unbounded sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

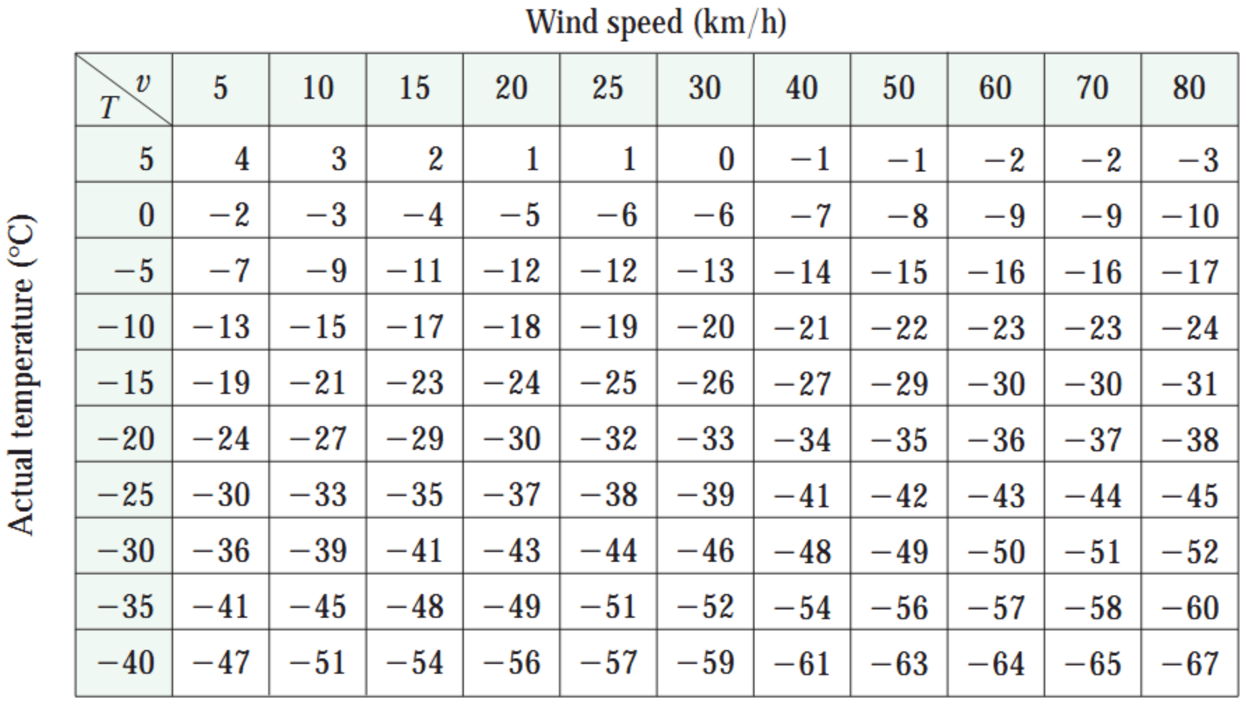
Example

Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

Example

In regions with severe winter weather, the wind-chill index is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of T and v, and we can write W=*f* (T,v). Table records values of W compiled by the National Weather Service of the US and the Meteorological Service of Canada.



Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

For instance, the table shows that if the temperature is −5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about −15°C with no wind. So

$$f(-5, 50) = -15$$

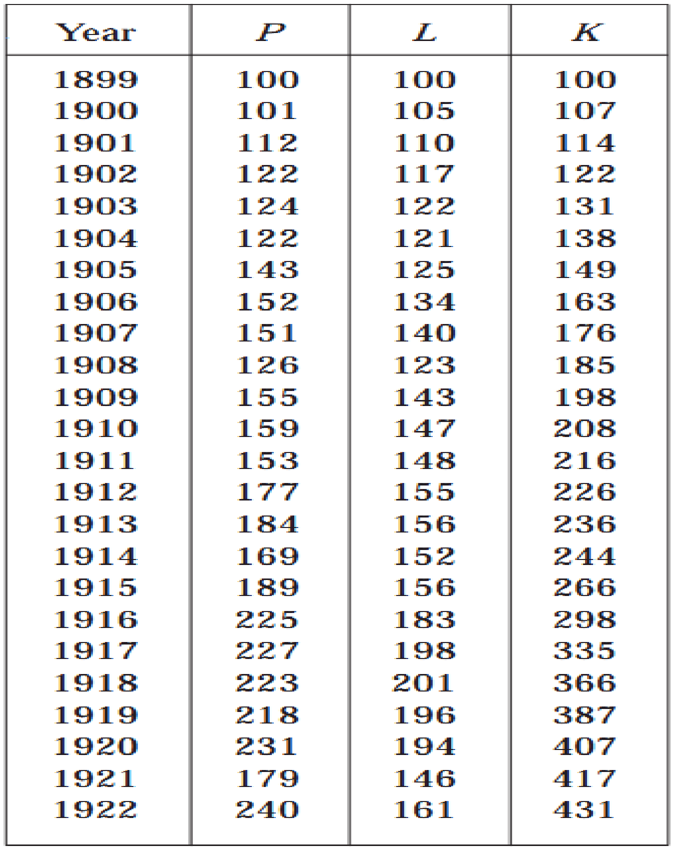
Example

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).

Cobb and Douglas used economic data published by the government to obtain Table. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.



Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Cobb and Douglas used the method of least squares to fit the data of Table to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

If we use the model given by the function in previous equation to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.
The production function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K): L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

Example

Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

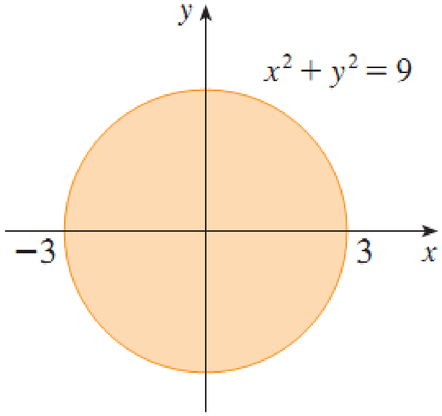
Solution

Domain is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

Range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$

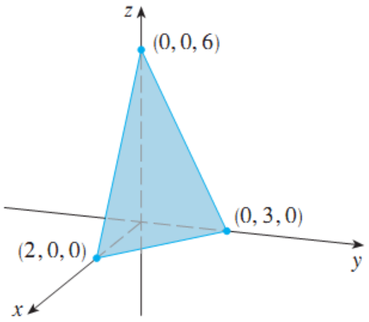


Graph
If is a function of two variables with domain D, then the graph of ***f*** is the set of all points (x,y,z) in $\mathbf{R}^3$ such that z = *f*(x,y) and (x,y) is in D.

Linear Function
A function of the form $f(x,y) = ax + by + c$ is called a linear function. The graph of such a function has the equation
$z = ax + by + c$ or $ax + by - z + c = 0$
so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

Example
Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution
The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts.
Putting $y = z = 0$ in the equation, we get x = 2 as the -intercept. Similarly, the y -intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant in Figure.



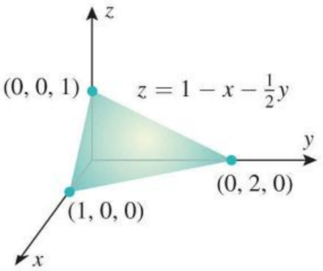
Example
In each part, describe the graph of the function in an xyz-coordinate system

(a) $f(x,y) = 1 - x - \frac{1}{2}y$ (b) $f(x,y) = \sqrt{1 - x^2 - y^2}$
(c) $f(x,y) = -\sqrt{x^2 + y^2}$

Solution
(a). By definition, the graph of the given function is the graph of the equation

$$z = 1 - x - \tfrac{1}{2}y$$

which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure).



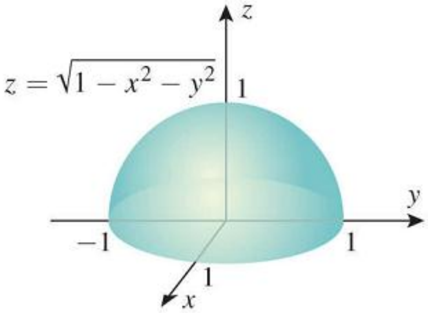
(b). By definition, the graph of the given function is the graph of the equation

$$z = \sqrt{1 - x^2 - y^2}$$

After squaring both sides, this can be rewritten as

$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since equation imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure).



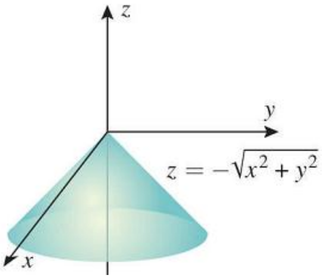
(c). The graph of the given function is the graph of the equation

$$z = -\sqrt{x^2 + y^2}$$

After squaring, we obtain

$$z^2 = x^2 + y^2$$

which is the equation of a circular cone. Since equation imposes the condition that $z \leq 0$, the graph is just the lower nappe of the cone (Figure).

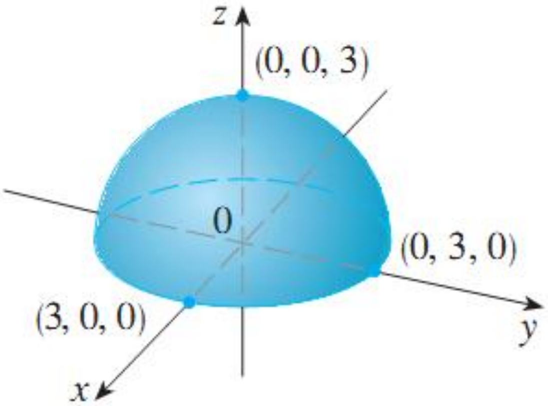


Example

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

The graph has equation $z = \sqrt{9 - x^2 - y^2}$.
We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$,
or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere.

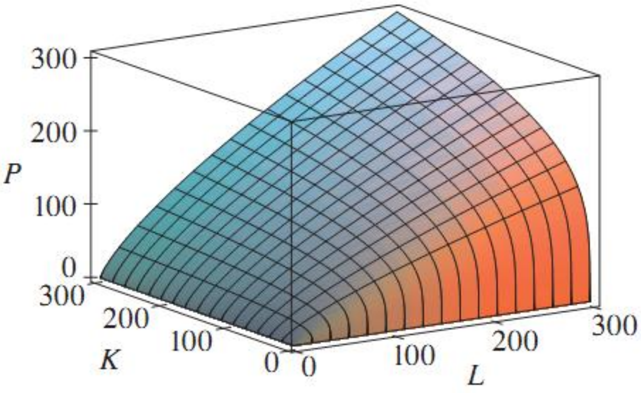


Example

Draw the graph of the Cobb-Douglas production function
$P(L, K) = 1.01L^{0.75}K^{0.25}$

Solution

Figure shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as is to be expected.



Example

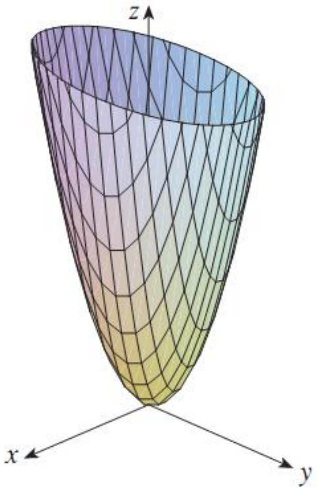
Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

Solution

Domain is R^2

The range of h is $[0, \infty)$ the set of all non-negative real numbers.

The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid. Horizontal traces are ellipses and vertical traces are parabolas.



Remark

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form contour lines, or level curves.

Level Curves

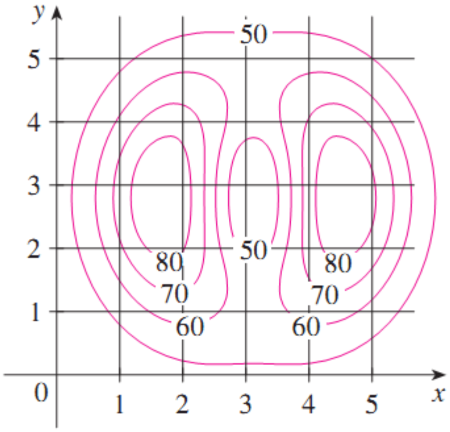
The level curves of a function f of two variables are the curves with equations $(x, y) = k$, where k is a constant (in the range of f).

Or A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k.

Or The set of points in the plane where a function f(x, y) has a constant value f(x, y) = c is called a **level curve** of f. The set of all points (x, y, f(x, y)) in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** z = f(x, y).

Example

A contour map for a function f is shown in Figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



Solution

The point (1, 3) lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1,3) \approx 73$$

Similarly, we estimate that

$$f(4,5) \approx 56$$

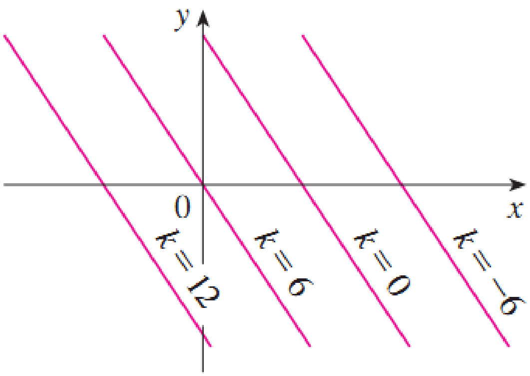
Example

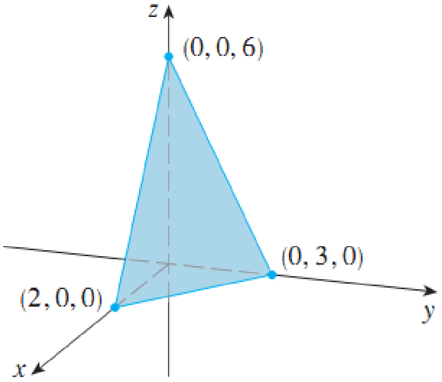
Sketch the level curves of the function $f(x,y) = 6 - 3x - 2y$ for the values $k = -6,0,6,12$

Solution

The level curves are $6 - 3x - 2y = k$ or $3x + 2y + (k-6) = 0$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6,0,6,12$ are $3x + 2y - 12 = 0, 3x + 2y - 6 = 0$ and $3x + 2y = 0$. They are sketched. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure on right).





Example

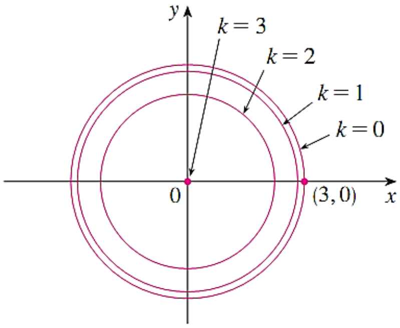
Sketch the level curves of the function

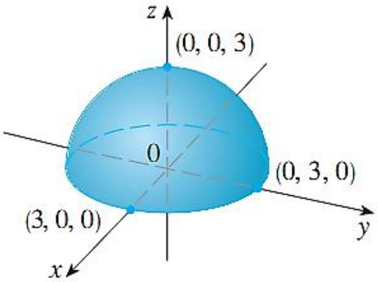
$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

Solution

The level curves are $\sqrt{9 - x^2 - y^2} = k$ or $x^2 + y^2 = 9 - k^2$

This is a family of concentric circles with center (0,0) and radius $\sqrt{9 - k^2}$. The cases $k = 0,1,2,3$ are shown in Figure. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure on right.





Example

Sketch the level curves of the function $f(x, y) = 4x^2 + y^2 + 1$

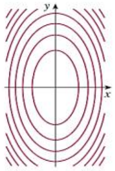
Solution

The level curves are

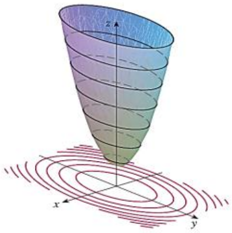
$$4x^2 + y^2 + 1 = k \qquad \text{or} \qquad \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1$$

which, for k > 1, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$.

Figure (a) shows a contour map of h drawn by a computer. Figure (b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure how the graph of h is put together from the level curves.



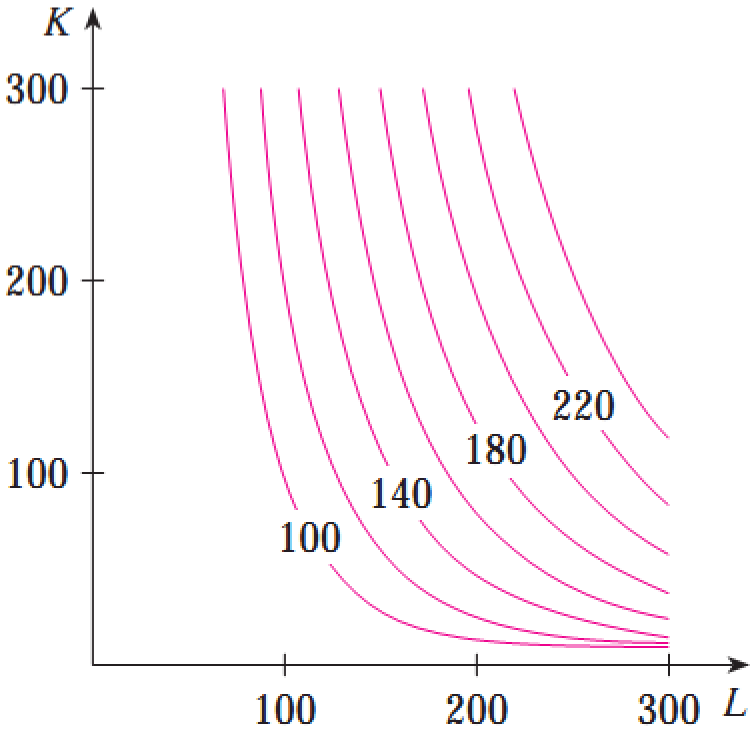
(a) Contour map



(b) Horizontal traces are raised level curves

Example
Plot level curves for the Cobb-Douglas production function.

Solution
In Figure



we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

Level curves are labeled with the value of the production P. For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of P = 140. We see that, for a fixed value of P, as L increases K decreases, and vice versa.

Example

Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution

The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

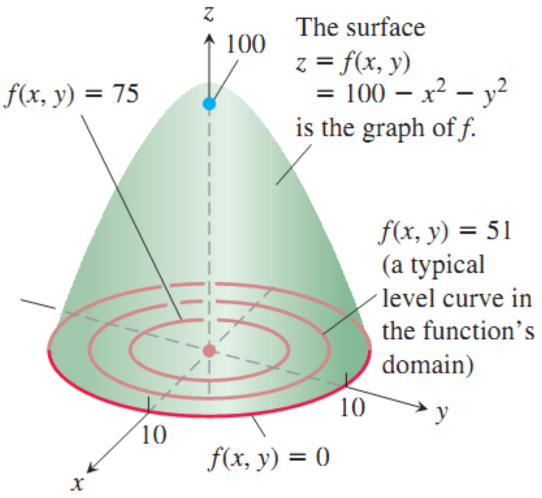
$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \qquad \text{or} \qquad x^2 + y^2 = 49$$
$$f(x, y) = 100 - x^2 - y^2 = 75, \qquad \text{or} \qquad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of f. ■



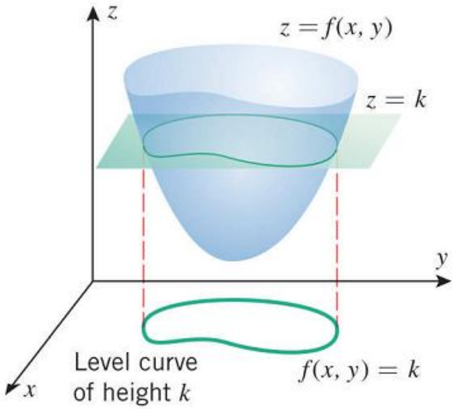
Contour Curve

The curve in space in which the plane z = c cuts a surface z = f(x, y) is made up of the points that represent the function value f(x, y) = c. It is called the contour curve f(x, y) = c to distinguish it from the level curve f(x, y) = c in the domain of f.

Remember

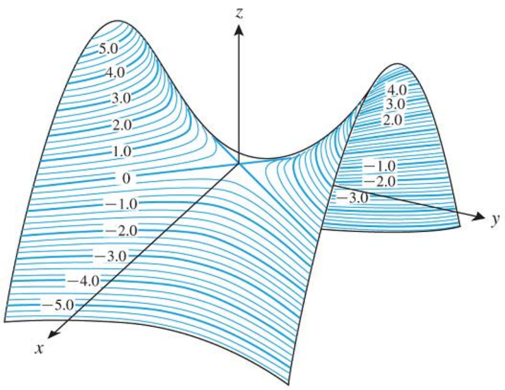
Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy-plane is called the **level curve of height k** or the **level curve with constant k** (Figure). A set of level curves for $z = f(x, y)$ is called a **contour plot** or **contour map** of *f*.

A **topographic** (or contour) map represents a three-dimensional landscape, such as a mountain range, by two-dimensional contour lines or curves of constant elevation.

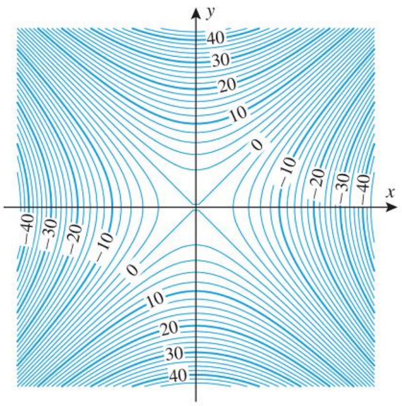


Example

The graph of the function $f(x, y) = y^2 - x^2$ in xyz-space is the hyperbolic paraboloid (saddle surface) shown in **Figure a**. The level curves have equations of the form $y^2 - x^2 = k$. For $k > 0$ these curves are hyperbolas opening along lines parallel to the y-axis; for $k < 0$ they are hyperbolas opening along lines parallel to the x-axis; and for $k = 0$ the level curve consists of the intersecting lines $y + x = 0$ and $y - x = 0$ (Figure b).



(a)



(b)

Example

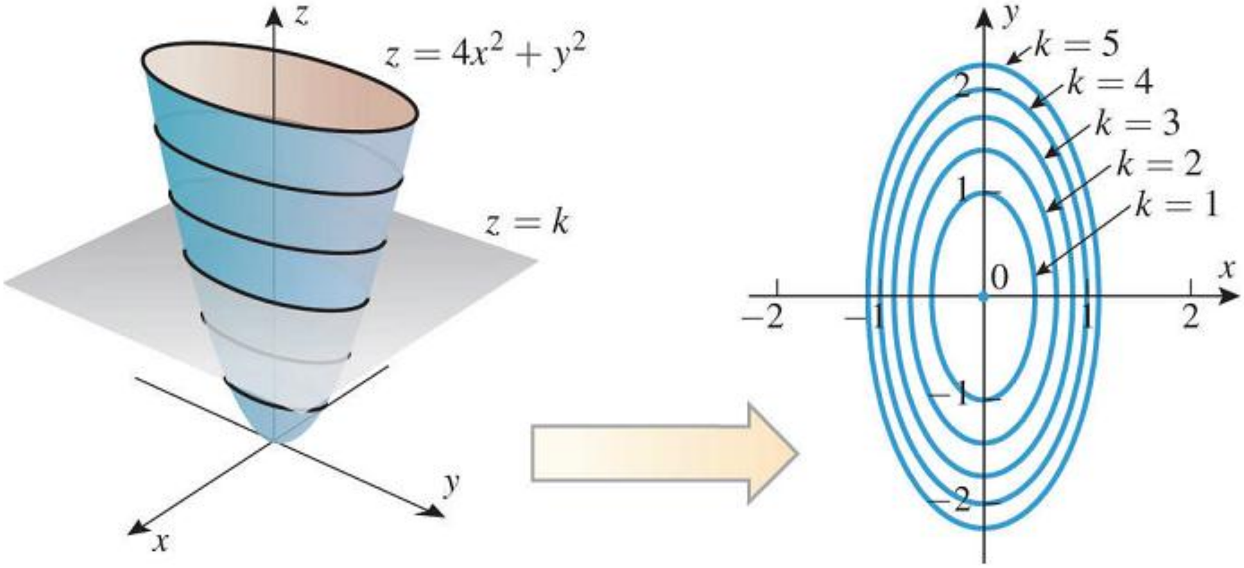
Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5.$

Solution

The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in the left part of Figure, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$.
If $k = 0$, then the graph is the single point (0, 0).
For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which represents a family of ellipses with x-intercepts $\pm\frac{\sqrt{k}}{2}$ and y-intercepts $\pm\sqrt{k}$.The contour plot for the specified values of k is shown in the right part of Figure.



Example

Let f(r, L) be the monthly payment on a 5-year car loan as a function of the interest rate r and the loan amount L. Figure is a contour plot of f(r, L). Use this plot in each part.

(a) Estimate the monthly payment on a loan of \$3000 at an interest rate of 7%.

(b) Estimate the monthly payment on a loan of \$5000 at an interest rate of 3%.

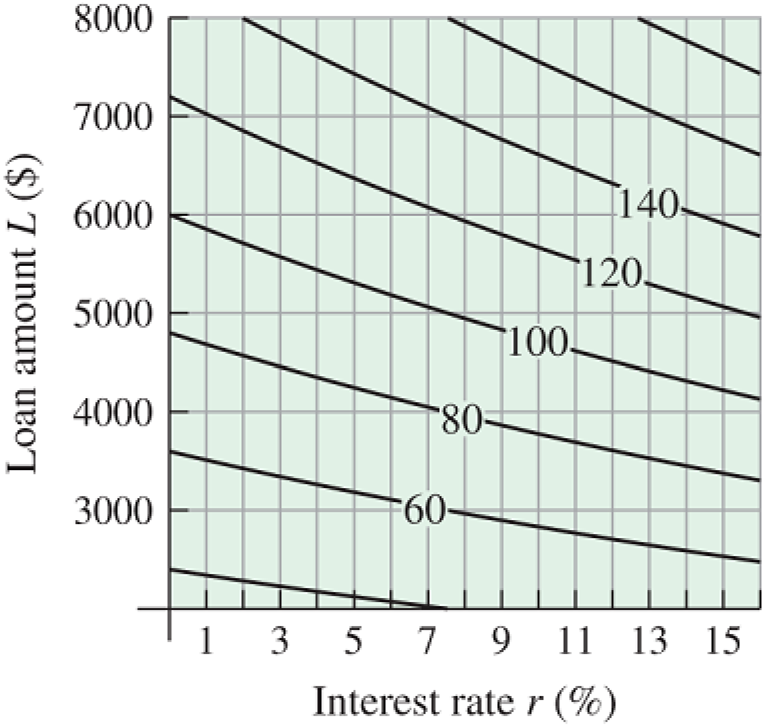
(c) Estimate the loan amount if the monthly payment is \$80 and the interest rate is 3%.

Solution

(a). Since the point (7, 3000) appears to lie on the contour labeled 60, we estimate the monthly payment to be \$60.

(b). Since the point (3, 5000) appears to be midway between the contours labeled 80 and 100, we estimate the monthly payment to be \$90.

(c). The vertical line x = 3 intersects the contour labeled 80 at a point whose L coordinate appears to be 4500. Hence, we estimate the loan amount to be \$4500.



Functions of Three or More Variables

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset R^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

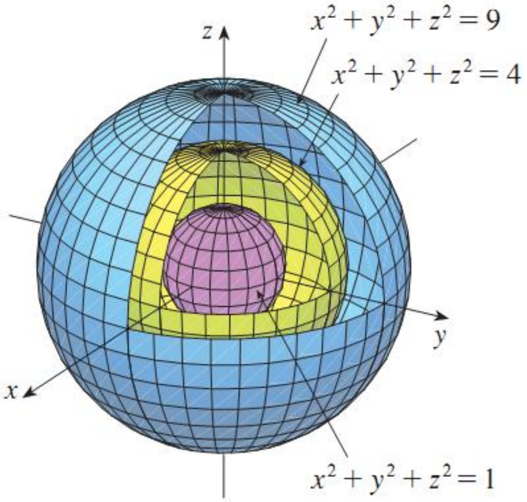
Example
Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a half-space consisting of all points that lie above the plane z = y.
Example
Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$
Solution
The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure)



Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed.

Level Surface of Functions of Three or More Variables

The set of points (x, y, z) in space where a function of three independent variables has a constant value f(x, y, z) = c is called a level surface of f.

Example

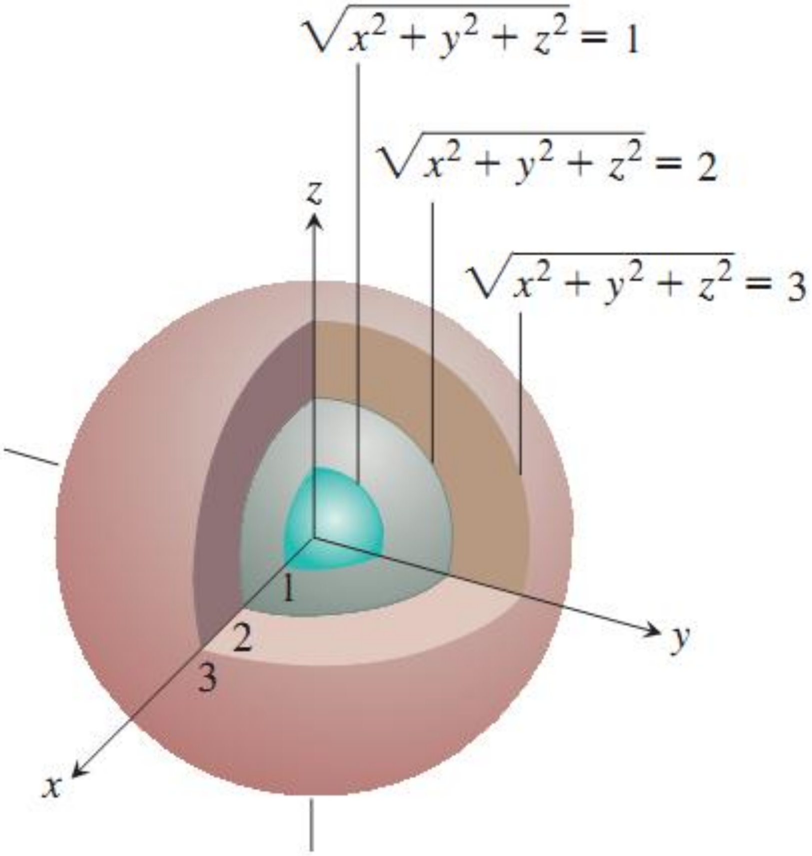
Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Solution

The value of f is the distance from the origin to the point (x, y, z). Each level surface $\sqrt{x^2 + y^2 + z^2} = c$, $c > 0$, is a sphere of radius c centered at the origin. Figure shows a cutaway view of three of these spheres. The level surface $\sqrt{x^2 + y^2 + z^2} = 0$ consists of the origin alone.

We are not graphing the function here; we are looking at level surfaces in the function's domain. The level surfaces show how the function's values change as we move through its domain. If we remain on a sphere of radius c centered at the origin, the function maintains a constant value, namely c. If we move from a point on one sphere to a point on another, the function's value changes. It increases if we move away from the origin and decreases if we move toward the origin. The way the values change depends on the direction we take.



Example

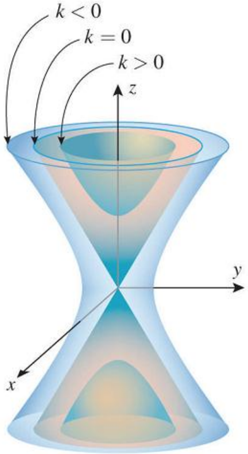
Find the level surfaces of the function $f(x,y,z) = z^2 - x^2 - y^2$

Solution

The level surfaces have equations of the form

$$z^2 - x^2 - y^2 = k$$

this equation represents a cone if k = 0, a hyperboloid of two sheets if k > 0, and a hyperboloid of one sheet if k < 0 (Figure).



Functions of n – Variables

A function of n variables, is a rule that assigns a number $z = f(x_1, x_2, x_3, \ldots, x_n)$ to an –tuple $(x_1, x_2, x_3, \ldots, x_n)$ of real numbers. We denote by $\mathbf{R}^{\mathbf{n}}$ the set of all such n – tuples.

For example, if a company uses different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, x_3, \ldots, x_n$:

$$C = f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Remember

In view of the one-to-one correspondence between points $(x_1, x_2, x_3, \ldots, x_n)$ in R^n and their position vectors $\boldsymbol{x} = \langle x_1, x_2, x_3, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of R^n:

1. As a function of real variables $x_1, x_2, x_3, \ldots, x_n$

2. As a function of a single point variable $(x_1, x_2, x_3, \ldots, x_n)$

3. As a function of a single vector variable $\boldsymbol{x} = \langle x_1, x_2, x_3, \ldots, x_n \rangle$

Limits

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write

$\text{lmt}_{(x,y)\to(a,b)}\ f(x,y) = L$

if for every number $\in > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if}\quad (x, y) \in D \quad\text{and}\quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad\text{then}\quad |f(x, y) - L| < \varepsilon$$

Remark

If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along a path C_1 and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_2 \neq L_2$, then limit does not exist.

Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad\text{and}\quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x, y) = kL \quad \text{(any number } k)$
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x, y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n}$,

   n a positive integer, and if n is even, we assume that $L > 0$.

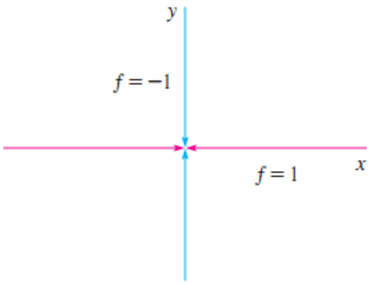
Examples

(a) $$\lim_{(x,\,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,\,y)\to(3,\,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Example

Show that $\lim_{(x,\,y)\to(0,\,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.



Solution

Given that $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

Along Horizontal Axis: put y = 0; $\lim_{(x,y)\to(x,0)} f(x,y) = \lim_{(x,y)\to(x,0)} \frac{x^2-y^2}{x^2+y^2} = 1$

Along Vertical Axis: put x = 0; $\lim_{(x,y)\to(0,y)} f(x,y) = \lim_{(x,y)\to(0,y)} \frac{x^2-y^2}{x^2+y^2} = -1$

Since we have obtained different limits along different paths, the given limit does not exist.

Example

Find $\lim_{(x,\,y)\to(0,\,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Solution

$$\lim_{(x,\,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,\,y)\to(0,0)} \frac{(x^2 - xy)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}$$

$$= \lim_{(x,\,y)\to(0,0)} \frac{x(x - y)\left(\sqrt{x} + \sqrt{y}\right)}{x - y}$$

$$= \lim_{(x,\,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right)$$

$$= 0\left(\sqrt{0} + \sqrt{0}\right) = 0$$

Example

If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution

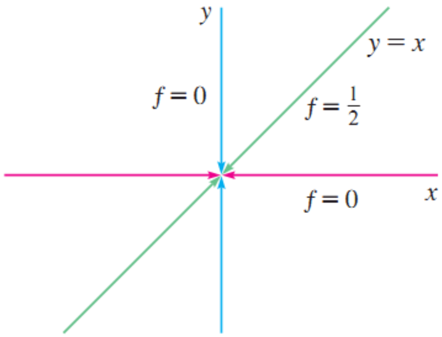
Given that $f(x,y) = \frac{xy}{x^2+y^2}$

Along Horizontal Axis: put y = 0; $\lim_{(x,y)\to(x,0)} f(x,y) = \lim_{(x,y)\to(x,0)} \frac{xy}{x^2+y^2} = 0$

Along Vertical Axis: put x = 0; $\lim_{(x,y)\to(0,y)} f(x,y) = \lim_{(x,y)\to(0,y)} \frac{xy}{x^2+y^2} = 0$

Along the line y = x; $\lim_{(x,y)\to(x,x)} f(x,y) = \lim_{(x,y)\to(x,x)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

Since we have obtained different limits along different paths, the given limit does not exist.



Example

If $f(x, y) = \frac{xy^2}{x^2 + y^4}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution

Given that $f(x,y) = \frac{xy^2}{x^2+y^4}$

Along the line y = mx; $\lim_{(x,y)\to(x,mx)} f(x,y) = \lim_{(x,y)\to(x,mx)} \frac{xy^2}{x^2+y^4} = \frac{m^2x}{1+m^4x^2}$

Thus f has the same limiting value along every nonvertical line through the origin. But that does not show that the given limit is 0, for if we now let $(x,y) \to (0,0)$ along the parabola $x = y^2$, we have

$\lim_{(x,y)\to(y^2,mx)} f(x,y) = \lim_{(x,y)\to(y^2,mx)} \frac{m^2x}{1+m^4x^2} = \frac{1}{2}$

Since different paths lead to different limiting values, the given limit does not exist.

Example

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Solution

Let $\in > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$$

that is, $$\text{if} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2+y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2+y^2) \leqslant 1$ and therefore

$$\frac{3x^2|y|}{x^2+y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2+y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| \leqslant 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

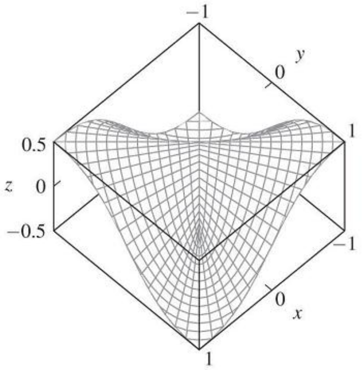
Example

Figure shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis
(b) the y-axis
(c) the line $y = x$
(d) the line $y = -x$
(e) the parabola $y = x^2$



Solution

Solution (a). The x-axis has parametric equations $x = t, y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=0)}} f(x, y) = \lim_{t\to 0} f(t, 0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure given below.

Solution (b). The y-axis has parametric equations $x = 0, y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} f(x, y) = \lim_{t\to 0} f(0, t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure given below.

Solution (c). The line $y = x$ has parametric equations $x = t, y = t$, with $(0, 0)$ corresponding to $t = 0$, so

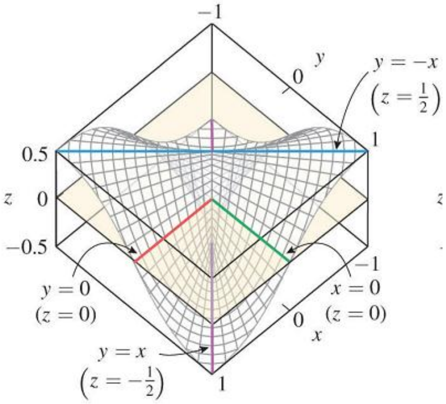
$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} f(x, y) = \lim_{t\to 0} f(t, t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure given below.

Solution (d). The line $y = -x$ has parametric equations $x = t, y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x, y) = \lim_{t\to 0} f(t, -t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure given below.



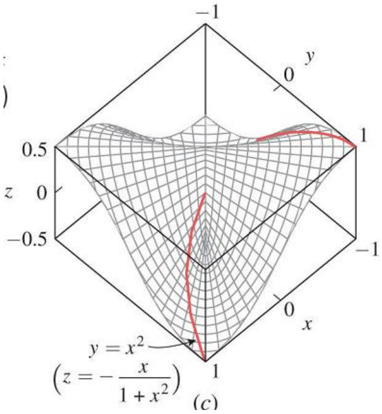
Solution (e). The parabola $y = x^2$ has parametric equations $x = t, y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2 + t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1 + t^2}\right) = 0$$

This is consistent with Figure, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1 + t^2}$$

superimposed on the surface.



Example

Evaluate

$$\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9]$$

Solution

$$\begin{aligned}\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] &= \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9 \\ &= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3\left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9 \\ &= 5(1)^3(4)^2 - 9 = 71 \;\blacktriangleleft\end{aligned}$$

Relationships between General Limits and Limits Along Smooth Curves

(a) If f(x, y)→L as (x, y)→(x_0, y_0), then f(x, y)→L as (x, y)→(x_0, y_0) along any smooth curve.

(b) If the limit of f(x, y) fails to exist as (x, y)→(x_0, y_0) along some smooth curve, or if f(x, y) has different limits as (x, y)→(x_0, y_0) along two different smooth curves, then the limit of f(x, y) does not exist as (x, y)→(x_0, y_0).

Example

Show that limit exists or not for

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2 + y^2}$$

Solution

The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2 + y^2}$$

does not exist because we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x = 0)}} -\frac{xy}{x^2 + y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y = x)}} -\frac{xy}{x^2 + y^2} = -\frac{1}{2}$$

Example

Find $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution

Let (r,θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2+y^2}$, so that $r \to 0^+$ if and only if $(x, y) \to (0, 0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0+} r^2 \ln r^2 \\
&= \lim_{r\to 0+} \frac{2\ln r}{1/r^2} && \boxed{\text{This converts the limit to an indeterminate form of type } \infty/\infty.} \\
&= \lim_{r\to 0+} \frac{2/r}{-2/r^3} && \boxed{\text{L'Hôpital's rule}} \\
&= \lim_{r\to 0+} (-r^2) = 0 \quad \blacktriangleleft
\end{aligned}$$

Continuity
A function f of two variable is said to be continuous at (a,b) if
$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
We say f is continuous on D if f is continuous at every point (a,b) in D.
In addition, if f is continuous at every point in an open set D, then we say that f is continuous on D, and if f is continuous at every point in the xy-plane, then we say that f is **continuous everywhere**.

Example

Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Solution
Since this is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

Example

Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Solution
The function f is discontinuous at (0,0) because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set

$$D = \{(x,y) \mid (x,y) \neq (0,0)\}$$

Example (Previously Solved)
Discuss the continuity of

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution
Here g is defined at (0,0) but is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

Example (Previously Solved)
Discuss the continuity of

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution

We know f is continuous for (x,y) ≠ (0,0) since it is equal to a rational function there. Also, we have

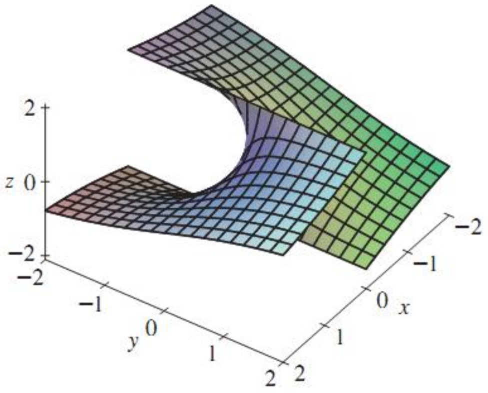
$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at (0,0), and so it is continuous on R^2.

Example
Where is the function h(x,y) = arctan(y/x) continuous?
Solution
The function $f(x, y) = \frac{y}{x}$ is a rational function and therefore continuous except on the line x = 0. The function g(t) = arctan(t) is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where . The graph in Figure shows the break in the graph of above the -axis.



The following theorem, which we state, illustrates some of the ways in which continuous functions can be combined to produce new continuous functions.

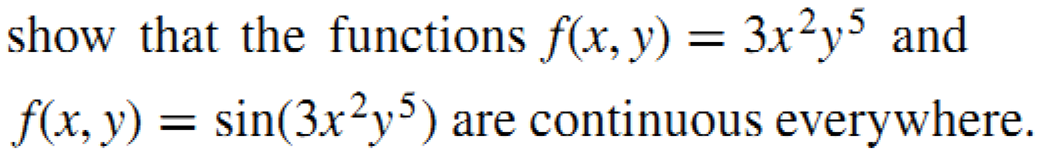
Theorem

(a) If g(x) is continuous at x_0 and h(y) is continuous at y_0, then f(x, y) = g(x)h(y) is continuous at (x_0, y_0).

(b) If h(x, y) is continuous at (x_0, y_0) and g(u) is continuous at $u = h(x_0, y_0)$, then the composition f (x, y) = g(h(x, y)) is continuous at (x_0, y_0).

(c) If f (x, y) is continuous at (x_0, y_0), and if x(t) and y(t) are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition f (x(t), y(t)) is continuous at t_0.

Example



show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution

The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by Theorem,

"If g(x) is continuous at x_0 and h(y) is continuous at y_0, then f(x, y) = g(x)h(y) is continuous at (x_0, y_0)."

the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane.

Since $3x^2y^5$ is continuous at every point in the xy-plane and sin u is continuous at every real number u, it follows from Theorem

"If h(x, y) is continuous at (x_0, y_0) and g(u) is continuous at $u = h(x_0, y_0)$, then the composition f (x, y) = g(h(x, y)) is continuous at (x_0, y_0)."

that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere.

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Example

Evaluate $\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2}$

Solution

Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = -\frac{2}{5}$

Example

Check the continuity of $f(x,y) = \frac{x^3y^2}{1-xy}$

Solution

Since the function $f(x,y) = \frac{x^3y^2}{1-xy}$ is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, f(x, y) is continuous everywhere except on the hyperbola xy = 1.

Example

Show that $f\ (x, y)\ =\ \sqrt{1\ -\ x^2 - y^2}$ is continuous at its domain.

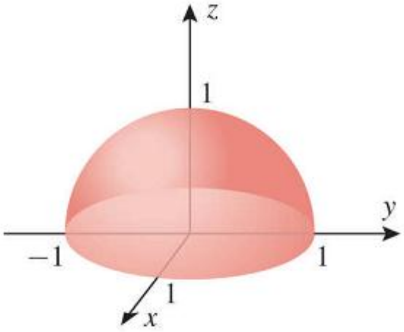
Solution

The graph of the function $f\ (x, y)\ =\ \sqrt{1\ -\ x^2 - y^2}$ is the upper hemi-sphere shown in Figure, and the natural domain of f is the closed unit disk $x^2 + y^2 \le 1$ The graph of f has no jumps, tears or holes, so it passes our "intuitive test" of continuity.

In this case the continuity at a point (x_0, y_0) on the boundary reflects the fact that

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{1 - x^2 - y^2} = \sqrt{1 - x_0^2 - y_0^2} = 0$$

when (x, y) is restricted to points on the closed unit disk $x^2 + y^2 \le 1$. It follows that f is continuous on its domain.

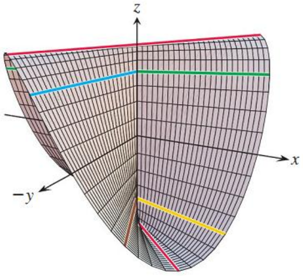


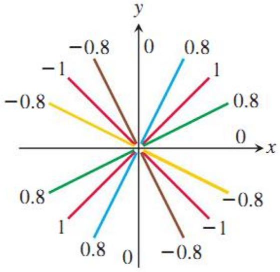
Example

Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure).





Solution

The function *f* is continuous at every point (x, y) except (0, 0) because its values at points other than (0, 0) are given by a rational function of x and y, and therefore at those points the limiting value is simply obtained by substituting the values of x and y into that rational expression.

At (0, 0), the value of *f* is defined, but *f* has no limit as (x, y) → (0, 0). The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m, the function *f* has a constant value on the “punctured” line y = mx, x≠0, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, *f* has this number as its limit as (x, y) approaches (0, 0) along the line:

$$\lim_{\substack{(x, y)\to(0,0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with each value of the slope m. There is therefore no single number we may call the limit of *f* as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous at the origin.

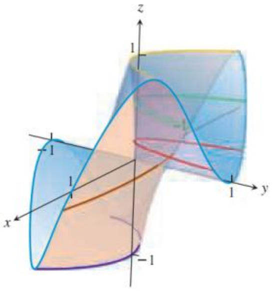
Two-Path Test for Nonexistence of a Limit

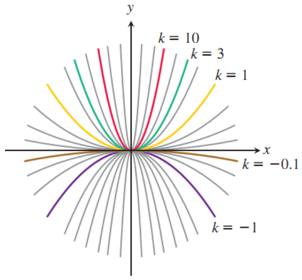
If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,\, y)\to(x\, ,\, y\,)} f(x, y)$ does not exist.

Example
Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

has no limit as (x, y) approaches (0, 0).





Solution
The limit cannot be found by direct substitution, which gives the indeterminate form 0 / 0. We examine the values of f along parabolic curves that end at (0, 0). Along the curve $y = kx^2$, x 0, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x,\,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,\,y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches (0, 0) along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches (0, 0) along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches (0, 0).

Continuity of Function with three Variables

A function f of three variable is said to be continuous at (a,b,c) if

$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b)$

We say f is continuous on D if f is continuous at every point (a,b,c) in D.

Or

For every number $\in > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y, z) \text{ is in the domain of } f \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$

$$\text{then} \quad |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z)\to(a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in R^3 except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

Remark

If we use the vector notation, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Partial Derivatives

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations for Partial Derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$

Solution

$$f_x(x, y) = 3x^2 + 2xy^3$$

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example

Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x,y) = 2x^3y^2 + 2y + 4x$.

Solution

Since

$$f_x(x,3) = \frac{d}{dx}[f(x,3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

we have $f_x(1,3) = 54 + 4 = 58$. Also, since

$$f_y(1,y) = \frac{d}{dy}[f(1,y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

we have $f_y(1,3) = 4(3) + 2 = 14$.

Example

Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y) = 2x^3y^2 + 2y + 4x$, and use those partial derivatives to compute $f_x(1,3)$ and $f_y(1,3)$.

Solution

$$f_x(x,y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

$$f_y(x,y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

$$f_x(1,3) = 6(1^2)(3^2) + 4 = 58 \quad \text{and} \quad f_y(1,3) = 4(1^3)3 + 2 = 14$$

Example

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

Example

Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Example

Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

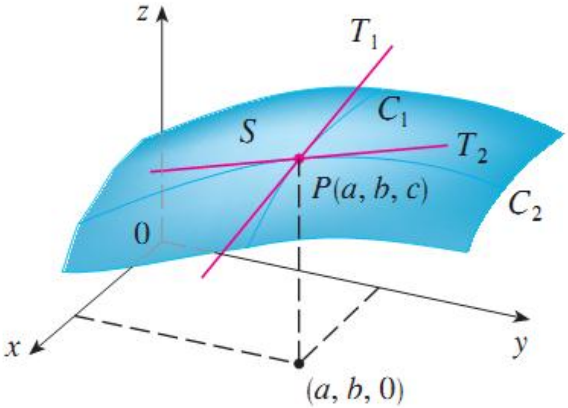
$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$

Interpretations of Partial Derivatives

The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes y = b and x = a.



Remark

Partial derivatives can also be interpreted as rates of change. If z = f(x,y) , then $\frac{\partial z}{\partial x}$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\frac{\partial z}{\partial y}$ represents the rate of change of z with respect to y when x is fixed.

Example

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

Solution

We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

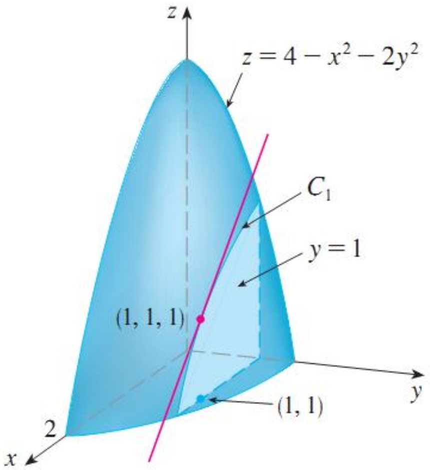
$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

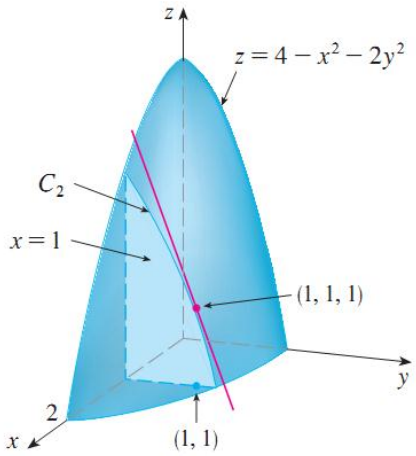
The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane y = 1 intersects it in the parabola $z = 2 - x^2$, y = 1. (See Figure 1.)

The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1,1) = -2$.

Similarly, the plane x = 1 intersects the paraboloid is the parabola $z = 3 - 2y^2$, x = 1 , and the slope of the tangent line at (1, 1, 1) is $f_y(1,1) = -4$.

(See Figure 2.)





Example

If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution

Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Solution

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Example

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

Solution

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4)$$

$$= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4)$$

$$= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5 y^2 \cos(xy^3) \quad \blacktriangleleft$$

Example

Recall that the wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of W with respect to v at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.

Solution

$$\frac{\partial W}{\partial v}(T, v) = 0 + 0 + (0.4275T - 35.75)(0.16)v^{0.16-1} = (0.4275T - 35.75)(0.16)v^{-0.84}$$

$$\frac{\partial W}{\partial v}(25, 10) = (-4.01)10^{-0.84} \approx -0.58 \ \frac{{}^\circ\mathrm{F}}{\mathrm{mi/h}}$$

Example

Let $f(x, y) = x^2 y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(1,-2)$.
(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(1,-2)$.

Solution

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1,-2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1,-2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y.

Example

Let

$$f(x,y)=\begin{cases}-\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(a) Show that $f_x(x, y)$and $f_y(x, y)$ exist at all points (x, y).

(b) Explain why f is not continuous at (0, 0).

Solution

$$f_x(x,y)=-\frac{(x^2+y^2)y-xy(2x)}{(x^2+y^2)^2}=\frac{x^2y-y^3}{(x^2+y^2)^2}$$

$$f_y(x,y)=-\frac{(x^2+y^2)x-xy(2y)}{(x^2+y^2)^2}=\frac{xy^2-x^3}{(x^2+y^2)^2}$$

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

This shows that f has partial derivatives at (0, 0) and the values of both partial derivatives are 0 at that point.

To explain why *f* is not continuous at (0, 0).

$$\lim_{(x,y)\to(0,0)}-\frac{xy}{x^2+y^2}$$

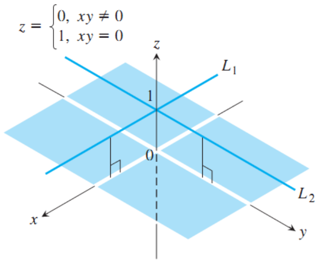
$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}}-\frac{xy}{x^2+y^2}=0 \quad\text{and}\quad \lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}}-\frac{xy}{x^2+y^2}=-\frac{1}{2}$$

Thus limit does not exist.

Hence, f is not continuous at (0, 0).

Example

Let



$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(a) Find the limit of f as (x, y) approaches (0, 0) along the line y = x.
(b) Prove that f is not continuous at the origin.
(c) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Solution

(a) Since f(x, y) is constantly zero along the line y = x (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(b) Since f(0, 0) = 1, the limit in part (a) is not equal to f(0, 0), which proves that f is not continuous at (0, 0).

(c) To find $\partial f/\partial x$ at (0, 0), we hold y fixed at y = 0. Then f(x, y) = 1 for all x, and the graph of f is the line L_1 in Figure. The slope of this line at any x is $\partial f/\partial x$ = 0. In particular, $\partial f/\partial x$ = 0 at (0, 0).
Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y, so $\partial f/\partial x$ = 0 at (0, 0).

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

Generally we may write

$$\frac{\partial u}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution

$$f_x = ye^{xy} \ln z$$

$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$

Example

If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then

$$f_x(x, y, z) = 3x^2y^2z^4 + 2y$$
$$f_y(x, y, z) = 2x^3yz^4 + 2x$$
$$f_z(x, y, z) = 4x^3y^2z^3 + 1$$
$$f_z(-1, 1, 2) = 4(-1)^3(1)^2(2)^3 + 1 = -31$$

Example

If $f(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$, then

$$f_\rho(\rho, \theta, \phi) = 2\rho \cos\phi \sin\theta$$
$$f_\theta(\rho, \theta, \phi) = \rho^2 \cos\phi \cos\theta$$
$$f_\phi(\rho, \theta, \phi) = -\rho^2 \sin\phi \sin\theta$$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f.

If $z = f(x,y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\frac{\partial^2 f}{\partial y \partial x}$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example

Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3$$

Remember

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example

Let $f(x, y) = y^2e^x + y$.

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x$$

Example

If $f(x, y) = x\cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y\,\partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x\,\partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$

Example

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Example

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution

$f_y = -4xyz + x^2$, $f_{yx} = -4yz + 2x$, $f_{yxy} = -4z$

$f_{yxyz} = -4.$

Clairaut′s Theorem/ The Mixed Derivative Theorem

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

In need we may also use

$$f_{xyy} = f_{yxy} = f_{yyx}$$

Example

Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Solution

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

Example

Find $\dfrac{\partial^2 w}{\partial x\,\partial y}$ if $w = xy + \dfrac{e^y}{y^2 + 1}$.

Solution

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y\,\partial x} = 1.$$

Partial Differential Equations

- Laplace's equation after Pierre Laplace (1749–1827). Solutions of this equation are called harmonic functions; they play a role in problems of heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- The wave equation describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

- Heat conduction equation

$$u_t = \alpha^2 u_{xx}$$

Example

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

Example

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

Solution

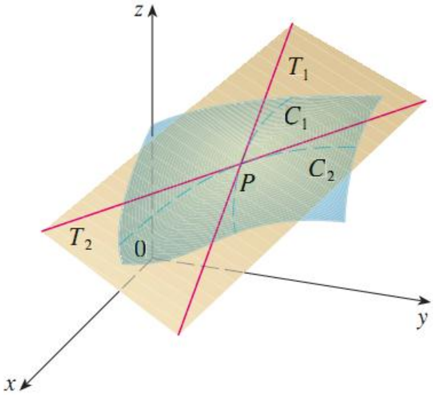
$$u_x = \cos(x - at) \qquad u_t = -a\cos(x - at)$$

$$u_{xx} = -\sin(x - at) \qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

Therefore u satisfies wave equation.

Tangent Planes

A tangent plane is a flat surface that touches a curved surface at a single point, called the point of tangency. It is a plane that contains all the tangent lines to a surface at that point.



Equation of Tangent Plane to a Surface

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the equation of the tangent plane at $(1,1,3)$ is

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$z = 4x + 2y - 3$$

Linearization/ Local Linear Approximations

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b).

Linear Approximations

The linear approximation of a function is approximating the value of the function at a point using a line.

Or the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent plane approximation of f at (a, b).

Example

Find the Linearization and Linear Approximation to the paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the equation of the tangent plane at $(1,1,3)$ is

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$z = 4x + 2y - 3$$

Then Linearization is

$$L(x, y) = 4x + 2y - 3$$

And Linear Approximation is

$$f(x, y) \approx 4x + 2y - 3$$

Example

Find the Linearization of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point $(3,2)$.

Solution

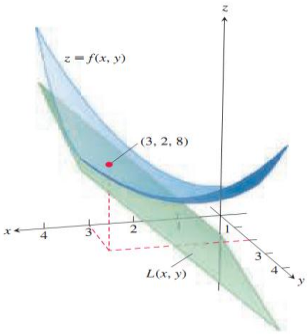
$$f(3,2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8$$

$$f_x(3,2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (2x - y)\Big|_{(3,2)} = 4$$

$$f_y(3,2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (-x + y)\Big|_{(3,2)} = -1,$$

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of *f* at (3, 2) is L(x, y) = 4x - y – 2



The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

satisfies the inequality

$$|E(x,y)| \le \frac{1}{2}M\left(|x - x_0| + |y - y_0|\right)^2.$$

To make $|E(x,y)|$ small for a given M, we just make $|x - x_0|$ 0 and $|y - y_0|$small.

Example

Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3\sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region

$$R: \quad |x - 2| \leq 0.01, \qquad |y - 1| \leq 0.02, \qquad |z| \leq 0.01.$$

Solution

$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$

Thus,

$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$

Since

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$

and $|-3\sin z| \leq 3\sin 0.01 \approx 0.03$, we may take $M = 2$ as a bound on the second partials. Hence, the error incurred by replacing f by L on R satisfies

$$|E| \leq \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016.$$

Example

Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(3, 4)$. Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the points $(3, 4)$ and $(3.04, 3.98)$.

Solution

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

with $f_x(3, 4) = \frac{3}{5}$ and $f_y(3, 4) = \frac{4}{5}$. Therefore, the local linear approximation to f at $(3, 4)$ is given by

$$L(x, y) = 5 + \tfrac{3}{5}(x - 3) + \tfrac{4}{5}(y - 4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \tfrac{3}{5}(0.04) + \tfrac{4}{5}(-0.02) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045$$

between the points $(3, 4)$ and $(3.04, 3.98)$. ◀

Increment

If $z = f(x, y)$, then the increment of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Differentiable Function/ Differentiability

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Or

A function f of two variables is said to be differentiable at (x_0, y_0) provided $f_x(x_0,y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Total Differential of a Function

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

in the linearization of f is called the **total differential of *f***.

Example

Prove that $f(x, y) = x^2 + y^2$ is differentiable at (0, 0).

Solution

The increment is

$$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = (\Delta x)^2 + (\Delta y)^2$$

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, we have $f_x(0, 0) = f_y(0, 0) = 0$, and (4) becomes

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0$$

Therefore, f is differentiable at (0, 0).

Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example

Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable. The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$$

Differentials

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the differential dz, also called the total differential, is defined by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Or

A function f of three variables is said to be differentiable at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0$$

Example

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution

(a)

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

Example

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1cm in each. Use differentials to estimate the maximum error in the calculated cone.

Solution

The volume V of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = \frac{2\pi rh}{3}\,dr + \frac{\pi r^2}{3}\,dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leqslant 0.1$, $|\Delta h| \leqslant 0.1$. To estimate the largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about 20π cm^3 $\approx$ 63 cm^3. ■

Functions of Three or More Variables

For such functions the linear approximation is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization L(x,y,z) is the right side of this expression.

If $w = f(x, y, z)$, then the increment of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The differential dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

Example

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution

If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \leqslant 0.2$, $|\Delta y| \leqslant 0.2$, and $|\Delta z| \leqslant 0.2$. To estimate the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

Remark

- If a function f of two variables is differentiable at each point of a region R in the xy- plane, then we say that f is differentiable on R; and if f is differentiable at every point in the xy-plane, then we say that f is differentiable everywhere. For a function f of three variables we have corresponding conventions.
- If a function is differentiable at a point, then it is continuous at that point.
- If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.
- $\mathbf{dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy}$ is called total differential of z at (x_0,y_0) or as the total differential of f at (x_0, y_0).

Example

Approximate the change in z = xy^2 from its value at (0.5, 1.0) to its value at (0.503,1.004). Compare the magnitude of the error in this approximation with the distance between the points (0.5, 1.0) and (0.503, 1.004).

Solution

For $z = xy^2$ we have $dz = y^2\,dx + 2xy\,dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1.0 = 0.004$ yields

$$dz = 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007$$

Since $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$

and the error in approximating Δz by dz has magnitude

$$|dz - \Delta z| = |0.007 - 0.007032048| = 0.000032048$$

Since the distance between (0.5, 1.0) and $(0.503, 1.004) = (0.5 + \Delta x, 1.0 + \Delta y)$ is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005$$

we have

$$\frac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.0064096 < \frac{1}{150}$$

Thus, the magnitude of the error in our approximation is less than $\frac{1}{150}$ of the distance between the two points.

Example

The length, width, and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

Solution

The diagonal D of a box with length x, width y, and height z is given by

$$D = \sqrt{x^2 + y^2 + z^2}$$

Let x_0, y_0, z_0, and $D_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ denote the actual values of the length, width, height, and diagonal of the box. The total differential dD of D at (x_0, y_0, z_0) is given by

$$dD = \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dx + \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dy + \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dz$$

If x, y, z, and $D = \sqrt{x^2 + y^2 + z^2}$ are the measured and computed values of the length, width, height, and diagonal, respectively, then

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta z = z - z_0$$

and

$$\left|\frac{\Delta x}{x_0}\right| \leq 0.05, \quad \left|\frac{\Delta y}{y_0}\right| \leq 0.05, \quad \left|\frac{\Delta z}{z_0}\right| \leq 0.05$$

$$\begin{aligned}\frac{\Delta D}{D_0} \approx \frac{dD}{D_0} &= \frac{1}{x_0^2 + y_0^2 + z_0^2}[x_0 \Delta x + y_0 \Delta y + z_0 \Delta z] \\ &= \frac{1}{x_0^2 + y_0^2 + z_0^2}\left[x_0^2 \frac{\Delta x}{x_0} + y_0^2 \frac{\Delta y}{y_0} + z_0^2 \frac{\Delta z}{z_0}\right]\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{dD}{D_0}\right| &= \frac{1}{x_0^2 + y_0^2 + z_0^2}\left|x_0^2 \frac{\Delta x}{x_0} + y_0^2 \frac{\Delta y}{y_0} + z_0^2 \frac{\Delta z}{z_0}\right| \\ &\leq \frac{1}{x_0^2 + y_0^2 + z_0^2}\left(x_0^2\left|\frac{\Delta x}{x_0}\right| + y_0^2\left|\frac{\Delta y}{y_0}\right| + z_0^2\left|\frac{\Delta z}{z_0}\right|\right) \\ &\leq \frac{1}{x_0^2 + y_0^2 + z_0^2}\left(x_0^2(0.05) + y_0^2(0.05) + z_0^2(0.05)\right) = 0.05\end{aligned}$$

we estimate the maximum percentage error in D to be 5%.

The Chain Rule (Case – I)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where x = g(t) and y = h(t)are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Since we often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Proof

A change of Δt in Δt produces changes of Δx in x and Δy in y. These, in turn, produce a change of Δz in z, and from we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. [If the functions ε_1 and ε_2 are not defined at $(0, 0)$, we can define them to be 0 there.] Dividing both sides of this equation by Δt, we have

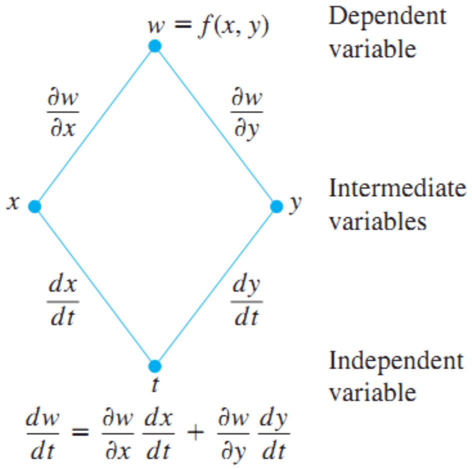
$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t + \Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\
&= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}
\end{aligned}$$

The dependency Diagram

To remember the Chain Rule, picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.



The dependency diagram on the preceding page provides a convenient way to remember the Chain Rule. The “true” independent variable in the composite function is t, whereas x and y are intermediate variables (controlled by t) and w is the dependent variable.

Example

Use the Chain Rule to find the derivative of w = xy with respect to t along the path x = cost, y = sint. What is the derivative’s value at t $= \frac{\pi}{2}$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t)$$

$$= (y)(-\sin t) + (x)(\cos t) = (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t.$$

Or

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}(2\cos 2t) = \cos 2t.$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2\frac{\pi}{2}\right) = \cos \pi = -1.$$

Example

Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of w(t) are changing along the path of a helix as t changes. What is the derivative's value at t = 0?

Solution

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1)\\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1\\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

$$\left.\frac{dw}{dt}\right|_{t=0} = 1 + \cos(0) = 2.$$

For a physical interpretation of change along a curve, think of an object whose position is changing with time t. If w = T(x, y, z) is the temperature at each point (x, y, z) along a curve C with parametric equations x = x(t), y = y(t), and z = z(t), then the composite function w = T(x(t), y(t), z(t)) represents the temperature relative to t along the curve.

The derivative dw/ dt is then the instantaneous rate of change of temperature due to the motion along the curve.

Example

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)\\ &= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r\end{aligned}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.$$

Example

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\\ &= (2x)(1) + (2y)(1)\\ &= 2(r - s) + 2(r + s)\\ &= 4r\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}\\ &= (2x)(-1) + (2y)(1)\\ &= -2(r - s) + 2(r + s)\\ &= 4s\end{aligned}$$

Example

Suppose that

$$z = x^2 y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find dz/dt, and check the result by expressing z as a function of t and differentiating directly.

Solution

By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)$$

$$= (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

Example

Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$$

Use the chain rule to find dw/dθ when θ = π/4.

Solution

By the chain rule

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x)(-\sin\theta) + \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y)(\cos\theta)$$

$$+ \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z)(\sec^2\theta)$$

When θ = π/4, we have

$$x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad z = \tan\frac{\pi}{4} = 1$$

Substituting $x = 1/\sqrt{2}, y = 1/\sqrt{2}, z = 1, \theta = \pi/4$ in the formula for $dw/d\theta$ yields

$$\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(2)(2)$$

$$= \sqrt{2} \blacktriangleleft$$

Example

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Solution

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$

Example

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Solution

If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

The Chain Rule for Partial Derivatives (Case – II)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where x = g(s,t) and y = h(s,t) are both differentiable functions of t. Then z is a differentiable function of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)$$

The Chain Rule (General Version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example

Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v$$

find ∂z/∂u and ∂z/∂v using the chain rule.

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right] e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u+v}{v}\right] e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right] e^{(2u+v)(u/v)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$

$$= \left[y - x\left(\frac{u}{v^2}\right)\right] e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right] e^{(2u+v)(u/v)}$$

$$= -\frac{2u^2}{v^2} e^{(2u+v)(u/v)}$$ ◀

Example

Given that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Use appropriate forms of the chain rule to find ∂w/∂u and ∂w/∂v.

Solution

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

Example

Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Use appropriate forms of the chain rule to find ∂w/∂ρ and ∂w/∂θ.

Solution

$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x \sin\phi \cos\theta + 2y \sin\phi \sin\theta - 2z \cos\phi \\
&= 2\rho \sin^2\phi \cos^2\theta + 2\rho \sin^2\phi \sin^2\theta - 2\rho \cos^2\phi \\
&= 2\rho \sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho \cos^2\phi \\
&= 2\rho(\sin^2\phi - \cos^2\phi) \\
&= -2\rho \cos 2\phi
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial \theta} &= (2x)(-\rho \sin\phi \sin\theta) + (2y)\rho \sin\phi \cos\theta \\
&= -2\rho^2 \sin^2\phi \sin\theta \cos\theta + 2\rho^2 \sin^2\phi \sin\theta \cos\theta \\
&= 0
\end{aligned}$$

This result is explained by the fact that w does not vary with θ. You can see this directly by expressing the variables x, y, and z in terms of ρ, φ, and θ in the formula for w. (Verify that $w = -\rho^2 \cos 2\varphi$.)

Example

Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find dw/dx.

Solution

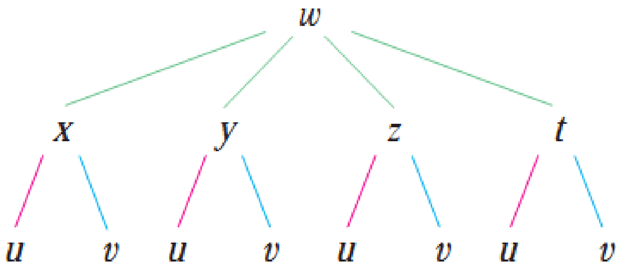
$$\begin{aligned}
\frac{dw}{dx} &= y + (x + z)\cos x + ye^x \\
&= \sin x + (x + e^x)\cos x + e^x \sin x
\end{aligned}$$

Example

Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Solution

We apply The Chain Rule (General Version) with n = 4 and m = 2. Figure shows the tree diagram.



Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\frac{\partial y}{\partial u}$. With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

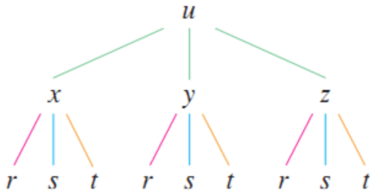
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Example

If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

Solution

With the help of the tree diagram in Figure, we have



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

Example

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Solution

Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0 \qquad \blacksquare$$

Example

If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

Solution

(a) The Chain Rule gives

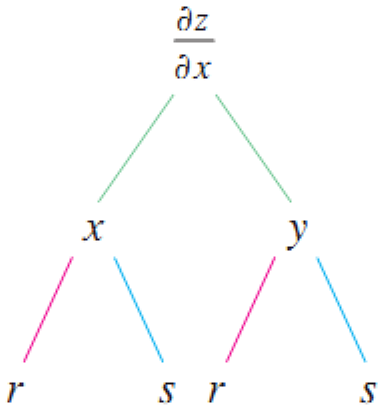
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Applying the Product Rule to the expression in part (a), we get

$$\boxed{5} \qquad \frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

But, using the Chain Rule again (see Figure),



we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\,\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\,\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\,\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\,\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Implicit Differentiation

Implicit differentiation makes use of the chain rule to differentiate a function which cannot be explicitly expressed in the form y= f(x).

It is defined as follows

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

The Implicit Function Theorem

The Implicit Function Theorem, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a,b) where, F(a,b) = 0, $F_y(a,b) \neq 0$ and F_x and F_y are continuous on the disk, then the equation F(x,y) = 0 defines y as a function of x near the point (a,b) and the derivative of this function is given by Equation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

We may also write as follows

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example

Find y' if $x^3 + y^3 = 6xy$.

Solution

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Example

Find $\partial z/\partial x$ assuming that the equation $yz - ln\, z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

Example

The plane x = 1 intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5).

Solution

The parabola lies in a plane parallel to the yz-plane, and the slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane x = 1 and ask for the slope at y = 2. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$.

Then, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Example

Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

Solution

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

Substituting the y- and z-coordinates of the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ in this expression, we find that the slope at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ is $-\frac{1}{2}$ and the slope at $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ is $\frac{1}{2}$.

Example

Given that

$$x^3 + y^2x - 3 = 0$$

find dy/dx, and check the result using implicit differentiation.

Solution

$$f(x, y) = x^3 + y^2x - 3,$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Example

Suppose that D = $\sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where x = 3and y = 4.

Solution

$$D^2 = x^2 + y^2$$

$$2D\frac{\partial D}{\partial x} = 2x \quad \text{and thus} \quad D\frac{\partial D}{\partial x} = x$$

$$5\left.\frac{\partial D}{\partial x}\right|_{x=3,y=4} = 3 \quad \text{or} \quad \left.\frac{\partial D}{\partial x}\right|_{x=3,y=4} = \frac{3}{5}$$

Thus, D is increasing at a rate of $\frac{3}{5}$ unit per unit increase in x at the point (3, 4).

Example

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

Example

If x, y, and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

Solution

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x \sin(y + 3z)\right] = x \frac{\partial}{\partial z}\sin(y + 3z)$$

$$= x \cos(y + 3z)\frac{\partial}{\partial z}(y + 3z)$$

$$= 3x \cos(y + 3z).$$

Example

If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find the value of $\frac{\partial R}{\partial R_2}$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution

To find $\frac{\partial R}{\partial R_2}$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2{}^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2{}^2} = \left(\frac{R}{R_2}\right)^2.$$

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3 + 2 + 1}{90} = \frac{6}{90} = \frac{1}{15}$$

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Thus at the given values, a small change in the resistance R_2 leads to a change in R about 1/9th as large.

Example

$$F(x, y) = y^2 - x^2 - \sin xy.$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

Example

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at (0, 0, 0) if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Solution

Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad \text{and} \qquad F_z = 2z + xye^{xz} + \cos y.$$

Since $F(0, 0, 0) = 0$, $F_z(0, 0, 0) = 1 \neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x, y, z) = 0$ defines z as a differentiable function of x and y near the point (0, 0, 0). From Equations (2),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At (0, 0, 0) we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

Directional Derivatives

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Geometrically, $D_u f(x_0, y_0)$ can be interpreted as the slope of the surface $z = f(x,y)$ in the direction of u at the point $(x_0, y_0, f(x_0, y_0))$

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Proof

If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\boxed{4} \qquad g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$
$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\, a + f_y(x, y)\, b$$

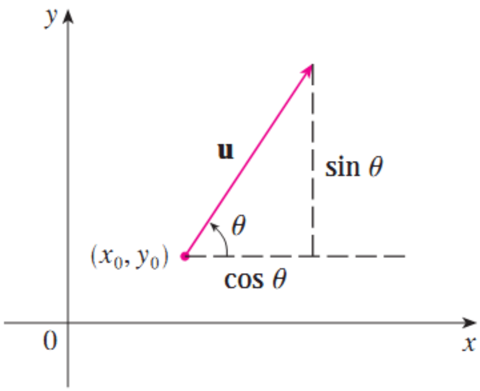
If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\boxed{5} \qquad g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b \qquad \blacksquare$$

If the unit vector **u** makes an angle θ with the positive -axis (as in Figure),



then we can write **u** = $\langle cos\theta, sin\theta\rangle$ and the formula in Previous Theorem becomes

$$D_{\mathbf{u}}\, f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Example

Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and **u** is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}\, f(1, 2)$?

Solution

$$\begin{aligned} D_{\mathbf{u}}\, f(x, y) &= f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6} \\ &= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2} \\ &= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore

$$D_{\mathbf{u}}\, f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}\,(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

Example

Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$.

Solution

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s\to 0} \frac{f\left(1 + s\cdot\frac{1}{\sqrt{2}}, 2 + s\cdot\frac{1}{\sqrt{2}}\right) - f(1, 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1\cdot 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$.

Example

Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Solution

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right)\right]_{s=0}$$

$$f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right) = \left(1 + \frac{\sqrt{3}s}{2}\right)\left(2 + \frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2$$

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds}\left[\frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2\right]_{s=0}$$

$$= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0} = \frac{1}{2} + \sqrt{3}$$

Since $\frac{1}{2} + \sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point (1, 2) in the direction of u, the function f(x, y) = xy will increase by about 2.23 times the distance moved.

Example
Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.
Solution

$$f_x(x, y) = ye^{xy}, \quad f_y(x, y) = xe^{xy}$$

$$f_x(-2, 0) = 0, \quad f_y(-2, 0) = -2$$

The unit vector u that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Thus

$$\begin{aligned} D_{\mathbf{u}}f(-2, 0) &= f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \quad \blacktriangleleft \end{aligned}$$

Example
Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
Solution

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 - z^3, \quad f_z(x, y, z) = -3yz^2 + 1$$

$$f_x(1, -2, 0) = -4, \quad f_y(1, -2, 0) = 1, \quad f_z(1, -2, 0) = 1$$

Since a is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then

$$D_{\mathbf{u}}f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$

The Gradient Vector

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector if

If $f(x, y) = \sin x + e^{xy}$,

Solution

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

$$\nabla f(0, 1) = \langle 2, 0 \rangle$$

Remark

- With the notation for the gradient vector, we can rewrite the directional derivative of a differentiable function as

  $$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

  This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.
- At (x, y), the surface z = f(x, y) has its maximum slope in the direction of the gradient, and the maximum slope is $\|\nabla f(x, y)\|$.
- At (x, y), the surface z = f(x, y) has its minimum slope in the direction that is opposite to the gradient, and the minimum slope is $-\|\nabla f(x, y)\|$.
- **Gradients are Normal to Level Curves**
  Assume that f(x, y) has continuous first-order partial derivatives in an open disk centered at (x_0, y_0) and that $\nabla f(x_0, y_0) \neq 0$.Then$\nabla f(x_0, y_0)$ is normal to the level curve of f through (x_0, y_0).
- **The Chain Rule for Paths/ The Derivative Along a Path:** we formulate it using gradient vector in this form: $\boldsymbol{\frac{d}{dt} f(r(t)) = \nabla f(r(t)).r'(t)}$

Example

Find the directional derivative of the function

$$f(x, y) = x^2 y^3 - 4y$$

at the point (2,−1) in the direction of the vector.

$$\mathbf{v} = 2\,\mathbf{i} + 5\mathbf{j}$$

Solution

We first compute the gradient vector at (2, −1):

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\mathbf{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Therefore, we have

$$D_{\mathbf{u}}\,f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

The directional derivative for the Functions of Three Variables

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}\,f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Directional Derivative in the Compact Form

If we use vector notation, then we can write the directional derivative in the compact form

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

The Gradient Vector for the Functions of Three Variables

If f is a function of three variables x,y and z, then the gradient of f is the vector function ∇f defined by

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Remark

With the notation for the gradient vector, we can rewrite the directional derivative of a differentiable function as

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Example

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution

(a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle\end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

$$\begin{aligned}D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) \\ &= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}\end{aligned}$$

Maximizing the Directional Derivative

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Proof

Using equation

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

We have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f.

Example

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5} \rangle$$

$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

(b) Here, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

Example

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Solution

The gradient of T is

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

$$= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k}$$

$$= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k})$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

Here the temperature increases fastest in the direction of the gradient vector

$\nabla T(1, 1, -2) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\tfrac{5}{8}\sqrt{41} \approx 4°\mathrm{C/m}$.

Theorem

Let f be a function of either two variables or three variables, and let P denote the point P(x_0, y_0) or P(x_0, y_0, z_0), respectively. Assume that f is differentiable at P.

(a) If ∇f = 0 at P, then all directional derivatives of f at P are zero.

(b) If ∇f ≠ 0 at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $||\nabla f||$ at P.

(c) If ∇f ≠ 0 at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\,||\nabla f||$ at P.

Example

Let f(x, y) = x^2e^y. Find the maximum value of a directional derivative at (−2, 0), and find the unit vector in the direction in which the maximum value occurs.

Solution

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

$$\nabla f(-2,0) = -4\mathbf{i} + 4\mathbf{j}$$

the maximum value of the directional derivative is

$$\|\nabla f(-2,0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of ∇f(−2, 0). The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2,0)}{\|\nabla f(-2,0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$f_x(2, 0) = \left(e^y - y\sin(xy)\right)\Big|_{(2,\,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = \left(xe^y - x\sin(xy)\right)\Big|_{(2,\,0)} = 2e^0 - 2\cdot 0 = 2.$$

$$\nabla f\big|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f\big|_{(2,\,0)} &= \nabla f\big|_{(2,\,0)} \cdot \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1. \end{aligned}$$

Example

A heat-seeking particle is located at the point (2, 3) on a flat metal plate whose temperature at a point (x, y) is $\mathbf{T(x, y) = 10 - 8x^2 - 2y^2}$

Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

Solution

Assume that the trajectory is represented parametrically by the equations

$$\mathbf{x = x(t),\ y = y(t)}$$

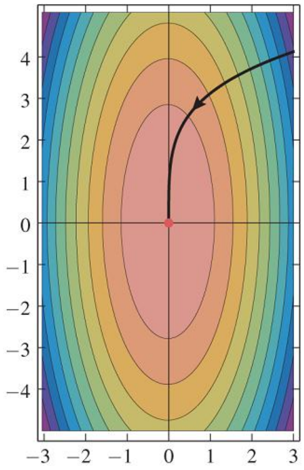
where the particle is at the point (2, 3) at time t = 0. Because the particle moves in the direction of maximum temperature increase, its direction of motion at time t is in the direction of the gradient of T(x, y), and hence its velocity vector v(t) at time t points in the direction of the gradient. Thus, there is a scalar k that depends on t such that

$$\mathbf{v}(t) = k\nabla T(x, y)$$

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = k(-16x\mathbf{i} - 4y\mathbf{j})$$

$$\frac{dx}{dt} = -16kx, \quad \frac{dy}{dt} = -4ky$$

$$\frac{dy}{dx} = \frac{-4ky}{-16kx} = \frac{y}{4x}$$



Thus, we can obtain the trajectory by solving the initial-value problem

$$\frac{dy}{dx} - \frac{y}{4x} = 0, \quad y(2) = 3$$

$$y = \frac{3}{\sqrt[4]{2}}x^{1/4}$$

The graph of the trajectory and a contour plot of the temperature function are shown in Figure.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1), and

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$\nabla f|_{(1,1)} = (x\mathbf{i} + y\mathbf{j})\Big|_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

Tangent Lines/Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Then tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has **normal vector** $\nabla F(x_0, y_0, z_0)$ is defined by the following equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

A tangent plane to a level surface serves as a flat plane that touches the surface at a single point, effectively acting as a linear approximation of the surface's behavior in the immediate vicinity of that point, allowing us to analyze local properties like the gradient and normal vector at that specific location on the surface; essentially, it provides a way to understand how the surface changes near a particular point by representing it with a flat plane that best fits the curvature at that point.

Normal Line

A normal line to a point (x,y) on a curve is the line that goes through the point (x,y) and is perpendicular to the tangent line. Since the normal line and tangent line are perpendicular, they will have slopes that are opposite reciprocals of each other.

Equation of the Normal Line

The equation of a normal line to a curve at a given point is $y=mx+b$, where m is the slope and b is the y-intercept. The slope of the normal line is the negative reciprocal of the curve's derivative at the point.

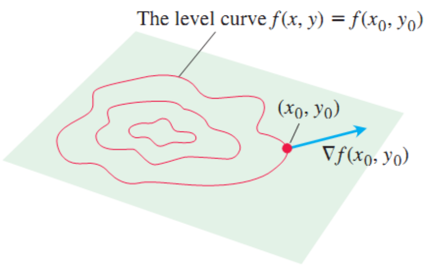
Symmetric Equations of the Normal Line

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Remember

At every point (x_0 , y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0 , y_0).

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0.$$



Example

Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

at the point $(-2, 1)$.

Solution

The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f\big|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)\Bigg|_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

Because this gradient vector is nonzero, the tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0$$

$$x - 2y = -4.$$

Example

$$f(x, y) = x - y \qquad g(x, y) = 3y$$

$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

   and

$$f\nabla g + g\nabla f = (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j})$$

$$= 3y\mathbf{i} + (3x - 6y)\mathbf{j}.$$

Example

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of **v** is obtained by dividing **v** by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of **v** is therefore

$$D_{\mathbf{u}}f|_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

Example

Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

(a) Find an equation of the tangent plane to the ellipsoid at the point (1, 2, 1).

(b) Find parametric equations of the line that is normal to the ellipsoid at the point (1, 2, 1).

(c) Find the acute angle that the tangent plane at the point (1, 2, 1) makes with the xy-plane.

Solution

(a) Using $F(x,y,z) = x^2 + 4y^2 + z^2$ with (1, 2, 1) we have

$$\nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\rangle = \langle 2x, 8y, 2z\rangle$$

$$\mathbf{n} = \nabla F(1,2,1) = \langle 2, 16, 2\rangle$$

Hence, the equation of the tangent plane is

$$2(x-1) + 16(y-2) + 2(z-1) = 0 \quad \text{or} \quad x + 8y + z = 18$$

(b) Since $\mathbf{n} = \langle 2, 16, 2\rangle$ at the point (1, 2, 1), it follows that parametric equations for the normal line to the ellipsoid at the point (1, 2, 1) are

$$x = 1 + 2t, \quad y = 2 + 16t, \quad z = 1 + 2t$$

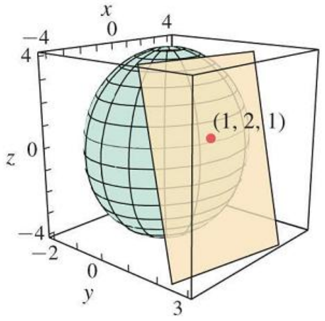
(c) To find the acute angle θ between the tangent plane and the xy-plane,

$$\mathbf{n}_1 = \mathbf{n} = \langle 2, 16, 2\rangle$$

$$\mathbf{n}_2 = \langle 0, 0, 1\rangle$$

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\|\,\|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83^\circ$$



Example

Find the equations of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution

The ellipsoid is the level surface (with k = 3) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then the equation of the tangent plane at $(-2,1,-3)$ is

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

Also, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\tfrac{2}{3}}$$

Significance of the Gradient Vector

- A gradient vector signifies the direction of the steepest ascent (or maximum rate of change) of a scalar field at a given point, essentially pointing in the direction where a function increases the fastest, with its magnitude representing the "steepness" of that increase.
- The gradient vectors always point to the direction where the function increases maximum. This property helps to find maxima/minima of the function using the steepest ascent/descent algorithm.

Tangent Planes to Surfaces of the Form z = f(x, y)

To find a tangent plane to a surface of the form z = f(x, y), we can use

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

With the function F(x, y, z) = z − f(x, y).

Example: Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point (2, 1, 4).

Solution: Let $F(x, y, z) = z - x^2y$.Then F(x, y, z) = 0 on the surface, so we can find the find the gradient of F at the point (2, 1, 4):

$$\nabla F(x, y, z) = -2xy\mathbf{i} - x^2\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, 1, 4) = -4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

the tangent plane has equation

$$-4(x - 2) - 4(y - 1) + 1(z - 4) = 0 \quad \text{or} \quad -4x - 4y + z = -8$$

and the normal line has equations

$$x = 2 - 4t, \qquad y = 1 - 4t, \qquad z = 4 + t$$

Example: Find parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ at the point (1,1,2)

Solution

$$x^2 + y^2 - z = 0 \quad \text{and} \quad 3x^2 + 2y^2 + z^2 - 9 = 0$$

$$F(x, y, z) = x^2 + y^2 - z \quad \text{and} \quad G(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, \quad \nabla G(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \nabla G(1, 1, 2) = 6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

Thus, a tangent vector at (1, 1, 2) to the curve of intersection is

$$\nabla F(1, 1, 2) \times \nabla G(1, 1, 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 6 & 4 & 4 \end{vmatrix} = 12\mathbf{i} - 14\mathbf{j} - 4\mathbf{k}$$

Since any scalar multiple of this vector will do just as well, we can multiply by 1/2 to reduce the size of the coefficients and use the vector of 6i − 7j − 2k to determine the direction of the tangent line. This vector and the point (1, 1, 2) yield the parametric equations

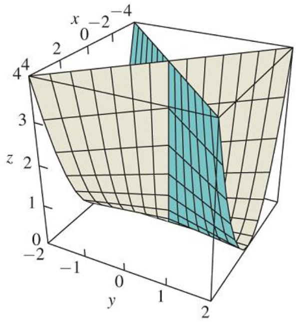
$$x = 1 + 6t, \quad y = 1 - 7t, \quad z = 2 - 2t$$

Example

Find an equation of the tangent plane to the parametric surface

$$x = uv, \quad y = u, \quad z = v^2$$

at the point where $u = 2$ and $v = -1$. This surface, called ***Whitney's umbrella***, is an example of a self-intersecting parametric surface



Solution

$$\mathbf{r} = uv\mathbf{i} + u\mathbf{j} + v^2\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) = v\mathbf{i} + \mathbf{j} \qquad \frac{\partial \mathbf{r}}{\partial v}(u, v) = u\mathbf{i} + 2v\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u}(2, -1) = -\mathbf{i} + \mathbf{j} \qquad \frac{\partial \mathbf{r}}{\partial v}(2, -1) = 2\mathbf{i} - 2\mathbf{k}$$

a normal to the surface at this point is

$$\frac{\partial \mathbf{r}}{\partial u}(2, -1) \times \frac{\partial \mathbf{r}}{\partial v}(2, -1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

Since any normal will suffice to find the tangent plane, it makes sense to multiply this vector by $-\frac{1}{2}$ and use the simpler normal $\mathbf{i} + \mathbf{j} + \mathbf{k}$. It follows from the given parametric equations that the point on the surface corresponding to $u = 2$ and $v = -1$ is $(-2, 2, 1)$, so the tangent plane at this point can be expressed in point-normal form as

$$(x + 2) + (y - 2) + (z - 1) = 0 \quad \text{or} \quad x + y + z = 1$$

Example

Find the tangent plane and normal line of the level surface

$f(x, y, z) = x^2 + y^2 + z - 9 = 0$ A circular paraboloid

at the point $P_0(1, 2, 4)$.

Solution

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

Example

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Solution

$f(x, y) = x\cos y - ye^x$

$$f_x(0, 0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x\sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0,$$

$$x - y - z = 0.$$

Example

The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E. Find parametric equations for the line tangent to E at the point P(1, 1, 3).

Solution

The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\Big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\Big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t.$$

Estimating the Change in *f* in a Direction **u**

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional}\\ \text{derivative}}}\ \underbrace{ds}_{\substack{\text{Distance}\\ \text{increment}}}$$

Example

Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution

We first find the derivative of f at P_0 in the direction of the vector

$$\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

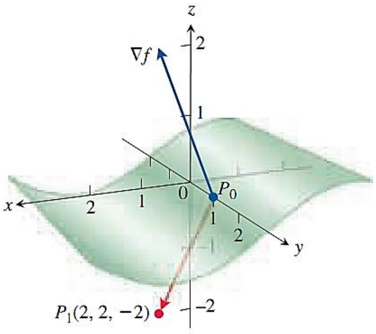
$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

$$\nabla f|_{(0,1,0)} = ((y\cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})\Big|_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving ds = 0.1 unit away from P_0 in the direction of u is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$



Extreme Value of a Function

An extreme value of a function is a maximum or minimum value of the function within a given interval. There are two types of extreme values: local and absolute.

Maximum Value of a Function

The "maximum value" of a function refers to the highest value that the function reaches across its entire domain, essentially the "peak" point on the graph of the function; it's the value where the function is greater than or equal to all other values it can produce.

Key points about maximum value:

Visual interpretation:

On a graph, the maximum value is the highest point on the curve representing the function.

Finding the maximum:

To find the maximum value, you typically need to calculate the derivative of the function, set it equal to zero to find critical points, and then evaluate the function at those points along with the endpoints of the domain to identify the highest value.

Local Maximum of a Function

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a local maximum value.

Absolute/ Global Maximum of a Function

A function of two variables has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points in in the domain of f.

Local vs. Global maximum:

Local maximum: A point where the function is higher than its immediate neighbors but might not be the highest overall.

Global maximum: The absolute highest value of the function across its entire domain.

Minimum Value of a Function

The "minimum value" of a function refers to the lowest point on the graph of that function, essentially the smallest output value the function can produce across its entire domain; it's the point where the function reaches its lowest possible value.

Key points about minimum value:

Visualizing:

When looking at a graph, the minimum value is the "lowest point" on the curve.

Finding with calculus:

To mathematically find the minimum value, you typically take the derivative of the function, set it equal to zero to find critical points, then test those points to see which one gives the lowest output.

Local Minimum of a Function

A function of two variables has a local minimum at $(\mathrm{a,b})$ if $f(x,y) \geq f(a,b)$ when $(\mathrm{x,y})$ is near $(\mathrm{a,b})$. [This means that $f(x,y) \geq f(a,b)$ for all points $(\mathrm{x,y})$ in some disk with center $(\mathrm{a,b})$.] The number $f(a,b)$ is called a local minimum value.

Absolute/ Global Minimum of a Function

A function of two variables has an absolute minimum at $(\mathrm{a,b})$ if $f(x,y) \geq f(a,b)$ for all points in in the domain of f.

Local vs. Global minimum:

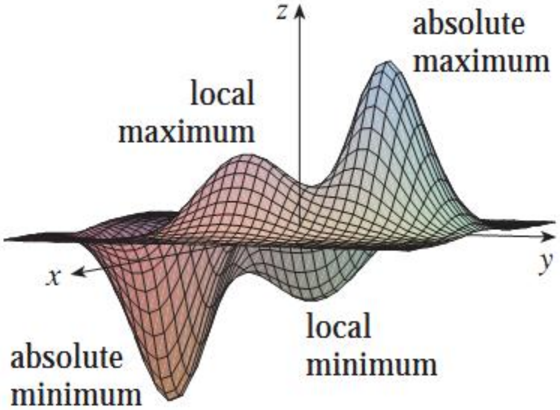
Local minimum: A point where the function is lower than its immediate neighbors, but might not be the lowest value overall.

Global minimum: The absolute lowest value the function takes on across its entire domain.

Relative Extremum Vs Absolute Extremum

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a relative extremum at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an absolute extremum at (x_0, y_0).

Remark
Look at the hills and valleys in the graph of f shown in Figure. There are two points (a, b) where f has a local maximum, that is, where $f(a,b)$ is larger than nearby values of $f(x,y)$. The larger of these two values is the absolute maximum. Likewise, has two local minima, where $f(a,b)$ is smaller than nearby values. The smaller of these two values is the absolute minimum.



Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(\mathrm{a}, \mathrm{b}) = 0$ and $f_y(\mathrm{a}, \mathrm{b}) = 0$.

Proof

Let $\mathrm{g}(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $\mathrm{g}(\mathrm{a}) = 0$ by Fermat's Theorem. But $\mathrm{g}'(\mathrm{a}) = f_x(x, b)$ and so $f_x(\mathrm{a}, \mathrm{b}) = 0$. Similarly, by applying Fermat's Theorem to the function $\mathrm{G}(y) = f(\mathrm{a}, y)$, we obtain $f_y(\mathrm{a}, \mathrm{b}) = 0$.

Critical / Stationary Point of a Function

A point (a, b) is called a critical point (or stationary point) of f if $f_x(\mathrm{a}, \mathrm{b}) = 0$ and $f_y(\mathrm{a}, \mathrm{b}) = 0$, or if one of these partial derivatives does not exist.

Remark

If has a local maximum or minimum at , then is a critical point of . However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

Saddle Point of a Function

Saddle points in a multivariable function are those critical points where the function attains neither a local maximum value nor a local minimum value. Saddle points mostly occur in multivariable functions. For example $(0,0)$ is a saddle point of $z = \ y^2 - x^2$.

Example
Find the critical points of

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

Solution

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

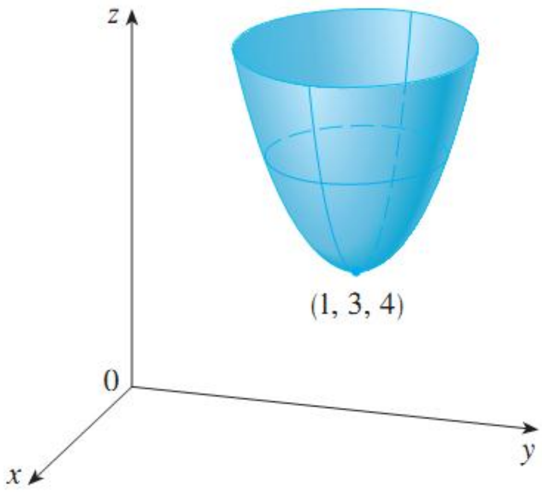
$$f_x(x, y) = 2x - 2 \qquad\qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

This can be confirmed geometrically from the graph of f which is the elliptic paraboloid with vertex (1,3,4) shown in Figure.



Example

Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

Solution

The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

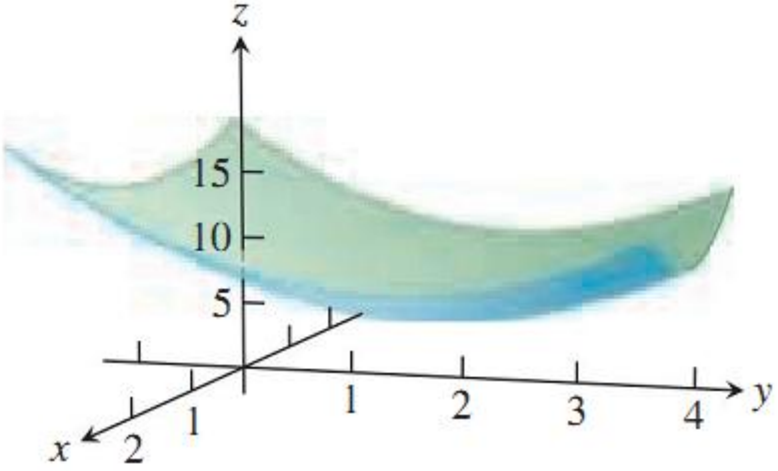
$f_x = 2x = 0$

and

$f_y = 2y - 4 = 0$.

The only possibility is the point (0, 2), where the value of f is 5.

Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point (0, 2) gives a local minimum (Figure).



Example

Find the extreme values of $f(x, y) = y^2 - x^2$

Solution

Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(\mathrm{a}, \mathrm{b}) = 0$ and $f_y(\mathrm{a}, \mathrm{b}) = 0$[that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [\, f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Remark

- In case (c) the point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b).
- If D = 0, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.
- To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Theorem

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(\mathrm{a},\mathrm{b}) = 0$ and $f_y(\mathrm{a},\mathrm{b}) = 0$[that is, (a,b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [\, f_{xy}(a, b)]^2$$

If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

Proof

We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k\rangle$. The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f &= D_{\mathbf{u}}(D_{\mathbf{u}} f) = \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

$$D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx} f_{yy} - f_{xy}^2) \tag{10}$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx} f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x, y) \geqslant f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum.

Example

Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Solution

We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

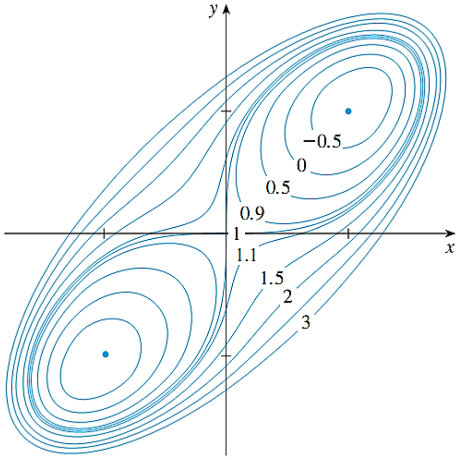
Next we calculate the second partial derivatives and : D(x, y):

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure



Example

Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution

The function is defined and differentiable for all x and y, and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

Or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8.$

Example

Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution

Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x\,(2 - x) = 0.$$

The two critical points are therefore (0, 0) and (2, 2).

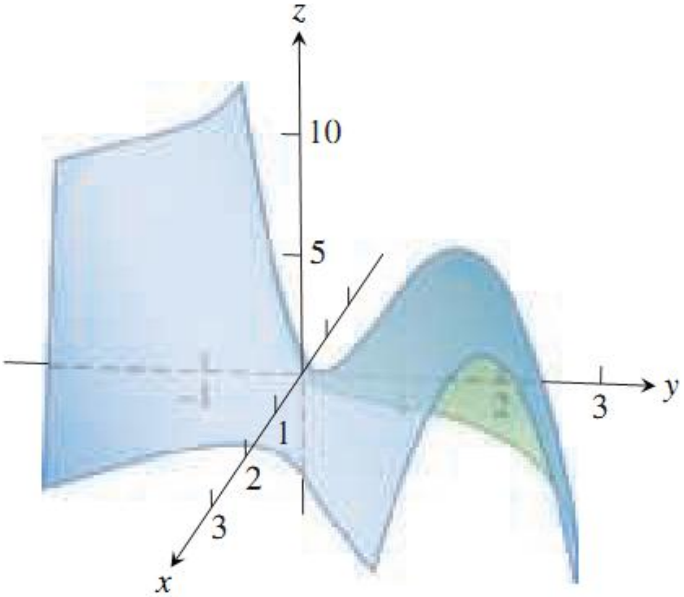
To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point (0, 0) we see that the value of the discriminant is the negative number -72, so the function has a saddle point at the origin. At the critical point (2, 2) we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point (2, 2) gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure



Example

Find the critical points of the function $f(x, y) = 10xye^{-(x2+y2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

Solution

First we find the partial derivatives f_x and f_y and set them simultaneously to zero in seeking the critical points:

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0,$$
$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$f_{xx} = -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)},$$
$$f_{xy} = f_{yx} = 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)},$$
$$f_{yy} = -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.$$

The following table summarizes the values needed by the Second Derivative Test.

Critical Point	f_{xx}	f_{xy}	f_{yy}	**Discriminant D**
$(0, 0)$	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

From the table we find that $D < 0$ at the critical point $(0, 0)$, giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$, each giving local minimum values.

Example

Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

Solution

The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$\boxed{4} \qquad 2x(10y - 5 - 2x^2) = 0$$

$$\boxed{5} \qquad 5x^2 - 4y - 4y^3 = 0$$

From Equation 4 we see that either

$$x = 0 \qquad \text{or} \qquad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

$$\boxed{6} \qquad x^2 = 5y - 2.5$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$\boxed{7} \qquad 4y^3 - 21y + 12.5 = 0$$

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or a rootfinder to locate these roots.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	-10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	-55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	-1.48	-5.87	-187.64	saddle point

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: The dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

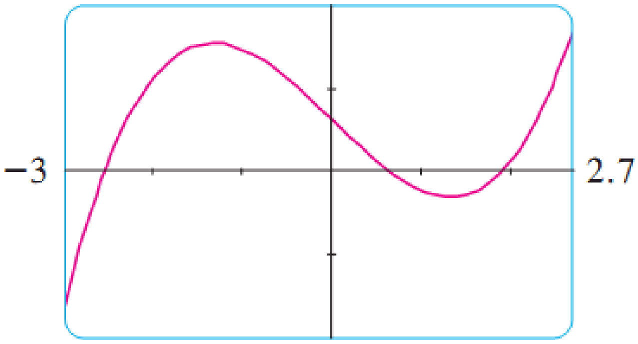


FIGURE 6

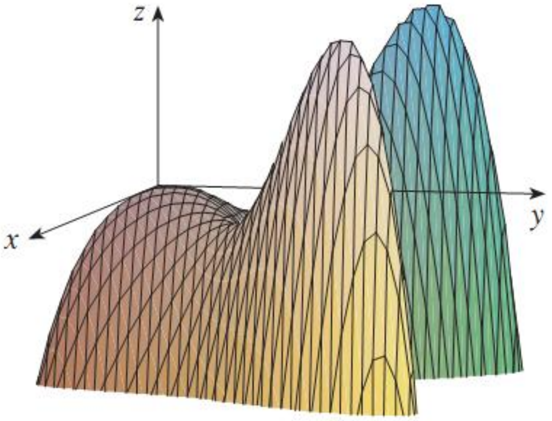


FIGURE 7

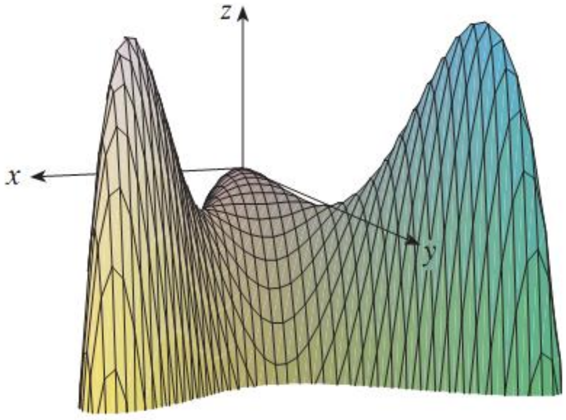


FIGURE 8

Example

Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution

Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f

$$6x - 2y = 0$$

satisfy the equations $\quad -2x + 2y - 8 = 0$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so (2, 6) is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

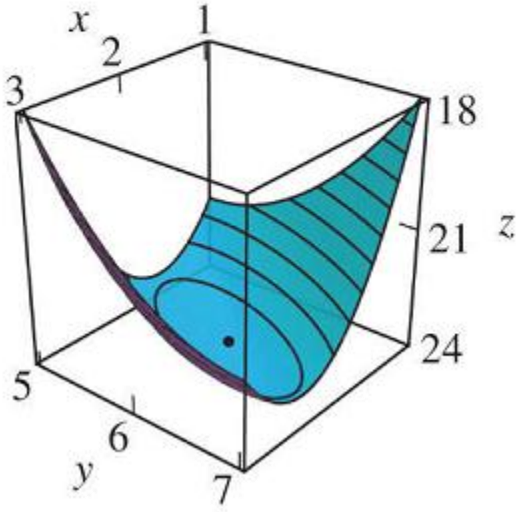
At the point (2, 6) we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at (2, 6). Figure shows a graph of f in the vicinity of the relative minimum.



Example

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution

Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \qquad (1)$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \qquad (2)$$

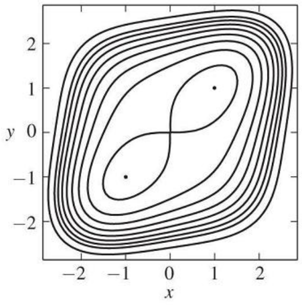
Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0, y = 1, y = -1$. Thus, the critical points of f are $(0, 0), (1, 1)$, and $(-1, -1)$.

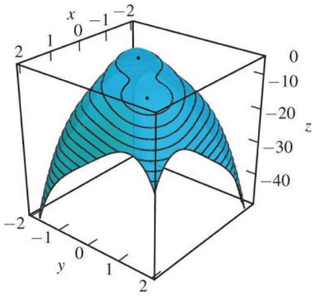
From (1), $$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points (1, 1) and (−1,−1), we have $D > 0$and $f_{xx} < 0$, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure.





Example

Find the shortest distance from the point $(1,0,-2)$ to the plane

$$x + 2y + z = 4$$

Solution

The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} = \sqrt{\left(\tfrac{5}{6}\right)^2 + \left(\tfrac{5}{3}\right)^2 + \left(\tfrac{5}{6}\right)^2} = \tfrac{5}{6}\sqrt{6}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$. ■

Example

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution

Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3. ■

Closed Set

A closed set in R^2 is one that contains all its boundary points.

Bounded Set

A bounded set in R^2 is one that is contained within some disk. In other words, it is finite in extent.

Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Procedure

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

The square region R whose points satisfy the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 1$ is a closed and bounded set in the xy-plane.

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

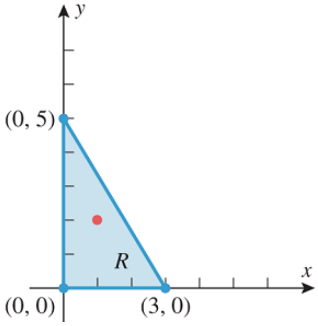
Solution

$$\frac{\partial f}{\partial x} = 3y - 6 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \quad \text{and} \quad 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so (1, 2) is the only critical point. As shown in Figure, this critical point is in the interior of R.



Next we want to determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

The line segment between (0, 0) and (3, 0): On this line segment we have $y = 0$, so (equation given) simplifies to a function of the single variable x,

$u(x) = f(x,0) = -6x + 7, \ 0 \leq x \leq 3$

This function has no critical points because $u'(x) = -6$ is nonzero for all x. Thus the extreme values of u(x) occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points (0, 0) and (3, 0) of R.

The line segment between (0, 0) and (0, 5): On this line segment we have $x = 0$, so (equation given) simplifies to a function of the single variable y,

$v(y) = f(0, y) = -3y + 7, 0 \leq y \leq 5$

This function has no critical points because v′(y) = −3 is nonzero for all y. Thus, the extreme values of v(y) occur at the endpoints y = 0and y = 5, which correspond to the points (0, 0) and (0, 5) of R.

The line segment between (3, 0) and (0, 5): In the xy-plane, an equation for this line segment is

$y = -\frac{5}{3}x + 5,\ 0 \leq x \leq 3$

so (equation given) simplifies to a function of the single variable x,

$$\begin{aligned} w(x) &= f\left(x, -\tfrac{5}{3}x + 5\right) = 3x\left(-\tfrac{5}{3}x + 5\right) - 6x - 3\left(-\tfrac{5}{3}x + 5\right) + 7 \\ &= -5x^2 + 14x - 8, \qquad 0 \leq x \leq 3 \end{aligned}$$

Since w′(x) = −10x + 14, the equation w′(x) = 0 yields x = 7
5 as the only critical point of
w. Thus, the extreme values of w occur either at the critical point $x = \frac{7}{5}$ or at the endpoints x = 0and x = 3. The endpoints correspond to the points (0, 5) and (3, 0) of R, and from $y = -\frac{5}{3}x + 5,\ 0 \leq x \leq 3$ the critical point corresponds to $\left(\frac{7}{5}, \frac{8}{3}\right)$.

Finally, Table lists the values of f(x, y) at the interior critical point and at the points on the boundarywhere an absolute extremumcan occur. Fromthe table we conclude that the absolute maximum value of f is f(0, 0) = 7 and the absolute minimum value is f(3, 0) = −11.

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	-11	-8	$\frac{9}{5}$	1

Example

Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution

Let

x = length of the box (in feet)

y = width of the box (in feet)

z = height of the box (in feet)

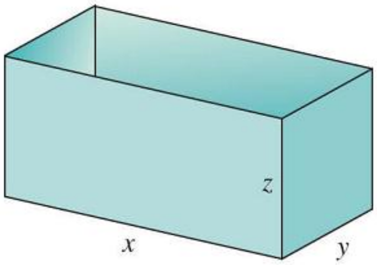
S = surface area of the box (in square feet)

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

S = xy + 2xz + 2yz

(Figure) subject to the volume requirement

$$xyz = 32$$



we obtain z = 32/xy, so

$$S = xy + \frac{64}{y} + \frac{64}{x}$$

which expresses S as a function of two variables. The dimensions x and y in this formula must be positive, but otherwise have no limitation, so our problem reduces to finding the absolute minimum value of S over the open first quadrant:
x > 0, y > 0. Because this region is neither closed nor bounded, we have no mathematical guarantee at this stage that an absolute minimum exists. However, if S has an absolute minimum value in the open first quadrant, then it must occur at a critical point of S. Thus, our next step is to find the critical points of S.

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0, \quad x - \frac{64}{y^2} = 0$$

$$y = \frac{64}{x^2} \quad \text{and} \quad x - \frac{64}{(64/x^2)^2} = 0$$

$$x\left(1 - \frac{x^3}{64}\right) = 0$$

The solutions of this equation are x = 0and x = 4.

Since we require x > 0, the only solution of significance is x = 4. Substituting this value into $y = \frac{64}{x^2}$ yields y = 4. We conclude that the point (x, y) = (4, 4) is the only critical point of S in the first quadrant. Since S = 48 if x = y = 4, this suggests we try to show that the minimum value of S on the open first quadrant is 48.
It immediately follows from Equation

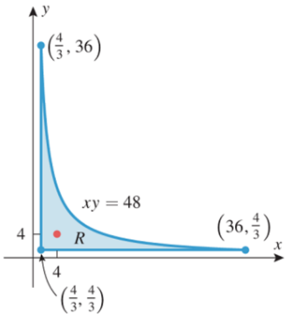
$$S = xy + \frac{64}{y} + \frac{64}{x}$$

that 48 < S at any point in the first quadrant for which at least one of the inequalities

$$xy > 48, \quad \frac{64}{y} > 48, \quad \frac{64}{x} > 48$$

is satisfied. Therefore, to prove that 48 ≤ S, we can restrict attention to the set of points in the first quadrant that satisfy the three inequalities

$$xy \leq 48, \quad \frac{64}{y} \leq 48, \quad \frac{64}{x} \leq 48$$



These inequalities can be rewritten as

$$xy \leq 48, \quad y \geq \frac{4}{3}, \quad x \geq \frac{4}{3}$$

and they define a closed and bounded region R within the first quadrant (Figure). The function S is continuous on R, so S has an absolute minimum value somewhere on R. Since the point (4, 4) lies within R, and 48 < S on the boundary of R (why?), the minimum value of S on R must occur at an interior point. It then follows that the mimimum value of S on R must occur at a critical point of S. Hence, the absolute minimum of S on R (and therefore on the entire open first quadrant) is S = 48 at the point (4, 4). Substituting x = 4and y = 4 into (xyz = 32) yields z = 2, so the box using the least material has a height of 2 ft and a square base whose edges are 4 ft long.

EXTREMUM PROBLEMS WITH CONSTRAINTS

The problem of minimizing
S = xy + 2xz + 2yz subject to the constraint xyz − 32 = 0
is a special case of the following general problem:

Three-Variable Extremum Problem with One Constraint

Maximize or minimize the function f(x, y, z) subject to the constraint g(x, y, z) = 0.

Two-Variable Extremum Problem with One Constraint

Maximize or minimize the function f(x, y) subject to the constraint g(x, y) = 0.

Example

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

Solution

Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in $\boxed{9}$, we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By the methods of Chapter 3, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f. ■

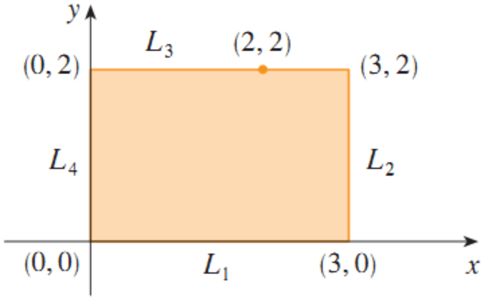


FIGURE 12

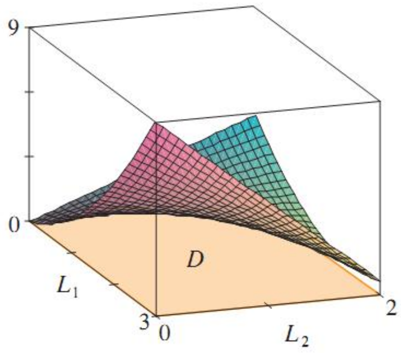


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

Example

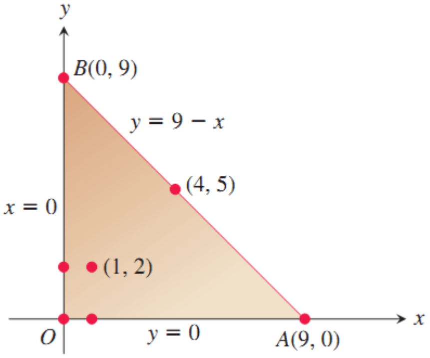
Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Solution

Since f is differentiable, the only places where f can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary (Figure).



(a) Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$

yielding the single point $(x, y) = (1, 2)$. The value of f there is

$$f(1, 2) = 7.$$

(b) Boundary points. We take the triangle one side at a time:

i) On the segment OA, $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$. Its extreme values (as we know from Chapter 4) may occur at the endpoints

$$x = 0 \quad \text{where} \quad f(0, 0) = 2$$
$$x = 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

or at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

ii) On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 4y - y^2.$$

As in part i), we consider $f(0, y)$ as a function of y defined on the closed interval $[0, 9]$. Its extreme values can occur at the endpoints or at interior points where $f'(0, y) = 0$. Since $f'(0, y) = 4 - 2y$, the only interior point where $f'(0, y) = 0$ occurs at $(0, 2)$, with $f(0, 2) = 6$. So the candidates for this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -43, \qquad f(0, 2) = 6.$$

iii) We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of the line segment AB. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

Setting $f'(x, 9 - x) = 16 - 4x = 0$ gives

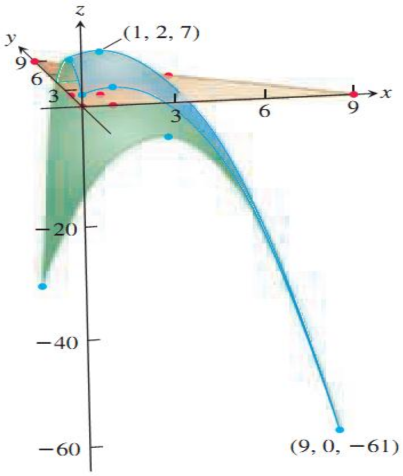
$$x = 4.$$

At this value of x,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$

Summary

We list all the function value candidates: 7, 2, -61, 3, -43, 6, -11. The maximum is 7, which f assumes at (1, 2). The minimum is -61, which f assumes at (9, 0). See Figure.



Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of f

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist)

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**

ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive**

The Orthogonal Gradient Theorem

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve, C: r(t) = x(t)i + y(t)j + z(t)k.

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0 .

Proof

We show that ∇f is orthogonal to the curve's tangent vector r′ at P_0 . The values of f on C are given by the composition $f(x(t), y(t), z(t))$, whose derivative with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \mathbf{r}'$$

At any point P_0 where f has a local maximum or minimum relative to its values on the curve, $df/dt = 0$, so

$$\nabla f \cdot \mathbf{r}' = 0$$

Corollary

At the points on a smooth curve r(t) = x(t)i + y(t)j where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve,

$$\nabla f \cdot \mathbf{r}' = 0$$

Lagrange Multipliers

In mathematical optimization, the **method of Lagrange multipliers** is a strategy for finding the local maxima and minima of a function subject to equation constraints (i.e., subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables). It is named after the mathematician Joseph-Louis Lagrange.

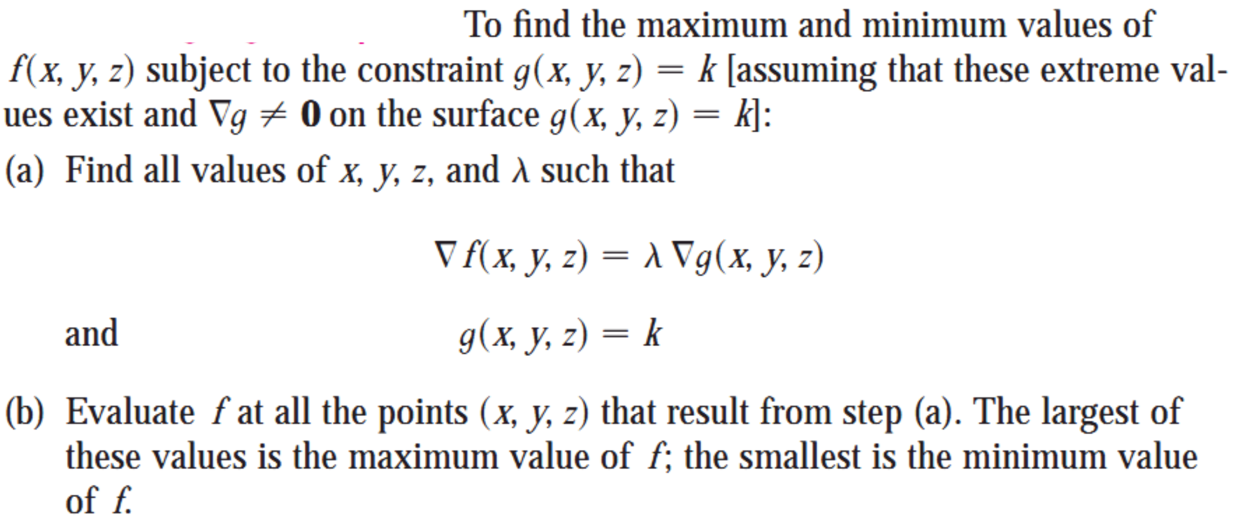
The basic idea is to convert a constrained problem into a form such that the derivative test of an unconstrained problem can still be applied.

If we have the following equation

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Then the number λ in Equation is called a Lagrange multiplier.

Method of Lagrange Multipliers



To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

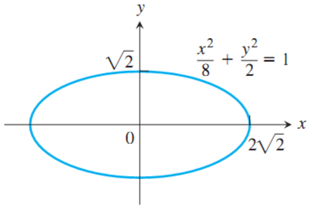
$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example

Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$



Solution

We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

Example

Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution

We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$
$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

Example
A rectangular box without a lid is to be made from 12 m^2 of cardboard.
Find the maximum volume of such a box.
Solution

Let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$V = xyz$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$yz = \lambda(2z + y) \tag{2}$$

$$xz = \lambda(2z + x) \tag{3}$$

$$xy = \lambda(2x + 2y) \tag{4}$$

$$2xz + 2yz + xy = 12 \tag{5}$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. Doing this, we have

$$\boxed{6} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \qquad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ and this would contradict $\boxed{5}$. Therefore, from $\boxed{6}$ and $\boxed{7}$, we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From $\boxed{7}$ and $\boxed{8}$ we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$.

Example

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution

We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$$\boxed{9} \qquad 2x = 2x\lambda$$

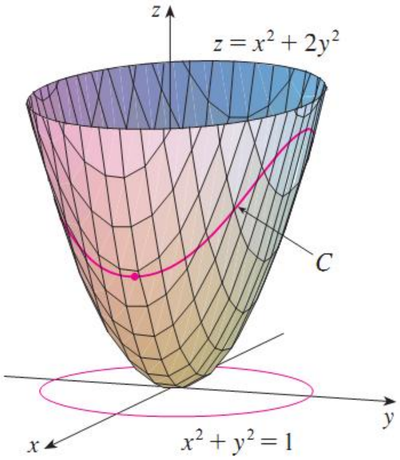
$$\boxed{10} \qquad 4y = 2y\lambda$$

$$\boxed{11} \qquad x^2 + y^2 = 1$$

From $\boxed{9}$ we have $x = 0$ or $\lambda = 1$. If $x = 0$, then $\boxed{11}$ gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from $\boxed{10}$, so then $\boxed{11}$ gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. Checking with Figure 2, we see that these values look reasonable.



Example

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution

According to the procedure, we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, te only critical point is (0,0). We compare the value of f at that point with the extreme values on the boundary;

$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$

Therefore the maximum value of f on the disk $x^2 + y^2 \leqslant 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$.

Example

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3,1,-1)$.

Solution

The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

$$\boxed{12} \qquad 2(x-3) = 2x\lambda$$

$$\boxed{13} \qquad 2(y-1) = 2y\lambda$$

$$\boxed{14} \qquad 2(z+1) = 2z\lambda$$

$$\boxed{15} \qquad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from $\boxed{12}$, $\boxed{13}$, and $\boxed{14}$, and then substitute these values into $\boxed{15}$. From $\boxed{12}$ we have

$$x - 3 = x\lambda \qquad \text{or} \qquad x(1-\lambda) = 3 \qquad \text{or} \qquad x = \frac{3}{1-\lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from $\boxed{12}$.] Similarly, $\boxed{13}$ and $\boxed{14}$ give

$$y = \frac{1}{1-\lambda} \qquad z = -\frac{1}{1-\lambda}$$

Therefore, from $\boxed{15}$, we have

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

which gives $(1-\lambda)^2 = \frac{11}{4}$, $1-\lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \quad \text{and} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$ and the farthest is $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$. ▬

Lagrange Multipliers (Two Constraints Form)

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing above Equation in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

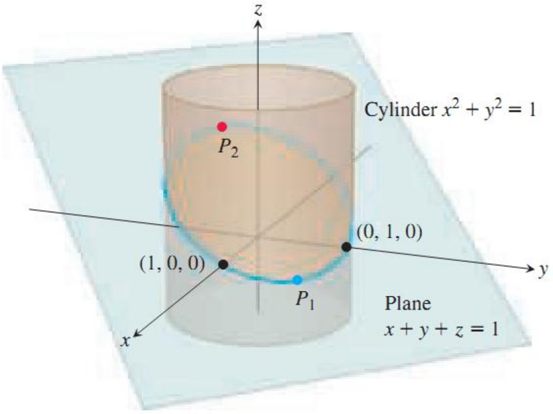
$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Example

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse Find the points on the ellipse that lie closest to and farthest from the origin.



Solution

We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2$$

(the square of the distance from (x, y, z) to the origin) subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0 \tag{3}$$

$$g_2(x, y, z) = x + y + z - 1 = 0. \tag{4}$$

The gradient equation in Equations (2) then gives

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2\lambda x + \mu)\mathbf{i} + (2\lambda y + \mu)\mathbf{j} + \mu\mathbf{k}$$

or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in Equations (5) yield

$$2x = 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z,$$

$$2y = 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z. \tag{6}$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Equations (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Figure 14.59.

If $x = y$, then Equations (3) and (4) give

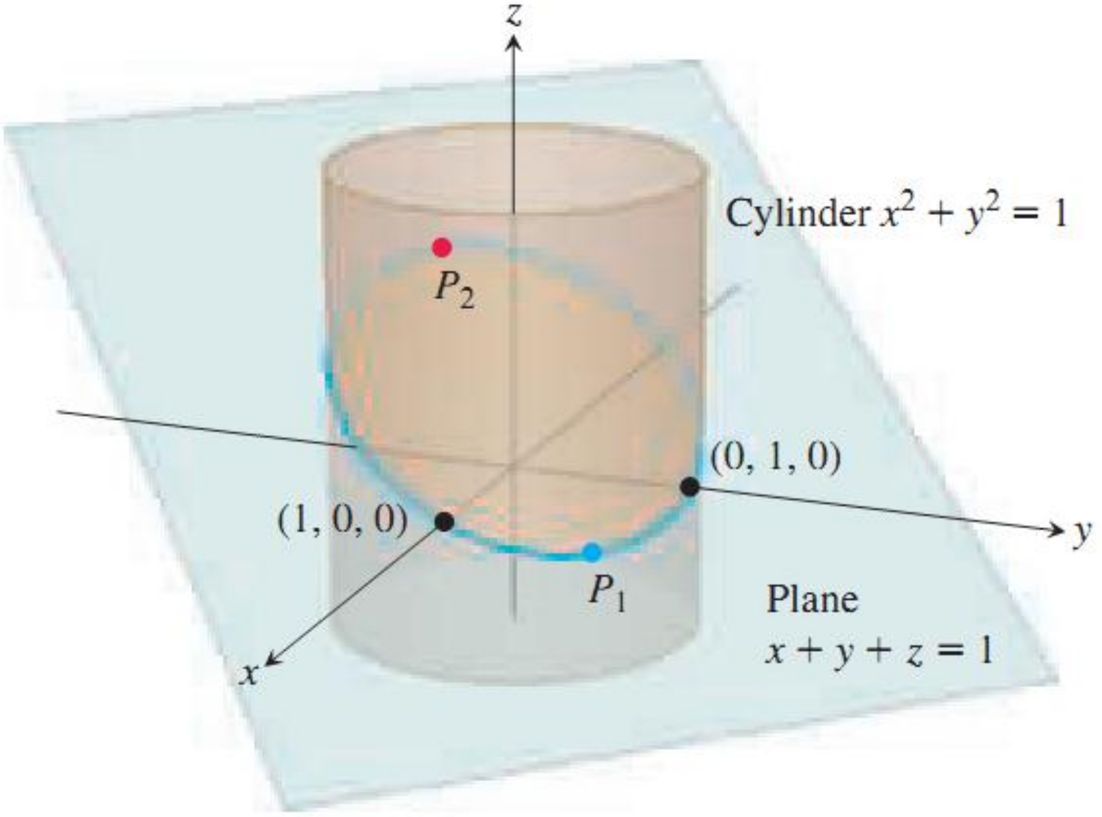
$$\begin{aligned} x^2 + x^2 - 1 &= 0 & \qquad x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

Here we need to be careful, however. Although P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2 . (See Figure)



Example

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution

We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

$$\boxed{17} \qquad 1 = \lambda + 2x\mu$$

$$\boxed{18} \qquad 2 = -\lambda + 2y\mu$$

$$\boxed{19} \qquad 3 = \lambda$$

$$\boxed{20} \qquad x - y + z = 1$$

$$\boxed{21} \qquad x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from $\boxed{19}$] in $\boxed{17}$, we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, $\boxed{18}$ gives $y = 5/(2\mu)$. Substitution in $\boxed{21}$ then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from $\boxed{20}$, $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$. ■

Example

At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution

The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize f(x, y) = xy subject to the constraint g(x, y) = $x^2 + y^2 - 1 = 0$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \quad \text{and} \quad x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y}$$

from which we obtain

$$\frac{y}{2x} = \frac{x}{2y}$$

or

$$y^2 = x^2$$

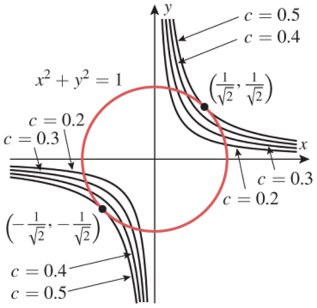
Substituting this in (g(x, y) = $x^2 + y^2 - 1 = 0$) yields

$$2x^2 - 1 = 0$$

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation $y^2 = x^2$ produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	−1/2	−1/2	1/2

Thus, the function $f(x, y) = xy$ has an absolute maximum of $\frac{1}{2}$ occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of $-\frac{1}{2}$ occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. Figure 13.9.3 shows some level curves $xy = c$ and the constraint curve in the vicinity of the maxima. A similar figure for the minima can be obtained using negative values of c for the level curves $xy = c$. ◀



Example

Use the method of Lagrange multipliers to

find the dimensions of a rectangle with perimeter p

and maximum area.

Solution

Let

x = length of the rectangle, y = width of the rectangle, A = area of the rectangle

We want to maximize A = xy on the line segment

$2x + 2y = p, \qquad 0 \leq x, y$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since f(x, y) = xy is a continuous function, it follows from the Extreme-Value Theorem that f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment.

If g(x, y) = 2x + 2y, then we have $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2\mathbf{i} + 2\mathbf{j}$

Noting that $\nabla g \neq 0$, it follows from that $y\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + 2\mathbf{j})$ at a constrained relative maximum. This is equivalent to the two equations

$y = 2\lambda \qquad \text{and} \qquad x = 2\lambda$

Eliminating λ from these equations we obtain x = y, which shows that the rectangle is actually a square. Using this condition and constraint $(2x + 2y = p, 0 \leq x, y)$, we obtain x = p/4, y = p/4.

Example

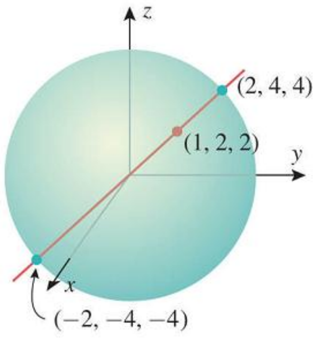
Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point (1, 2, 2).

Solution

To avoid radicals, we will find points on the sphere that minimize and maximize the square of the distance to (1, 2, 2). Thus, we want to find the relative extrema of $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$ subject to the constraint $x^2 + y^2 + z^2 = 36$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Thus, $\nabla g = 0$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq 0$ at any point of the sphere ($x^2 + y^2 + z^2 = 36$), and hence the constrained relative extrema must occur at points where

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$



That is,

$2(x - 1)\mathbf{i} + 2(y - 2)\mathbf{j} + 2(z - 2)\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$

which leads to the equations

$2(x - 1) = 2x\lambda, 2(y - 2) = 2y\lambda, 2(z - 2) = 2z\lambda$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite as $\frac{x-1}{x} = \lambda, \frac{y-2}{y} = \lambda, \frac{z-2}{z} = \lambda$. The first two equations imply that $\frac{x-1}{x} = \frac{y-2}{y}$ from which it follows that $y = 2x$ Similarly, the first and third equations imply that $z = 2x$

Substituting in the constraint equation ($x^2 + y^2 + z^2 = 36$), we obtain $9x^2 = 36$ or $x = \pm 2$.

Substituting these values in ($y = 2x$) and ($z = 2x$) yields two points: (2,4,4) and (−2,−4,−4)

Since $f(2,4,4) = 9$and $f(-2,-4,-4) = 81$, it follows that (2, 4, 4) is the point on the sphere closest to (1, 2, 2), and (−2,−4,−4) is the point that is farthest (Figure)

Example

Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution

The problem is to minimize the surface area $S = xy + 2xz + 2yz$ subject to the volume constraint $xyz = 32$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz$, then

$\nabla f = (y + 2z)i + (x + 2z)j + (2x + 2y)k$ and $\nabla g = yzi + xzj + xyk$

It follows that $\nabla g \neq 0$ at any point on the surface $xyz = 32$, since x, y, and z are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$(y + 2z)i + (x + 2z)j + (2x + 2y)k = \lambda(yzi + xzj + xyk)$

This condition yields the three equations

$y + 2z = \lambda yz$, $x + 2z = \lambda xz$, $2x + 2y = \lambda xy$

Because x, y,and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \quad \frac{1}{z} + \frac{2}{x} = \lambda, \quad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations, $y = x$ and from the first and third equations, $z = \frac{1}{2}x$
Substituting in the volume constraint ($xyz=32$) yields $\frac{1}{2}x^3 = 32$

This equation, yields $x = 4$, $y = 4$, $z = 2$

Introduction to Taylor's theorem for multivariable functions

Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a,b)}$$
$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\Big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)}$$
$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,\, b+ck)}.$$

Taylor's Formula for $f(x, y)$ at the Origin

$$f(x, y) = f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+ \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y\frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right)$$
$$+ \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,\, cy)}$$

Example

Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

Solution

We take $n = 2$

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+ \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})\Big|_{(cx,\, cy)}.$$

Calculating the values of the partial derivatives,

$$f(0,0) = \sin x \sin y \Big|_{(0,0)} = 0, \qquad f_{xx}(0,0) = -\sin x \sin y \Big|_{(0,0)} = 0,$$

$$f_x(0,0) = \cos x \sin y \Big|_{(0,0)} = 0, \qquad f_{xy}(0,0) = \cos x \cos y \Big|_{(0,0)} = 1,$$

$$f_y(0,0) = \sin x \cos y \Big|_{(0,0)} = 0, \qquad f_{yy}(0,0) = -\sin x \sin y \Big|_{(0,0)} = 0,$$

we have the result

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}\left(x^2(0) + 2xy(1) + y^2(0)\right), \quad \text{or} \quad \sin x \sin y \approx xy.$$

The error in the approximation is

$$E(x,y) = \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,\,cy)}.$$

The third derivatives never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x| \le 0.1$ and $|y| \le 0.1$. Hence

$$|E(x,y)| \le \frac{1}{6}\left((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3\right) = \frac{8}{6}(0.1)^3 \le 0.00134$$

(rounded up). The error will not exceed 0.00134 if $|x| \le 0.1$ and $|y| \le 0.1$.

MULTIPLE INTEGRALS

Definite Integral

If $f(x)$ is defined for $a \leqslant x \leqslant b$,

Then definite integral is defined as follows

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Where $\sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$ is the Riemann sum.

Volumes of the Solid

The volume of the solid that lies under the graph of f and above the rectangle R is given as follows

$$V = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Using

$$\iint\limits_R f(x, y)\, dA = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

We have

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint\limits_R f(x, y)\, dA$$

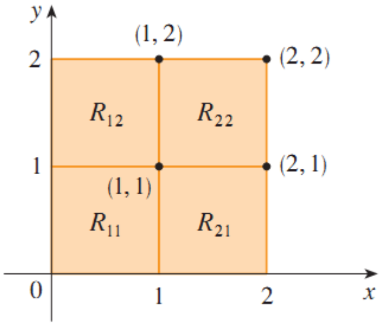
The sum, $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$ is called a double Riemann sum and is used as an approximation to the value of the double integral.

Example

Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

Solution

The squares are shown in Figure.



The paraboloid is the graph of

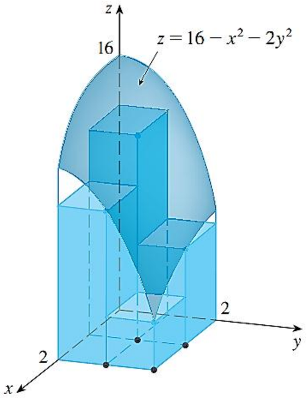
$f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

$$= f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

This is the volume of the approximating rectangular boxes shown in Figure.



Example

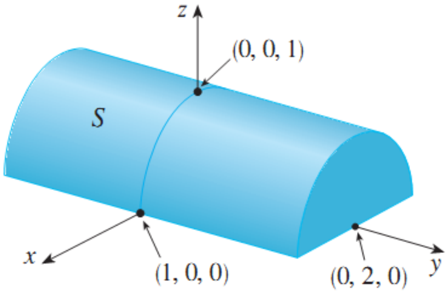
If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

Solution

The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$



Double Integrals over Rectangles

The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

Corresponding function is called **integrable**, if this limit exists.

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$.

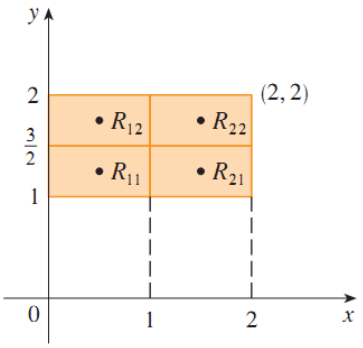
Solution

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\ &= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A \\ &= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \Delta A \\ &= \left(-\tfrac{67}{16}\right)\tfrac{1}{2} + \left(-\tfrac{139}{16}\right)\tfrac{1}{2} + \left(-\tfrac{51}{16}\right)\tfrac{1}{2} + \left(-\tfrac{123}{16}\right)\tfrac{1}{2} \\ &= -\tfrac{95}{8} = -11.875\end{aligned}$$

Thus we have

$$\iint_R (x - 3y^2)\, dA \approx -11.875$$



Properties of Double Integrals

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\iint_R c f(x, y)\, dA = c \iint_R f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

Iterated/Repeated Integrals

An iterated integral is the result of applying multiple integrals to a function with multiple variables. It's a nested integral that's used to evaluate double integrals over rectangles.

Formulae are given as follows

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

Example

Evaluate the iterated integral.

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx$$

Solution

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2}x^2\, dx = \left.\frac{x^3}{2}\right]_0^3 = \frac{27}{2}$$

Example
Evaluate the iterated integral.

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy$$

Solution

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \left[\int_0^3 x^2 y\, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y\, dy = 9\, \frac{y^2}{2} \Big]_1^2 = \frac{27}{2}$$

Example
Evaluate the iterated integral.

$$\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$$

Solution

$$\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx = \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy \right] dx$$

$$= \int_1^3 (40y - xy^2)\Big]_{y=2}^{4} dx$$

$$= \int_1^3 [(160 - 16x) - (80 - 4x)]\, dx$$

$$= \int_1^3 (80 - 12x)\, dx$$

$$= (80x - 6x^2)\Big]_1^3 = 112$$

Example

Evaluate the iterated integral.

$$\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$$

Solution

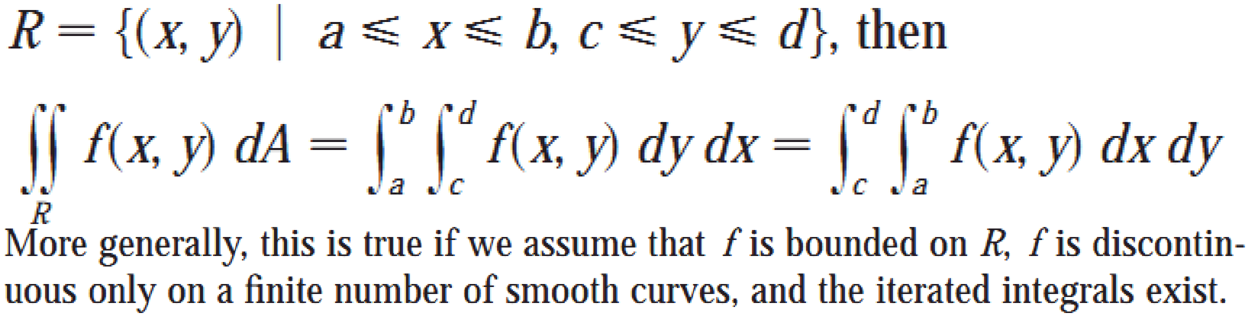
$$\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\, dx \right] dy$$

$$= \int_2^4 (40x - x^2y)\Big]_{x=1}^{3}\, dy \qquad = \int_2^4 [(120 - 9y) - (40 - y)]\, dy$$

$$= \int_2^4 (80 - 8y)\, dy \;= (80y - 4y^2)\Big]_2^4 = 112$$

Fubini's Theorem

If f is continuous on the rectangle

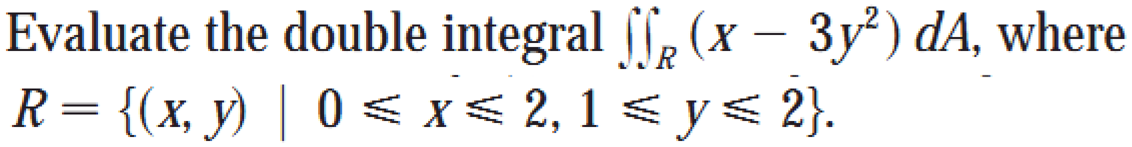


$R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

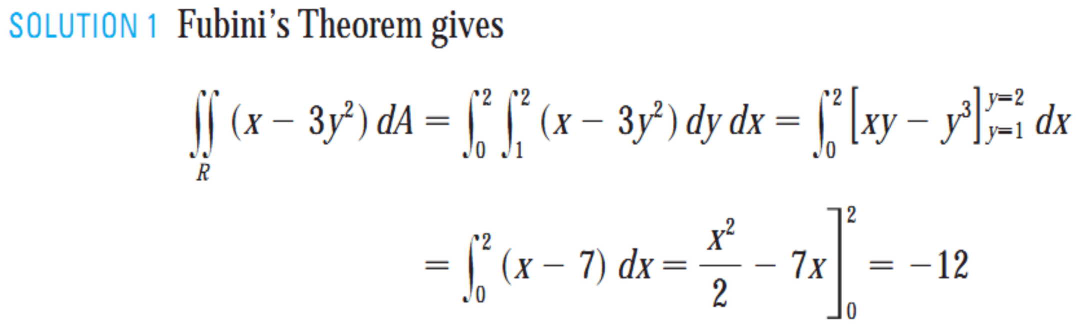
More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example



Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$.

Solution



SOLUTION 1 Fubini's Theorem gives

$$\iint\limits_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx = \int_0^2 \Big[xy - y^3\Big]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx = \frac{x^2}{2} - 7x \Big]_0^2 = -12$$

SOLUTION 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3\Big]_1^2 = -12 \qquad \blacksquare$$

Example

Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

Solution

SOLUTION 1 If we first integrate with respect to x, we get

$$\iint_R y\sin(xy)\,dA = \int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy = \int_0^\pi \left[-\cos(xy)\right]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$

SOLUTION 2 If we reverse the order of integration, we get

$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

To evaluate the inner integral, we use integration by parts with

$$u = y \qquad dv = \sin(xy)\,dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

and so $$\int_0^\pi y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

If we now integrate the first term by parts with $u = -1/x$ and $dv = \pi \cos \pi x\, dx$, we get $du = dx/x^2$, $v = \sin \pi x$, and

$$\int \left(-\frac{\pi \cos \pi x}{x} \right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

Therefore
$$\int \left(-\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2} \right) dx = -\frac{\sin \pi x}{x}$$

and so
$$\int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx = \left[-\frac{\sin \pi x}{x} \right]_1^2$$

$$= -\frac{\sin 2\pi}{2} + \sin \pi = 0$$

Example

Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

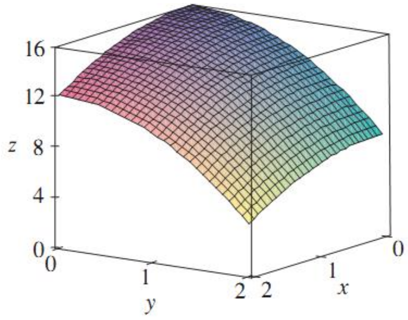
Solution

We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$.

$$V = \iint_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$

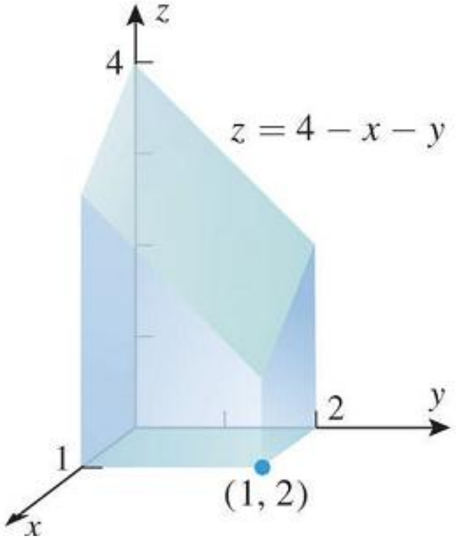
$$= \int_0^2 \left[16x - \tfrac{1}{3}x^3 - 2y^2 x \right]_{x=0}^{x=2} dy$$

$$= \int_0^2 \left(\tfrac{88}{3} - 4y^2 \right) dy = \left[\tfrac{88}{3}y - \tfrac{4}{3}y^3 \right]_0^2 = 48$$



Example

Use a double integral to find the volume of the solid that is bounded above by the plane z = 4 − x − y and below by the rectangle R = [0, 1] × [0, 2] (Figure).



Solution

$$V = \iint_R (4 - x - y)\,dA = \int_0^2 \int_0^1 (4 - x - y)\,dx\,dy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5$$

Example

Calculate the volume under the plane z = 4 - x - y over the rectangular region $R: 0 \leq x \leq 2, 0 \leq y \leq 1$ in the xy-plane.

Solution

$$\text{Volume} = \int_{x=0}^{x=2} A(x)\,dx = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=1}(4-x-y)\,dy\right)dx$$

$$= \int_{x=0}^{x=2}\left[4y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2}\left(\frac{7}{2} - x\right)dx$$

$$= \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = 5.$$

Example

Evaluate the double integral

$$\iint_R y^2 x\,dA$$

over the rectangle R = {(x, y): −3 ≤ x ≤ 2, 0 ≤ y ≤ 1}.

Solution

$$\iint_R y^2 x\,dA = \int_0^1 \int_{-3}^{2} y^2 x\,dx\,dy = \int_0^1 \left[\frac{1}{2}y^2 x^2\right]_{x=-3}^{2} dy$$

$$= \int_0^1 \left(-\frac{5}{2}y^2\right)dy = -\frac{5}{6}y^3\Big]_0^1 = -\frac{5}{6} \quad ◄$$

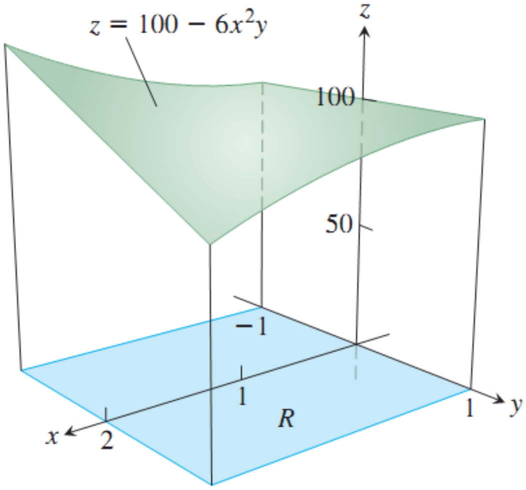
Example

Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution

Figure displays the volume beneath the surface.



$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_0^2 (100 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \left[100x - 2x^3y \right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \left[200y - 8y^2 \right]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_0^2 \left[100y - 3x^2y^2 \right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 \left[(100 - 3x^2) - (-100 - 3x^2) \right] dx$$

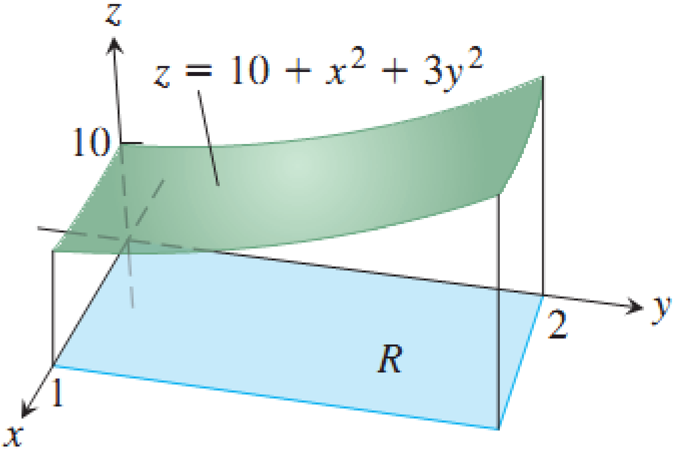
$$= \int_0^2 200\, dx = 400.$$

Example

Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, 0 \le y \le 2$.

Solution

The surface and volume are shown in Figure.



The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}.$$

Remember

$$\iint\limits_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

Example

If $R = [0, \pi/2] \times [0, \pi/2]$, then,

$$\iint\limits_R \sin x \cos y\, dA = \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy$$

$$= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} = 1 \cdot 1 = 1$$

Double Integrals over General Regions/ Iterated Integrals with Nonrectangular Regions

If F is integrable over R, then we define the **double integral of** f **over** D by

$$\iint\limits_D f(x, y)\, dA = \iint\limits_R F(x, y)\, dA$$

Where

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

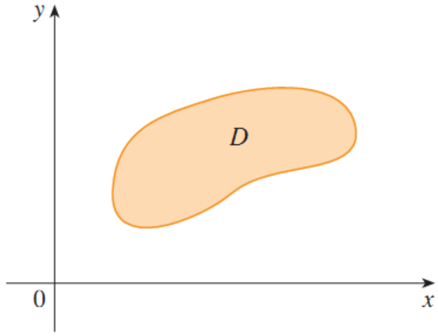


FIGURE 1

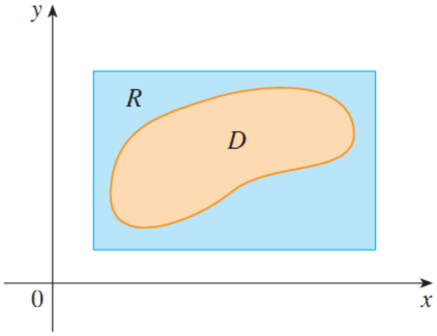


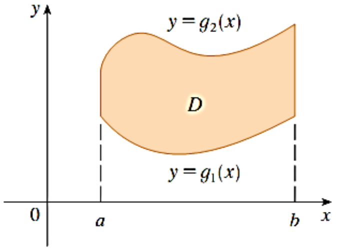
FIGURE 2

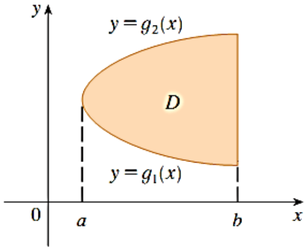
Plane Region of Type I

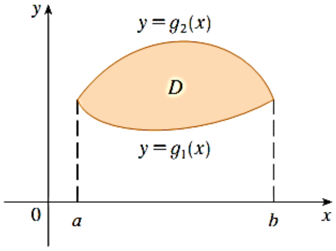
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure .







Integration Formula for Plane Region of Type I

If f is continuous on a type I region D such that

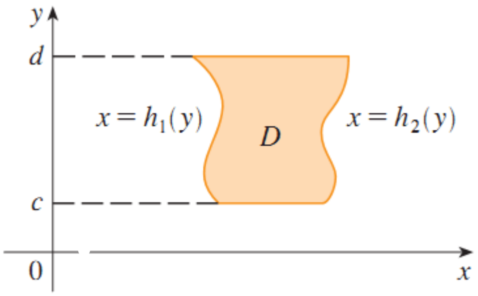
$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

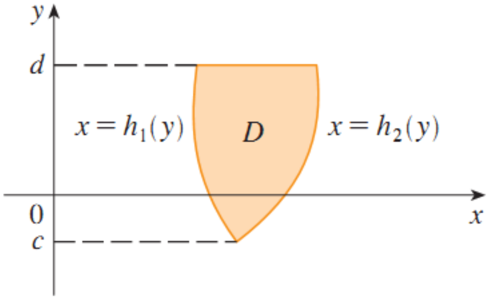
then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Plane Region of Type II

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$





Integration Formula for Plane Region of Type II

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example

Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$.

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

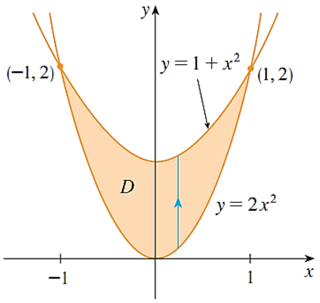
$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}$$



Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above D is

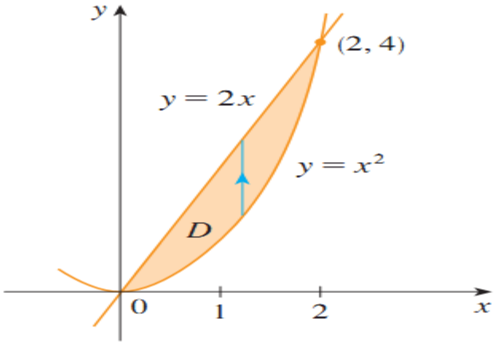
$$V = \iint_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx$$

$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Bigg]_0^2 = \frac{216}{35}$$



Example

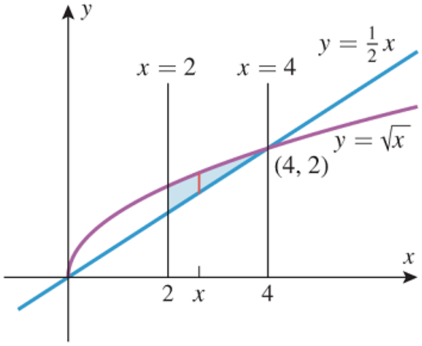
Evaluate $\iint\limits_R xy\, dA$

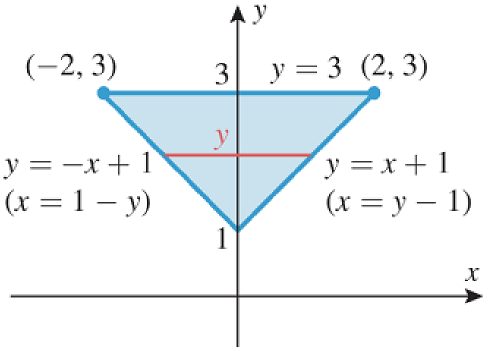
over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution

$$\iint\limits_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$ ◀





Example

Evaluate $\iint\limits_R (2x - y^2)\, dA$

over the triangular region R enclosed between the lines y = −x + 1, y = x + 1, and y = 3.

Solution(second figure above)

$$\iint\limits_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

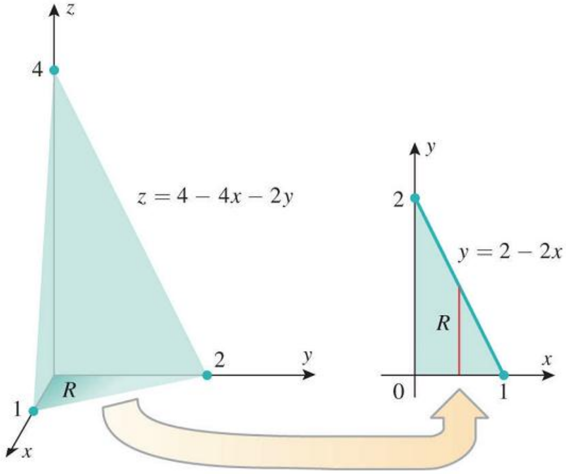
$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy$$

$$= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$ ◀

Example

Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution



The region R is bounded by the x-axis, the y-axis, and the line y = 2 − 2x [setting z = 0], so that treating R as a type I region yields

$$V = \iint_R (4 - 4x - 2y)\,dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\,dx = \frac{4}{3}$$

Example

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\,dy\,dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\,dx = 8(2\pi) = 16\pi$$

Example

Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

we prefer to express D as a type II region:

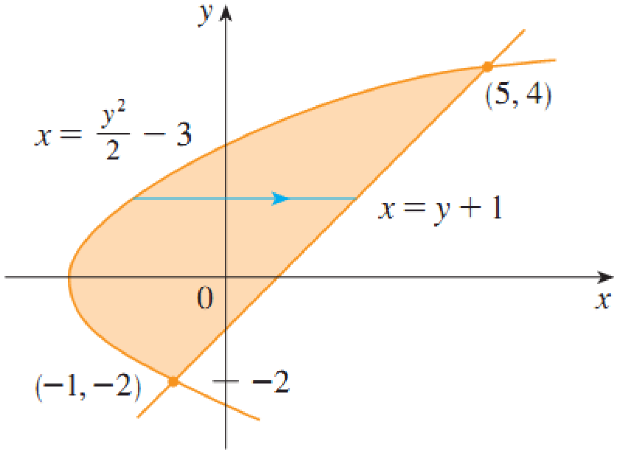
$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy = \int_{-2}^{4} \left[\frac{x^2}{2}\, y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2} \int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \tfrac{1}{2} \int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\,\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$



Example

Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x/2 \leqslant y \leqslant 1 - x/2\}$$

Therefore

$$\begin{aligned}
V &= \iint_D (2 - x - 2y)\, dA \\
&= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\
&= \int_0^1 \Big[2y - xy - y^2\Big]_{y=x/2}^{y=1-x/2}\, dx \\
&= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx \\
&= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{x^3}{3} - x^2 + x \Big]_0^1 = \frac{1}{3}
\end{aligned}$$

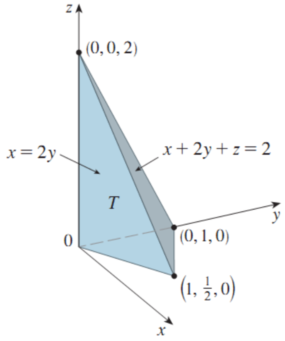


FIGURE 13

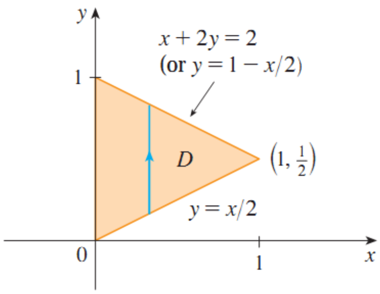


FIGURE 14

Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

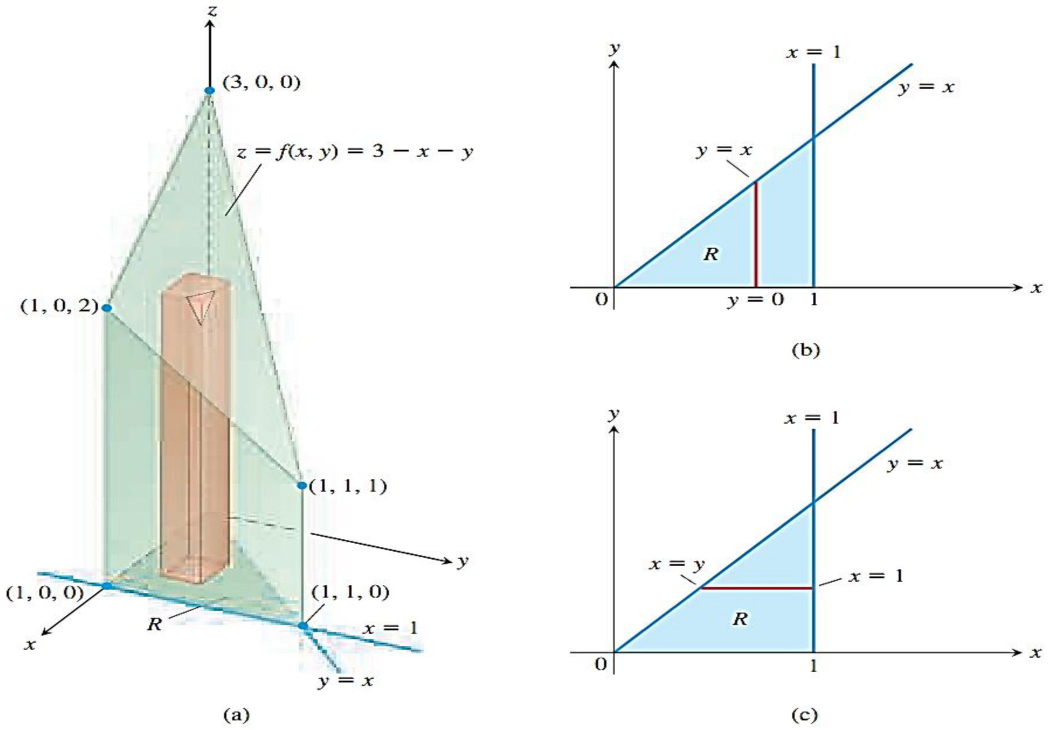
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines y = x and x = 1 and whose top lies in the plane $z = f(x, y) = 3 - x - y$.

Solution

See Figure. For any x between 0 and 1, y may vary from y = 0 to y = x (Figure b).



(a) (b) (c)

Hence,

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

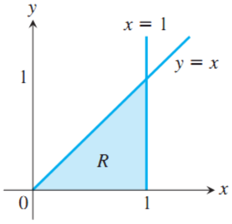
$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

Example

Calculate

$$\iint_R \frac{\sin x}{x}\,dA,$$



where R is the triangle in the xy-plane bounded by the x-axis, the line y = x, and the line x = 1.

Solution

The region of integration is shown in Figure. If we integrate first with respect to y and next with respect to x, then because x is held fixed in the first integration, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\,dy\right) dx = \int_0^1 \left[y\,\frac{\sin x}{x}\right]_{y=0}^{y=x} dx = \int_0^1 \sin x\,dx = -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy,$$

we run into a problem because $\int ((\sin x)/x)\,dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you irst choose doesn't work, try the other. Some-times neither order will work, and then we may need to use numerical approximations.

Example

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Solution

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

where $$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

Or

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

Then

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

$$= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin(y^2)\, dy = -\tfrac{1}{2} \cos(y^2)\Big]_0^1 = \tfrac{1}{2}(1 - \cos 1)$$

Example

estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

Solution

Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$

and therefore $e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$

Iterated Integrals with Nonconstant Limits of Integration

Example

Evaluate

$$\int_0^1 \int_{-x}^{x^2} y^2 x \, dy \, dx$$

Solution

$$\int_0^1 \int_{-x}^{x^2} y^2 x \, dy \, dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x \, dy \right] dx = \int_0^1 \left. \frac{y^3 x}{3} \right]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \Big]_0^1 = \frac{13}{120}$$

Example

Evaluate

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y \, dx \, dy$$

Solution

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y \, dx \, dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y \, dx \right] dy = \int_0^{\pi/3} \left. \frac{x^2}{2} \sin y \right]_{x=0}^{\cos y} dy$$

$$= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y \sin y \right] dy = -\frac{1}{6} \cos^3 y \Big]_0^{\pi/3} = \frac{7}{48} \quad ◀$$

Example(reversing the order of integration)

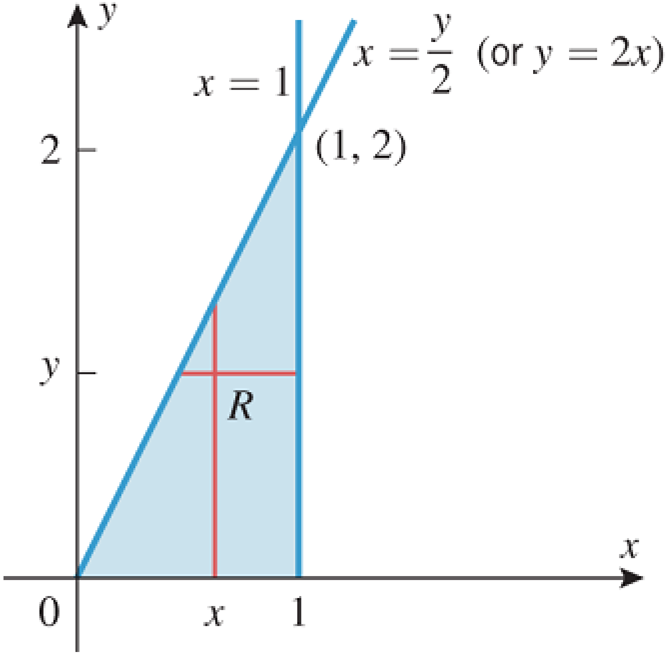
Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$

cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

Solution

For the inside integration, y is fixed and x varies from the line x = y/2 to the line x = 1 (Figure).



For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \iint_R e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\, dx = e^{x^2}\Big]_0^1 = e - 1$$

Example

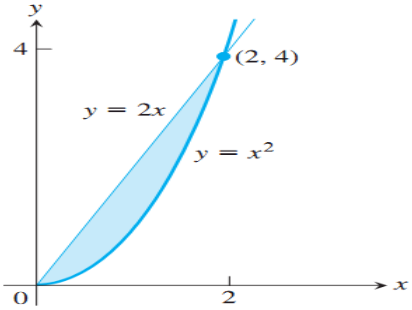
Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

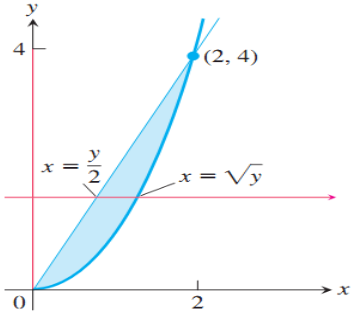
and write an equivalent integral with the order of integration reversed.

Solution

The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$



To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$



The integral is $$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8.

Example

Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution

Figure 15.18a shows the surface and the “wedgelike” solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\,dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

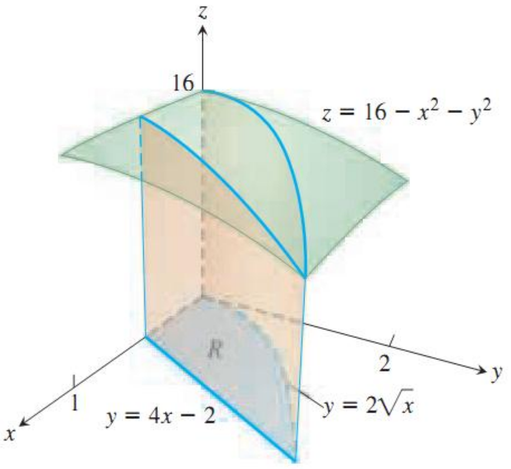
$$\iint_R (16 - x^2 - y^2)\,dA$$

$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\,dx\,dy$$

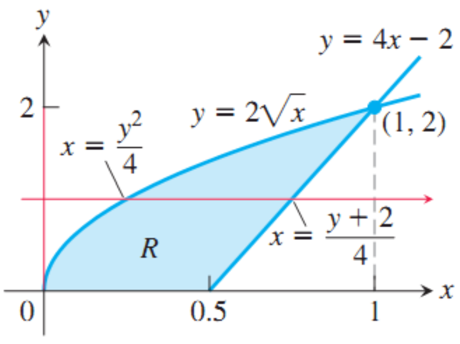
$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y + 2) - \frac{(y + 2)^3}{3\cdot 64} - \frac{(y + 2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4. \qquad ■$$



(a)

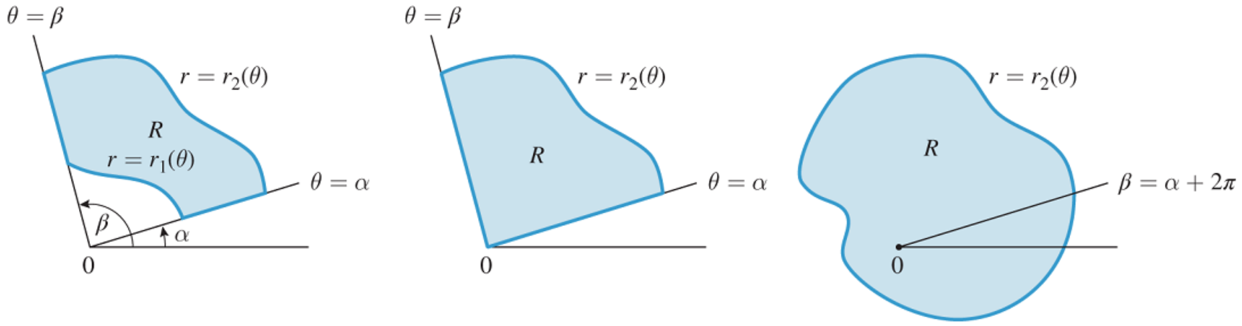


(b)

Simple Polar Region

A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \leq \beta$ (ii) $\beta - \alpha \leq 2\pi$ (iii) $0 \leq r_1(\theta) \leq r_2(\theta)$

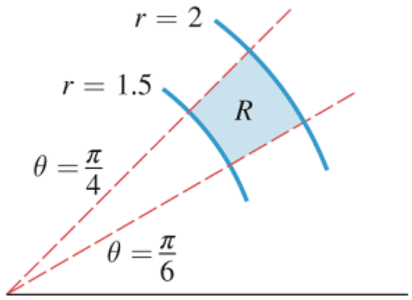


(a) (b) (c)

Polar Rectangle

A polar rectangle is a simple polar region for which the bounding polar curves are circular arcs. For example, Figure shows the polar rectangle R given by

$$1.5 \leq r \leq 2, \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}$$



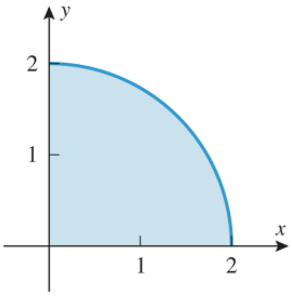
Example

Discuss quarter disk in rectangular as well as polar coordinates.

Solution

Quarter disk in rectangular coordinates;

$$0 \leq y \leq \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$



Quarter disk in polar coordinates;

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq \pi/2$$

Double Integrals in Polar Coordinates

If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example

Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

The region R can be described as

$$R = \{(x, y) \mid y \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$$

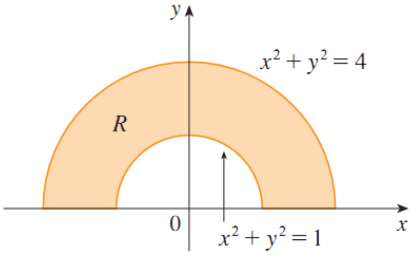
$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$

$$= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta = \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta$$

$$= \int_0^\pi \left[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\Big]_0^\pi = \frac{15\pi}{2}$$



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Example

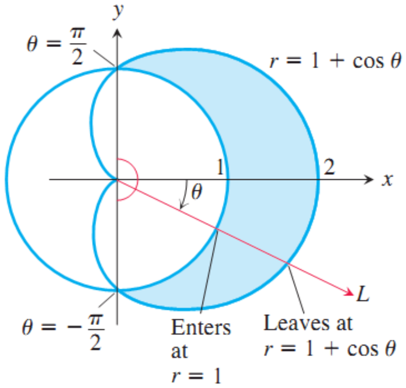
Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

Required limits of integration is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.



Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta.$$

Example

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

Changing Cartesian Integrals into Polar Integrals

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

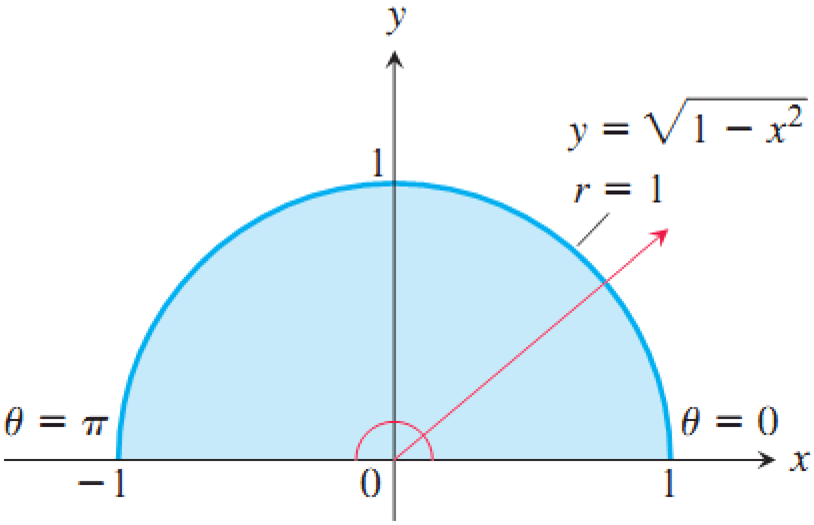
Example

Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$

Solution



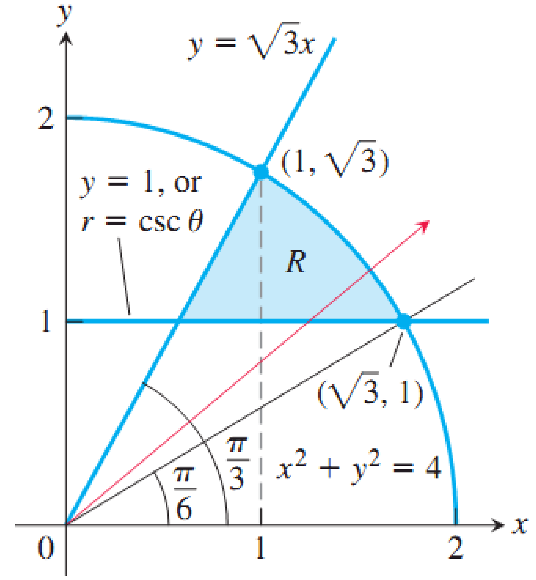
$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^{1} e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$

Example

Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution



$$\iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta$$

$$= \int_{\pi/6}^{\pi/3}\frac{1}{2}\left[4 - \csc^2\theta\right] d\theta$$

$$= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3}$$

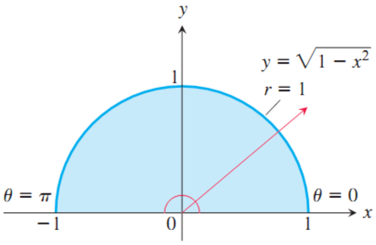
$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}.$$

Example

Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Solution



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

Example

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

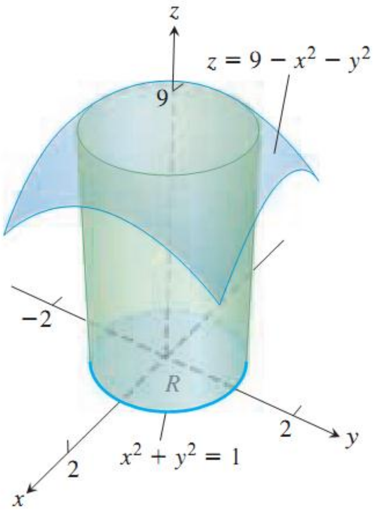
Solution

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$



Example

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution

If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

which is not easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\, dx$.

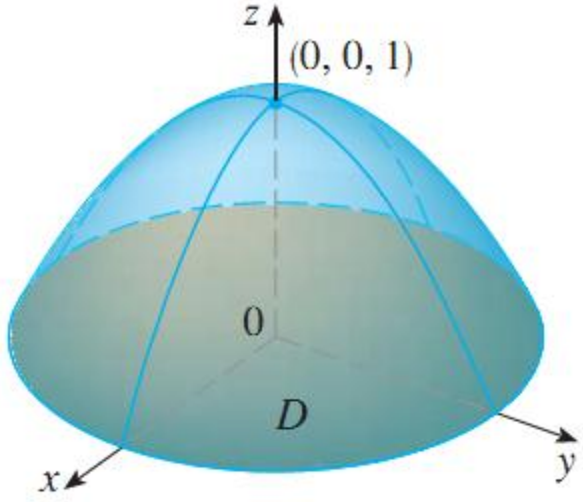
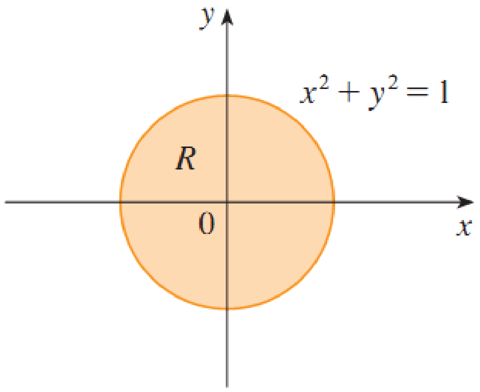


FIGURE 6



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

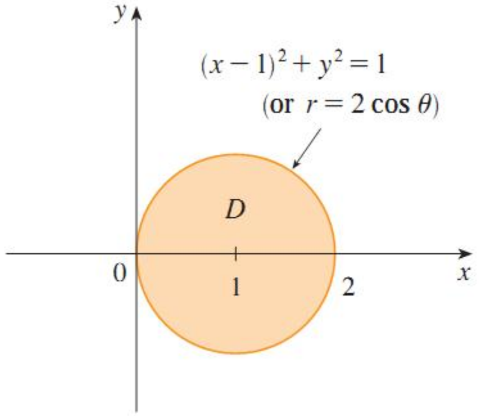
Example

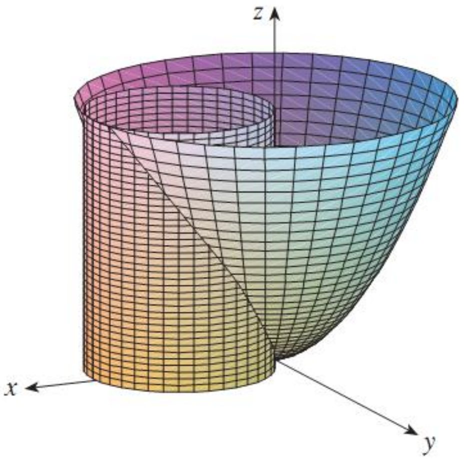
Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution

The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$(x - 1)^2 + y^2 = 1$





In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \left\{(r, \theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant r \leqslant 2\cos\theta\right\}$$

and, by Formula , we have

$$V = \iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\pi/2} \left[1 + 2\cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)\right] d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$ ■

Example

Evaluate

$$\iint_R \sin\theta \, dA$$

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

Solution

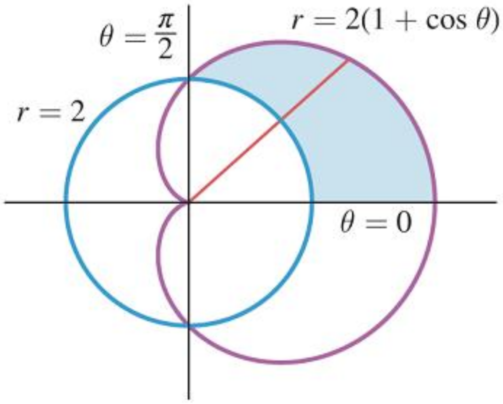
$$\iint_R \sin\theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2 \int_0^{\pi/2} [(1 + \cos\theta)^2 \sin\theta - \sin\theta] \, d\theta$$

$$= 2 \left[-\frac{1}{3}(1 + \cos\theta)^3 + \cos\theta\right]_0^{\pi/2}$$

$$= 2 \left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3} \blacktriangleleft$$



Example

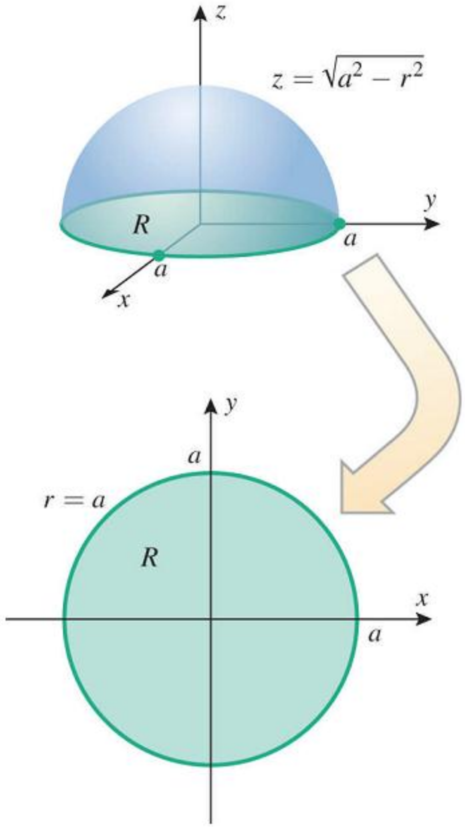
The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

Solution

$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3}a^3\, d\theta$$

$$= \left[\frac{2}{3}a^3\theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3$$ ◀



Remember

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example

Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution

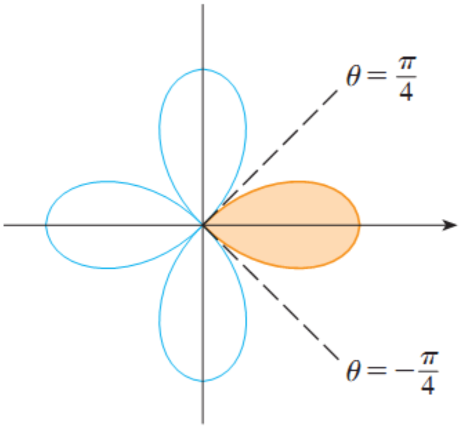
From the sketch of the curve in Figure, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\}$$

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta$$

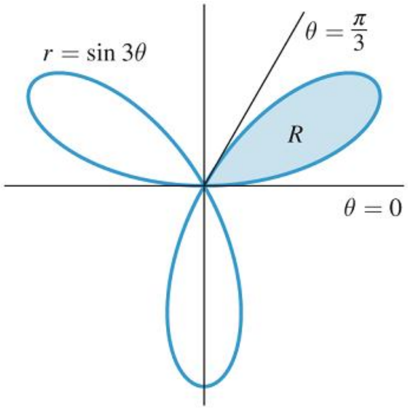
$$= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= \tfrac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\, d\theta = \tfrac{1}{4}\left[\theta + \tfrac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$



Example

Use a polar double integral to find the area enclosed by the three-petaled rose r = sin 3θ.



Solution

$$A = 3\iint_R dA = 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta$$

$$= \frac{3}{2}\int_0^{\pi/3}\sin^2 3\theta\,d\theta = \frac{3}{4}\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta$$

$$= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi \quad \blacktriangleleft$$

Example

Use polar coordinates to evaluate $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}(x^2+y^2)^{3/2}\,dy\,dx.$

Solution

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}(x^2+y^2)^{3/2}\,dy\,dx = \iint_R (x^2+y^2)^{3/2}\,dA$$

$$= \int_0^{\pi}\int_0^{1}(r^3)r\,dr\,d\theta = \int_0^{\pi}\frac{1}{5}\,d\theta = \frac{\pi}{5}$$

Remember

Lamina

An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a lamina (Figure). A lamina is called **homogeneous** if its composition is uniform throughout and **inhomogeneous** otherwise.

Density of a lamina

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

Mass of a lamina

$$m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA$$

Total Charge Per Unit Area

$$m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA$$

Example

Charge is distributed over the triangular region D in Figure so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter ($\mathrm{C/m^2}$). Find the total charge.

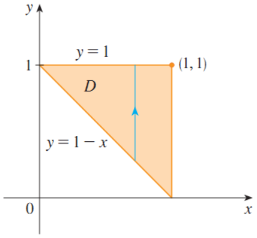
Solution

$$Q = \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \int_0^1 \left[x \frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} [1^2 - (1 - x)^2]\, dx$$

$$= \tfrac{1}{2} \int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}$$

Thus the total charge is $\frac{5}{24}$ C.



Remember

Moment of the entire lamina about the x-axis:

$$M_x = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D y\rho(x, y) \, dA$$

Moment of the entire lamina about the y-axis:

$$M_y = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x\rho(x, y) \, dA$$

Center of Mass

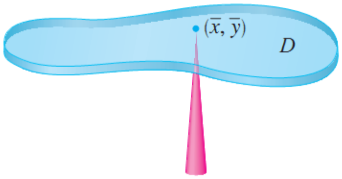
The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y) \, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y) \, dA$$

where the mass m is given by $$m = \iint_D \rho(x, y) \, dA$$

The physical significance of the center of mass

The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus the lamina balances horizontally when supported at its center of mass.



Example'

A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density function δ(x, y) = xy. Find its total mass.

Solution

$$M = \iint_R \delta(x, y) \, dA = \iint_R xy \, dA = \int_0^1 \int_0^{-x+1} xy \, dy \, dx$$

$$= \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{-x+1} dx = \int_0^1 \left[\frac{1}{2} x^3 - x^2 + \frac{1}{2} x \right] dx = \frac{1}{24} \text{ (unit of mass)}$$

Mass and First Moment Formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta \, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x\,\delta\, dV, \qquad M_{xz} = \iiint\limits_D y\,\delta\, dV, \qquad M_{xy} = \iiint\limits_D z\,\delta\, dV$$

Center of mass: $\bar{x} = \dfrac{M_{yz}}{M}, \quad \bar{y} = \dfrac{M_{xz}}{M}, \quad \bar{z} = \dfrac{M_{xy}}{M}$

TWO-DIMENSIONAL PLATE

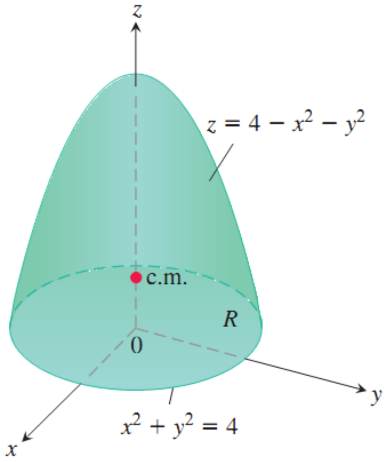
Mass: $M = \iint\limits_R \delta \, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x\,\delta\, dA, \quad M_x = \iint\limits_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \quad \bar{y} = \dfrac{M_x}{M}$

Example’

Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$



Solution

By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$M_{xy} = \iint\limits_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint\limits_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx$$

$$= \frac{\delta}{2}\iint\limits_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2 r\,dr\,d\theta \qquad \text{Polar coordinates simplify the integration.}$$

$$= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

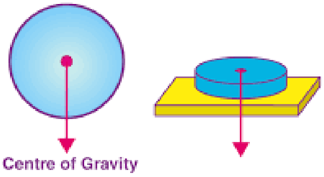
A similar calculation gives the mass

$$M = \iint\limits_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$.

CENTER OF GRAVITY OF AN INHOMOGENEOUS LAMINA

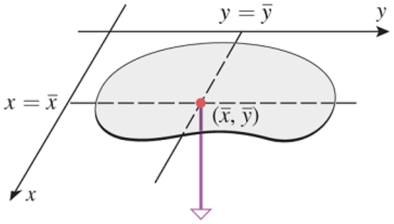
The Centre of gravity is **an imaginary point in a body of matter** where the total weight of the body is thought to be concentrated.



Problem

Suppose that a lamina with a continuous density function $\delta(x, y)$ occupies a region R in a horizontal xy-plane. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity.

Solution



$$\Delta M_k \approx \delta(x_k^*, y_k^*)\Delta A_k$$

$$\sum_{k=1}^{n} (x_k^* - \bar{x})\Delta M_k = \sum_{k=1}^{n} (x_k^* - \bar{x})\delta(x_k^*, y_k^*)\Delta A_k \approx 0$$

$$\sum_{k=1}^{n} (y_k^* - \bar{y})\Delta M_k = \sum_{k=1}^{n} (y_k^* - \bar{y})\delta(x_k^*, y_k^*)\Delta A_k \approx 0$$

$$\lim_{n\to+\infty} \sum_{k=1}^{n} (x_k^* - \bar{x})\delta(x_k^*, y_k^*)\Delta A_k = 0$$

$$\lim_{n\to+\infty} \sum_{k=1}^{n} (y_k^* - \bar{y})\delta(x_k^*, y_k^*)\Delta A_k = 0$$

$$\iint_R (x - \bar{x})\delta(x, y)\, dA = \iint_R x\delta(x, y)\, dA - \bar{x} \iint_R \delta(x, y)\, dA = 0$$

$$\iint_R (y - \bar{y})\delta(x, y)\, dA = \iint_R y\delta(x, y)\, dA - \bar{y} \iint_R \delta(x, y)\, dA = 0$$

After simplification we have

Center of Gravity $(\bar{x}, \bar{y})$ *of a Lamina*

$$\bar{x} = \frac{\displaystyle\iint_R x\delta(x,y)\,dA}{\displaystyle\iint_R \delta(x,y)\,dA}, \qquad \bar{y} = \frac{\displaystyle\iint_R y\delta(x,y)\,dA}{\displaystyle\iint_R \delta(x,y)\,dA}$$

In both formulas the denominator is the mass M of the lamina. The numerator in the formula for $\bar{x}$ is denoted by M_y and is called the **first moment of the lamina about the y-axis**; the numerator of the formula for $\bar{y}$ is denoted by M_x and is called the **first moment of the lamina about the x-axis**. Thus, Formulas can be expressed as

Alternative Formulas for Center of Gravity $(\bar{x}, \bar{y})$ *of a Lamina*

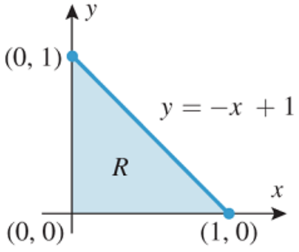
$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R}\iint_R x\delta(x,y)\,dA$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R}\iint_R y\delta(x,y)\,dA$$

Example

Find the center of gravity of the triangular lamina with vertices $(0,0)$, $(0,1)$, and $(1,0)$ and density function $\delta(x,y) = xy$.

Solution



$$M = \iint_R \delta(x,y)\,dA = \iint_R xy\,dA = \frac{1}{24}$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\delta(x,y)\,dA = \iint_R x^2y\,dA = \int_0^1 \int_0^{-x+1} x^2y\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right) dx = \frac{1}{60}$$

and the moment about the x-axis is

$$M_x = \iint_R y\delta(x,y)\,dA = \iint_R xy^2\,dA = \int_0^1 \int_0^{-x+1} xy^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3}xy^3\right]_{y=0}^{-x+1} dx = \int_0^1 \left(-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\right) dx = \frac{1}{60}$$

$$\bar{x} = \frac{M_y}{M} = \frac{1/60}{1/24} = \frac{2}{5}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/60}{1/24} = \frac{2}{5}$$

so the center of gravity is $\left(\frac{2}{5}, \frac{2}{5}\right)$.

The centroid of the lamina /The centroid of the region R

The center of gravity of a homogeneous lamina is called the centroid of the lamina or sometimes the centroid of the region R.

$$\bar{x} = \frac{\displaystyle\iint_R x\,dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R x\,dA \qquad \bar{y} = \frac{\displaystyle\iint_R y\,dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R y\,dA$$

Centroid of the Object

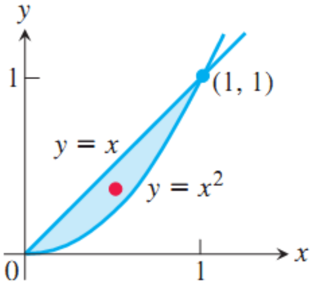
When the density of a solid object or plate is constant, the center of mass is called the centroid of the object.

Example

Find the centroid of the region in the first quadrant that is bounded above by the line y = x and below by the parabola y = x^2.

Solution

We sketch the region and include enough detail to determine the limits of integration (Figure).



We then set δ equal to 1 and evaluate the appropriate formulas;

$$M = \int_0^1 \int_{x^2}^{x} 1\,dy\,dx = \int_0^1 \Big[y \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x\,dy\,dx = \int_0^1 \Big[xy \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

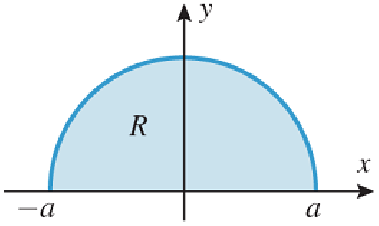
From these values of M, M_x, and M_y, we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $(1/2, 2/5)$.

Example

Find the centroid of the semicircular region in Figure



Solution

By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance.

$$\bar{y} = \frac{1}{\text{area of } R} \iint_R y\,dA = \frac{1}{\frac{1}{2}\pi a^2} \iint_R y\,dA$$

$$= \frac{1}{\frac{1}{2}\pi a^2} \int_0^{\pi} \int_0^{a} (r \sin\theta) r\,dr\,d\theta$$

Evaluating in polar coordinates

$$= \frac{1}{\frac{1}{2}\pi a^2} \int_0^{\pi} \left[\frac{1}{3} r^3 \sin\theta\right]_{r=0}^{a} d\theta$$

$$= \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{1}{3}a^3\right) \int_0^{\pi} \sin\theta\,d\theta = \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{2}{3}a^3\right) = \frac{4a}{3\pi}$$

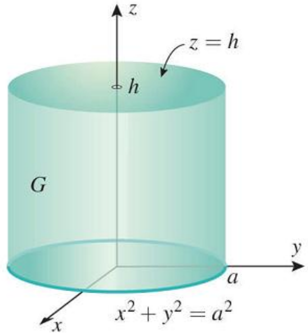
so the centroid is $\left(0, \dfrac{4a}{3\pi}\right)$.

Center of Gravity and Centroid of a Solid

Center of Gravity $(\bar{x}, \bar{y}, \bar{z})$ ***of a Solid G***	***Centroid*** $(\bar{x}, \bar{y}, \bar{z})$ ***of a Solid G***
$\bar{x} = \dfrac{1}{M} \iiint_G x\delta(x, y, z)\,dV$	$\bar{x} = \dfrac{1}{V} \iiint_G x\,dV$
$\bar{y} = \dfrac{1}{M} \iiint_G y\delta(x, y, z)\,dV$	$\bar{y} = \dfrac{1}{V} \iiint_G y\,dV$
$\bar{z} = \dfrac{1}{M} \iiint_G z\delta(x, y, z)\,dV$	$\bar{z} = \dfrac{1}{V} \iiint_G z\,dV$

Example

Find the mass and the center of gravity of a cylindrical solid of height h and radius a (Figure), assuming that the density at each point is proportional to the distance between the point and the base of the solid.



Solution

Since the density is proportional to the distance z from the base, the density function has the form δ(x, y, z) = kz, where k is some (unknown) positive constant of proportionality. The mass of the solid is

$$M = \iiint_G \delta(x,y,z)\,dV = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h kz\,dz\,dy\,dx$$

$$= k\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{2}h^2\,dy\,dx$$

$$= kh^2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$$

$$= \tfrac{1}{2}kh^2\pi a^2$$

Interpret the integral as the area of a semicircle.

Without additional information, the constant k cannot be determined. However, as we will now see, the value of k does not affect the center of gravity.

$$\bar{z} = \frac{1}{M}\iiint_G z\delta(x,y,z)\,dV = \frac{1}{\frac{1}{2}kh^2\pi a^2}\iiint_G z\delta(x,y,z)\,dV$$

$$= \frac{1}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h z(kz)\,dz\,dy\,dx$$

$$= \frac{k}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{1}{3}h^3\,dy\,dx$$

$$= \frac{\frac{1}{3}kh^3}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}2\sqrt{a^2-x^2}\,dx$$

$$= \frac{\frac{1}{3}kh^3\pi a^2}{\frac{1}{2}kh^2\pi a^2} = \frac{2}{3}h$$

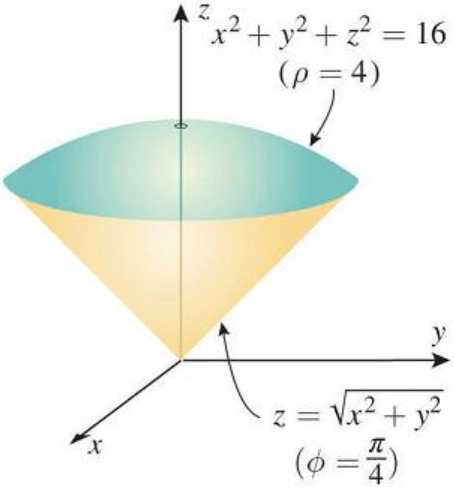
This yields $\bar{x} = \bar{y} = 0.$
However, this is evident by inspection, since it follows from the symmetry of the solid and the form of its density function that the center of gravity is on the z-axis. Thus, the center of gravity is $(0,0,\frac{2}{3}h)$.

Example'

Find the centroid of the solid G bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = 16.$

Solution

The solid G is sketched in Figure.



In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2+y^2}$ is

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta}$$

which simplifies to

$$\rho \cos \phi = \rho \sin \phi$$

Dividing both sides of this equation by ρ cos φ yields tan φ = 1, from which it follows that

$$\varphi = \pi/4$$

$$\begin{aligned} V = \iiint_G dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin \phi \right]_{\rho=0}^{4} d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{64}{3} \sin \phi \, d\phi \, d\theta \\ &= \frac{64}{3} \int_0^{2\pi} [-\cos \phi]_{\phi=0}^{\pi/4} d\theta = \frac{64}{3} \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2} \right) d\theta \\ &= \frac{64\pi}{3}(2 - \sqrt{2}) \end{aligned}$$

By symmetry, the centroid $(\bar{x}, \bar{y}, \bar{z})$ is on the z-axis, so $\bar{x} = \bar{y} = 0$. In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone

$z = \sqrt{x^2 + y^2}$ is $\phi = \pi/4$,

$$\bar{z} = \frac{1}{V} \iiint_G z \, dV = \frac{1}{V} \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^4}{4}\cos\phi\sin\phi\right]_{\rho=0}^{4} d\phi\, d\theta$$

$$= \frac{64}{V}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos\phi\, d\phi\, d\theta = \frac{64}{V}\int_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_{\phi=0}^{\pi/4} d\theta$$

$$= \frac{16}{V}\int_0^{2\pi} d\theta = \frac{32\pi}{V} = \frac{3}{2(2-\sqrt{2})}$$

The centroid of G is

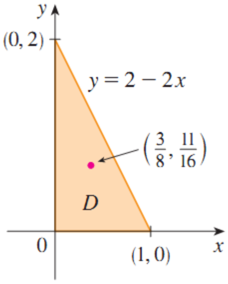
$$(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \frac{3}{2(2-\sqrt{2})}\right) \approx (0, 0, 2.561)$$

Example'

Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

Solution

The triangle is shown in Figure. (Note that the equation of the upper boundary is y = 2 - 2x.)



The mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2)\, dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x, y)\, dA = \tfrac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \left[xy + 3x^2 y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{3}{2} \int_0^1 (x - x^3)\, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

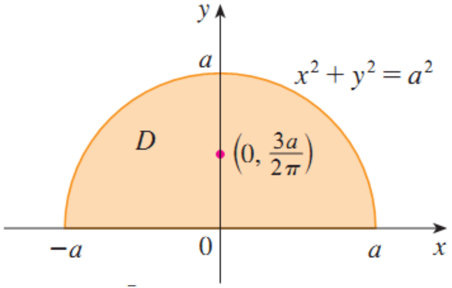
The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Example'

The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Solution

Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$.



Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$. Thus the mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \iint_D K\sqrt{x^2 + y^2}\, dA$$

$$= \int_0^{\pi}\int_0^a (Kr)\, r\, dr\, d\theta = K\int_0^{\pi} d\theta \int_0^a r^2\, dr = K\pi \frac{r^3}{3}\Big]_0^a = \frac{K\pi a^3}{3}$$

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$. The y-coordinate is given by

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \frac{3}{K\pi a^3} \int_0^{\pi} \int_0^{a} r \sin\theta\, (Kr)\, r\, dr\, d\theta$$

$$= \frac{3}{\pi a^3} \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} r^3\, dr = \frac{3}{\pi a^3} \Big[-\cos\theta\Big]_0^{\pi} \left[\frac{r^4}{4}\right]_0^{a}$$

$$= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}$$

Therefore the center of mass is located at the point $(0, 3a/(2\pi))$.

Moment of Inertia

The moment of inertia (also called the second moment) of a particle of mass m about an axis is defined to be $\mathbf{I = mr^2}$, where r is the distance from the particle to the axis.

The moment of inertia of the lamina about the x-axis:

$$I_x = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

The moment of inertia of the lamina about the y-axis:

$$I_y = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA$$

The Moment of Inertia about the Origin, also called the Polar Moment of Inertia:

$$I_0 = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2)\, \rho(x, y)\, dA$$

Note that $I_0 = I_x + I_y$.

Moment of Inertia for Three-Dimensional Solid

About the x-axis: $$I_x = \iiint (y^2 + z^2)\,\delta\, dV \qquad \delta = \delta(x, y, z)$$

About the y-axis: $$I_y = \iiint (x^2 + z^2)\,\delta\, dV$$

About the z-axis: $$I_z = \iiint (x^2 + y^2)\,\delta\, dV$$

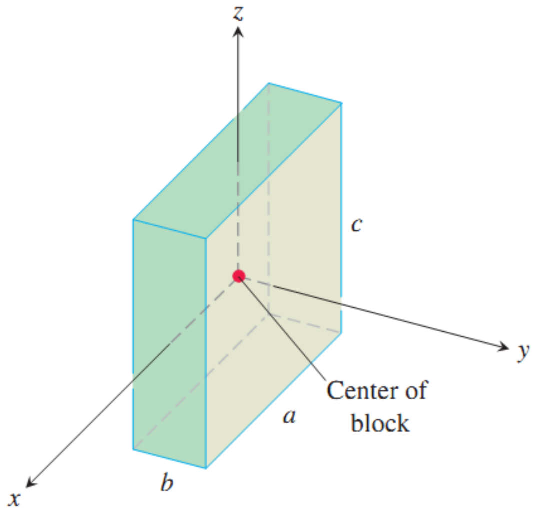
About a line L: $$I_L = \iiint r^2(x, y, z)\,\delta\, dV$$

$r(x, y, z) =$ distance from the point (x, y, z) to line L

Example'

Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in

Figure



Solution

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2 + z^2)\,\delta\, dx\, dy\, dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z since δ is constant. The rectangular solid consists of eight symmetric

pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$
\begin{aligned}
I_x &= 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2}(y^2+z^2)\,\delta\,dx\,dy\,dz = 4a\delta\int_0^{c/2}\int_0^{b/2}(y^2+z^2)\,dy\,dz \\
&= 4a\delta\int_0^{c/2}\left[\frac{y^3}{3}+z^2y\right]_{y=0}^{y=b/2}dz \\
&= 4a\delta\int_0^{c/2}\left(\frac{b^3}{24}+\frac{z^2b}{2}\right)dz \\
&= 4a\delta\left(\frac{b^3c}{48}+\frac{c^3b}{48}\right) = \frac{abc\delta}{12}(b^2+c^2) = \frac{M}{12}(b^2+c^2). \qquad M = abc\delta
\end{aligned}
$$

Similarly,

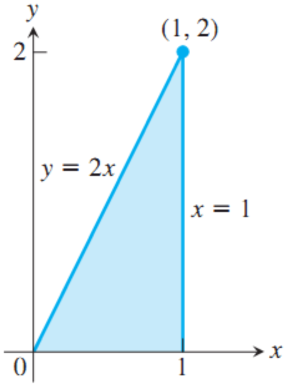
$$I_y = \frac{M}{12}(a^2+c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2+b^2).$$

Example

A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. Find the plate's moments of inertia about the coordinate axes and the origin.

Solution

$$
\begin{aligned}
I_x &= \int_0^1\int_0^{2x} y^2\,\delta(x,y)\,dy\,dx = \int_0^1\int_0^{2x}(6xy^2+6y^3+6y^2)\,dy\,dx \\
&= \int_0^1\left[2xy^3+\frac{3}{2}y^4+2y^3\right]_{y=0}^{y=2x}dx = \int_0^1(40x^4+16x^3)\,dx \\
&= \left[8x^5+4x^4\right]_0^1 = 12.
\end{aligned}
$$



Similarly, the moment of inertia about the y-axis is

$$I_y = \int_0^1\int_0^{2x} x^2\,\delta(x,y)\,dy\,dx = \frac{39}{5}.$$

$$I_0 = I_x + I_y$$

$$I_0 = 12 + \frac{39}{5} = \frac{60+39}{5} = \frac{99}{5}.$$

Remark

The mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

The moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \frac{\pi\rho a^4}{2} = \tfrac{1}{2}(\rho\pi a^2)a^2 = \tfrac{1}{2}ma^2$$

Example

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution

The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant a$. Let's compute I_0 first:

$$\begin{aligned} I_0 &= \iint_D (x^2 + y^2)\rho \, dA = \rho \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta \\ &= \rho \int_0^{2\pi} d\theta \int_0^a r^3 \, dr = 2\pi\rho \left[\frac{r^4}{4} \right]_0^a = \frac{\pi\rho a^4}{2} \end{aligned}$$

Instead of computing I_x and I_y directly, we use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry of the problem). Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}$$

Remember

If we increase the mass or the radius of the disk, we thereby increase the moment of inertia. In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

The Radius Of Gyration Of A Lamina ‘

The radius of gyration of a lamina about an axis is the number R such that $I = mR^2$ where m is the mass of the lamina and I is the moment of inertia about the given axis.

Equation $I = mR^2$ says that if the mass of the lamina were concentrated at a distance R from the axis, then the moment of inertia of this “point mass” would be the same as the moment of inertia of the lamina.

In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the -axis and the radius of gyration $\bar{\bar{x}}$ with respect to the -axis are given by the equations

$$m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y$$

Thus $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes. (Note the analogy with the center of mass.)

Example

Find the radius of gyration about the x-axis of the disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution

As noted, the mass of the disk is $m = \rho\pi a^2$, so

$$\bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{\frac{1}{4}\pi\rho a^4}{\rho\pi a^2} = \frac{a^2}{4}$$

Therefore the radius of gyration about the x-axis is $\bar{\bar{y}} = \frac{1}{2}a$, which is half the radius of the disk.

Area Calculated as a Double Integral

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA$$

Example

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution

The region R may be treated equally well as type I (Figure a) or type II (Figure b). Treating R as type I yields

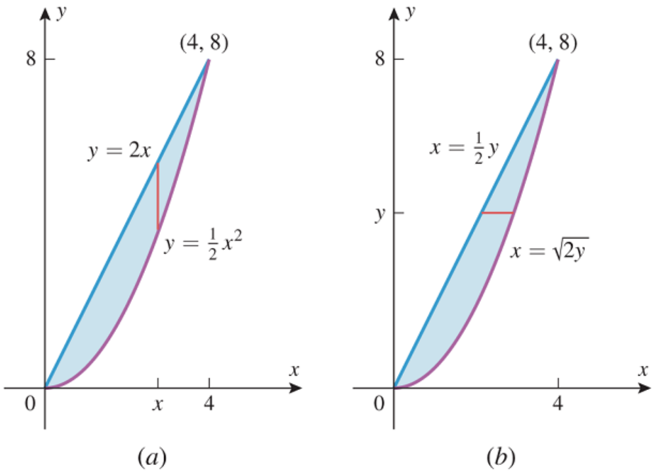
$$\text{area of } R = \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\, dx = \int_0^4 [y]_{y=x^2/2}^{2x}\, dx$$

$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

Treating R as type II yields

$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\, dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\, dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}$$



Example

Find the area of the region R bounded by y = x and y = x^2 in the first quadrant.

Solution

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Example

Find the area of the region R enclosed by the parabola y = x^2 and the line y = x + 2.

Solution

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$

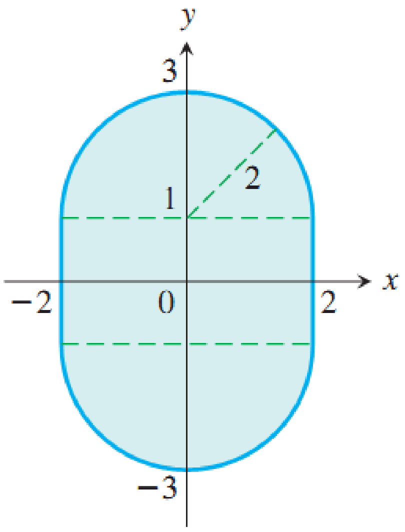
$$A = \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}.$$

Example

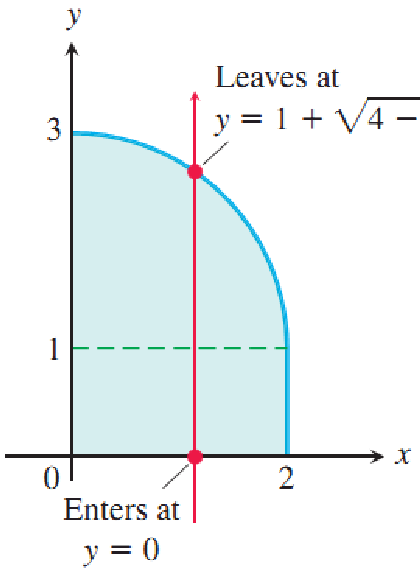
Find the area of the playing field described by $R: -2 \le x \le 2, -1 - \sqrt{4 - x^2} \le y \le 1 + \sqrt{4 - x^2}$, using

(a) Fubini's Theorem **(b)** Simple geometry.

Solution



(a)



(b)

(a) From the symmetries observed in the figure, we see that the area of R is 4 times its area in the first quadrant. As shown in Figure 15.21b, a vertical line at x enters this part of the region at $y = 0$ and exits at $y = 1 + \sqrt{4 - x^2}$. Therefore, using Fubini's Theorem, we have

$$\begin{aligned} A &= \iint_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\, dx \\ &= 4\int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx \\ &= 4\left[x + \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 \\ &= 4\left(2 + 0 + 2\cdot\frac{\pi}{2} - 0\right) = 8 + 4\pi. \end{aligned}$$

Integral Table Formula 45

(b) The region R consists of a rectangle mounted on two sides by half disks of radius 2. The area can be computed by summing the area of the 4×2 rectangle and the area of a circle of radius 2, so

$$A = 8 + \pi 2^2 = 8 + 4\pi.$$

■

Average Value

$$\textbf{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\, dA.$$

Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle
$R: 0 \le x \le \pi, 0 \le y \le 1.$

Solution

The value of the integral of f over R is

$$\int_0^{\pi}\int_0^1 x\cos xy\, dy\, dx = \int_0^{\pi}\left[\sin xy\right]_{y=0}^{y=1} dx$$

$$\int x \cos xy\, dy = \sin xy + C$$

$$= \int_0^{\pi} (\sin x - 0)\, dx = -\cos x\Big]_0^{\pi} = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

Surface Area

The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

If we use the alternative notation for partial derivatives, we can rewrite Formula as follows:

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Notice the similarity between the surface area and the arc length formula:

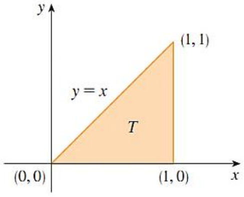
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Solution

The region is shown in Figure and is described by T



$$T = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\}$$

Using Formula with $f(x, y) = x^2 + 2y$, we get

$$A = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx$$

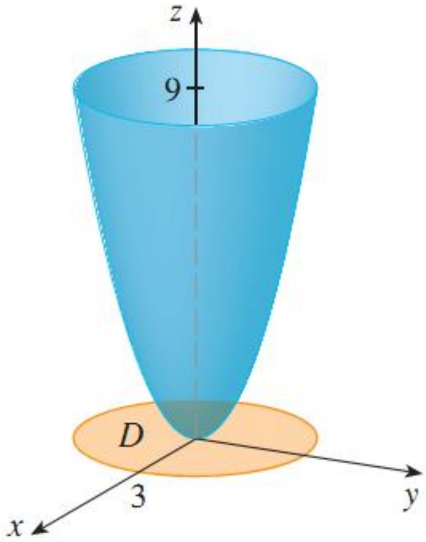
$$= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 5)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(27 - 5\sqrt{5}\right)$$

Example

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3.



$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates, we obtain

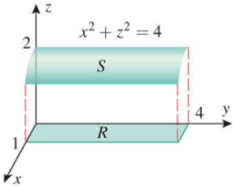
$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \tfrac{1}{8}\sqrt{1 + 4r^2}\,(8r)\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

Example

Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy-plane whose coordinates satisfy $0 \le x \le 1$ and $0 \le y \le 4$.

Solution



$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

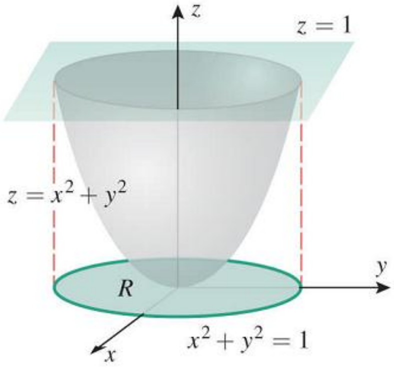
$$= \iint_R \sqrt{\left(-\frac{x}{\sqrt{4 - x^2}}\right)^2 + 0 + 1}\, dA = \int_0^4 \int_0^1 \frac{2}{\sqrt{4 - x^2}}\, dx\, dy$$

$$= 2\int_0^4 \left[\sin^{-1}\left(\frac{1}{2}x\right)\right]_{x=0}^{1} dy = 2\int_0^4 \frac{\pi}{6}\, dy = \frac{4}{3}\pi \blacktriangleleft$$

Example

Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

Solution (conversion to polar form)



$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$S = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_{r=0}^{1} d\theta$$

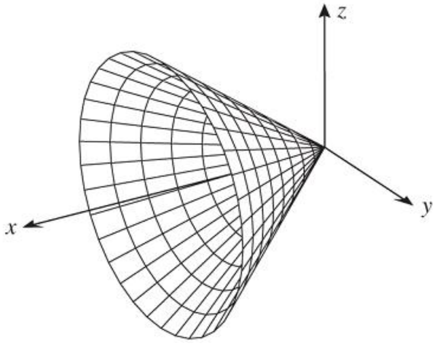
$$= \int_0^{2\pi} \frac{1}{12}(5\sqrt{5} - 1)\, d\theta = \frac{1}{6}\pi(5\sqrt{5} - 1) \blacktriangleleft$$

Surface Area of Parametric Surfaces

$$S = \iint\limits_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

Example

It follows that the parametric equations x = u, y = u cos v, z = u sin v represent the cone that results when the line y = x in the xy-plane is revolved about the x-axis. Use Formula to find the surface area of that portion of the cone for which $0 \leq u \leq 2$and$0 \leq v \leq 2\pi$.



Solution

$$\mathbf{r} = u\mathbf{i} + u\cos v\mathbf{j} + u\sin v\mathbf{k} \quad (0 \leq u \leq 2,\ \ 0 \leq v \leq 2\pi)$$

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + \cos v\mathbf{j} + \sin v\mathbf{k} \qquad \frac{\partial \mathbf{r}}{\partial v} = -u\sin v\mathbf{j} + u\cos v\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix} = u\mathbf{i} - u\cos v\mathbf{j} - u\sin v\mathbf{k}$$

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{u^2 + (-u\cos v)^2 + (-u\sin v)^2} = |u|\sqrt{2} = u\sqrt{2}$$

$$S = \iint\limits_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA = \int_0^{2\pi} \int_0^2 \sqrt{2}u\, du\, dv = 2\sqrt{2}\int_0^{2\pi} dv = 4\pi\sqrt{2}$$

Triple Integrals

The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Fubini's Theorem for Triple Integrals

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Example

Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

Solution

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\Bigg]_0^3 = \frac{27}{4}$$

Example

Evaluate the triple integral

$$\iiint\limits_G 12xy^2z^3\,dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution

$$\iiint\limits_G 12xy^2z^3\,dV = \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3\,dz\,dy\,dx$$

$$= \int_{-1}^{2}\int_0^3 \left[3xy^2z^4\right]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_0^3 48xy^2\,dy\,dx$$

$$= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3}\,dx = \int_{-1}^{2} 432x\,dx$$

$$= 216x^2\Big]_{-1}^{2} = 648 \blacktriangleleft$$

Example

Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint\limits_G z\,dV$$

Solution

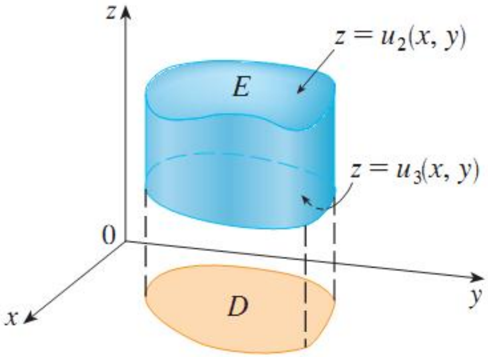
$$\iiint\limits_G z\,dV = \int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1\int_0^y \frac{1}{2}z^2\Big]_{z=0}^{\sqrt{1-y^2}}\,dx\,dy$$

$$= \int_0^1\int_0^y \frac{1}{2}(1-y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1-y^2)x\Big]_{x=0}^{y}\,dy$$

$$= \frac{1}{2}\int_0^1 (y-y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{8} \blacktriangleleft$$

Triple Integral over a General Bounded Region E of Type I
If E is a plane region of Type I

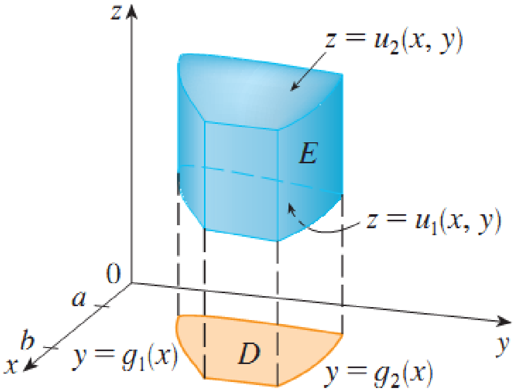
$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$



Then

$$\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA$$

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure),



Then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

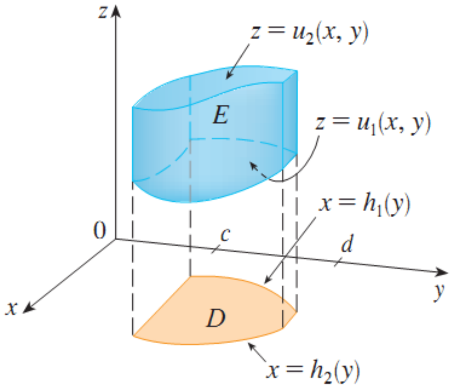
And integral is of the form

$$\iiint\limits_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

Triple Integral over a General Bounded Region E of Type II

If E is a plane region of Type II

$$E = \{(x, y, z) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y),\ u_1(x, y) \leq z \leq u_2(x, y)\}$$



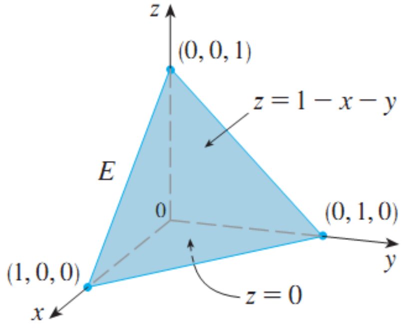
Then

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

Example

Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution



When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure) and one of its projection D onto the xy-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula . Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure , and we have

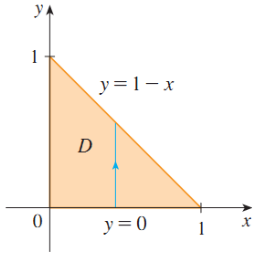
$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$



Triple Integral over a Solid Region

A solid region is of type 2 if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

Then

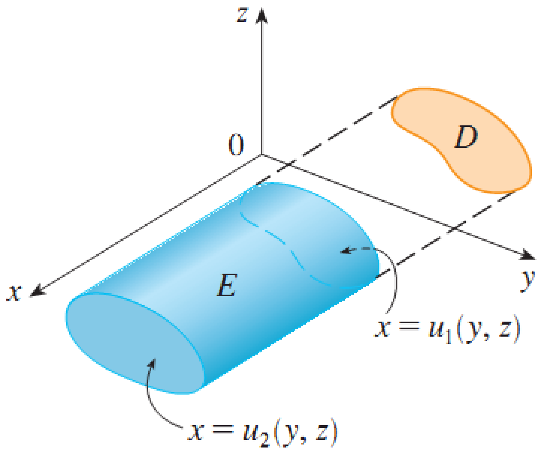
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA$$

Also A solid region is of type 3 if it is of the form

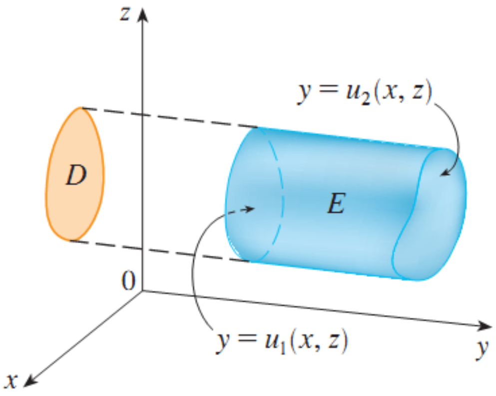
$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA$$



A Type 2 Region



A Type 3 Region

Example

Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution

The solid E is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

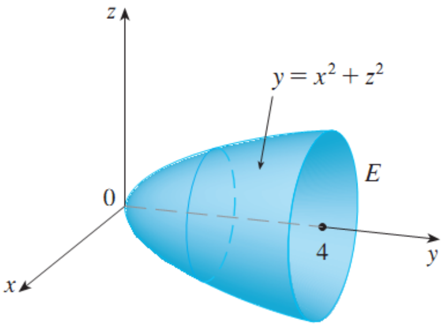


FIGURE 9
Region of integration

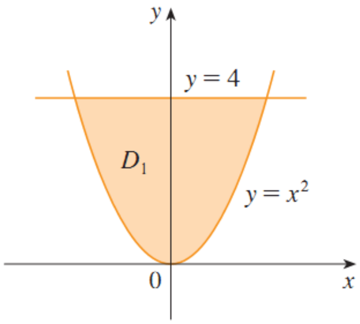


FIGURE 10
Projection onto xy-plane

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz-plane is the disk $x^2 + z^2 \leqslant 4$ shown in Figure 11.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right] dA = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

Although this integral could be written as

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\begin{aligned}\iiint_E \sqrt{x^2 + z^2}\, dV &= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr \\ &= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15}\end{aligned}$$

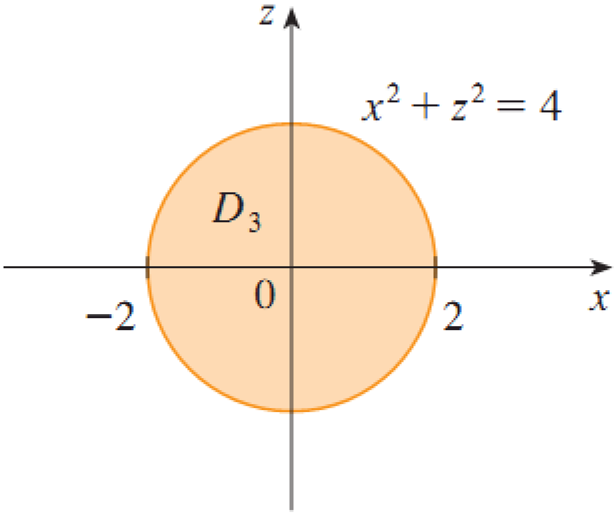


FIGURE 11

Projection onto xz-plane

Example

Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Solution

$$\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx = \iiint\limits_E f(x, y, z)\, dV$$

where $E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2, 0 \leqslant z \leqslant y\}$. This description of E enables us to write projections onto the three coordinate planes as follows:

$$\begin{aligned}
&\text{on the } xy\text{-plane:} && D_1 = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\} \\
& && = \left\{(x, y) \mid 0 \leqslant y \leqslant 1, \sqrt{y} \leqslant x \leqslant 1\right\} \\
&\text{on the } yz\text{-plane:} && D_2 = \{(x, y) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y\} \\
&\text{on the } xz\text{-plane:} && D_3 = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant x^2\}
\end{aligned}$$

From the resulting sketches of the projections in Figure 12 we sketch the solid E in Figure 13. We see that it is the solid enclosed by the planes $z = 0$, $x = 1$, $y = z$ and the parabolic cylinder $y = x^2$ $\left(\text{or } x = \sqrt{y}\right)$.

If we integrate first with respect to x, then z, and then y, we use an alternate description of E:

$$E = \left\{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant y, \sqrt{y} \leqslant x \leqslant 1\right\}$$

Thus

$$\iiint\limits_E f(x, y, z)\, dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z)\, dx\, dz\, dy$$

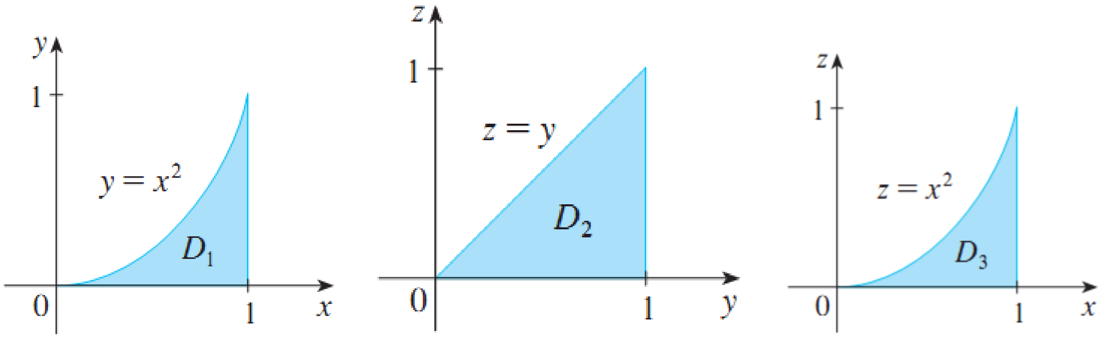


FIGURE 12 Projections of E

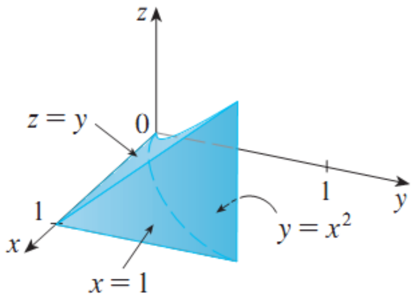


FIGURE 13
The solid E

Example

Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution

The tetrahedron T and its projection D onto the xy-plane are shown in Figures 14 and 15. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

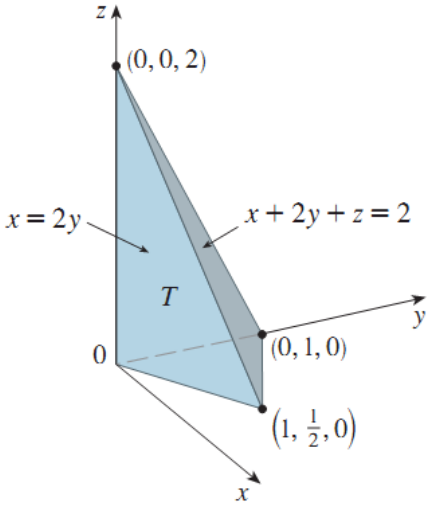


FIGURE 14

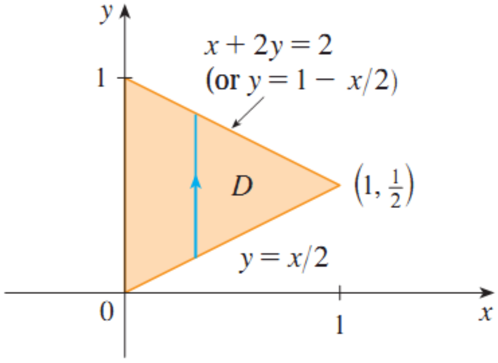


FIGURE 15

Therefore we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \frac{1}{3}$$

Example

Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution

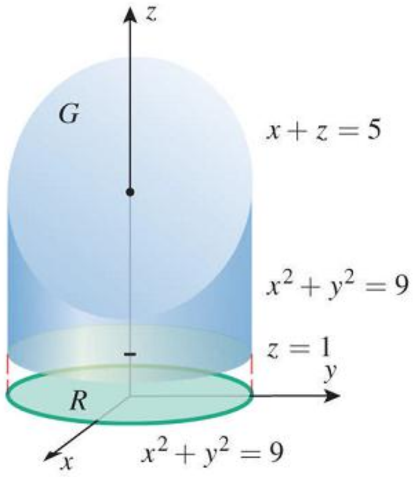
$$\text{volume of } G = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big]_{z=1}^{5-x} dy\,dx$$

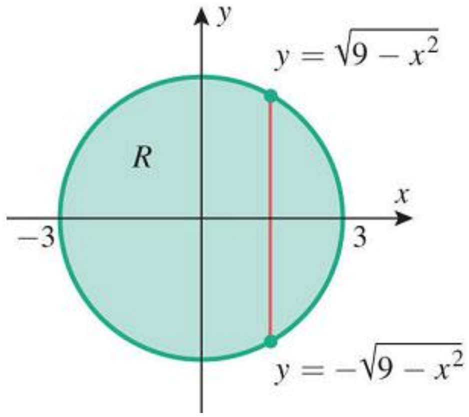
$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx = \int_{-3}^{3} (8-2x)\sqrt{9-x^2}\,dx$$

$$= 8\int_{-3}^{3} \sqrt{9-x^2}\,dx - \int_{-3}^{3} 2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi$$ ◀





Example

Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad \text{and} \quad z = 6 - 7x^2 - y^2$$

Solution

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz \right] dA$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

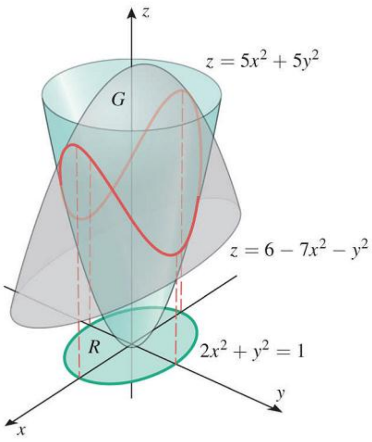
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\, dy\, dx$$

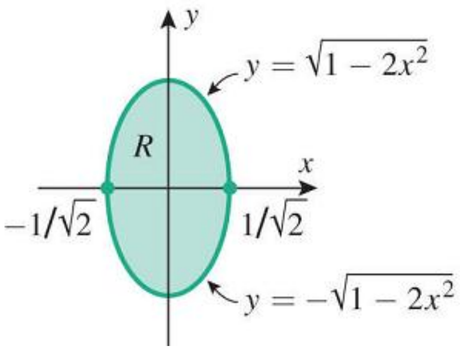
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[6(1 - 2x^2)y - 2y^3 \right]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 8 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)^{3/2}\, dx = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4 \theta\, d\theta = \frac{3\pi}{\sqrt{2}}$$

Let $x = \frac{1}{\sqrt{2}} \sin \theta$.

Use the Wallis cosine formula





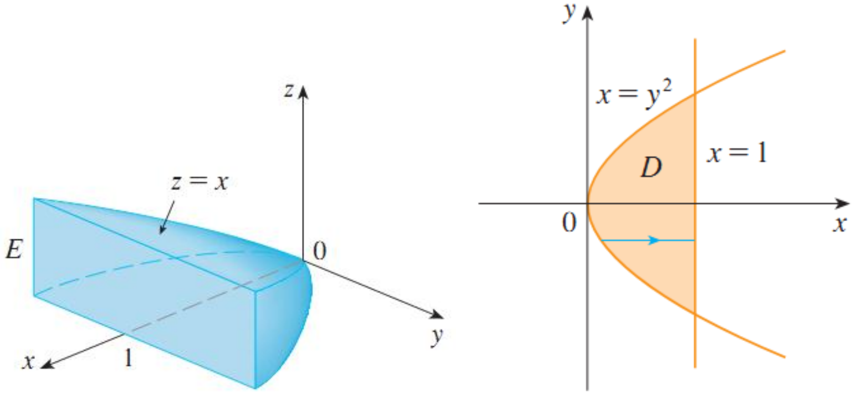
Example

Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

Solution

The solid E and its projection onto the xy-plane are shown in Figure 16. The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant x\}$$



Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint\limits_E \rho\, dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x \rho\, dz\, dx\, dy$$

$$= \rho\int_{-1}^{1}\int_{y^2}^{1} x\, dx\, dy = \rho\int_{-1}^{1}\left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy$$

$$= \frac{\rho}{2}\int_{-1}^{1}(1 - y^4)\, dy = \rho\int_0^1 (1 - y^4)\, dy$$

$$= \rho\left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$M_{yz} = \iiint\limits_E x\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} x\rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2 \, dx \, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3} \right]_{x=y^2}^{x=1} dy$$

$$= \frac{2\rho}{3} \int_{0}^{1} (1 - y^6) \, dy = \frac{2\rho}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \frac{4\rho}{7}$$

$$M_{xy} = \iiint\limits_E z\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z\rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2} \right]_{z=0}^{z=x} dx \, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2 \, dx \, dy$$

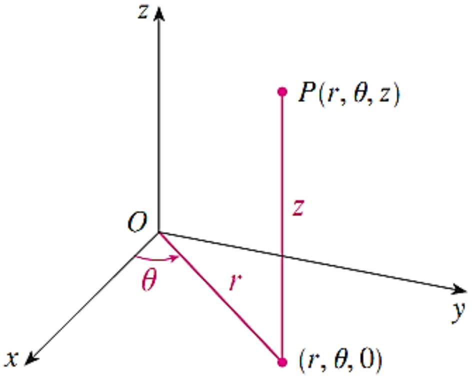
$$= \frac{\rho}{3} \int_{0}^{1} (1 - y^6) \, dy = \frac{2\rho}{7}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14} \right)$$

Triple Integrals in Cylindrical Coordinates

In the cylindrical coordinate system, a point **P** in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of **P** onto the xy-plane and z is the directed distance from the xy-plane to P. (See Figure).



To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

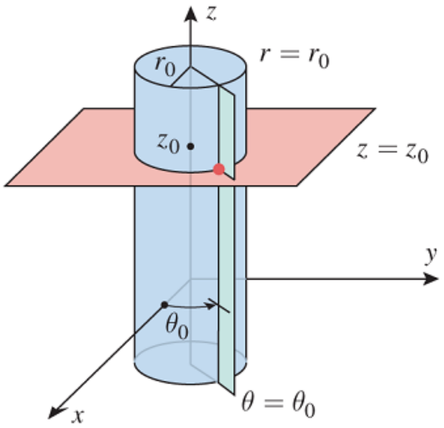
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Remember

The sphere with center the origin and radius c has the simple equation $r = c$ (see Figure). This is the reason for the name "spherical" coordinates. The equation represents a right circular cylinder centered on the z-axis

The graph of the equation $\theta = c$ is a vertical half-plane hinged on the z-axis (see Figure),

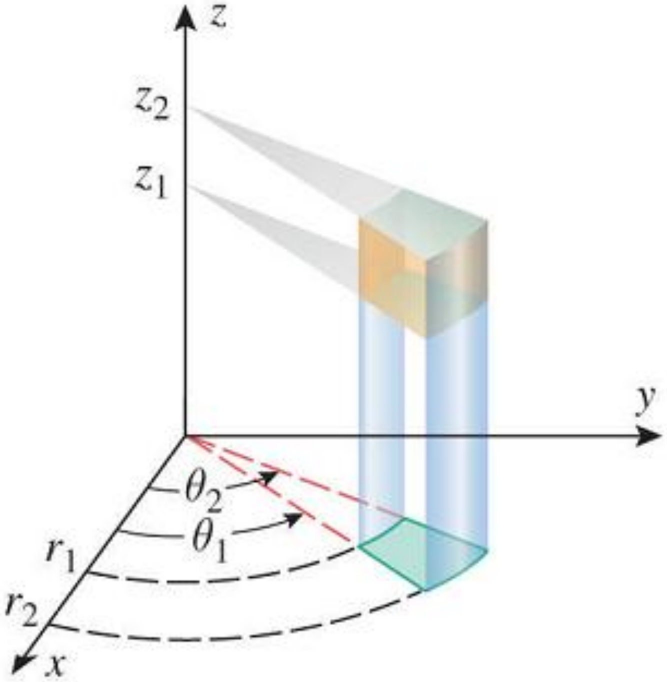
The equation $z = c$ represents a horizontal plane (see Figure).



Cylindrical Wedges or Cylindrical Elements of Volume

A cylindrical wedge is a solid enclosed between six surfaces of the following form:

two cylinders (blue)	$r = r_1,$	$r = r_2$	$(r_1 < r_2)$
two vertical half-planes (yellow)	$\theta = \theta_1,$	$\theta = \theta_2$	$(\theta_1 < \theta_2)$
two horizontal planes (gray)	$z = z_1,$	$z = z_2$	$(z_1 < z_2)$



The dimensions $\theta_2 - \theta_1$, $r_2 - r_1$,and $z_2 - z_1$ are called the central angle, thickness, and height of the wedge.

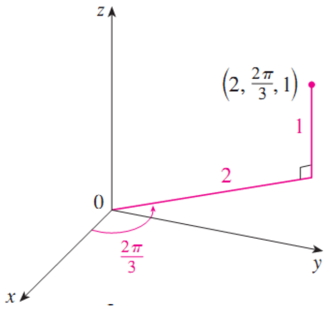
Example

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution

The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure



its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Thus the point is $\left(-1, \sqrt{3}, 1\right)$ in rectangular coordinates.

(b) From Equations 2 we have

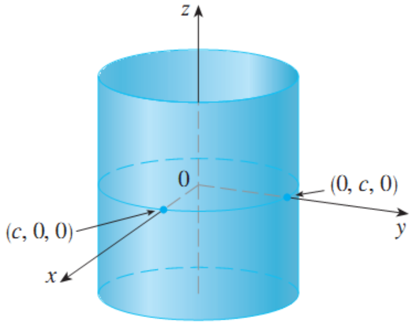
$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $\left(3\sqrt{2}, 7\pi/4, -7\right)$. Another is $\left(3\sqrt{2}, -\pi/4, -7\right)$. As with polar coordinates, there are infinitely many choices.

Importanc



Importance

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$. (See Figure .) This is the reason for the name “cylindrical” coordinates.

Example

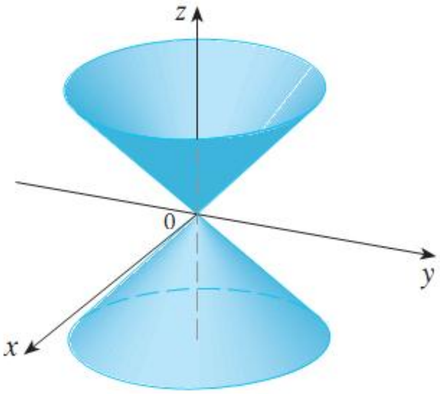
Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution

The equation says that the z-value, or height, of each point on the surface is the same as r, the distance from the point to the -axis. Because θ doesn't appear, it can vary. So any horizontal trace in the plane $z = k(k > 0)$ is a circle of radius k. These traces suggest that the surface is a cone. This prediction can be confirmed by converting the equation into rectangular coordinates. We have

$$z^2 = r^2 = x^2 + y^2$$

This is a circular cone whose axis is the z-axis (see Figure).

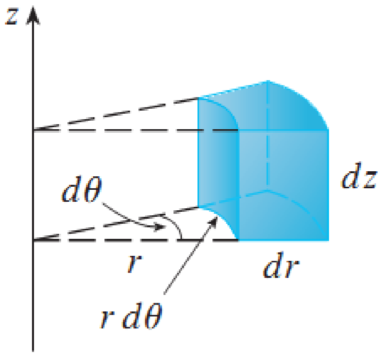


Formula for Triple Integration in Cylindrical Coordinates

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Example

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



Solution

In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant.

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta = K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

Example

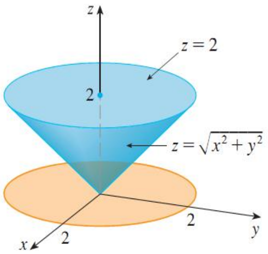
Evaluate $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.$

Solution

This iterated integral is a triple integral over the solid region

$$E = \{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ -\sqrt{4 - x^2} \leqslant y \leqslant \sqrt{4 - x^2},\ \sqrt{x^2 + y^2} \leqslant z \leqslant 2\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \leqslant 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$.



This region has a much simpler description in cylindrical coordinates:

$$E = \left\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2,\ r \leqslant z \leqslant 2\right\}$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2 + y^2)\,dz\,dy\,dx = \iiint_E (x^2 + y^2)\,dV$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2 - r)\,dr$$

$$= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \tfrac{16}{5}\pi$$

Example

Use cylindrical coordinates to evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

Solution

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx = \iiint_G x^2\,dV$$

$$= \iint_R \left[\int_0^{9-r^2} r^2\cos^2\theta\,dz\right] dA = \int_0^{2\pi}\int_0^3\int_0^{9-r^2}(r^2\cos^2\theta)r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos^2\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \left[zr^3\cos^2\theta\right]_{z=0}^{9-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9r^3 - r^5)\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\left(\frac{9r^4}{4} - \frac{r^6}{6}\right)\cos^2\theta\right]_{r=0}^{3} d\theta$$

$$= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{243}{4}\int_0^{2\pi}\frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{243\pi}{4} \blacktriangleleft$$

Example

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution

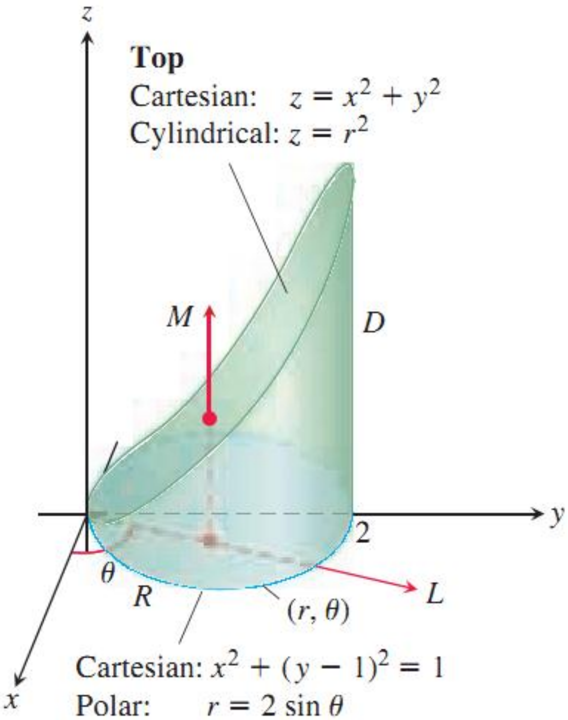
The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r\sin\theta = 0$$

$$r = 2\sin\theta.$$



Then

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Example

Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^5}{2}dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}d\theta = \frac{32\pi}{3}.$$

The value of M is

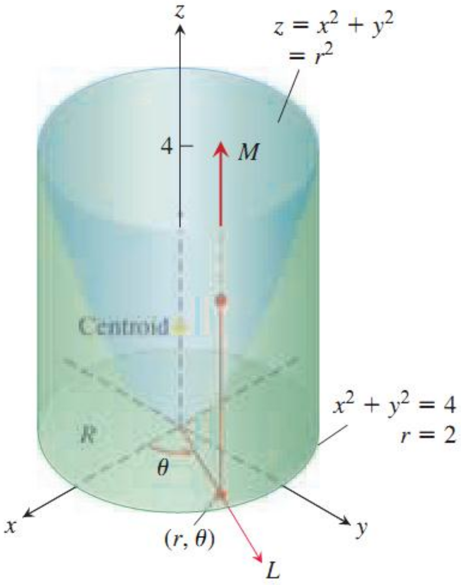
$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \Big[z\Big]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

Therefore,

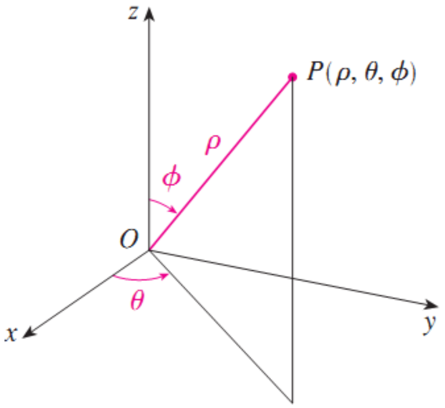
$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies on the z-axis, outside the solid.



Spherical Coordinates

It simplifies the evaluation of triple integrals over regions bounded by spheres or cones. The spherical coordinates (ρ, θ, φ) of a point in space are shown in Figure, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive z-axis and the line segment OP. Note that $\rho \geq 0$ and $0 \leq \varphi \leq \pi$.

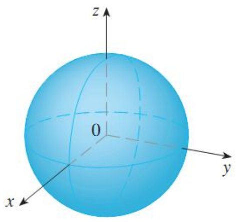


The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

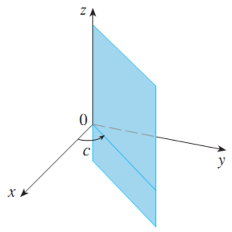
Example

The sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure);

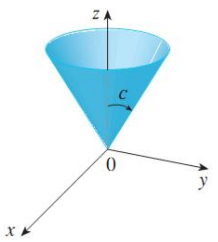
This is the reason for the name “spherical” coordinates. the first equation represents a sphere centered at the origin.



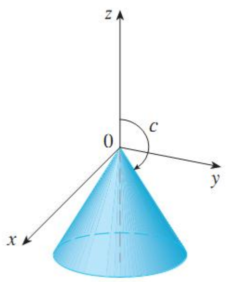
The graph of the equation $\theta = c$ is a vertical half-plane hinged on the z-axis (see Figure),



The equation $\varphi = c$ represents a half-cone with the z-axis or horizontal plane as its axis (see Figure). It is a right circular cone nappe with its vertex at the origin and its line of symmetry along the z-axis for $\varphi = \pi/2$, and is the xy-plane if $\varphi = \pi/2$



$0 < c < \pi/2$

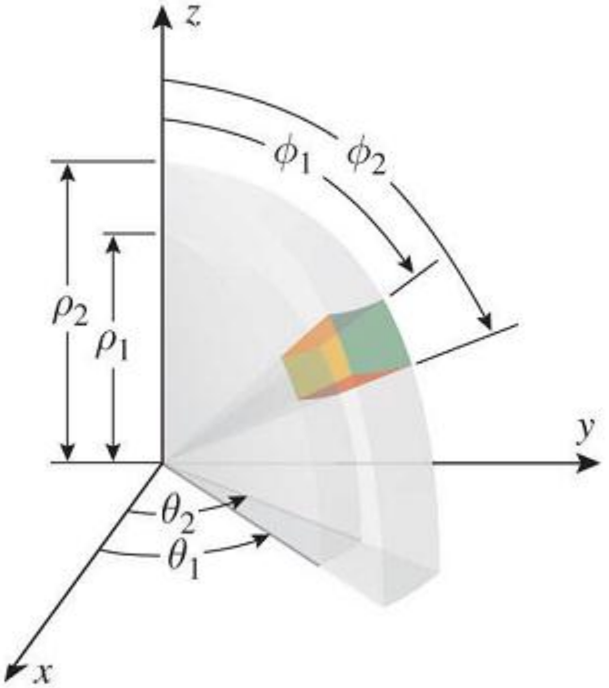


$\pi/2 < c < \pi$

Spherical Wedges or Spherical Elements of Volume

A spherical wedge is a solid enclosed between six surfaces of the following form:

two spheres (green)	$\rho = \rho_1,$	$\rho = \rho_2$	$(\rho_1 < \rho_2)$
two vertical half-planes (yellow)	$\theta = \theta_1,$	$\theta = \theta_2$	$(\theta_1 < \theta_2)$
nappes of two circular cones (pink)	$\phi = \phi_1,$	$\phi = \phi_2$	$(\phi_1 < \phi_2)$



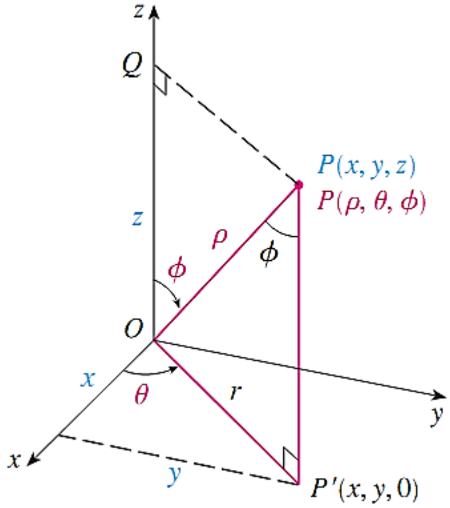
We will refer to the numbers $\rho_2 - \rho_1$, $\theta_2 - \theta_1$, and $\varphi_2 - \varphi_1$ as the dimensions of a spherical wedge.

The relationship between rectangular and spherical coordinates

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$



Example

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

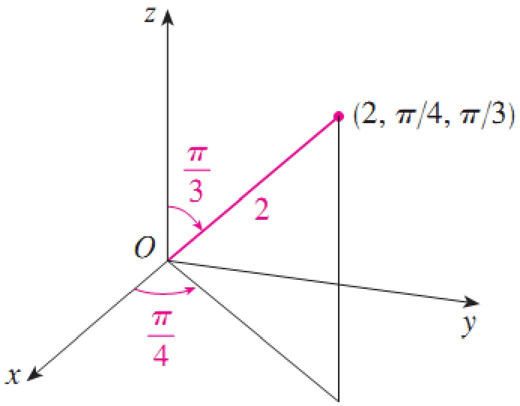
Solution

$$x = \rho \sin \phi \cos \theta = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates.



Example

The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Solution

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$

(Note that $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$.) Therefore spherical coordinates of the given point are $(4, \pi/2, 2\pi/3)$.

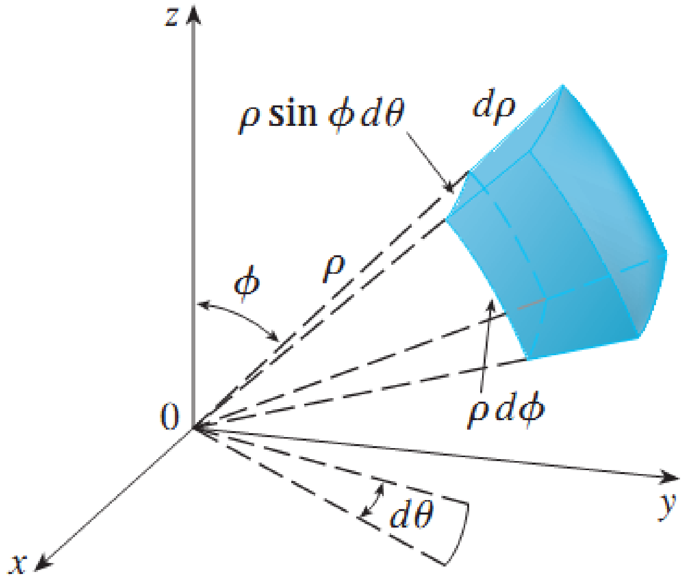
Triple Integration in Spherical Coordinates

$$\iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$



Example

Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

Solution

Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

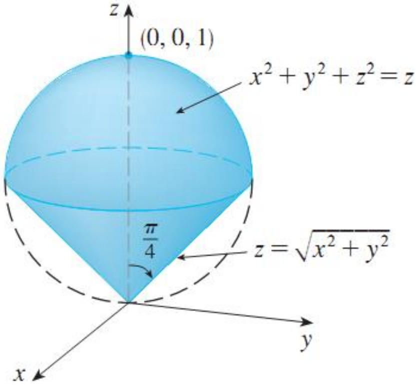
$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin \phi \ d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \sin \phi \, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} \, d\rho$$

$$= \left[-\cos \phi\right]_0^{\pi} (2\pi) \left[\tfrac{1}{3} e^{\rho^3}\right]_0^1 = \tfrac{4}{3}\pi(e - 1)$$

Example

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



Solution

Notice that the sphere passes through the origin and has center $\left(0, 0, \frac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho\cos\phi \qquad \text{or} \qquad \rho = \cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\,\cos^2\theta + \rho^2\sin^2\phi\,\sin^2\theta} = \rho\sin\phi$$

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \rho \leqslant \cos\phi\}$$

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2\sin\phi\ d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\cos^3\phi\,d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$$

Example

Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.

Solution

$$V = \iiint\limits_G dV = \iint\limits_R \left[\int_0^{\sqrt{25-r^2}} dz \right] dA$$

$$V = \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^3 [rz]_{z=0}^{\sqrt{25-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^3 r\sqrt{25 - r^2}\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25 - r^2)^{3/2} \right]_{r=0}^{3} d\theta$$

$$\boxed{\begin{aligned} u &= 25 - r^2 \\ du &= -2r\,dr \end{aligned}}$$

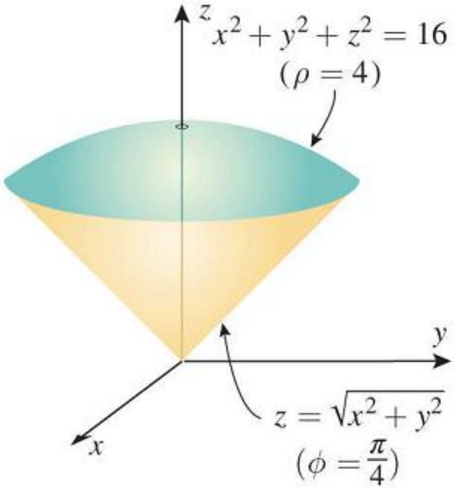
$$= \int_0^{2\pi} \frac{61}{3}\,d\theta = \frac{122}{3}\pi$$ ◄

Example

Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Solution

The solid G is sketched in Figure.



In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2 + y^2}$ is

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta}$$

which simplifies to

$$\rho \cos \phi = \rho \sin \phi$$

Dividing both sides of this equation by $\rho \cos \varphi$ yields $\tan \varphi = 1$, from which it follows that

$\varphi = \pi/4$

$$\begin{aligned} V = \iiint_G dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin \phi \right]_{\rho=0}^{4} d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{64}{3} \sin \phi \, d\phi \, d\theta \\ &= \frac{64}{3} \int_0^{2\pi} [-\cos \phi]_{\phi=0}^{\pi/4} \, d\theta = \frac{64}{3} \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2} \right) d\theta \\ &= \frac{64\pi}{3}(2 - \sqrt{2}) \approx 39.26 \blacktriangleleft \end{aligned}$$

Example

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$

Solution

$$x^2 + y^2 + (z - 1)^2 = 1$$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

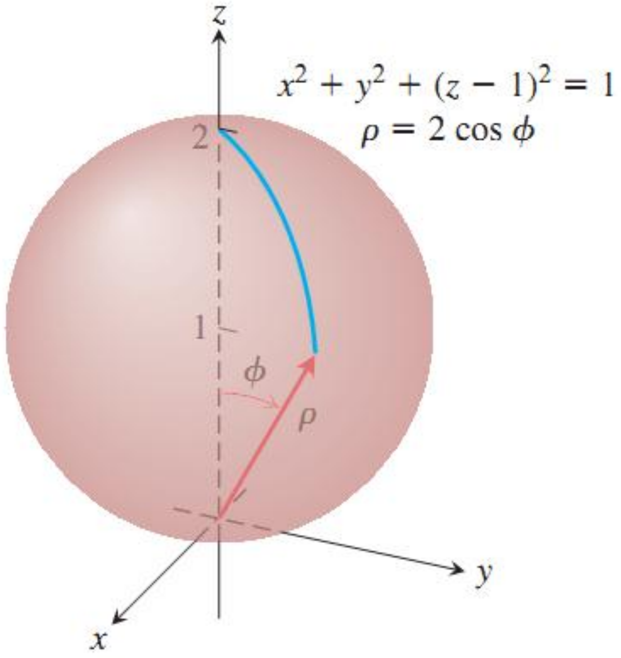
$$\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi.$$

The angle ϕ varies from 0 at the north pole of the sphere to $\pi/2$ at the south pole; the angle θ does not appear in the expression for ρ, reflecting the symmetry about the z-axis



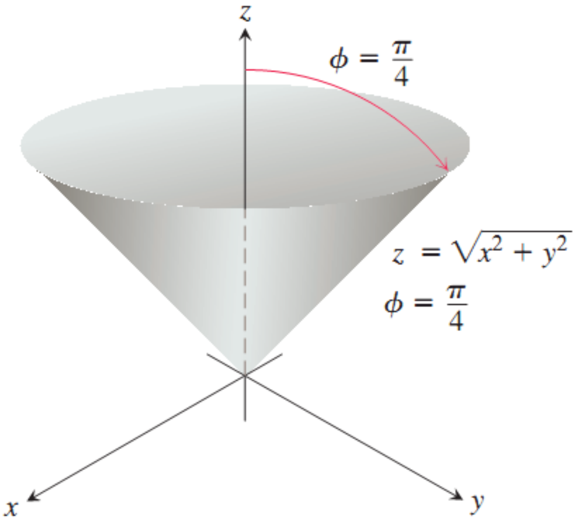
Example

Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.

Solution

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \rho \cos \phi &= \sqrt{\rho^2 \sin^2 \phi} \\ \rho \cos \phi &= \rho \sin \phi \\ \cos \phi &= \sin \phi \\ \phi &= \frac{\pi}{4}. \end{aligned}$$

$\rho = 4$	Sphere, radius 4, center at origin
$\phi = \frac{\pi}{3}$	Cone opening up from the origin, making an angle of $\pi/3$ radians with the positive z-axis
$\theta = \frac{\pi}{3}.$	Half-plane, hinged along the z-axis, making an angle of $\pi/3$ radians with the positive x-axis



Example

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

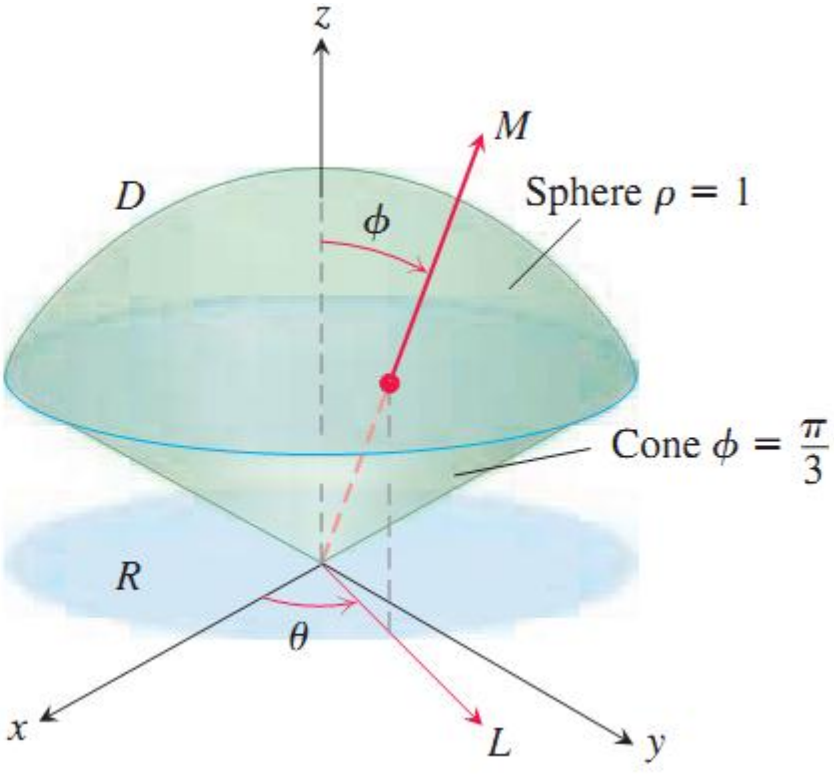
Solution

The volume is $V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3} \right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} \cos\phi \right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3} \right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$



Example

A solid of constant density $\delta = 1$ occupies the region D (cone with $\varphi = \frac{\pi}{3}$). Find the solid's moment of inertia about the z-axis.

Solution

In rectangular coordinates, the moment is

$$I_z = \iiint\limits_D (x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint\limits_D (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \iiint\limits_D \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta.$$

$$I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi \, d\phi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3} (1 - \cos^2\phi) \sin\phi \, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi} \frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.$$

Coordinate Conversion Formulas

CYLINDRICAL TO RECTANGULAR	SPHERICAL TO RECTANGULAR	SPHERICAL TO CYLINDRICAL
$x = r\cos\theta$	$x = \rho \sin\phi \cos\theta$	$r = \rho \sin\phi$
$y = r\sin\theta$	$y = \rho \sin\phi \sin\theta$	$z = \rho \cos\phi$
$z = z$	$z = \rho \cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

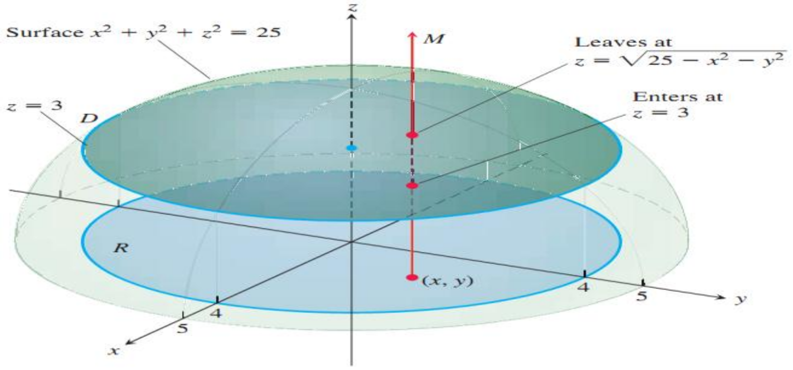
$$\begin{aligned} dV &= dx\, dy\, dz \\ &= dz\, r\, dr\, d\theta \\ &= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{aligned}$$

Triple Integrals in Rectangular Coordinates

Example

Let S be the sphere of radius 5 centered at the origin, and let D be the region under the sphere that lies above the plane $z = 3$. Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the region D.

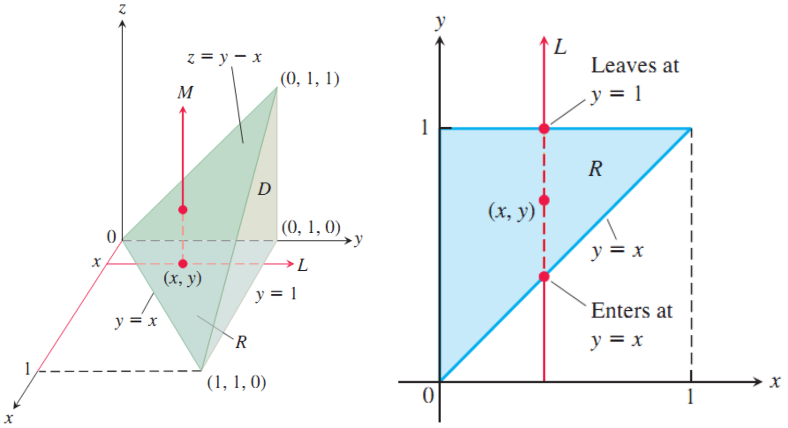
Solution



$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} F(x, y, z)\, dz\, dy\, dx.$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D whose vertices are (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dz\, dy\, dx$.

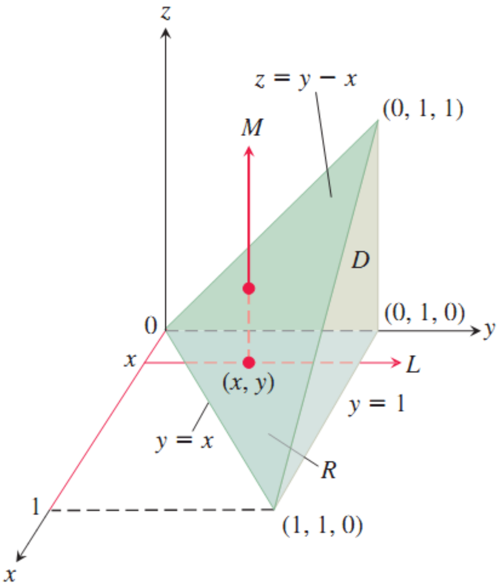


Solution

$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_{0}^{1} \int_{x}^{1} \int_{0}^{y-x} F(x, y, z)\, dz\, dy\, dx.$$

Example

Find the volume of the tetrahedron D whose vertices are (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1) by integrating F(x, y, z) = 1 over the region using the order dz dy dx. Then do the same calculation using the order dy dz dx.



Solution

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx$$

$$= \int_0^1 \int_x^1 (y - x)\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 - xy \right]_{y=x}^{y=1} dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2} x^2 \right) dx$$

$$= \left[\frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3 \right]_0^1$$

$$= \frac{1}{6}.$$

another way

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx$$

$$= \int_0^1 \left[(1 - x) z - \frac{1}{2} z^2 \right]_{z=0}^{z=1-x} dx$$

$$= \int_0^1 \left[(1 - x)^2 - \frac{1}{2} (1 - x)^2 \right] dx$$

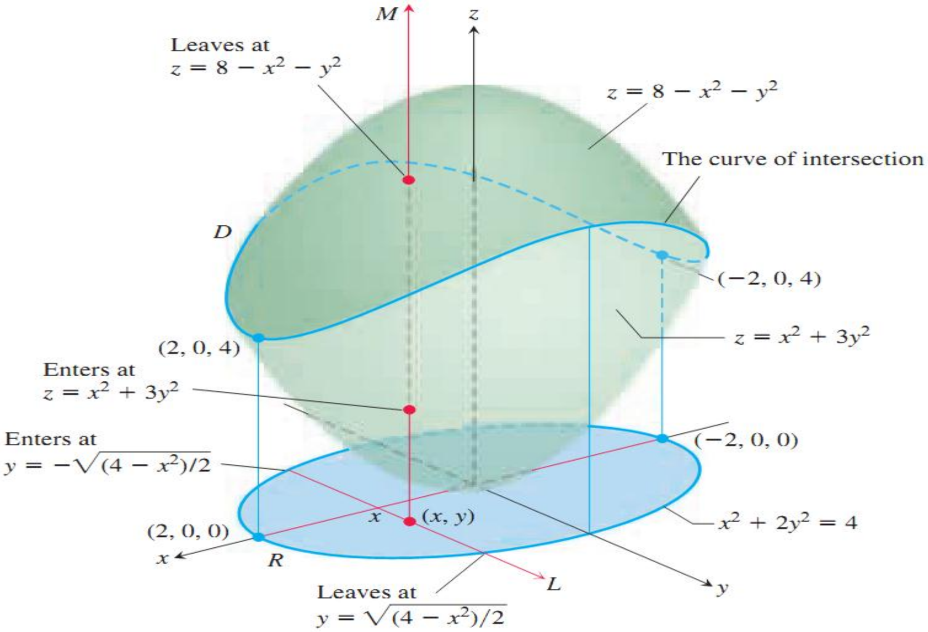
$$= \frac{1}{2} \int_0^1 (1 - x)^2\, dx$$

$$= -\frac{1}{6} (1 - x)^3 \Big]_0^1 = \frac{1}{6}.$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution



$$V = \iiint_D dz\,dy\,dx$$ Integrand is 1 when computing volume.

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$ Substitute limits of integration.

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx$$ Integrate over z and evaluate.

$$= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$ Integrate over y.

$$= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right) dx$$ Evaluate.

$$= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx$$

$$= 8\pi\sqrt{2}.$$ After integration with the substitution $x = 2\sin u$ ■

Average Value in Space

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV.$$

Example

Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution

Volume is calculated as follows

$$\int_0^2 \int_0^2 \int_0^2 xyz\, dx\, dy\, dz = \int_0^2 \int_0^2 \left[\frac{x^2}{2} yz\right]_{x=0}^{x=2} dy\, dz = \int_0^2 \int_0^2 2yz\, dy\, dz$$

$$= \int_0^2 \left[y^2 z\right]_{y=0}^{y=2} dz = \int_0^2 4z\, dz = \left[2z^2\right]_0^2 = 8.$$

$$\begin{array}{c}\text{Average value of} \\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz\, dV = \left(\frac{1}{8}\right)(8) = 1.$$

Rectangular to Spherical Coordinates

Triple integrals can be converted from rectangular coordinates to spherical coordinates by making the substitution x = ρ sin φ cos θ, y = ρ sin φ sin θ, z = ρ cos φ.
The two integrals are related by the equation

$$\iiint\limits_G f(x,y,z)\,dV = \iiint\limits_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Example

Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

Solution

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$= \iiint\limits_G z^2\sqrt{x^2+y^2+z^2}\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^5\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{32}{3}\cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\pi/2} d\theta = \frac{32}{9}\int_0^{2\pi} d\theta = \frac{64}{9}\pi \blacktriangleleft$$

Remember

$x = x(t), \quad y = y(t)$	A curve in the plane
$x = x(t), \quad y = y(t), \quad z = z(t)$	A curve in 3-space
$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$	A surface in 3-space

Substitutions/ Change of Variables in Multiple Integrals/ Integration by Substitution/ u-Substitution/ Reverse Chain Rule

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u, we can write the Substitution Rule as

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$.

Change of Variables by a Transformation

change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write, $x = x(u, v) \qquad y = y(u, v)$

C^1 Transformation

A transformation is ^{1} if all its first partial derivatives exist and are continuous.

Existence of inverse Transformation

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane

$x = g(u, v) \qquad y = h(u, v)$ **transformation**

$u = G(x, y) \qquad v = H(x, y)$ **inverse transformation**

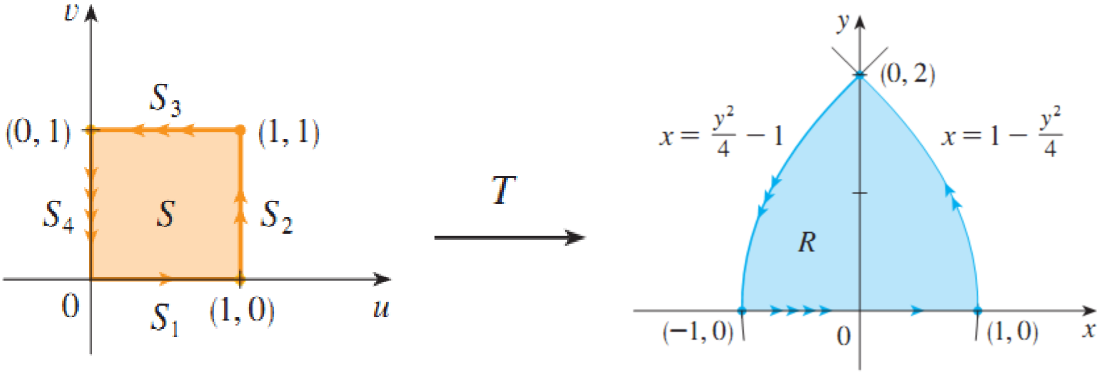
Example

A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

Solution



The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \leqslant u \leqslant 1)$. (See Figure) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leqslant x \leqslant 1$. Thus S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2, is $u = 1$ $(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

$$x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0$$

Finally, S_4 is given by $u = 0$ $(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure) bounded by the x-axis and the parabolas **given by Equations above.**

Example

Let T be the transformation from the uv-plane to the xy-plane defined by the equations

$$x = \tfrac{1}{4}(u+v), \quad y = \tfrac{1}{2}(u-v) \qquad (3)$$

(a) Find $T(1,3)$.

(b) Sketch the constant v-curves corresponding to $v = -2, -1, 0, 1, 2$.

(c) Sketch the constant u-curves corresponding to $u = -2, -1, 0, 1, 2$.

(d) Sketch the image under T of the square region in the uv-plane bounded by the lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$.

Solution

(*a*). Substituting $u = 1$ and $v = 3$ in (3) yields $T(1,3) = (1,-1)$.

Solutions (b and c). In these parts it will be convenient to express the transformation equations with u and v as functions of x and y. From (3)

$$4x = u + v, \quad 2y = u - v$$

Combining these equations gives

$$4x + 2y = 2u, \quad 4x - 2y = 2v$$

or

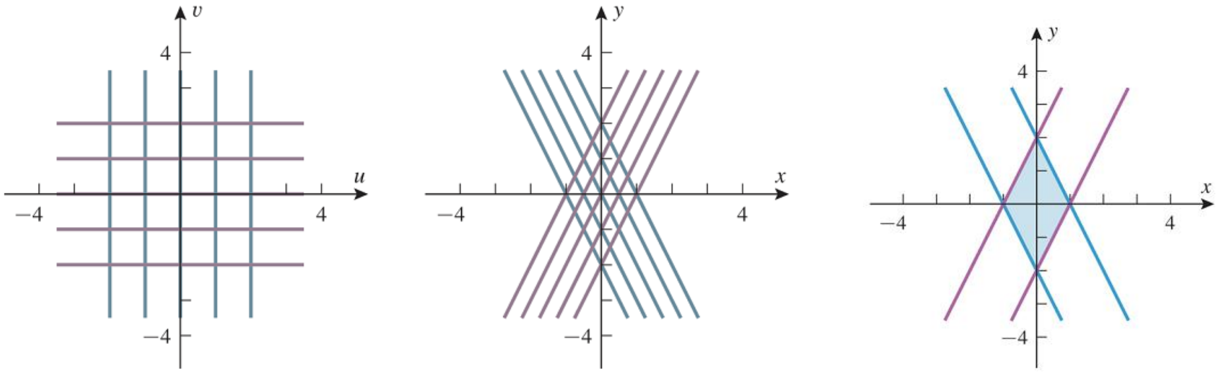
$$2x + y = u, \quad 2x - y = v$$

Thus, the constant v-curves corresponding to $v = -2, -1, 0, 1$, and 2 are

$$2x - y = -2, \quad 2x - y = -1, \quad 2x - y = 0, \quad 2x - y = 1, \quad 2x - y = 2$$

and the constant u-curves corresponding to $u = -2, -1, 0, 1$, and 2 are

$$2x + y = -2, \quad 2x + y = -1, \quad 2x + y = 0, \quad 2x + y = 1, \quad 2x + y = 2$$



Solution (d). The image of a region can often be found by finding the image of its boundary. In this case the images of the boundary lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$ enclose the diamond-shaped region in the xy-plane shown in Figure **3 above**

Jacobian of the Transformation (change of variables in multiple integrals)

The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variables in a Double Integral

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

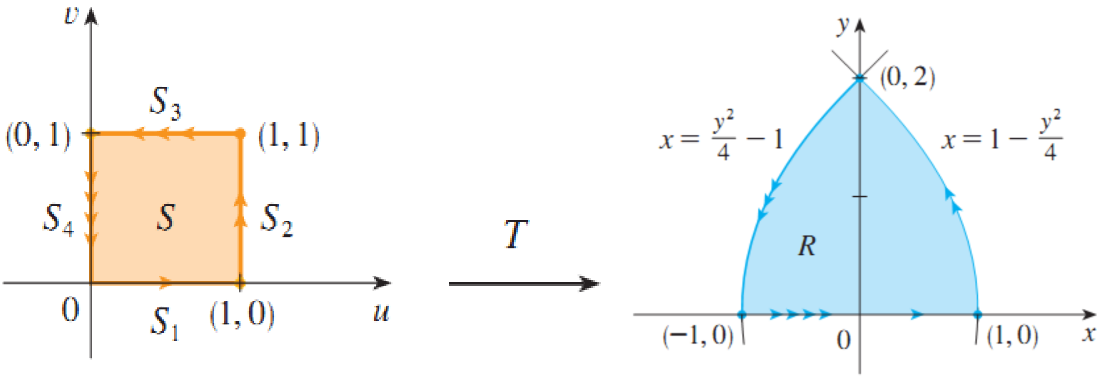
$$\iint_R f(x, y)\, dA = \iint_S f\big(x(u, v), y(u, v)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Example

Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

Solution

The region R is pictured in Figure



$T(S) = R$, where S is the square $[0, 1] \times [0, 1]$.

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

$$\iint_R y\,dA = \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\,du\,dv$$

$$= 8 \int_0^1 \int_0^1 (u^3v + uv^3)\,du\,dv = 8 \int_0^1 \left[\tfrac{1}{4}u^4v + \tfrac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v + 4v^3)\,dv = \left[v^2 + v^4\right]_0^1 = 2$$

Example

Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

Solution

Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function: $u = x + y \qquad v = x - y$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane.

$$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

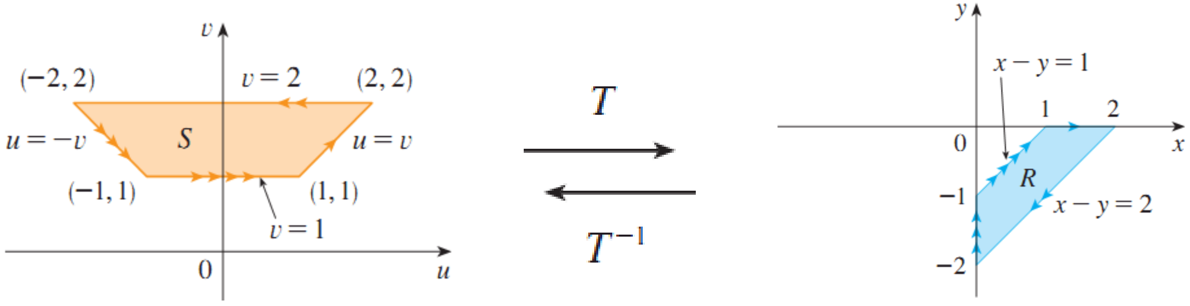
To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus the region S is the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ shown in Figure



Since $S = \{(u, v) \mid 1 \leqslant v \leqslant 2,\ -v \leqslant u \leqslant v\}$

$$\iint\limits_R e^{(x+y)/(x-y)}\, dA = \iint\limits_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right) du\, dv = \tfrac{1}{2} \int_1^2 \left[v e^{u/v} \right]_{u=-v}^{u=v} dv$$

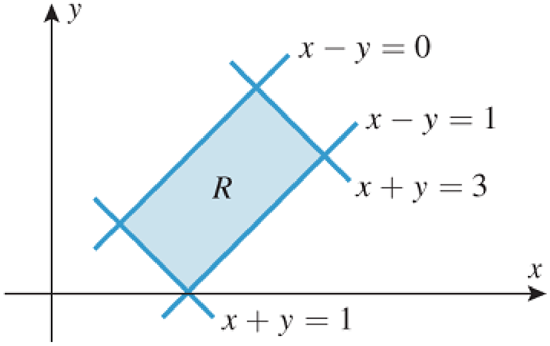
$$= \tfrac{1}{2} \int_1^2 (e - e^{-1}) v\, dv = \tfrac{3}{4}(e - e^{-1})$$

Example

Evaluate

$$\iint_R \frac{x-y}{x+y}\,dA$$

where R is the region enclosed by $x-y=0$, $x-y=1$, $x+y=1$, and $x+y=3$



Solution

This integral would be tedious to evaluate directly because the region R is oriented in such a way that we would have to subdivide it and integrate over each part separately. However, the occurrence of the expressions x − y and x + y in the equations of the boundary suggests that the transformation

$u = x + y, \quad v = x - y$

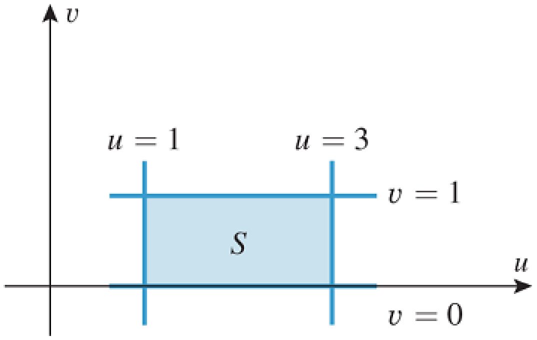
would be helpful, since with this transformation the boundary lines

$x + y = 1, \quad x + y = 3, \quad x - y = 0, \quad x - y = 1$

are constant u-curves and constant v-curves corresponding to the lines

$u = 1, \quad u = 3, \quad v = 0, \quad v = 1$

in the uv-plane. These lines enclose the rectangular region S shown in Figure.



To find the Jacobian $\partial(x, y)/\partial(u, v)$ of this transformation, we first solve $u = x + y$, $v = x - y$ for x and y in terms of u and v. This yields

$$x = \tfrac{1}{2}(u+v), \quad y = \tfrac{1}{2}(u-v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{4} - \tfrac{1}{4} = -\tfrac{1}{2}$$

$$\iint\limits_R \frac{x-y}{x+y}\,dA = \iint\limits_S \frac{v}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

$$= \iint\limits_S \frac{v}{u}\left|-\frac{1}{2}\right| dA_{uv} = \frac{1}{2}\int_0^1 \int_1^3 \frac{v}{u}\,du\,dv$$

$$= \frac{1}{2}\int_0^1 v \ln|u| \Big]_{u=1}^{3} dv$$

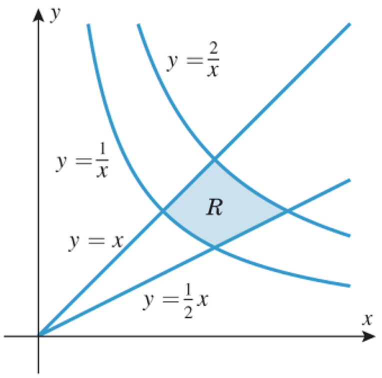
$$= \frac{1}{2}\ln 3 \int_0^1 v\,dv = \frac{1}{4}\ln 3 \blacktriangleleft$$

Example

Evaluate

$$\iint\limits_R e^{xy}\,dA$$

where R is the region enclosed by the lines $y = \frac{1}{2}x$ and $y = x$ and the hyperbolas $y = 1/x$ and $y = 2/x$



Solution
We look for a transformation in which the boundary curves in the xy-plane become constant v-curves and constant u-curves. For this purpose we rewrite the four boundary curves as

$$\frac{y}{x}=\frac{1}{2},\quad \frac{y}{x}=1,\quad xy=1,\quad xy=2$$

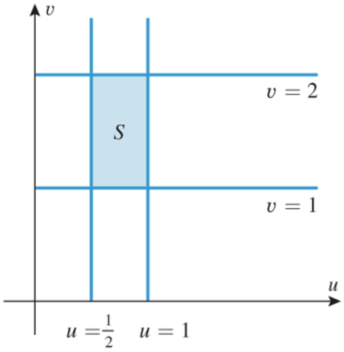
which suggests the transformation

$$u=\frac{y}{x},\quad v=xy$$

With this transformation the boundary curves in the xy-plane are constant u-curves and constant v-curves corresponding to the lines

$$u=\tfrac{1}{2},\quad u=1,\quad v=1,\quad v=2$$

in the uv-plane. These lines enclose the region S shown in Figure.



To find the Jacobian ∂(x, y)/∂(u, v) of this transformation, we first solve

$$u=\frac{y}{x},\quad v=xy$$

for x and y in terms of u and v. This yields

$$x=\sqrt{v/u},\quad y=\sqrt{uv}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\dfrac{1}{2u}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2\sqrt{uv}}\\[2mm] \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\sqrt{\dfrac{u}{v}}\end{vmatrix}=-\frac{1}{4u}-\frac{1}{4u}=-\frac{1}{2u}$$

$$\iint_R e^{xy}\,dA=\iint_S e^{v}\left|-\frac{1}{2u}\right|\,dA_{uv}=\frac{1}{2}\iint_S \frac{1}{u}e^{v}\,dA_{uv}$$

$$=\frac{1}{2}\int_1^2\int_{1/2}^{1}\frac{1}{u}e^{v}\,du\,dv=\frac{1}{2}\int_1^2 e^{v}\ln|u|\Big]_{u=1/2}^{1}\,dv$$

$$=\frac{1}{2}\ln 2\int_1^2 e^{v}\,dv=\frac{1}{2}(e^2-e)\ln 2 \;◀$$

Example

Find the Jacobian for the polar coordinate transformation x = r cos θ, y = r sin θ, and write the Cartesian integral $\iint f(x,y)\,dx\,dy$ as a polar integral.

Solution

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Since we assume $r \geq 0$ when integrating in polar coordinates, $|J(r,\theta)| = |r| = r$, so

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example

Evaluate

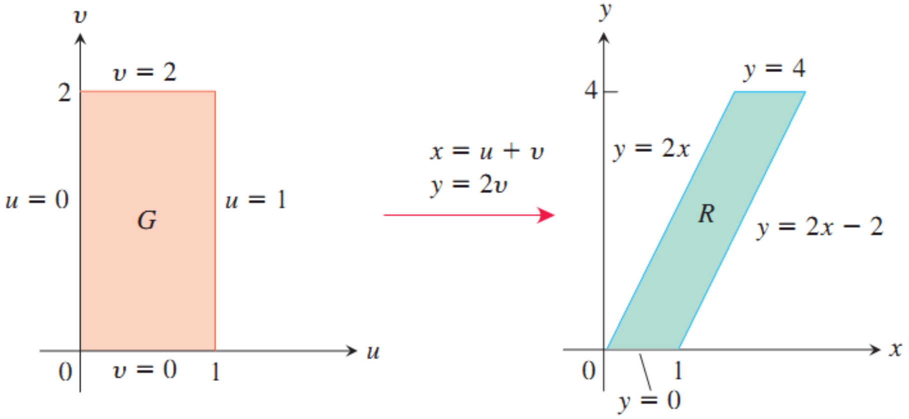
$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by applying the transformation

$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2}$$

and integrating over an appropriate region in the uv-plane.

Solution



$$x = u + v, \qquad y = 2v.$$

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u + v) & \dfrac{\partial}{\partial v}(u + v) \\[2ex] \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\,dx\,dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u\,|J(u, v)|\,du\,dv$$

$$= \int_0^2 \int_0^1 (u)(2)\,du\,dv = \int_0^2 \Big[u^2\Big]_0^1 dv = \int_0^2 dv = 2.$$

Example

Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x + y}\,(y - 2x)^2\,dy\,dx.$$

Solution

$u = x + y$ and $v = y - 2x$.

Routine algebra produces x and y as functions of u and v:

$$x = \frac{u}{3} - \frac{v}{3}, \qquad y = \frac{2u}{3} + \frac{v}{3}.$$

xy-equations for the boundary of R	**Corresponding uv-equations for the boundary of G**	**Simplified uv-equations**
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & -\dfrac{1}{3} \\ \dfrac{2}{3} & \dfrac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

$$\int_0^1 \int_0^{1-x} \sqrt{x + y}\,(y - 2x)^2\, dy\, dx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2 \left|J(u, v)\right|\, dv\, du$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv\, du = \frac{1}{3}\int_0^1 u^{1/2} \left[\frac{1}{3} v^3\right]_{v=-2u}^{v=u} du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}(u^3 + 8u^3)\, du = \int_0^1 u^{7/2}\, du = \frac{2}{9} u^{9/2}\Big]_0^1 = \frac{2}{9}.$$

Example

Evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$$

Solution

$u = \sqrt{xy}$ and $v = \sqrt{y/x}$

$u^2 = xy$ and $v^2 = y/x$, which imply that $u^2 v^2 = y^2$ and $u^2/v^2 = x^2$

$x = \frac{u}{v}$ and $y = uv$

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

$$\iint\limits_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint\limits_G v e^u \frac{2u}{v}\, du\, dv = \iint\limits_G 2ue^u\, du\, dv.$$

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du.$$

Note the order of integration.

$$\begin{aligned} \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du &= 2\int_1^2 \Big[vue^u \Big]_{v=1}^{v=2/u} du \\ &= 2\int_1^2 (2e^u - ue^u)\, du \\ &= 2\int_1^2 (2 - u)e^u\, du \\ &= 2\Big[(2 - u)e^u + e^u \Big]_{u=1}^{u=2} \\ &= 2(e^2 - (e + e)) = 2e(e - 2). \end{aligned}$$

Integrate by parts.

Change of Variables in a Triple Integral

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint_R f(x, y, z)\, dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Example

Derive the formula for triple integration in spherical coordinates.

Solution

Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\,(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta)$$

$$- \rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = \left|-\rho^2 \sin\phi\right| = \rho^2 \sin\phi$$

$$\iiint_R f(x, y, z)\, dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big)\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Example

Find the volume of the region G enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution

To evaluate this integral, we make the change of variables

$x = au, \quad y = bv, \quad z = cw$

which maps the region S in uvw-space enclosed by a sphere of radius 1 into the region G in xyz-space. This can be seen by noting that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{becomes} \quad u^2 + v^2 + w^2 = 1$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$V = \iiint_G dV = \iiint_S \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dV_{uvw} = abc \iiint_S dV_{uvw} = \tfrac{4}{3}\pi abc.$$

Volume enclosed by a sphere of radius 1, which we know to be $\frac{4}{3}\pi$

Example

Evaluate

$$\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

by applying the transformation

$$u=(2x-y)/2,\qquad v=y/2,\qquad w=z/3$$

and integrating over an appropriate region in *uvw*-space.

Solution

$$x=u+v,\qquad y=2v,\qquad z=3w.$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D:

xyz*-equations for the boundary of *D	**Corresponding *uvw*-equations for the boundary of *G***	**Simplified *uvw*-equations**
$x=y/2$	$u+v=2v/2=v$	$u=0$
$x=(y/2)+1$	$u+v=(2v/2)+1=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

$$J(u,v,w)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{vmatrix}=6.$$

$$\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

$$=\int_0^1\int_0^2\int_0^1 (u+w)\,\left|J(u,v,w)\right|\,du\,dv\,dw$$

$$=\int_0^1\int_0^2\int_0^1 (u+w)(6)\,du\,dv\,dw = 6\int_0^1\int_0^2\left[\frac{u^2}{2}+uw\right]_0^1 dv\,dw$$

$$=6\int_0^1\int_0^2\left(\frac{1}{2}+w\right)dv\,dw = 6\int_0^1\left[\frac{v}{2}+vw\right]_0^2 dw = 6\int_0^1 (1+2w)\,dw$$

$$=6\left[w+w^2\right]_0^1 = 6(2) = 12.$$

LAPLACE TRANSFORM

Because of their simplicity, Laplace transforms are frequently used to solve a wide class of partial differential equations. Like other transforms, Laplace transforms are used to determine particular solutions. In solving partial differential equations, the general solutions are difficult, if not impossible, to obtain. The transform technique sometimes offers a useful tool for finding particular solutions. The Laplace transform is closely related to the complex Fourier transform, so the Fourier integral formula can be used to define the Laplace transform and its inverse.

INTEGRAL TRANSFORMATION

Consider a set $K(x,y) = \{f(x); f\ is\ function\ of\ x\ over\ [a,b]\}$ then integral transformation is defined as

$T\{f(x)\} = F(y) = \int_a^b f(x)K(x,y)dx$ where $K(x,y)$ is kernel of T.

LAPLACE TRANSFORMATION

If $f(t)$ is defined for all values of $t > 0$, then the Laplace transform of $f(t)$ is denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$ and is defined by the integral

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-\mathrm{st}} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-\mathrm{st}} f(t)dt$$

If $F(s)$ is laplace transform of $f(t)$ then $f(t)$ is called the <u>INVERSE LAPLACE TRANSFORM</u> of $F(s)$ i.e. $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Question:

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

Question: Show that $\mathcal{L}\{c\} = \frac{c}{s}$ where ‘c’ is constant.

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}} f(t)dt$

Then $\mathcal{L}\{c\} = \int_0^\infty e^{-\mathrm{st}}\, cdt = c\int_0^\infty e^{-\mathrm{st}}\, dt = c\left|-\frac{e^{-st}}{s}\right|_0^\infty = \frac{c}{s}$

Question: Show that $\mathcal{L}\{e^{\mathrm{at}}\} = \frac{1}{s-a}$ where ‘a’ is constant.

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}} f(t)dt$

Then $\mathcal{L}\{e^{\mathrm{at}}\} = \int_0^\infty e^{-\mathrm{st}}\, e^{\mathrm{at}} dt = \int_0^\infty e^{-(s-a)\mathrm{t}}\, dt = \left|-\frac{e^{-(s-a)\mathrm{t}}}{(s-a)}\right|_0^\infty = \frac{1}{s-a}$

Question: Show that $\mathcal{L}\{t^{\mathrm{n}}\} = \frac{n!}{s^{n+1}}$ where ‘$n > 0$’

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\mathrm{st}} f(t)dt$

Then for n = 1;

$$\mathcal{L}\{t\} = \int_0^\infty e^{-\mathrm{st}}\, tdt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} dt = \left|-\frac{te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-\mathrm{st}}\, dt = \frac{1}{s}$$

In above $te^{-\mathrm{st}} \rightarrow 0$ as $t \rightarrow \infty$

for n = 2;

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-\mathrm{st}}\, t^2\, dt = \left|-\frac{t^2\, e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} 2t\, dt = \left|-\frac{t^2\, e^{-st}}{s}\right|_0^\infty +$$

$\frac{2}{s}\int_0^\infty e^{-st}\, dt = \frac{2}{s^3}$ In this part $t^2\, e^{-st}, te^{-st} \rightarrow 0$ as $t \rightarrow \infty$

And in general

$$\mathcal{L}\{t^{\mathrm{n}}\} = \int_0^\infty e^{-\mathrm{st}}\, t^{\mathrm{n}}\, dt = \left|-\frac{t^{\mathrm{n}}\, e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} nt^{\mathrm{n}-1}\, dt$$

$$\mathcal{L}\{t^{\mathrm{n}}\} = \left|-\frac{t^{\mathrm{n}}\, e^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{\mathrm{n}-1}\, dt = \frac{n}{s}\mathcal{L}\{t^{\mathrm{n}-1}\} =$$

$$\frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\ldots\ldots\ldots..\frac{2}{s}\cdot\frac{1}{s}\mathcal{L}\{t^0\}$$

$$\mathcal{L}\{t^{\mathrm{n}}\} = \frac{(n-1)(n-1)(n-1)\ldots\ldots\ldots.3.2.1}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^n}\cdot\frac{1}{s}$$

Hence $\mathcal{L}\{t^{\mathrm{n}}\} = \frac{n!}{s^{n+1}}$ where ‘$n > 0$’

THE ERROR FUNCTION

39. Prove: $\mathcal{L}\{\operatorname{erf}\sqrt{t}\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\} = \frac{1}{s\sqrt{s+1}}$.

Using infinite series, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\} &= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right)du\right\} \\
&= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\cdot 2!} - \frac{t^{7/2}}{7\cdot 3!} + \cdots\right)\right\} \\
&= \frac{2}{\sqrt{\pi}}\left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5\cdot 2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7\cdot 3!\,s^{9/2}} + \cdots\right\} \\
&= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{s^{7/2}} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{s^{9/2}} + \cdots \\
&= \frac{1}{s^{3/2}}\left\{1 - \frac{1}{2}\frac{1}{s} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{s^2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{s^3} + \cdots\right\} \\
&= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}}
\end{aligned}$$

Question: Show that $\mathcal{L}\{Sinat\} = \frac{a}{s^2+a^2}$

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\text{st}} f(t)dt$

Then $\mathcal{L}\{Sinat\} = \int_0^\infty e^{-\text{st}} Sinatdt$

$\therefore \int_0^\infty e^{\text{at}} Sinbtdt = \frac{e^{\text{at}}}{a^2+b^2}[aSinbt - bCosbt]$ therefore

$\mathcal{L}\{Sinat\} = \left|\frac{e^{-\text{st}}}{s^2+a^2}[-sSinat - aCosat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-a)\right] = \frac{a}{s^2+a^2}$

Question: Show that $\mathcal{L}\{Cosat\} = \frac{s}{s^2+a^2}$

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\text{st}} f(t)dt$

Then $\mathcal{L}\{Cosat\} = \int_0^\infty e^{-\text{st}} Cosatdt$

$\therefore \int_0^\infty e^{\text{at}} Cosbtdt = \frac{e^{\text{at}}}{a^2+b^2}[aCosbt + bSinbt]$ therefore

$\mathcal{L}\{Cosat\} = \left|\frac{e^{-\text{st}}}{s^2+a^2}[-sCosat + aSinat]\right|_0^\infty = \left[0 - \frac{e^0}{s^2+a^2}(-s)\right] = \frac{s}{s^2+a^2}$

Question: Show that $\mathcal{L}\{Sinhat\} = \frac{a}{s^2-a^2}$

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\text{st}} f(t)dt$

Then $\mathcal{L}\{Sinhat\} = \int_0^\infty e^{-\text{st}} \left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\left[\int_0^\infty e^{-\text{st}} e^{at}dt - \int_0^\infty e^{-\text{st}} e^{-at}dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{a}{s^2-a^2}$

Question: Show that $\mathcal{L}\{Coshat\} = \frac{s}{s^2-a^2}$

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\text{st}}\, f(t)dt$

Then $\mathcal{L}\{Coshat\} = \int_0^\infty e^{-\text{st}} \left(\frac{e^{at}+e^{-at}}{2}\right) dt = \frac{1}{2}\left[\int_0^\infty e^{-\text{st}}\, e^{at}dt + \int_0^\infty e^{-\text{st}}\, e^{-at}dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt + \int_0^\infty e^{-(s+a)t}\, dt\right]$

$\mathcal{L}\{Sinhat\} = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{(s+a)}\right|_0^\infty = \frac{s}{s^2-a^2}$

FUNCTION OF EXPONENTIAL ORDER: A function f (t) is said to be of exponential order as $t \to \infty$ if there exist real constants M and c such that $|f(t)| \leq Me^{\text{ct}}$ for $0 \leq t < \infty$.

FUNCTION OF CLASS 'A': A function f (t) which is peicewise continuous and is of exponential order is said to be function of class A.

EXISTENCE THEOREM OF LAPLACE TRANSFORMATION:

Let f be piecewise continuous in the interval $[0, T]$ for every positive T, and let f be of exponential order, that is, $f(t) = O(e^{\text{at}})$ as $t \to \infty$ for some $a > 0$. Then, the Laplace transform of $f(t)$ exists for $Res > a$.

OR sufficient condition for the existence of Laplace transformation is that it should be a function of class A.

Proof: Since f is piecewise continuous and of exponential order, we have

$$|\mathcal{L}\{f(t)\}| = \left|\int_0^\infty e^{-\text{st}}\, f(t)dt\right| \leq \int_0^\infty e^{-\text{st}}\, |f(t)|dt \leq \int_0^\infty e^{-\text{st}}\, Me^{\text{at}}\, dt = M\int_0^\infty e^{-(s-a)\text{t}}\, dt$$

$|\mathcal{L}\{f(t)\}| \leq \frac{M}{s-a}$ Thus the Laplace transform of $f(t)$ exists for $Res > a$.

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points. Hence we may say that the inverse of a given transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical.

Remark: $F(s) = s^2$ is not L.T. of any piecewise continuous function of exponential order, because s^2 does not approaches to zero as $s \to \infty$ i.e. $\mathcal{L}^{-1}\{s^2\}$ does not exists.

Question: Show that $\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ where α is any real.

Solution: Since $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-\mathrm{st}} f(t)dt$

$$\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-\mathrm{st}}\, t^{\alpha}\, dt = \int_0^{\infty} e^{-\mathrm{u}} \left(\frac{u}{s}\right)^{\alpha} \frac{1}{s} du = \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-\mathrm{u}}\, u^{\alpha}\, du \,\ldots\ldots\ldots\text{(i)}$$

Since by definition of Gamma function we have

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\mathrm{u}}\, u^{\alpha-1}\, du \Rightarrow \Gamma(\alpha+1) = \int_0^{\infty} e^{-\mathrm{u}}\, u^{\alpha}\, du \quad (i) \Rightarrow \mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

USEFUL RESULTS:

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ then $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}$
- $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}$

Question: Find $\mathcal{L}\left\{t^{1/2}\right\}$ and $\mathcal{L}\left\{t^{-1/2}\right\}$

Solution: Since $\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

Put $\alpha = \frac{1}{2}$ Then $\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ now using $\mathcal{L}\{t^{\alpha}\} = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}$

we have $\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s.s^{\frac{1}{2}}}$

Then $\mathcal{L}\left\{t^{1/2}\right\} = \frac{1}{2s}\frac{\sqrt{\pi}}{\sqrt{s}}$ as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ thus $\mathcal{L}\left\{t^{1/2}\right\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$

Put $\alpha = -\frac{1}{2}$ Then $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$ now we have $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$

Then $\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}}$ as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ thus $\mathcal{L}\left\{t^{-1/2}\right\} = \sqrt{\frac{\pi}{s}}$

Question: Find $\mathcal{L}\{t^{k/2}\}$ where 'k' is an odd positive integer. $\mathcal{L}\{t^{5/2}\}=?$
Solution: Suppose k = m + 1 where 'm' is any positive integer.
Then using $\mathcal{L}\{t^{\alpha}\}=\frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\}=\mathcal{L}\left\{t^{\frac{2m+1}{2}}\right\}=\mathcal{L}\left\{t^{m+\frac{1}{2}}\right\}=\frac{m+\frac{1}{2}}{s}\mathcal{L}\left\{t^{m+\frac{1}{2}-1}\right\}=\frac{m+\frac{1}{2}}{s}.\frac{m-\frac{1}{2}}{s}.\mathcal{L}\left\{t^{m-\frac{1}{2}-1}\right\}$$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\}=\frac{2m+1}{2s}.\frac{2m-1}{2s}.\frac{2m-3}{2s}\dots\dots..\frac{3}{2s}.\frac{1}{2s}.\mathcal{L}\left\{t^{-\frac{1}{2}}\right\}=\frac{(2m+1).(2m+1).(2m+1)\dots\dots\dots..3.1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}}$$

$$\mathcal{L}\left\{t^{\frac{k}{2}}\right\}=\frac{(2m+1).(2m+1-2).(2m+1-4)\dots\dots\dots..3.1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}}=\frac{(k).(k-2).(k-4)\dots\dots\dots..3.1}{(2)^{\frac{k+1}{2}}}\sqrt{\frac{\pi}{s^{k+2}}}$$

Where we use $2m+1=k\Rightarrow m=(k-1)/2$

If k = 5 then $\mathcal{L}\left\{t^{\frac{5}{2}}\right\}=\frac{5.3.1}{(2)^{\frac{5+1}{2}}}\sqrt{\frac{\pi}{s^{5+2}}}=\frac{15}{(2)^{3}}\sqrt{\frac{\pi}{s^{7}}}$

Example

$$\mathcal{L}(t^{n+1})=\int_0^{\infty}e^{-st}t^{n+1}\,dt=-\frac{1}{s}e^{-st}t^{n+1}\Big|_0^{\infty}+\frac{n+1}{s}\int_0^{\infty}e^{-st}t^{n}\,dt.$$

$$\mathcal{L}(t^{n+1})=\frac{n+1}{s}\mathcal{L}(t^{n})=\frac{n+1}{s}\cdot\frac{n!}{s^{n+1}}=\frac{(n+1)!}{s^{n+2}}.$$

PROPERTIES OF LAPLACE TRANSFORMS

LINEARITY PROPERTY: THE LAPLACE TRANSFORMATION $\mathcal{L}$ IS LINEAR.

Proof. Let $u(t)=af(t)+bg(t)$ where a and b are constants.
We have, by definition

$$\mathcal{L}\{u(t)\}=\int_0^{\infty}e^{-st}f(t)dt=\int_0^{\infty}e^{-st}[af(t)+bg(t)]dt$$

$$\mathcal{L}\{u(t)\}=a\int_0^{\infty}e^{-st}f(t)dt+b\int_0^{\infty}e^{-st}g(t)dt=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$$

$\mathcal{L}\{af(t)+bg(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$ hence proved.

1st SHIFTING PROPERTY (s- SHIFTING / 1st TRANSLATION THEOREM):

If $F(s)$ is the laplace transformation of $f(t)$ Then $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$
Proof. By definition, we have

$$\mathcal{L}\{e^{at}f(t)\}=\int_0^{\infty}e^{-st}e^{at}f(t)dt=\int_0^{\infty}e^{-(s-a)t}f(t)dt=F(s-a)$$

This result also known as 1st shifting theorem or 1st translation theorem.

EXAMPLES:

i. If $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ then $\mathcal{L}\{t^2 e^{\mathrm{t}}\} = \frac{2}{(s-1)^3}$
ii. If $\mathcal{L}\{Sinwt\} = \frac{w}{s^2+w^2}$ then $\mathcal{L}\{e^{\mathrm{at}} Sinwt\} = \frac{w}{(s-a)^2+w^2}$
iii. If $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2}$ then $\mathcal{L}\{e^{\mathrm{at}} Coswt\} = \frac{s-a}{(s-a)^2+w^2}$
iv. If $\mathcal{L}\{t^{\mathrm{n}}\} = \frac{n!}{s^{n+1}}$ then $\mathcal{L}\{e^{\mathrm{at}} t^{\mathrm{n}}\} = \frac{n!}{(s-a)^{n+1}}$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\}$

Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{\mathrm{at}} f(t)\} = F(s-a) \Rightarrow e^{\mathrm{at}} f(t) = e^{\mathrm{at}} \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$
Thus $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2-1^2}\right\} = e^{-\mathrm{t}} \mathcal{L}^{-1}\left\{\frac{s}{s^2-1^2}\right\} = e^{-\mathrm{t}} Cosht$

Question: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\}$

Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{\mathrm{at}} f(t)\} = F(s-a) \Rightarrow e^{\mathrm{at}} f(t) = e^{\mathrm{at}} \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$
Thus $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-1^2}\right\} = e^{-\mathrm{t}} \mathcal{L}^{-1}\left\{\frac{1}{s^2-1^2}\right\} = e^{-\mathrm{t}} Sinht$

Question: Find $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\}$

Answer: in this question we will use the first shifting theorem according to which
$\mathcal{L}\{e^{\mathrm{at}} f(t)\} = F(s-a) \Rightarrow e^{\mathrm{at}} f(t) = e^{\mathrm{at}} \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-a)\}$
Thus $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{2}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\}\left\{\frac{2}{s}\right\}$
$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = e^{-\mathrm{t}} \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - e^{-2\mathrm{t}} \mathcal{L}^{-1}\left\{\frac{2}{s}\right\}$
$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = 3e^{-\mathrm{t}} - 2e^{-2\mathrm{t}}$ since $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

Question: Find $\mathcal{L}^{-1}\left\{\frac{3s-137}{s^2+2s+401}\right\}$

Answer:

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$

SCALING PROPERTY: If $F(s)$ is the laplace transformation of (t) , then
$\mathcal{L}\,[f\ (at)] = \frac{1}{a}\ F\ (\frac{s}{a})$ with a > 0

Proof. By definition we have
$\mathcal{L}\,\{f(at)\} = \int_0^{\infty} e^{-\mathrm{st}}\ f(at)dt = \frac{1}{a}\int_0^{\infty} e^{-\left(\frac{s}{a}\right)\mathrm{t}'}\ f(t')dt' = \frac{1}{a}\ F\ (\frac{s}{a})$
putting $at = t'$ This result also known as Rule of Scale.

EXAMPLES:

i. If $\mathcal{L}\{Cost\} = \frac{s}{s^2+1}$ then $\mathcal{L}\{Coswt\} = \frac{s}{s^2+w^2} = \frac{1}{w}\left[\frac{s/w}{(s/w)^2+1}\right]$

ii. If $\mathcal{L}\{e^{\mathrm{t}}\} = \frac{1}{s-1}$ then $\mathcal{L}\{e^{\mathrm{at}}\} = \frac{1}{s-a} = \frac{1}{a}\left[\frac{1}{\left(\frac{s}{a}-1\right)}\right]$

DIFFERENTIATION PROPERTY:

Let f be continuous and f' piecewise continuous, in $0 \le t \le T$ for all $T > 0$. Let f also be of exponential order as $t \to \infty$ Then, the Laplace transform of $f'\ (t)$ exists and is given by

$$\mathcal{L}[f'\ (t)] = s\mathcal{L}[f(t)] - f\ (0) = sF(s) - f\ (0)$$

Proof. If $f(t)$ is continuous and $f'(t)$ is sectionally continuous on the interval $[0, \infty)$ and both are of exponential order then
$\mathcal{L}\,\{f'(t)\} = \int_0^{\infty} e^{-\mathrm{st}}\ f'(t)dt = |e^{-\mathrm{st}} f(t)|_0^{\infty} - (-s)\int_0^{\infty} e^{-\mathrm{st}}\ f(t)dt$
$\mathcal{L}\,\{f'(t)\} = [0 - f(0)] + s\mathcal{L}\,\{f(t)\}$
$\mathcal{L}[f'\ (t)] = s\mathcal{L}[f(t)] - f\ (0) = sF(s) - f\ (0)$
If f' and f'' satisfy the same conditions imposed on f and f' respectively, then, the Laplace transform of $f''\ (t)$ can be obtained immediately by applying the preceding theorem; that is

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f\,'(0) = s^2F(s) - sf(0) - f'\ (0)$$

Proof. If $f(t), f'(t)$ are continuous and $f''(t)$ is sectionally continuous on the interval $[0, \infty)$ and all are of exponential order then
$\mathcal{L}\,\{f''(t)\} = \int_0^{\infty} e^{-\mathrm{st}}\ f''(t)dt = |e^{-\mathrm{st}} f'(t)|_0^{\infty} - (-s)\int_0^{\infty} e^{-\mathrm{st}}\ f'(t)dt$
$\mathcal{L}\,\{f''(t)\} = [0 - f'(0)] + s\mathcal{L}\,\{f'(t)\} = -f'(0) + s[sF(s) - f\ (0)]$
$\mathcal{L}[f''(t)] = s\mathcal{L}[f(t)] - f\,'(0) = s^2F(s) - sf(0) - f'\ (0)$

Clearly, the Laplace transform of $f^n(t)$ can be obtained in a similar manner by successive application.The result may be written as
$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - \cdots \ldots \ldots - sf^{n-2}(0) - f^{n-1}(0)$

Example

Let $f(t) = t\sin\omega t$. Then $f(0)=0$, $f'(t) = \sin\omega t + \omega t\cos\omega t$, $f'(0)=0$, $f'' = 2\omega\cos\omega t - \omega^2 t\sin\omega t$. Hence

$$\mathcal{L}(f'') = 2\omega\frac{s}{s^2+\omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f), \qquad \text{thus} \qquad \mathcal{L}(f) = \mathcal{L}(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

Example

Let $f(t) = \cos\omega t$. Then $f(0) = 1, f'(0) = 0, f''(t) = -\omega^2\cos\omega t$.

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f). \qquad \text{By algebra,} \qquad \mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

Similarly, let $g = \sin\omega t$. Then $g(0) = 0$, $g' = \omega\cos\omega t$.

$$\mathcal{L}(g') = s\mathcal{L}(g) = \omega\mathcal{L}(\cos\omega t). \qquad \text{Hence,} \qquad \mathcal{L}(\sin\omega t) = \frac{\omega}{s}\mathcal{L}(\cos\omega t) = \frac{\omega}{s^2+\omega^2}.$$

INTEGRATION PROPERTY :

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$$

PROOF:

Consider $g(\tau) = \int_0^t f(\tau)\,d\tau \Rightarrow g'(\tau) = f(t) \Rightarrow \mathcal{L}[g'(\tau)] = \mathcal{L}[f(t)]$

$\Rightarrow sG(s) - g(0) = \mathcal{L}[f(t)] \Rightarrow s\mathcal{L}[g(\tau)] - 0 = \mathcal{L}[f(t)]$

$\Rightarrow \mathcal{L}[g(\tau)] = \frac{F(s)}{s} \qquad \Rightarrow \mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$

Question: Solve the initial value problem $u' - 2u = 0$ with $u(0) = 1$

Answer: Given $u' - 2u = 0$

$\Rightarrow \mathcal{L}\{u'\} - 2\mathcal{L}\{u\} = 0 \Rightarrow sU(s) - u(0) - 2U(s) = 0$

Using $u(0) = 1 \quad \Rightarrow sU(s) - 1 - 2U(s) = 0 \Rightarrow U(s) = \frac{1}{s-2}$

$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \qquad \Rightarrow u(t) = e^{2t}$ required answer.

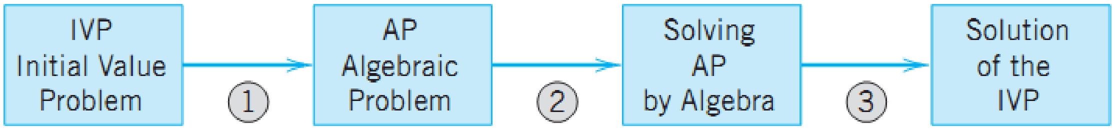
Example

$$\frac{1}{s(s^2+\omega^2)} \text{ and } \frac{1}{s^2(s^2+\omega^2)}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin\omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{\sin\omega t}{\omega}\,d\tau = \frac{1}{\omega^2}(1-\cos\omega t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}.$$

Remember
The process of solving an ODE using the Laplace transform method consists of three steps:
Step 1. The given ODE is transformed into an algebraic equation, called the subsidiary equation.
Step 2. The subsidiary equation is solved by purely algebraic manipulations.
Step 3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.



Question:
Discuss how the Laplace transform method solves ODEs and initial value problems.
Solution
We consider an initial value problem

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are constant.
After taking laplace

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

divide by $s^2 + as + b$

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Apply inverse to get solution.

Question:

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Answer:

$$y'' - y = t,$$

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

$$Y = (s + 1)Q + \frac{1}{s^2}Q = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}.$$ where $Q = 1/(s^2 - 1)$

$$Y = \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

Question:

Solve the initial value problem $u'' + 4u' + 3u = 0$ with $u(0) = 1, u'(0) = 0$

Answer

Given $u'' + 4u' + 3u = 0$

$\Rightarrow \mathcal{L}\{u''\} + 4\,\mathcal{L}\{u'\} + 3\mathcal{L}\{u\} = 0$

$\Rightarrow s^2U(s) - su(0) - u'(0) + 4sU(s) - 4u(0) + 3U(s) = 0$

$\Rightarrow s^2U(s) - s + 4sU(s) - 4 + 3U(s) = 0$ since $u(0) = 1, u'(0) = 0$

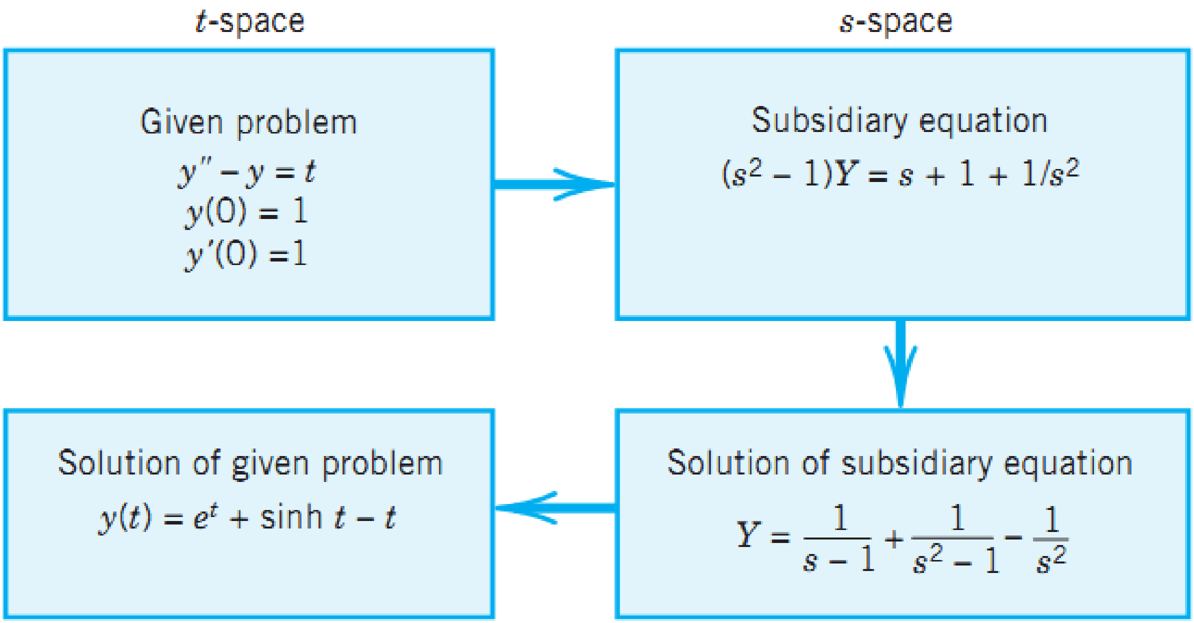
$\Rightarrow U(s) = \frac{s+4}{s^2+4s+2} \Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+2}\right\}$

$\Rightarrow u(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1} - \frac{1/2}{s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1/2}{s+3}\right\}$

$\Rightarrow u(t) = e^{-t}\mathcal{L}^{-1}\left\{\frac{3/2}{s}\right\} - e^{-3t}\mathcal{L}^{-1}\left\{\frac{1/2}{s}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}$ since $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

Steps of the Laplace transform method



Advantages of the Laplace Method

1. Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
2. Initial values are automatically taken care of.
3. Complicated inputs r(t) (right sides of linear ODEs) can be handled very Efficiently.

Question:

Solve the initial value problem

$$y'' + y' + 9y = 0. \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

Answer:

$$y'' + y' + 9y = 0.$$

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, \qquad \text{thus} \qquad (s^2 + s + 9)Y = 0.16(s + 1).$$

$$Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}.$$

Hence by the first shifting theorem and the formulas for cos and sin

$$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16 \cos \sqrt{\frac{35}{4}}\, t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin \sqrt{\frac{35}{4}}\, t\right)$$

$$= e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t).$$

Question (Shifted Data Problem):

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution

We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$.

$$s^2\tilde{Y} - s \cdot \tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

$$\begin{aligned}\tilde{y} &= \mathcal{L}^{-1}(\tilde{Y}) = 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t} \\ &= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}.\end{aligned}$$

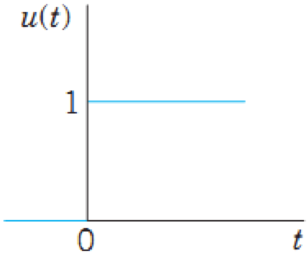
Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin\tilde{t} = \dfrac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

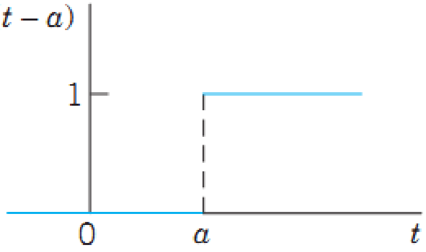
$$y = 2t - \sin t + \cos t.$$

UNIT STEP FUNCTION/ HEAVISIDE FUNCTION:

A real valued function $H: R \to R$ is defined as

$$H(t-a) = \begin{cases} 1 & ; \quad t \geq a \\ 0 & ; \quad t < a \end{cases} \qquad \text{When } a = 0 \quad ; \quad H(t) = \begin{cases} 1 & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases}$$





LAPLACE TRANSFORM OF STEP FUNCTION:

The Heaviside unit step function is defined by

$$H\,(t-a)=\begin{cases}0 & t<a\\ 1 & t\geq a\end{cases}\quad \text{where } a\geq 0$$

Now, we will find its Laplace transform.

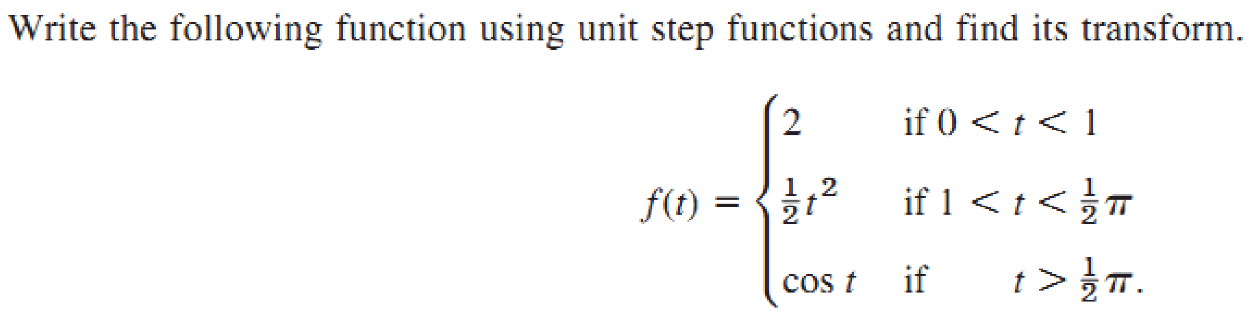
$$\mathcal{L}\{H\,(t-a)\}=\int_0^\infty e^{-st}\,H\,(t-a)\,dt$$

$$\mathcal{L}\{H\,(t-a)\}=\int_0^a e^{-st}\,H\,(t-a)\,dt+\int_a^\infty e^{-st}\,H\,(t-a)\,dt$$

$$\mathcal{L}\{H\,(t-a)\}=\int_0^a e^{-st}\,.0\,dt+\int_a^\infty e^{-st}\,.1\,dt$$

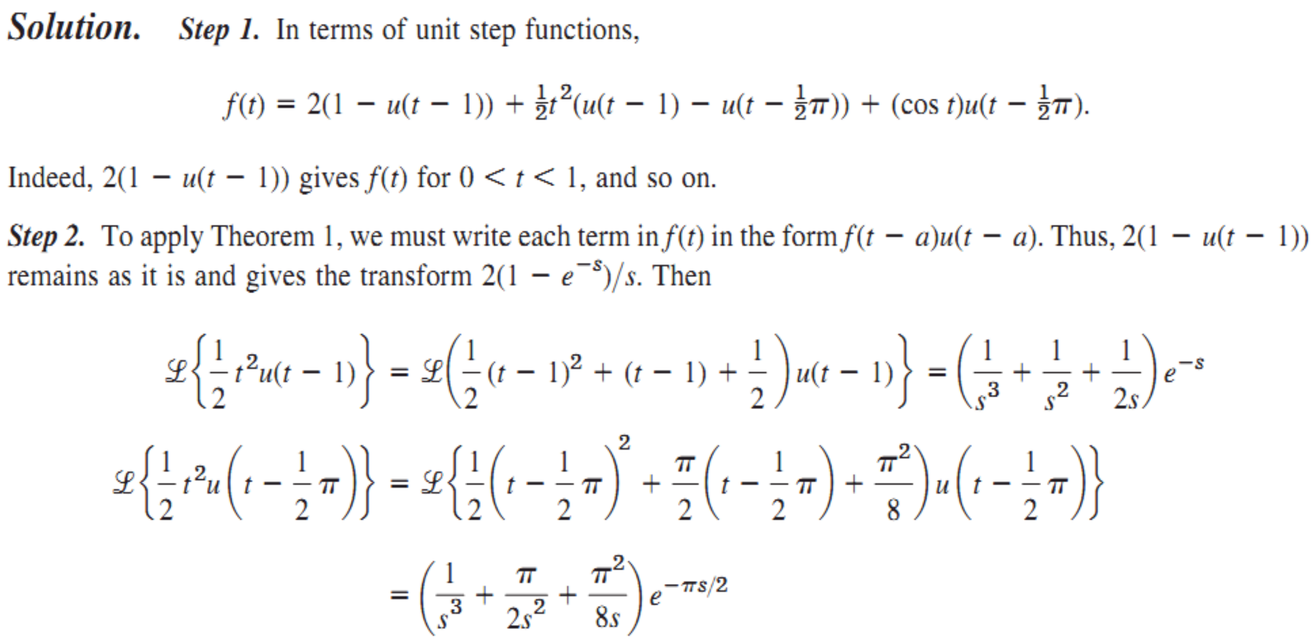
$$\mathcal{L}\{H\,(t-a)\}=\int_a^\infty e^{-st}\,dt=\left|\frac{e^{-st}}{-s}\right|_a^\infty=\frac{e^{-as}}{s}\qquad ;\ s>0$$

Example



Write the following function using unit step functions and find its transform.

$$f(t)=\begin{cases}2 & \text{if } 0<t<1\\ \frac{1}{2}t^2 & \text{if } 1<t<\frac{1}{2}\pi\\ \cos t & \text{if } \qquad t>\frac{1}{2}\pi.\end{cases}$$



Solution. ***Step 1.*** In terms of unit step functions,

$$f(t)=2(1-u(t-1))+\tfrac{1}{2}t^2(u(t-1)-u(t-\tfrac{1}{2}\pi))+(\cos t)u(t-\tfrac{1}{2}\pi).$$

Indeed, $2(1-u(t-1))$ gives $f(t)$ for $0<t<1$, and so on.

Step 2. To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1-u(t-1))$ remains as it is and gives the transform $2(1-e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=\mathcal{L}\left(\frac{1}{2}(t-1)^2+(t-1)+\frac{1}{2}\right)u(t-1)\Bigg\}=\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2+\frac{\pi}{2}\left(t-\frac{1}{2}\pi\right)+\frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\Bigg\}$$

$$=\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)\,u\left(t-\frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t-\frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right\} = -\frac{1}{s^2+1}e^{-\pi s/2}.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t-a)$ is inconvenient, replace it by

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

by writing $f(t-a) = g(t)$, hence $f(t) = g(t+a)$ and then again writing f for g. Thus,

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2+t+\frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)$$

as before. Similarly for $\mathcal{L}\{\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\}$. Finally,

$$\mathcal{L}\left\{\cos t\, u\left(t-\frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t+\frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}.$$

Example (Application of Both Shifting Theorems. Inverse Transform)

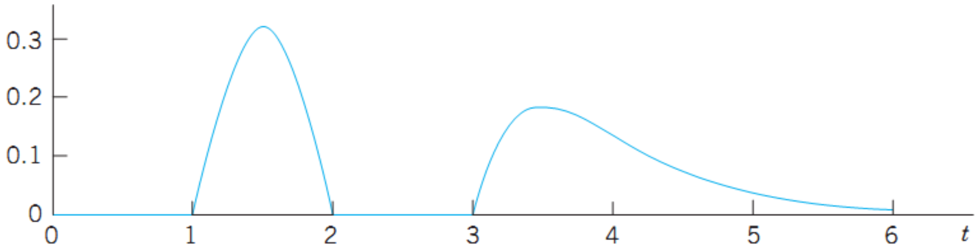
Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

Solution. Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and te^{-2t} because $1/s^2$ has the inverse t, so that $1/(s+2)^2$ has the inverse te^{-2t} by the first shifting theorem . Hence by the second shifting theorem (t-shifting),

$$f(t) = \frac{1}{\pi}\sin(\pi(t-1))\,u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))\,u(t-2) + (t-3)e^{-2(t-3)}\,u(t-3).$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so that the first and second terms cancel each other when $t > 2$. Hence we obtain $f(t) = 0$ if $0 < t < 1$, $-(\sin \pi t)/\pi$ if $1 < t < 2$, 0 if $2 < t < 3$, and $(t-3)e^{-2(t-3)}$ if $t > 3$. See Fig.



CONVOLUTION FUNCTION / FAULTUNG FUNCTION OF LAPLACE TRANSFORMATION.

The function $(f \, * \, g)\,(t) \; = \; \int_0^t f\;(t \, - \, \xi)\; g\;(\xi)\; d\xi$ is called the convolution of the functions f and g regarding laplace transformation.

THE CONVOLUTION SATISFIES THE FOLLOWING PROPERTIES:

1. $f \, * \, g \; = \; g \, * \, f$ (commutative).
2. $f \, * \, (g \, * \, h) \; = \; (f \, * \, g) \, * \, h$ (associative).
3. $f \, * \, (\alpha g \, + \, \beta h) \; = \; \alpha\,(f \, * \, g) + \beta\,(f \, * \, h)$ (distributive),

where α and β are constants.

USEFUL RESULT:

$$(f \, * \, g)(t) = \int_0^t f\;(t \, - \, \xi)g\;(\xi)d\xi = \int_0^\infty H\;(t - \xi)f\;(t \, - \, \xi)\; g\;(\xi)\; d\xi$$

CONVOLUTION / FAULTUNG THEOREM OF LAPLACE TRANSFORMATION

If $F(s)$ and $G(s)$are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then the Laplace transform of the convolution $(f * \, g)\;(t)$ is the product $F(s)G(s)$

OR $\mathcal{L}^{-1}\{F(s)G(s)\} = f * \, g \Rightarrow \mathcal{L}\{f * \, g\} = F(s)G(s)$

PROOF: By definition, we have

$$\mathcal{L}\{f * \, g\} = \int_0^\infty e^{-st}\;(f * \, g)\; dt$$

$$\mathcal{L}\{f * \, g\} = \int_0^\infty e^{-st} \int_0^t f\;(t \, - \, \xi)\; g\;(\xi)\; d\xi \;\; dt$$

$$\mathcal{L}\{f * \, g\} = \int_0^\infty e^{-st} \int_0^t f(\xi)g\;(t \, - \, \xi)\; d\xi \;\; dt \qquad \text{since } f \, * \, g \; = \; g \, * \, f$$

$$\mathcal{L}\{f * \, g\} = \int_0^\infty e^{-\mathrm{st}} \left[\int_0^\infty H\;(t - \xi)f(\xi)g(t \, - \, \xi)\; d\xi\right] dt$$

By reversing the order of integration, we have

$$\mathcal{L}\{f * \, g\} = \int_0^\infty \left[\int_0^\infty e^{-\mathrm{st}} H\;(t - \xi)g(t \, - \, \xi)\; dt\right] f(\xi)d\xi$$

If we introduce the new variable $\eta \; = \; (t \, - \, \xi)$ in the inner integral, we obtain

$$\mathcal{L}\{f * \, g\} = \int_0^\infty f\;(\xi)\; d\xi \left[\int_{-\xi}^\infty e^{-s(\,\xi+\eta)} H\;(\eta)g(\eta)\; d\eta\right]$$

$$\mathcal{L}\{f * \, g\} =$$

$$\int_0^\infty f\;(\xi)\; d\xi \left[\int_{-\xi}^0 e^{-s(\,\xi+\eta)} H\;(\eta)g(\eta)\; d\eta + \int_0^\infty e^{-s(\,\xi+\eta)} H\;(\eta)g(\eta)\; d\eta\right]$$

$\mathcal{L}\{f * \, g\} = \int_0^\infty f(\xi)\; d\xi \left[\int_{-\xi}^0 e^{-s(\,\xi+\eta)} 0.\, g(\eta)\; d\eta + \int_0^\infty e^{-s(\,\xi+\eta)}.\, 1.\, g(\eta)\; d\eta\right]$ by step function

$$\mathcal{L}\{f * \, g\} = \int_0^\infty f(\xi)\; d\xi \left[\int_0^\infty e^{-s(\,\xi+\eta)} g(\eta)\; d\eta\right]$$

$$\mathcal{L}\{f * \, g\} = \int_0^\infty e^{-s\xi} f(\xi)d\xi \; \int_0^\infty e^{-s\eta} g\;(\eta)\; d\eta$$

$$\mathcal{L}\{f * \, g\} = F(s)G(s)$$

Example

Let $H(s) = 1/[(s - a)s]$. Find $h(t)$.

Solution. $1/(s - a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t - \tau) \equiv 1$ we thus obtain from (1) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1\, d\tau = \frac{1}{a}(e^{at} - 1).$$

To check, calculate

$$H(s) = \mathcal{L}(h)(s) = \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{a} \cdot \frac{a}{s^2 - as} = \frac{1}{s-a} \cdot \frac{1}{s} = \mathcal{L}(e^{at})\mathcal{L}(1).$$

Example

Let $H(s) = 1/(s^2 + \omega^2)^2$. Find $h(t)$.

Solution. The inverse of $1/(s^2 + \omega^2)$ is $(\sin \omega t)/\omega$.

$$h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t - \tau)\, d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos (2\omega\tau - \omega t)]\, d\tau$$

$$= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t}$$

$$= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right]$$

Example (Repeated Complex Factors. Resonance)

In an undamped mass–spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system. Then the model is

$$y'' + \omega_0^2 y = K \sin \omega_0 t$$

where $\omega_0^2 = k/m$, k is the spring constant, and m is the mass of the body attached to the spring. We assume $y(0) = 0$ and $y'(0) = 0$, for simplicity. Then the subsidiary equation is

$$s^2 Y + \omega_0^2 Y = \frac{K\omega_0}{s^2 + \omega_0^2}. \qquad \text{Its solution is} \qquad Y = \frac{K\omega_0}{(s^2 + \omega_0^2)^2}.$$

$H(s) = 1/(s^2 + \omega^2)^2$. The inverse of $1/(s^2 + \omega^2)$ is $(\sin \omega t)/\omega$.

$$h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t - \tau)\, d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos (2\omega\tau - \omega t)]\, d\tau$$

$$= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t}$$

$$= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right]$$

$$y(t) = \frac{K\omega_0}{2\omega_0^2}\left(-t \cos \omega_0 t + \frac{\sin \omega_0 t}{\omega_0}\right) = \frac{K}{2\omega_0^2}(-\omega_0 t \cos \omega_0 t + \sin \omega_0 t).$$

Example (Response of a Damped Vibrating System to a Single Square Wave)

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}, \qquad y(0) = y'(0) = 0.$$

Solution by Convolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t-\tau) \cdot 1 \, d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau = e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)}.$$

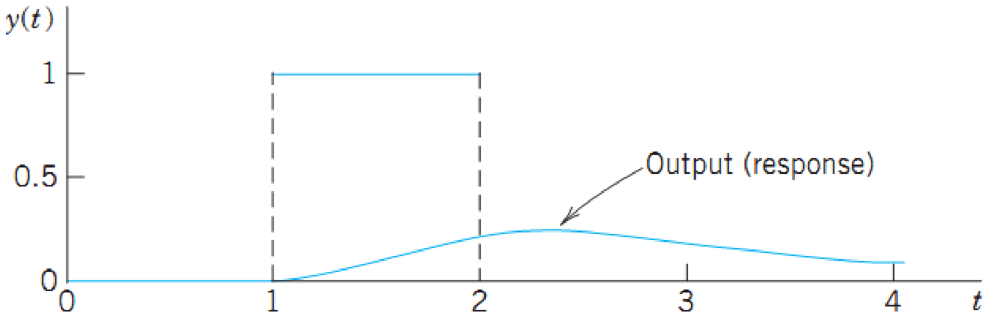
Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to t. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right).$$

Figure shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero, and finally decreases to zero in a monotone fashion.



Integral Equations

Equations in which the unknown function appears in an integral (and perhaps also outside of it).

Example (A Volterra Integral Equation of the Second Kind)

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad \text{and gives the answer} \qquad y(t) = t + \frac{t^3}{6}.$$

Example

Solve the Volterra integral equation

$$y(t) - \int_0^t (1+\tau)\,y(t-\tau)\,d\tau = 1 - \sinh t.$$

Solution. By (1) we can write $y - (1+t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}, \qquad \text{hence} \qquad Y(s)\cdot\frac{s^2-s-1}{s^2} = \frac{s^2-1-s}{s(s^2-1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2-1} \qquad \text{and the solution is} \qquad y(t) = \cosh t.$$ ■

Problem: Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\}$

Solution: Here we have $H(s) = F(s)G(s)$

then taking $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Rightarrow f(t) = t$

$G(s) = \frac{3}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\right\} \Rightarrow g(t) = Sin3t$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi = \int_0^t (t-\xi)\, Sin3(\xi)\, d\xi$$

$$h(t) = \int_0^t t\, Sin3(\xi)\, d\xi - \int_0^t \xi\, Sin3(\xi) d\xi = \left|-\frac{tCos3\xi}{3} + \frac{\xi Cos3\xi}{3} - \frac{Sin3\xi}{9}\right|_0^t$$

$$h(t) = -\frac{Sin3t}{9} + \frac{t}{3} = \frac{1}{9}(3t - Sin3t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\}$$

Problem: Use covolution theorem to find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$

Solution: Here we have $H(s) = F(s)G(s)$

then taking $F(s) = \frac{s}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)}\right\} \Rightarrow f(t) = Cos3t$

$G(s) = \frac{1}{(s^2+9)} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)}\right\} \Rightarrow g(t) = \frac{1}{3}Sin3t$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi = \frac{1}{3}\int_0^t Cos3(t-\xi)\, Sin3(\xi)\, d\xi$$

$$h(t) = \frac{1}{3}\int_0^t (Cos3tCos3\,\xi + Sin3tSin3\,\xi)\, Sin3(\xi)\, d\xi$$

$$h(t) = \frac{1}{3}\int_0^t Cos3tCos3\xi Sin3\xi + Sin3tSin^2 3\xi\, d\xi$$

$$h(t) = \frac{1}{6}Cos3t\int_0^t 2Cos3\xi Sin3\xi d\xi + \frac{1}{3}Sin3t\int_0^t Sin^2 3\xi\, d\xi$$

$$h(t) = \frac{1}{6}Cos3t\int_0^t Sin6\xi d\xi + \frac{1}{6}Sin3t\int_0^t \left(\frac{1-Cos6\xi}{2}\right)\, d\xi$$

$$h(t) = \frac{1}{6}Cos3t\left|-\frac{Cos6\xi}{6}\right|_0^t + \frac{1}{6}Sin3t\left|\xi - \frac{Sin6\xi}{6}\right|_0^t$$

$$h(t) = \frac{1}{36}Cos3t(1 - Cos6t) + \frac{1}{6}Sin3t\left(t - \frac{Sin6t}{6}\right)$$

$$h(t) = \frac{1}{36}Cos3t - \frac{1}{36}Cos3tCos6t + \frac{1}{6}tSin3t - \frac{1}{36}Sin3tSin6t$$

$$h(t) = -\frac{1}{36}[Cos3tCos6t + Sin3tSin6t] + \frac{1}{6}tSin3t + \frac{1}{36}Cos3t$$

$$h(t) = -\frac{1}{36}[Cos(6t - 3t)] + \frac{1}{6}tSin3t + \frac{1}{36}Cos3t$$

$$h(t) = -\frac{1}{36}Cos3t + \frac{1}{6}tSin3t + \frac{1}{36}Cos3t = -\frac{1}{6}tSin3t = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$$

Problem:

Use convolution theorem to find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

Solution: Here we have $H(s) = F(s)G(s) = \frac{1}{s^2+6s+13} = \frac{1}{(s+3+2i)(s+3-2i)}$

$$F(s) = \frac{1}{s+3+2i} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3+2i}\right\} \Rightarrow f(t) = e^{-(3+2i)t}$$

$$G(s) = \frac{1}{s+3-2i} \Rightarrow \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3-2i}\right\} \Rightarrow g(t) = e^{-(3-2i)t}$$

Now using Convolution theorem

$$h(t) = f * g = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t e^{-(3+2i)\xi}\, e^{-(3-2i)(t-\xi)}d\xi$$

$$h(t) = e^{-(3-2i)t}\int_0^t e^{-(3+2i)\xi}\, e^{(3-2i)\xi}d\xi$$

$$h(t) = e^{-(3-2i)t}\int_0^t e^{-4i\xi}\, d\xi$$

$$h(t) = e^{-(3-2i)t}\left|\frac{e^{-4i\xi}}{-4i}\right|_0^t = \frac{e^{-(3-2i)t}}{-4i}\left|e^{-4it} - e^0\right| = \frac{e^{-(3-2i)t}}{-4i}\left|e^{-4it} - 1\right|$$

$$h(t) = \frac{e^{-3t}}{2}\left[\frac{e^{-2it} - e^{2it}}{-2i}\right] = \frac{e^{-3t}}{2}\left[\frac{e^{2it} - e^{-2it}}{2i}\right]$$

$$h(t) = \frac{e^{-3t}}{2}Sin2t$$

Problem:

Use covolution theorem to calculate laplace transform of

$f(t) = \int_0^t (t - \beta)^3 e^{\beta} Sin\beta d\beta$

Solution:

Let $f(t) = g * h = \int_0^t (t - \beta)^3 e^{\beta} Sin\beta d\beta$ …………..(i)

Comparing with $g * h = \int_0^t g\,(t - \beta) h\,(\beta) d\beta$ …………..(ii) we get

$g\,(t - \beta) = (t - \beta)^3 \Rightarrow g\,(t\,) = t^3$ and $h\,(\beta) = e^{\beta} Sin\beta \Rightarrow h\,(t) = e^t Sint$

Now $\mathcal{L}\{f(t)\} = \mathcal{L}\{g * h\} = F(s)G(s) = \mathcal{L}\{g(t)\}.\mathcal{L}\{h(t)\} = \mathcal{L}\{t^3\}.\mathcal{L}\{e^t Sint\}$

$\mathcal{L}\{f(t)\} = \frac{3!}{s^{3+1}} . \frac{1}{(s-1)^2+1^2} = \frac{6}{s^4(s^2-2s+1)}$

Example (Unusual Properties of Convolution)

$f * 1 \neq f$ in general. For instance,

$$t * 1 = \int_0^t \tau \cdot 1\, d\tau = \frac{1}{2} t^2 \neq t.$$

The Gaussian Integral

Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ or $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

Solution: consider $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$ and $I = \int_{-\infty}^{\infty} e^{-y^2}\, dy$

then multiplying both $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy$

Now using polar coordinates

$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, rdrd\theta = \int_0^{2\pi} d\theta \left(-\frac{1}{2}\right) \int_0^{\infty} e^{-r^2}\, (-2r)dr = \pi \Rightarrow I = \sqrt{\pi}$

$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \Rightarrow 2\int_0^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

Theorem (Differentiation of Transforms):

If $f(t)$ is a function of exponential order 'c' then

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)\ ; s > a$

Proof: Consider $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-\text{st}}\, f(t)dt$

Differentiating w.r.to 's'

$\Rightarrow \frac{d}{ds} F(s) = (-1) \int_0^\infty e^{-\text{st}}\, tf(t)dt = (-1)\mathcal{L}\{tf(t)\} \Rightarrow (-1) \frac{d^1}{ds^1} F(s) = \mathcal{L}\{t^1 f(t)\}$

Again differentiating w.r.to 's'

$\Rightarrow \frac{d^2}{ds^2} F(s) = (-1)(-1) \int_0^\infty e^{-\text{st}}\, (-t)\, tf(t)dt = (-1)^2 \int_0^\infty e^{-\text{st}}\, t^2 f(t)dt = (-1)^2 \mathcal{L}\{t^2 f(t)\}$

$\Rightarrow (-1)^2 \frac{d^2}{ds^2} F(s) = \mathcal{L}\{t^2 f(t)\}$

Continuing this process, we get the required

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)\ ; s > a \qquad \therefore (-1)^n = (-1)^{-n}$

REMARK: $\mathcal{L}\{t^{-n} f(t)\} = \frac{d^n}{ds^n} F(s)$

Example

$$\mathcal{L}(t \sin \beta t) = \frac{2\beta s}{(s^2 + \beta^2)^2}.$$

$$\mathcal{L}(t \cos \beta t) = -\frac{(s^2 + \beta^2) - 2s^2}{(s^2 + \beta^2)^2} = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2}.$$

$$\mathcal{L}\left(t \cos \beta t \pm \frac{1}{\beta} \sin \beta t\right) = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2} \pm \frac{1}{s^2 + \beta^2}.$$

Laplace Transformation of Logrithmic Function:

Show that $\mathcal{L}\{lnt\} = \frac{1}{s}(\Gamma'(1) - lns)$

Solution: by using definition

$\mathcal{L}\{lnt\} = \int_0^\infty e^{-\mathrm{st}}\ lntdt = \int_0^\infty e^{-\mathrm{u}}\ ln\left(\frac{u}{s}\right)\frac{du}{s}$ by putting st = u

$\mathcal{L}\{lnt\} = \frac{1}{s}\int_0^\infty e^{-\mathrm{u}}\ lnudu - \frac{1}{s}\int_0^\infty e^{-\mathrm{u}}\ lnsdu = \frac{1}{s}(I) - \frac{1}{s}lns\int_0^\infty e^{-\mathrm{u}}\ du$

$\mathcal{L}\{lnt\} = \frac{1}{s}(I) - \frac{1}{s}lns(1) = \frac{1}{s}(I) - \frac{1}{s}lns$ …………..(i)

Now consider $I = \int_0^\infty e^{-\mathrm{u}}\ lnudu$

Since $\Gamma(\alpha) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha-1}\ du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha}\ du \Rightarrow \Gamma'(1) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha}\,lnu\,du$

Put $\alpha = 0 \Rightarrow \Gamma'(1) = \int_0^\infty e^{-\mathrm{u}}\,lnu\,du = I$

Thus $\mathcal{L}\{lnt\} = \frac{1}{s}(\Gamma'(1) - lns)$

where $\Gamma'(1) \approx 0.57721$ is called Euler's constant.

The Gamma Function:

Gamma function can be defined as follows $\Gamma(\alpha) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha-1}\ du$

Useful Results:

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

  Proof: since $\Gamma(\alpha) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha-1}\ du \Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha}\ du$

  $\Rightarrow \Gamma(\alpha+1) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha}\ du = \left|u^{\alpha}\,\frac{e^{-\mathrm{u}}}{-1}\right|_0^\infty - \int_0^\infty \left|\frac{e^{-\mathrm{u}}}{-1}\right| \alpha u^{\alpha-1}\ du$

  $\Rightarrow \Gamma(\alpha+1) = 0 + \alpha\int_0^\infty e^{-\mathrm{u}}\,u^{\alpha-1}\ du = \alpha\Gamma(\alpha)$
- $\Gamma(1) = 1$ we can prove it using $\Gamma(\alpha) = \int_0^\infty e^{-\mathrm{u}}\,u^{\alpha-1}\ du$ with $\alpha = 1$
- $\Gamma(\alpha+1) = \alpha!$

  Proof: since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

  $put\ \ \alpha = 1 \Rightarrow \Gamma(2) = 1.\Gamma(1) = 1.1 = 1!$

  $put\ \ \alpha = 2 \Rightarrow \Gamma(3) = 2.\Gamma(2) = 2.1 = 2!$

  $put\ \ \alpha = 3 \Rightarrow \Gamma(4) = 3.\Gamma(3) = 3.2.1 = 3!$

  : :

  Then $\Gamma(\ \alpha) = \ \alpha - 1! \Rightarrow \Gamma(\alpha+1) = \alpha!$

Second Shifting/ Time Shifting (Translation) Theorem:

If $F(s)$ and $H(s)$ are the Laplace transforms of $f(t)$ and $h(t)$ respectively, then

$\mathcal{L}[H\,(t-a)\,f\,(t-a)] = e^{-as}\,F(s) = e^{-as}\,\mathcal{L}\{f\,(t)\}$

Or $\mathcal{L}^{-1}\{e^{-as}\,F(s)\} = H\,(t-a)\,f\,(t-a)$

Proof: By definition

$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_0^\infty e^{-st}\,H\,(t-a)\,f\,(t-a)\,dt$

$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_0^a e^{-st}\,H\,(t-a)f\,(t-a)\,dt + \int_a^\infty e^{-st}\,H\,(t-a)f\,(t-a)\,dt$

$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_a^\infty e^{-st}\,f\,(t-a)\,dt$

Introducing the new variable $\xi = t-a$, we obtain

$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = \int_0^\infty e^{-(\xi+a)s}\,f\,(\xi)\,d\xi = e^{-as}\int_0^\infty e^{-\xi s}\,f\,(\xi)\,d\xi$

$\mathcal{L}\{H\,(t-a)\,f\,(t-a)\} = e^{-as}\mathcal{L}\{\,f\,(t\,)\} = e^{-as}F(s)$

REMARK:

1st Shifting theorem enables us to calculate Laplace transform of the function of the form $e^{kt}\,f(t)$ where the 2nd Shifting theorem in similar way enables us to calculate inverse Laplace transform of the function of the form $e^{-as}F(s)$

Corollary:

Prove that $\mathcal{L}\{p(t)f(t)\} = P(-D)F(s)$ where $p(t)$ is a polynomial in 't'.

Solution:

Since $p(t) = a_0 + a_1t + a_2t^2 + \ldots\ldots\ldots\ldots.. + a_nt^n = \sum_{i=1}^n a_it^i$ Then

$\mathcal{L}\{p(t)f(t)\} = \mathcal{L}\left\{\sum_{i=1}^n a_it^i\,f(t)\right\} = \sum_{i=1}^n a_i\,\mathcal{L}\left\{t^if(t)\right\} = \sum_{i=1}^n a_i\,(-1)^i\,\frac{d^i}{ds^i}F(s)$

$\mathcal{L}\{p(t)f(t)\} = \sum_{i=1}^n a_i\,(-1)^iD^iF(s) = \sum_{i=1}^n a_i\,(-D)^iF(s) = P(-D)F(s)$

Theorem

Let $f\,(t)$ be a piecewise continuous function for $t \geq 0$ and of exponential order. If $f\,(t)$ is periodic with period T then show that

$$\mathcal{L}\{f\,(t\,)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f\,(t\,)\,dt$$

Proof: By definition, we have

$$\mathcal{L}\{f\,(t\,)\} = \int_0^\infty e^{-st}\, f\,(t\,)\,dt$$

$$\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt + \int_T^\infty e^{-st}\, f\,(t\,)\,dt$$

In the 2nd integral on the right put $t = u + T \Rightarrow dt = du$

$$\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt + \int_0^\infty e^{-s(u+T\,)}\, f\,(u+T\,)\,du$$

$$\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt + e^{-sT}\int_0^\infty e^{-su}\, f\,(u+T\,)\,du$$

Since given function is periodic with period T therefore $f\,(u+T\,) = f\,(u)$

$$\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt + e^{-sT}\int_0^\infty e^{-su}\, f\,(u\,)\,du$$

$$\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt + e^{-sT}\mathcal{L}\{f\,(u\,)\}$$

$$\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt + e^{-sT}\mathcal{L}\{f\,(t\,)\}$$

$$(1-e^{-sT})\mathcal{L}\{f\,(t\,)\} = \int_0^T e^{-st}\, f\,(t\,)\,dt$$

$\mathcal{L}\{f\,(t\,)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f\,(t\,)\,dt$ As required the result.

Theorem (Integration of Transforms)

If $\mathcal{L}\{f\,(t\,)\} = F(s)$ then $\mathcal{L}\left\{\frac{f\,(t\,)}{t}\right\} = \int_s^\infty F(s)\,ds$ **or** $\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f\,(t\,)}{t}$

Proof: By definition, we have

$$\mathcal{L}\{f\,(t\,)\} = F(s) = \int_0^\infty e^{-st}\, f\,(t)\,dt$$

$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st}\, f\,(t)\,dt\right] ds$ integrating.

$\int_s^\infty F(s)\,ds = \int_0^\infty f\,(t)\left[\int_s^\infty e^{-st}\,ds\right]dt$ changing the order of integration.

$$\int_s^\infty F(s)\,ds = \int_0^\infty f\,(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty \frac{f\,(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{f\,(t\,)}{t}\right\}$$

Hence $\mathcal{L}\left\{\frac{f\,(t\,)}{t}\right\} = \int_s^\infty F(s)\,ds$

or $\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f\,(t\,)}{t}$

Example (Differentiation and Integration of Transforms)

Find the inverse transform of $\ln\left(1 + \frac{\omega^2}{s^2}\right) = \ln \frac{s^2 + \omega^2}{s^2}$.

Solution. Denote the given transform by $F(s)$. Its derivative is

$$F'(s) = \frac{d}{ds}(\ln(s^2 + \omega^2) - \ln s^2) = \frac{2s}{s^2 + \omega^2} - \frac{2s}{s^2}.$$

Taking the inverse transform and using (1), we obtain

$$\mathscr{L}^{-}\{F'(s)\} = \mathscr{L}^{-1}\left\{\frac{2s}{s^2 + \omega^2} - \frac{2}{s}\right\} = 2\cos \omega t - 2 = -tf(t).$$

Hence the inverse $f(t)$ of $F(s)$ is $f(t) = 2(1 - \cos \omega t)/t$.

Alternatively, if we let

$$G(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}, \qquad \text{then} \qquad g(t) = \mathscr{L}^{-1}(G) - 2(\cos \omega t - 1).$$

From this and (6) we get, in agreement with the answer just obtained,

$$\mathscr{L}^{-1}\left\{\ln \frac{s^2 + \omega^2}{s^2}\right\} = \mathscr{L}^{-1}\left\{\int_s^\infty G(s)\, ds\right\} = -\frac{g(t)}{t} = \frac{2}{t}(1 - \cos \omega t),$$

the minus occurring since s is the lower limit of integration.

In a similar way we obtain formula

$$\mathscr{L}^{-1}\left\{\ln\left(1 - \frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1 - \cosh at).$$

Special Linear ODEs with Variable Coefficients

Let $\mathcal{L}(y) = Y$. Then $\mathcal{L}(y') = sY - y(0)$

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}.$$

Similarly, $\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$

$$\mathcal{L}(ty'') = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2\frac{dY}{ds} + y(0).$$

Example (Laguerre's Equation. Laguerre Polynomials)

Laguerre's ODE is

$$ty'' + (1 - t)y' + ny = 0.$$

We determine a solution of equation with n = 0,1,2,….

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0.$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken to be zero), and taking exponentials, we get

$$\frac{dY}{Y} = -\frac{n + 1 - s}{s - s^2}\,ds = \left(\frac{n}{s - 1} - \frac{n + 1}{s}\right)ds \qquad \text{and} \qquad Y = \frac{(s - 1)^n}{s^{n+1}}.$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and prove **Rodrigues's formula**

$$l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}),$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called Laguerre polynomials and are usually denoted by L_n.

Example

Solve the following

$$\mathcal{L}(l_n) = \frac{(s-1)^n}{s^{n+1}} = Y.$$

Solution

We know that

$$l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}),$$

Using first shifting theorem

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}},$$

$$\mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n!s^n}{(s+1)^{n+1}}$$

because the derivatives up to the order $n-1$ are zero at 0. Now make another shift and divide by $n!$ to get

$$\mathcal{L}(l_n) = \frac{(s-1)^n}{s^{n+1}} = Y.$$

Example:

Find the general solution of the differential equation evaluate

$y''(t) + k^2y(t) = f(t)$

Solution

Given that $\quad y''(t) + k^2y(t) = f(t)$

$\Rightarrow \mathcal{L}\{y''(t)\} + k^2\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) + k^2Y(s) = F(s)$

$\Rightarrow s^2Y(s) + k^2Y(s) = F(s) + sy(0) + y'(0)$

$\Rightarrow Y(s) = \frac{c_1+c_2s+F(s)}{s^2+k^2}$ where we use $y'(0) = c_1, y(0) = c_2$

Now $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{c_1}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{c_2s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{c_1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{c_1}{k}Sinkt + c_2Coskt + \frac{1}{k}Sinkt * f(t)$

$\Rightarrow y(t) = \frac{c_1}{k}Sinkt + c_2Coskt + \frac{1}{k}\int_0^t e^{-st}\, Sink(t-\xi)f(\xi)\, d\xi$

Example:

Solve the IVP $y''(t) + ty'(t) - y(t) = 0$ with $y(0) = 0, y'(0) = 1$

Solution

Given that $y''(t) + ty'(t) - y(t) = 0$

$\Rightarrow \mathcal{L}\{y''(t)\} + \mathcal{L}\{ty'(t)\} - \mathcal{L}\{y(t)\} = 0$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) + \left(-\frac{d}{ds}\right)\mathcal{L}\{y'(t)\} - Y(s) = 0$

$\Rightarrow s^2Y(s) - 1 - \left(\frac{d}{ds}\right)\{sY(s) - y(0)\} - Y(s) = 0$

where we use $y(0) = 0, y'(0) = 1$

$\Rightarrow s^2Y(s) - 1 - sY'(s) - Y(s) - Y(s) = 0$ where we use $y(0) = 0, y'(0) = 1$

$\Rightarrow Y'(s) + \frac{2-s^2}{s}Y(s) = -\frac{1}{s}$ this will have in $I.F = s^2e^{-\frac{s^2}{2}}$

Thus $\Rightarrow Y(s) = \frac{1}{s^2} + ce^{\frac{s^2}{2}} \Rightarrow Y(s) = \frac{1}{s^2}$ when $s \to \infty \; then \; c = 0$

Now $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Rightarrow y(t) = t$

Example:

Slove the IVP $u'' - au = f(t)$ with $u(0) = u_0, u'(0) = u_1$

Solution: Given that $u'' - au = f(t)$

$\Rightarrow \mathcal{L}\{u''\} - a\mathcal{L}\{u\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2U(s) - su(0) - u'(0) - aU(s) = F(s)$

$\Rightarrow s^2U(s) - su_0 - u_1 - aU(s) = F(s)$ where we use $u(0) = u_0, u'(0) = u_1$

$\Rightarrow (s^2 - a)U(s) = F(s) + su_0 + u_1$

$\Rightarrow U(s) = \frac{F(s)}{s^2-a} + u_0.\frac{s}{s^2-a} + u_1.\frac{1}{s^2-a}$

Now $\Rightarrow \mathcal{L}^{-1}\{U(s)\} = u(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2-a}\right\} + u_0\mathcal{L}^{-1}\left\{\frac{s}{s^2-a}\right\} + u_1\mathcal{L}^{-1}\left\{\frac{1}{s^2-a}\right\}$

$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = u(t) = \frac{1}{\sqrt{a}}\mathcal{L}^{-1}\left\{F(s).\frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right\} + u_0\mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{a})^2}\right\} + \frac{u_1}{\sqrt{a}}\mathcal{L}^{-1}\left\{\frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right\}$

$\Rightarrow u(t) = \frac{1}{\sqrt{a}}\mathcal{L}^{-1}\{f(t) * Sinh\sqrt{a}t\} + u_0Cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}}Sinh\sqrt{a}t$

$\Rightarrow u(t) = \frac{1}{\sqrt{a}}\int_0^t e^{-st}\, Sinh\sqrt{a}(t - \xi)f(\xi)\, d\xi + u_0Cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}}Sinh\sqrt{a}t$

Example:

Use Laplace Transformation method to solve BVP

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}; 0<x<a; \quad 0\le t<\infty$

$u\,(0,t)=1,\ u\,(1,t)=1 \ \ ; t>0\ , u\,(x,0)=1+Sin\pi x$

Solution:

Given $\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\Rightarrow \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\}=\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\}\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s)=s\,U(x,s)-u(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s)=s\,U(x,s)-(1+Sin\pi x)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s)-s\,U(x,s)=-1-Sin\pi x$ …………(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s)=U_c(x,s)+U_p(x,s)$ …………(ii)

For Chractristic (auxiliary) solution

$(i)\Rightarrow (D^2-s)U(x,s)=-1-Sin\pi x\Rightarrow D^2-s=0\Rightarrow D=\pm\sqrt{s}$

Then $U_c(x,s)=c_1e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}$

For Particular solution

Consider $U_p(x,s)=\frac{-1-Sin\pi x}{D^2-s}=\frac{-e^{0x}}{D^2-s}-img\frac{e^{i\pi x}}{D^2-s}=\frac{-1}{0^2-s}-\frac{Sin\pi x}{(i\pi)^2-s}=\frac{1}{s}-\frac{Sin\pi x}{-\pi^2-s}$

Then $U_p(x,s)=\frac{1}{s}+\frac{Sin\pi x}{\pi^2+s}$

$(ii)\Rightarrow U(x,s)=U_c(x,s)+U_p(x,s)=c_1e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}+\frac{1}{s}+\frac{Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s)=c_1e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}+\frac{1}{s}+\frac{Sin\pi x}{\pi^2+s}$ …………(iii)

Now using BC's

$u\,(0,t)=1\Rightarrow \mathcal{L}\{u\,(0,t)\}=\mathcal{L}\{1=t^0\}\Rightarrow U(0,s)=\frac{1}{s}$

$u\,(1,t)=1\Rightarrow \mathcal{L}\{u\,(1,t)\}=\mathcal{L}\{1=t^0\}\Rightarrow U(1,s)=\frac{1}{s}$

$(iii)\Rightarrow U(0,s)=\frac{1}{s}=c_1e^0+c_2e^0+\frac{1}{s}+\frac{Sin(0)}{\pi^2+s}\Rightarrow c_1+c_2+\frac{1}{s}=\frac{1}{s}\Rightarrow c_1=-c_2$

$(iii)\Rightarrow U(1,s)=\frac{1}{s}=c_1e^{\sqrt{s}(1)}+c_2e^{-\sqrt{s}(1)}+\frac{1}{s}+\frac{Sin\pi}{\pi^2+s}$

$\Rightarrow c_1e^{\sqrt{s}}+c_2e^{-\sqrt{s}}+\frac{1}{s}-\frac{1}{s}=0$

$\Rightarrow c_1e^{\sqrt{s}}+c_2e^{-\sqrt{s}}=0\Rightarrow -c_2e^{\sqrt{s}}+c_2e^{-\sqrt{s}}=0$ $\quad\therefore c_1=-c_2$

$\Rightarrow c_2\left[e^{-\sqrt{s}}-e^{\sqrt{s}}\right]=0\Rightarrow c_2=0\ ,\ \left[e^{-\sqrt{s}}-e^{\sqrt{s}}\right]\neq 0$

$\Rightarrow c_2=0\Rightarrow c_1=0$ $\quad\therefore c_1=-c_2$

$(iii)\Rightarrow U(x,s)=\frac{1}{s}+\frac{Sin\pi x}{\pi^2+s}$ $\quad\therefore c_1=c_2=0$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{Sin\pi x}{\pi^2+s}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+Sin\pi x\mathcal{L}^{-1}\left\{\frac{1}{s-(-\pi^2)}\right\}$

$\Rightarrow u\,(x,t)=1+Sin\pi xe^{-\pi^2t}$ required solution.

Example:

Use Laplace Transformation method to solve BVP

$u_{tt}\ (x,t) = \alpha^2 u_{xx}\ (x,t); t > 0, x > 0$

$u\ (x,0) = u_t\ (x,0) = 0,\ u\ (0,t) = \ f(\ t)\ , \lim_{x\to\infty} u\ (x,t) = 0$

Solution:

Given $u_{tt}\ (x,t) = \alpha^2 u_{xx}\ (x,t) \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\}$

$\Rightarrow s^2\ U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow s^2\ U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) \Rightarrow s^2\ U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - \frac{s^2}{\alpha^2}\ U(x,s) = 0$

This is Homogeneous DE of 2nd order therefore

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm \frac{s}{a}$

Then $U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$ ………..(i)

Now using BC's

$u\ (0,t) = f(\ t) \Rightarrow \mathcal{L}\{u\ (0,t)\} = \mathcal{L}\{f(\ t)\} \Rightarrow U(0,s) = F(\ s)$

$\lim_{x\to\infty} u\ (x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u\ (x,t)\} = 0 \Rightarrow \lim_{x\to\infty} U\ (x,s) = 0$

$(i) \Rightarrow U(0,s) = F(\ s) = c_1 e^{\frac{s}{a}(0)} + c_2 e^{-\frac{s}{a}(0)} \Rightarrow c_1 + c_2 = F(\ s)$

$(i) \Rightarrow \lim_{x\to\infty} U\ (x,s) = 0 = \lim_{x\to\infty} \left[c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}\right] = c_1 e^{\infty} + c_2 e^{-\infty}$

$\Rightarrow c_1 = 0 \quad then \quad c_2 = F(\ s) \qquad \therefore c_1 + c_2 = F(\ s)$

Thus $(i) \Rightarrow U(x,s) = F(\ s) e^{-\frac{s}{a}x}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{F(\ s) e^{-\frac{s}{a}x}\right\}$

$\Rightarrow u\ (x,t) = H\left(\ t - \frac{x}{a}\right) f\left(\ t - \frac{x}{a}\right)$

where $H\left(\ t - \frac{x}{a}\right) f\left(\ t - \frac{x}{a}\right) = \begin{cases} 0 & t < \frac{x}{a} \\ f(\ t) & t \geq \frac{x}{a} \end{cases}$

Example:

Use Laplace Transformation method to solve BVP

$u_{tt}(x,t) = \alpha^2 u_{xx}\ (x,t) - g$

$u\ (x,0) = u_t\ (x,0) = 0,\ u\ (0,t) = 0\ , \lim_{x\to\infty} u_x\ (x,t) = 0$

Solution: Given $u_{tt}\ (x,t) = \alpha^2 u_{xx}\ (x,t) - g \Rightarrow \mathcal{L}\{u_{tt}\} = \alpha^2 \mathcal{L}\{u_{xx}\} - g\mathcal{L}\{1\}$

$\Rightarrow s^2\ U(x,s) - su(x,0) - u_t(x,0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow s^2\ U(x,s) - (0) - (0) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow s^2\ U(x,s) = \alpha^2 \frac{\partial^2}{\partial x^2} U(x,s) - \frac{g}{s}$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - \frac{s^2}{\alpha^2}\ U(x,s) = \frac{g}{\alpha^2 s}$ …………(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ …………(ii)

For Chractristic (auxiliary) solution

$\Rightarrow \left(D^2 - \frac{s^2}{\alpha^2}\right) U(x,s) = 0 \Rightarrow D^2 - \frac{s^2}{\alpha^2} = 0 \Rightarrow D = \pm\frac{s}{a}$

Then $U_c(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$

For Particular solution

Consider $U_p(x,s) = \frac{\frac{g}{\alpha^2 s}}{D^2-\frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}e^{0x}}{0^2-\frac{s^2}{\alpha^2}} = \frac{\frac{g}{\alpha^2 s}}{-\frac{s^2}{\alpha^2}} = -\frac{g}{s^3}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} - \frac{g}{s^3}$ …………(iii)

Now using BC's

$u\ (0,t) = 0 \Rightarrow \mathcal{L}\{u\ (0,t)\} = 0 \Rightarrow U(0,s) = 0$

$\lim_{x\to\infty} u_x\ (x,t) = 0 \Rightarrow \mathcal{L}\{\lim_{x\to\infty} u_x\ (x,t)\} = 0 \Rightarrow \lim_{x\to\infty}\frac{\partial}{\partial x}U\ (x,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1 e^0 + c_2 e^{-0} - \frac{g}{s^3} \Rightarrow c_1 + c_2 = \frac{g}{s^3}$

$(iii) \Rightarrow \lim_{x\to\infty}\frac{\partial}{\partial x}U\ (x,s) = 0 = \lim_{x\to\infty}\left[c_1\frac{s}{a}e^{\frac{s}{a}x} - \frac{s}{a}c_2 e^{-\frac{s}{a}x}\right] = c_1\frac{s}{a}e^{\infty} + c_2\frac{s}{a}e^{-\infty}$

$\Rightarrow c_1\frac{s}{a}e^{\infty} = 0 \Rightarrow c_1 = 0\ \ since\ \frac{s}{a}e^{\infty} \neq 0,\ \ then\ \ \ c_2 = \frac{g}{s^3}\qquad \therefore c_1 + c_2 = \frac{g}{s^3}$

Thus $(iii) \Rightarrow U(x,s) = \frac{g}{s^3}e^{-\frac{s}{a}x} - \frac{g}{s^3}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \frac{g}{2!}\mathcal{L}^{-1}\left\{e^{-\frac{x}{a}s}.\frac{2!}{s^{2+1}}\right\} - \frac{g}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$\Rightarrow u\ (x,t) = \frac{g}{2}H\left(\ t - \frac{x}{a}\right)\left(\ t - \frac{x}{a}\right)^2 - \frac{g}{2}(t^2)$

$\Rightarrow u\ (x,t) = \frac{g}{2}\left[H\left(\ t - \frac{x}{a}\right)\left(\ t - \frac{x}{a}\right)^2 - (t^2)\right]$

where $H\left(\ t - \frac{x}{a}\right)\left(\ t - \frac{x}{a}\right)^2 = \begin{cases} 0 & \text{t} < \frac{x}{a} \\ t^2 & \text{t} \geq \frac{x}{a} \end{cases}$

Example:

Use Laplace Transformation method to solve BVP

$u_{xx}\ (x,t) = u_{tt}\ (x,t); t > 0;\ 0 < x < 1$

$u\ (0,t) = 0 = u\ (1,t),\ u\ (x,0) = Sin\pi x\ \ ,u_t\ (x,0) = -Sin\pi x$

Solution:

Given $u_{xx}\ (x,t) = u_{tt}\ (x,t) \Rightarrow \mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s^2\ U(x,s) - su(x,0) - u_t(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) = s^2\ U(x,s) - sSin\pi x + Sin\pi x$

$\Rightarrow \frac{\partial^2}{\partial x^2}U(x,s) - s^2\ U(x,s) = -sSin\pi x + Sin\pi x$ ....(i)

Which is non – homogeneous 2nd order DE with solution

$U(x,s) = U_c(x,s) + U_p(x,s)$ ....(ii)

For Chractristic (auxiliary) solution

$\Rightarrow (D^2 - s^2)U(x,s) = 0 \Rightarrow D^2 - s^2 = 0 \Rightarrow D = \pm s$

Then $U_c(x,s) = c_1e^{sx} + c_2e^{-sx}$

For Particular solution

Consider

$U_p(x,s) = \frac{(1-s)Sin\pi x}{D^2-s^2} = (1-s)img\frac{e^{i\pi x}}{D^2-s^2} = (1-s)\frac{Sin\pi x}{(i\pi)^2-s} = (1-s)\frac{Sin\pi x}{-\pi^2-s}$

$U_p(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$(ii) \Rightarrow U(x,s) = U_c(x,s) + U_p(x,s) = c_1e^{sx} + c_2e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow U(x,s) = c_1e^{sx} + c_2e^{-sx} + \frac{(s-1)Sin\pi x}{\pi^2+s}$ ....(iii)

Now using BC's $u\ (0,t) = 0 \Rightarrow \mathcal{L}\{u\ (0,t)\} = 0 \Rightarrow U(0,s) = 0$

$u\ (1,t) = 0 \Rightarrow \mathcal{L}\{u\ (1,t)\} = 0 \Rightarrow U(1,s) = 0$

$(iii) \Rightarrow U(0,s) = 0 = c_1e^0 + c_2e^{-0} + \frac{(s-1)Sin\pi(0)}{\pi^2+s} \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$(iii) \Rightarrow U(1,s) = 0 = c_1e^s + c_2e^{-s} + \frac{(s-1)Sin\pi}{\pi^2+s}$

$\Rightarrow c_1e^s + c_2e^{-s} = 0 \Rightarrow c_1e^s - c_1e^{-s} = 0$

$\Rightarrow c_1(e^s - e^{-s}) = 0 \Rightarrow c_1 = 0\ \ as\ (e^s - e^{-s}) \neq 0 \Rightarrow c_2 = 0$

Thus $(iii) \Rightarrow U(x,s) = \frac{(s-1)Sin\pi x}{\pi^2+s}$

$\Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = Sin\pi x\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} - \frac{Sin\pi x}{\pi}\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$\Rightarrow u\ (x,t) = Sin\pi xCos\pi t - \frac{Sin\pi x}{\pi}Sin\pi t = Sin\pi x\left[Cos\pi t - \frac{Sin\pi x}{\pi}\right]$

The Kronecker Delta Function:

It is denoted by δ_{ij} and can be defined as follows;

$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

Dirac Delta Function

The Dirac delta function, δ(x), is a mathematical function that represents a point source or impulse. It's defined as:

$$\delta(t-a) = \begin{cases} \infty & if\ t = a \\ 0 & if\ t \neq a \end{cases}$$

The Dirac delta function is often used in physics, engineering, and signal processing to model instantaneous events or point sources. It's a generalized function, also known as a distribution, rather than a traditional function. It is named after Paul Dirac, a British physicist who introduced it in the 1920s.

Properties:

i. $\int_{-\infty}^{\infty} \delta(t)dt = 1$

ii. $\int_{0}^{\infty} \delta(t-a)dt = 1$

iii. For any continuous function $f(t)$; $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$

iv. $\delta(t) = \delta(-t)$

v. $\delta(at) = \frac{1}{a}\delta(t) \quad ; a > 0$

vi. SHIFTING PROPERTY: For any continuous function $f(t)$;

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

vii. If $\delta(t)$ is continuous differentiable. Then $\int_{-\infty}^{\infty} f(t)\delta'(t)dt = -f'(0)$

Remark:

i. Dirac delta function can be regarded as the generalization of Kronecker delta function. It strictly speaking a “generalized function” or “distribution function” or “ a unit impulse function”

ii. In kronecker delta function δ_{ij} the indecis i,j, are integral variables, whereas in passing to direc delta function they become real continuous variables.

1st SHIFTING PROPERTY OF DIRAC DELTA FUNCTION:

For any continuous function $f(x)$; $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

Where $f(x)$ is analytic (regualar or continuous function) at x = 0

Proof: Since $\delta(x)$ has singularity at x = 0, the limits $-\infty$ and ∞ of the integration may be changed to (or replace by) $0-\epsilon$ and $0+\epsilon$ where ϵ is a small positive number.

Since $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} f(x)\delta(x)dx$

Moreover, since $f(x)$ is continuous at x = 0. We obtain in $\lim\limits_{\epsilon\to 0}$ follow;

$f(0-\epsilon) = f(0+\epsilon) = f(0)$

Therefore $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)\lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} f(x)\delta(x)dx$

since $\delta(x)$ has singularity at x = 0. Therefore

$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0).1 = f(0)$

2nd SHIFTING PROPERTY OF DIRAC DELTA FUNCTION:

For any continuous function $f(x)$; $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

Where $f(x)$ is analytic (regualar or continuous function) at x = a

Proof: Consider $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx$

Set $x-a=t$ and write $f(t+a) = g(t) \Rightarrow f(a) = g(0)$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(t+a)\delta(t)dt = \int_{-\infty}^{\infty} g(t)\delta(t)dt$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = g(0)$ by 1st shifting property

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$ by hypothesis

Question: Show that $\delta(-x) = \delta(x)$

Let $\int_{-\infty}^{\infty}\delta(x)dx = 1$ …………(i)

$\int_{\infty}^{-\infty}\delta(-x)(-dx) = 1$ replacing x with $-x$

$\int_{-\infty}^{\infty}\delta(-x)dx = 1$ …………(ii) replacing x with $-x$

From (i) and (ii) $\delta(-x) = \delta(x)$

Question: Show that $\delta'(x) = -\delta'(-x)$

Since we know that $\delta(-x) = \delta(x)$

$\frac{\partial}{\partial(-x)}[\delta(-x)] = \frac{\partial}{\partial x}[\delta(x)]$

$\delta'(-x) = -\delta'(x)$

Hence $\delta'(x) = -\delta'(-x)$

Impulse

In mechanics, the integral of a force acting over a time interval $a \leq t \leq a + k$ is called the impulse of the force.

In the context of the Dirac delta function, an impulse can be represented mathematically as:

$f(t) = \delta(t - t_0)$

This represents an instantaneous impulse at time t_0, where $\delta(t - t_0)$ is the Dirac delta function.

The Dirac delta function is often used to represent impulses in signal processing, control systems, and physics.

Question

Find Laplace transformation of direct delta function.

Solution

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

To obtain the Laplace transform of $\delta(t - a)$, we write

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\,\frac{1-e^{-ks}}{ks}.$$

We now take the limit as $k \to 0$. By l'Hôpital's rule the quotient on the right has the limit 1 (differentiate the numerator and the denominator separately with respect to k, obtaining se^{-ks} and s, respectively, and use $se^{-ks}/s \to 1$ as $k \to 0$). Hence the right side has the limit e^{-as}. This suggests defining the transform of $\delta(t - a)$ by this limit, that is,

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

Example (Mass–Spring System Under a Square Wave)
Determine the response of the damped mass–spring system under a square wave, modeled by

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution

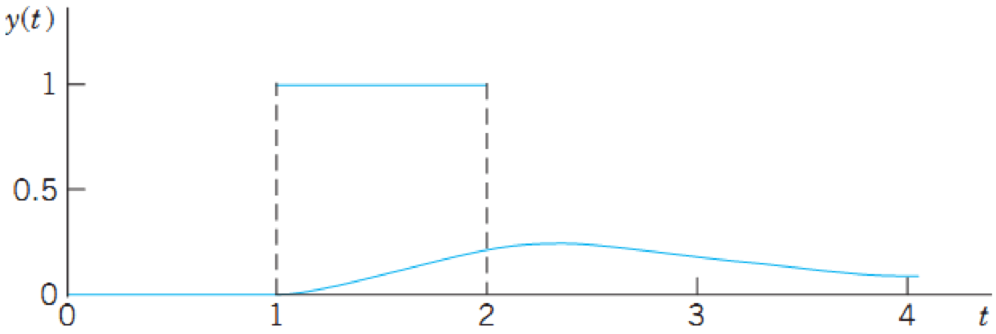
$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2),$$

$$s^2Y + 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}). \qquad \text{Solution} \qquad Y(s) = \frac{1}{s(s^2+3s+2)}(e^{-s} - e^{-2s}).$$

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}.$$

$$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

$$\begin{aligned} y &= \mathcal{L}^{-1}(F(s)e^{-s} - F(s)e^{-2s}) \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \\ &= \begin{cases} 0 & (0 < t < 1) \\ \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{-(t-2)} + \tfrac{1}{2}e^{-2(t-1)} - \tfrac{1}{2}e^{-2(t-2)} & (t > 2). \end{cases} \end{aligned}$$



Example (Hammerblow Response of a Mass–Spring System)
Find the response of the system in with the square wave replaced by a unit impulse at time t = 1.

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution

We now have the ODE and the subsidiary equation

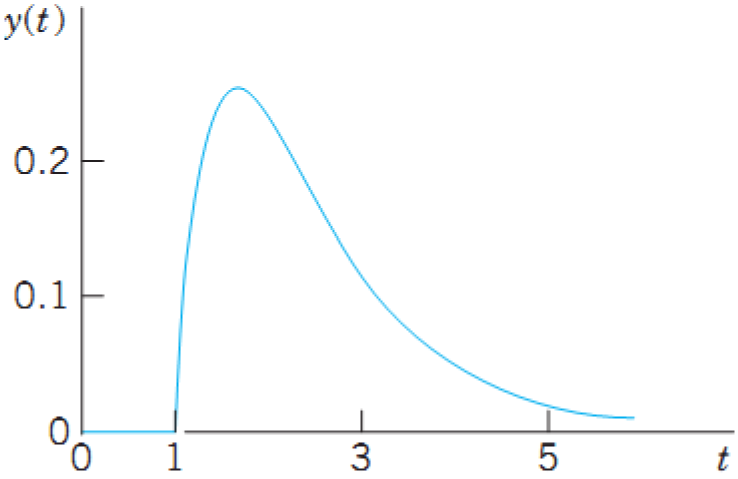
$$y'' + 3y' + 2y = \delta(t-1), \qquad \text{and} \qquad (s^2 + 3s + 2)Y = e^{-s}.$$

Solving algebraically gives

$$Y(s) = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}.$$

the inverse is

$$y(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } \quad t > 1. \end{cases}$$



Example (Four-Terminal RLC-Network)

Find the output voltage response in Fig. 135 if $R = 20\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, the input is $\delta(t)$ (a unit impulse at time $t = 0$), and current and charge are zero at time $t = 0$.

Solution. To understand what is going on, note that the network is an *RLC*-circuit to which two wires at *A* and *B* are attached for recording the voltage $v(t)$ on the capacitor. Recalling from Sec. 2.9 that current $i(t)$ and charge $q(t)$ are related by $i = q' = dq/dt$, we obtain the model

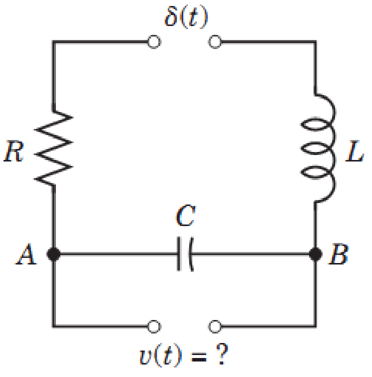
$$Li' + Ri + \frac{q}{C} = Lq'' + Rq' + \frac{q}{C} = q'' + 20q' + 10{,}000q = \delta(t).$$

From (1) and (2) in Sec. 6.2 and (5) in this section we obtain the subsidiary equation for $Q(s) = \mathcal{L}(q)$

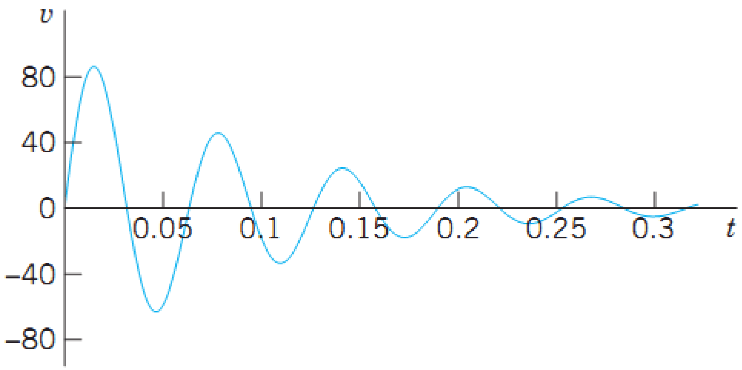
$$(s^2 + 20s + 10{,}000)Q = 1. \qquad \text{Solution} \qquad Q = \frac{1}{(s+10)^2 + 9900}.$$

By the first shifting theorem in Sec. 6.1 we obtain from Q damped oscillations for q and v; rounding $9900 \approx 99.50^2$, we get (Fig. 135)

$$q = \mathcal{L}^{-1}(Q) = \frac{1}{99.50} e^{-10t} \sin 99.50t \qquad \text{and} \qquad v = \frac{q}{C} = 100.5e^{-10t} \sin 99.50t.$$



Network



Voltage on the capacitor

Example (Unrepeated Complex Factors. Damped Forced Vibrations)
Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval

$$y'' + 2y' + 2y = r(t), \quad r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1, \quad y'(0) = -5.$$

Solution

$$y'' + 2y' + 2y = r(t), \quad r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1, \quad y'(0) = -5.$$

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\,\frac{2}{s^2 + 4}(1 - e^{-\pi s}).$$

We collect the Y-terms, $(s^2 + 2s + 2)Y$, take $-s + 5 - 2 = -s + 3$ to the right, and solve,

$$Y = \frac{20}{(s^2 + 4)(s^2 + 2s + 2)} - \frac{20e^{-\pi s}}{(s^2 + 4)(s^2 + 2s + 2)} + \frac{s - 3}{s^2 + 2s + 2}. \tag{6}$$

For the last fraction we get from Table 6.1 and the first shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{s + 1 - 4}{(s + 1)^2 + 1}\right\} = e^{-t}(\cos t - 4 \sin t). \tag{7}$$

In the first fraction in (6) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2 + 4)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 4} + \frac{Ms + N}{s^2 + 2s + 2}.$$

Multiplication by the common denominator gives

$$20 = (As + B)(s^2 + 2s + 2) + (Ms + N)(s^2 + 4).$$

We determine A, B, M, N. Equating the coefficients of each power of s on both sides gives the four equations

$$\text{(a)} \quad [s^3]: \quad 0 = A + M \qquad\qquad \text{(b)} \quad [s^2]: \quad 0 = 2A + B + N$$

$$\text{(c)} \quad [s]: \quad 0 = 2A + 2B + 4M \qquad\qquad \text{(d)} \quad [s^0]: \quad 20 = 2B + 4N.$$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (6) has the representation

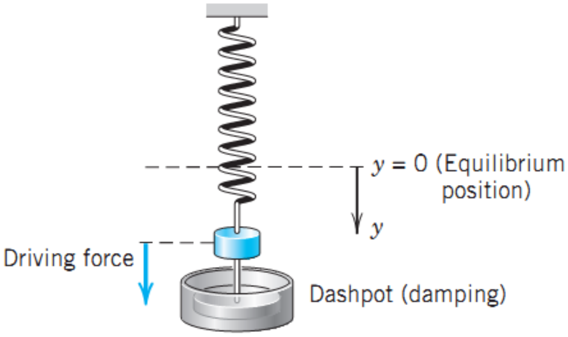
$$\frac{-2s - 2}{s^2 + 4} + \frac{2(s + 1) + 6 - 2}{(s + 1)^2 + 1}. \quad \text{Inverse transform:} \quad -2 \cos 2t - \sin 2t + e^{-t}(2 \cos t + 4 \sin t).$$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi.$$

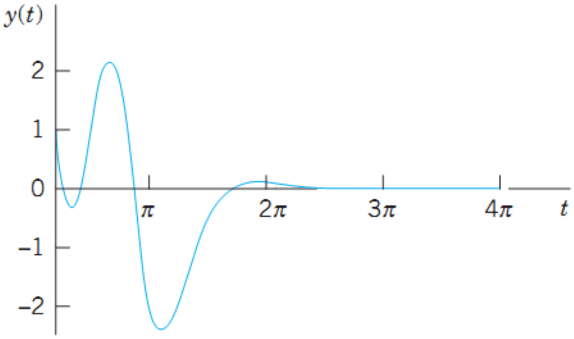
$$+2\cos(2t - 2\pi) + \sin(2t - 2\pi) - e^{-(t-\pi)}[2\cos(t - \pi) + 4\sin(t - \pi)]$$

$$= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t).$$

$$y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi.$$

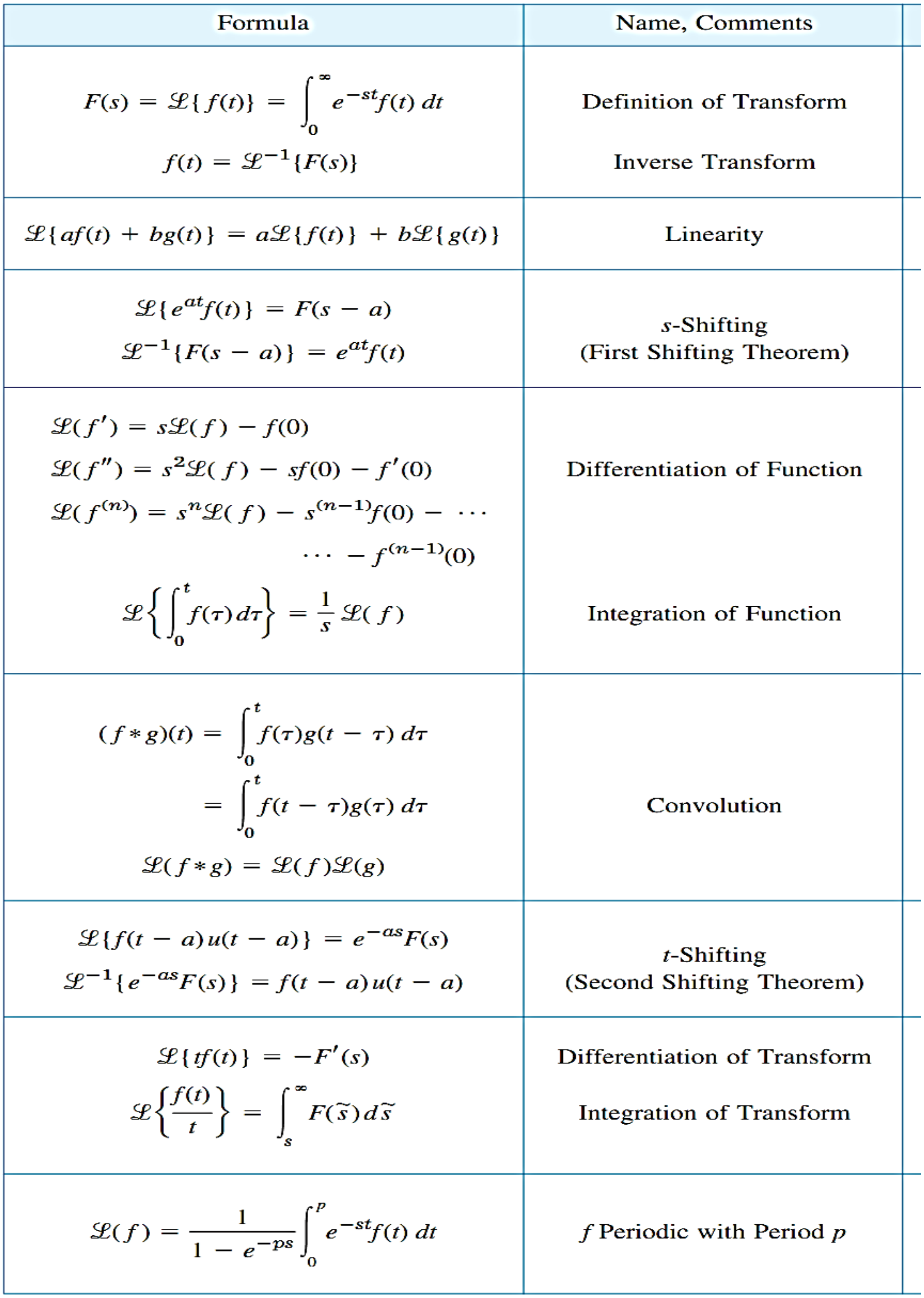


Mechanical system



Output (solution)

Laplace Transform: General Formulas

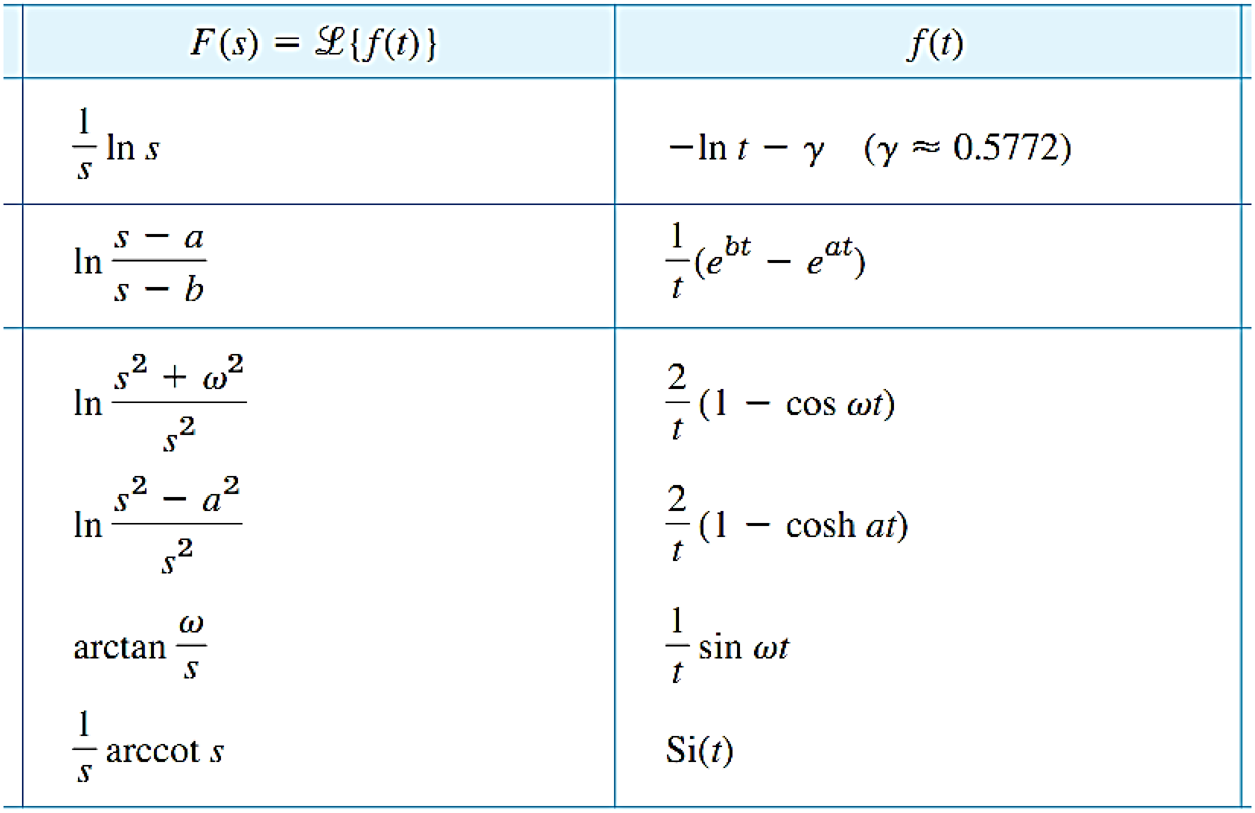


Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	*s*-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ $= \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	*t*-Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$	Differentiation of Transform
$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Integration of Transform
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	*f* Periodic with Period *p*

Table of Laplace Transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
1	$1/s$	1
2	$1/s^2$	t
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
7	$\dfrac{1}{s-a}$	e^{at}
8	$\dfrac{1}{(s-a)^2}$	te^{at}
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!}t^{n-1}e^{at}$
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}t^{k-1}e^{at}$
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(e^{at} - e^{bt})$
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(ae^{at} - be^{bt})$
13	$\dfrac{1}{s^2 + \omega^2}$	$\dfrac{1}{\omega}\sin \omega t$
14	$\dfrac{s}{s^2 + \omega^2}$	$\cos \omega t$
15	$\dfrac{1}{s^2 - a^2}$	$\dfrac{1}{a}\sinh at$
16	$\dfrac{s}{s^2 - a^2}$	$\cosh at$
17	$\dfrac{1}{(s-a)^2 + \omega^2}$	$\dfrac{1}{\omega}e^{at}\sinh \omega t$
18	$\dfrac{s-a}{(s-a)^2 + \omega^2}$	$e^{at}\cos \omega t$
19	$\dfrac{1}{s(s^2 + \omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$
20	$\dfrac{1}{s^2(s^2 + \omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$

$F(s) = \mathscr{L}\{f(t)\}$	$f(t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega} \sin \omega t$
$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$
$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$
$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$
$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$
$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$
$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$
$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$
$\dfrac{1}{\sqrt{s+a}\,\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2} t\right)$
$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1+2at)$
$\dfrac{1}{(s^2-a^2)^k} \quad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$
$\dfrac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$
$\dfrac{1}{\sqrt{s}} e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$
$\dfrac{1}{s^{3/2}} e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$
$e^{-k\sqrt{s}} \quad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$



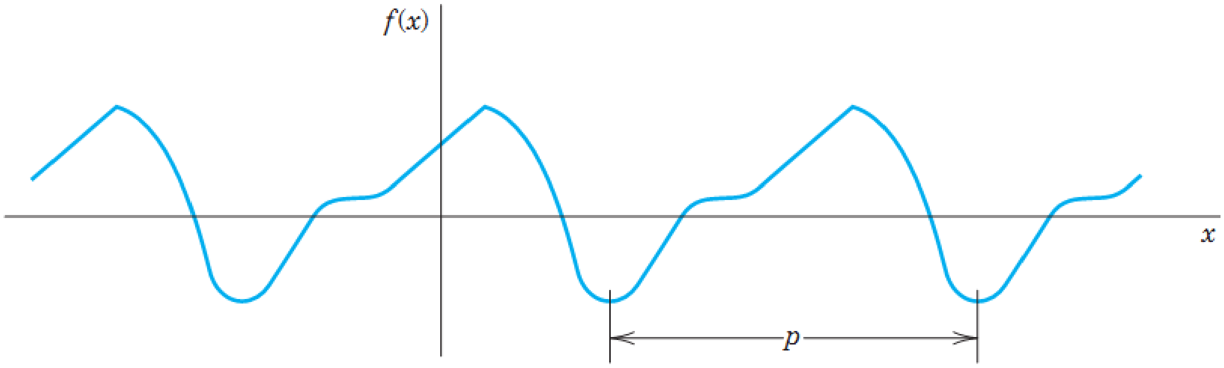
$F(s) = \mathscr{L}\{f(t)\}$	$f(t)$
$\frac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$
$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$
$\ln \frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$
$\ln \frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$
$\arctan \frac{\omega}{s}$	$\frac{1}{t}\sin \omega t$
$\frac{1}{s}\operatorname{arccot} s$	$\mathrm{Si}(t)$

FOURIER TRANSFORM

Periodic Function

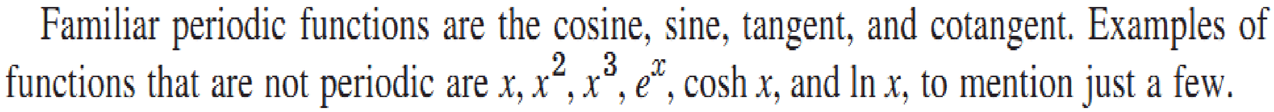
A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of $f(x)$, such that $f(x+p)=f(x)\ ;\ \forall x$

The graph of a periodic function has the characteristic that it can be obtained by periodic repetition of its graph in any interval of length p.



Example

The function $f(x)=tanx$ is a periodic function that is not defined for all real x but undefined for some points (more precisely, countably many points), that is $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\dots$



Familiar periodic functions are the cosine, sine, tangent, and cotangent. Examples of functions that are not periodic are x, x^2, x^3, e^x, cosh x, and ln x, to mention just a few.

Fundamental Period

The smallest positive period is often called the fundamental period.

Remember

If $f(x)$ has period p, it also has the period 2p because

$$f(x+2p)=f\big((x+p)+p\big)=f(x+p)=f(x)\ ;\ \forall x$$

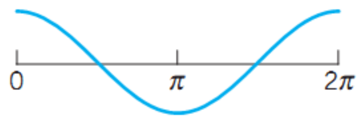
$$f(x+np)=f(x)\ ;\ \forall x$$

Furthermore if $f(x)$ and $g(x)$ have period p, then $af(x)+bg(x)$ with any constants a and b also has the period p.

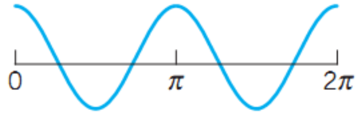
Functions $f(x)$ of period 2π

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots$$

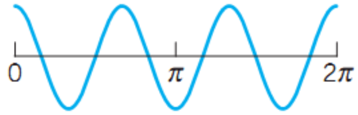
All these functions have the period 2π. They form the so-called **trigonometric system**.



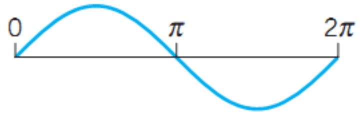
cos x



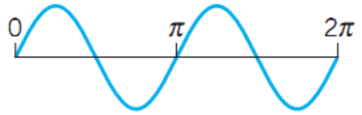
cos $2x$



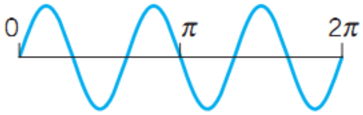
cos $3x$



sin x



sin $2x$



sin $3x$

Trigonometric Series

The series to be obtained will be a **trigonometric series**, that is, a series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series. We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

Fourier Series

A series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where Fourier coefficients of $f(x)$, given by the Euler formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, \cdots.$$

Fourier Series

A trigonometric series with any piecewise continuous periodic function $f\ (x)$ of period 2π and of the form $f\ (x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \text{kx} \ + b_k \sin \text{kx})$
is called the Fourier Series of a real valued function $f(x)$ where the symbol $\sim$ indicates an association of a_0, a_k, and b_k to f in some unique manner.
Where
$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx$
And are called Fourier Coefficiets.
We may also write $f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \text{kx} \ + b_k \sin \text{kx})$

Complex Form Of Fourier Series

Fourier Series expansion for in complex form is given as follows
$f\ (x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$; $-\pi < x < \pi$ Where
$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$
OR $f\ (x) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi x}{l}}$ Where $c_k = \frac{1}{2l}\int_{-l}^{l} f(y)e^{-i\frac{\pi y}{l}}dy$
Example (just read) :Find the Fourier series expansion for the function f(x) = x+ x^2, $-\pi < x < \pi$
Solultion: Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\pi^2}{3}$
$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{4}{k^2}Cosk\pi = \frac{4}{k^2}(-1)^k$; $k = 1,2,3, \dots \dots$
$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = -\frac{2}{k}Cosk\pi = -\frac{2}{k}(-1)^k$; $k = 1,2,3, \dots \dots$
Therefore, the Fourier series expansion for f is
$f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \text{kx} \ + b_k \sin \text{kx})$
$f\ (x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(\frac{4}{k^2}(-1)^k \cos \text{kx} \ - \frac{2}{k}(-1)^k \sin \text{kx})$
$f\ (x) = \frac{\pi^2}{3} - \ 4\cos x + 2sinx \ + \ cos\ 2x \ - \ sin\ 2x - \dots .$
Example (just read): Find the Fourier series expansion for the function
$f\ (x) = \begin{cases} -\pi & ; -\pi \ < \ x \ < \ 0 \\ x & ; 0 \ < \ x \ < \ \pi \end{cases}$
Solultion: Here
$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] = -\frac{\pi}{2}$
$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Coskxdx + \int_{0}^{\pi} f(x)Coskxdx\right]$
$a_k = \frac{1}{k^2\pi}(Cosk\pi - 1) = \frac{1}{k^2\pi}[(-1)^k \ - 1]$; $k = 1,2,3, \dots \dots$
$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)Sinkxdx + \int_{0}^{\pi} f(x)Sinkxdx\right]$

$b_k = \frac{1}{k}(1 - 2Cosk\pi) = \frac{1}{k}[1 - 2(-1)^k]\,;\ \ k = 1,2,3, \dots \dots$

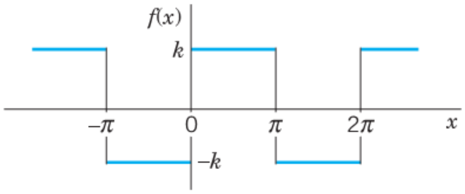
Therefore, the Fourier series expansion for f is

$f\ (x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \mathrm{kx}\ + b_k \sin \mathrm{kx})$

$f\ (x) = -\frac{\pi}{4}\ + \sum_{k=1}^{\infty}\left[\frac{1}{k^2\pi}[(-1)^k\ - 1] \cos \mathrm{kx} + \frac{1}{k}[1 - 2(-1)^k] \sin \mathrm{kx})\right]$

Example (Periodic Rectangular Wave)

Find the Fourier coefficients of the periodic function in Figure.



The formula is

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ \ \ k & \text{if} \quad \ \ 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

Solution

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\Bigg|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\Bigg|_{0}^{\pi}\right].$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right).$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right). \qquad \text{etc.}$$

Their graphs in Figure seem to indicate that the series is convergent and has the sum $f(x)$, the given function.

We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and k of our function, at these points. This is typical.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.

Orthogonal Functions/Systems of Orthogonal Functions

A sequence of functions $\{\varphi_n(x)\}$ is said to be orthogonal with respect to the weight function q (x) on the interval [a, b] if

$$\int_a^b \phi_m(x)\,\phi_n(x)\,q(x)\,dx = 0, \qquad m \neq n.$$

Theorem (Orthogonality of the Trigonometric System)

The trigonometric system is orthogonal on the interval $-\pi \leqq x \leqq \pi$ (hence also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity); *that is, the integral of the product of any two functions in* (3) *over that interval is* 0, *so that for any integers n and m,*

(a) $$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = 0 \qquad (n \neq m)$$

(b) $$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0 \qquad (n \neq m)$$

(c) $$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0 \qquad (n \neq m \text{ or } n = m).$$

Proof

$$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x\, dx$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x\, dx.$$

Since $m \neq n$ (integer!), the integrals on the right are all 0.

$$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin (n+m)x\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin (n-m)x\, dx = 0 + 0.$$

Theorem (Convergence of a Fourier Series)

Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $-\pi \leq x \leq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Proof

We prove convergence, but only for a continuous function having continuous first and second derivatives. And we do not prove that the sum of the series is because these proofs are much more advanced;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad\qquad n = 1, 2, \cdots$$

Integrating by parts, we obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{f(x)\sin nx}{n\pi}\bigg|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\, dx.$$

The first term on the right is zero. Another integration by parts gives

$$a_n = \frac{f'(x)\cos nx}{n^2\pi}\bigg|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\, dx.$$

The first term on the right is zero because of the periodicity and continuity of $f'(x)$. Since f'' is continuous in the interval of integration, we have

$$|f''(x)| < M$$

for an appropriate constant M. Furthermore, $|\cos nx| \leqq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx\, dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M\, dx = \frac{2M}{n^2}.$$

Similarly, $|b_n| < 2\,M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete.

Point – wise Convergence of a Fourier Series

An infinite series $\sum_{n=1}^{\infty} f_n(x)$ is called pointwise convergent in a<x<b to f (x) if it converges to f (x) for each x in a<x<b. In other words, for each x in a<x<b, we have

$$|f(x) - s_n(x)| \to 0 \qquad as \quad n \to \infty$$

where $s_n(x)$ is the nth partial sum defined by $s_n(x) = \sum_{k=1}^{n} f_k(x)$

Uniform Convergence of a Fourier Series

An infinite series $\sum_{n=1}^{\infty} f_n(x)$ is called pointwise convergent in a<x<b to f (x) if

$$\max_{a \le x \le b} |f(x) - s_n(x)| \to 0 \qquad as \quad n \to \infty$$

Evidently, uniform convergence implies pointwise convergence, but the converse is not necessarily true.

Mean Square Convergence of a Fourier Series

An infinite series $\sum_{n=1}^{\infty} f_n(x)$converges in the mean-square (or L^2) sense to f (x) in $a \le x \le b$ if

$$\int_a^b |f(x) - s_n(x)|^2\, dx \to 0 \qquad as \quad n \to \infty$$

It is noted that uniform convergence is stronger than both pointwise convergence and mean-square convergence.

Fourier Series with the function of period 2L

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

with the Fourier coefficients of $f\ (x)$ given by the Euler formulas

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

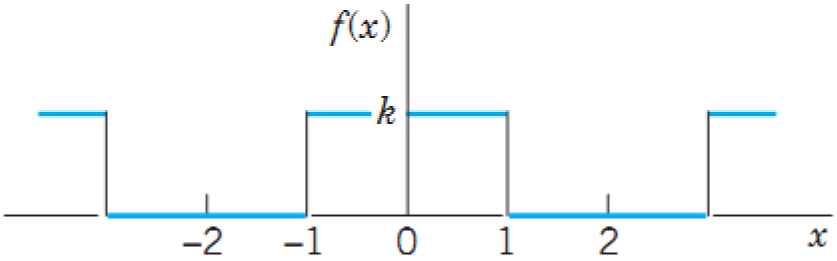
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots$$

Example (Periodic Rectangular Wave)

Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < \ \ 1 \\ 0 & \text{if} \qquad 1 < x < \ \ 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$



Solution

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = k/2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0 \text{ for } n = 1, 2, \cdots$$

Hence the Fourier series is a Fourier cosine series (that is, it has no sine terms)

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right)$$

Example (Periodic Rectangular Wave, Change of Scale)

Find the Fourier series of the function

$$f(x) = \begin{cases} -k & \text{if} \quad -2 < x < 0 \\ k & \text{if} \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

Solution

Since L = 2, we have $v = \frac{\pi x}{L} = \frac{\pi x}{2}$

Since the a_n are zero, the Fourier series of $f(x)$ is

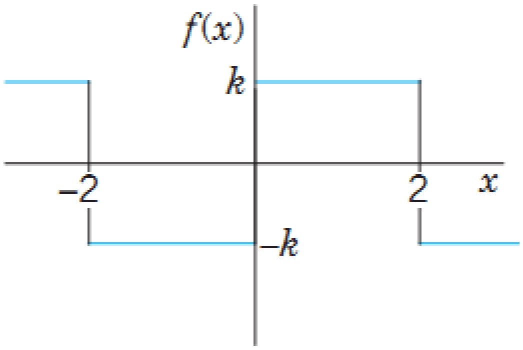
$$\frac{4k}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right)$$

replace v instead of x, we have

$$g(v) = \frac{4k}{\pi}\left(\sin v + \frac{1}{3}\sin 3v + \frac{1}{5}\sin 5v + \cdots\right)$$

the present Fourier series

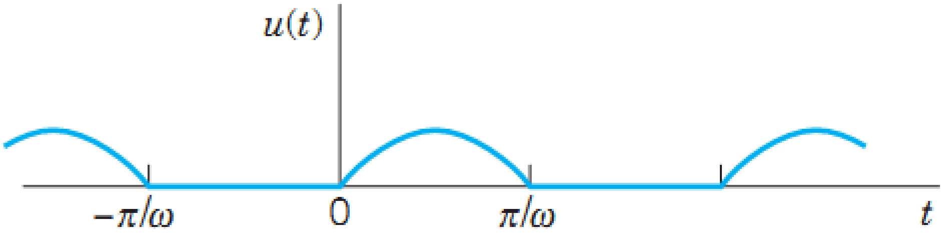
$$f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right).$$



Example (Half-Wave Rectifier)

A sinusoidal voltage $E\sin\omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Figure). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E\sin\omega t & \text{if } \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$



Solution

Since $u = 0$ when $-L < t < 0$

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E \sin \omega t \, dt = \frac{E}{\pi}$$ with t instead of x,

Using formulae

$$\begin{cases} \sin x \sin y = \tfrac{1}{2}[-\cos(x+y) + \cos(x-y)] \\ \cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)] \\ \sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)] \end{cases}$$ with $x = \omega t$ and $y = n\omega t$,

We have

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E \sin \omega t \cos n\omega t \, dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega} [\sin(1+n)\omega t + \sin(1-n)\omega t]\, dt.$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \cdots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right).$$

If n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, \cdots).$$

Using

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$ we have $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently

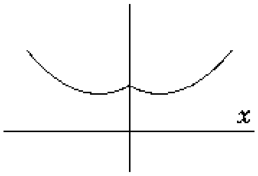
$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin \omega t - \frac{2E}{\pi}\left(\frac{1}{1 \cdot 3}\cos 2\omega t + \frac{1}{3 \cdot 5}\cos 4\omega t + \cdots\right)$$

Simplifications: Even and Odd Functions

Even Functions and Fourier cosine series

If $f(x)$ is an even function, that is $f(-x) = f(x)$, its Fourier series reduces to a Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$



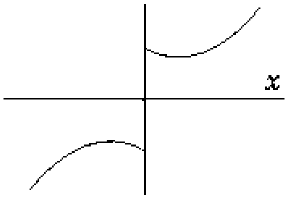
with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

Odd Functions and Fourier sine series

If $f(x)$ is an odd function, that is $f(-x) = -f(x)$, its Fourier series reduces to a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$



With coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Remember

$$\text{(a)} \quad \int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx \qquad \text{for even } g$$

$$\text{(b)} \quad \int_{-L}^{L} h(x)\,dx = 0 \qquad \text{for odd } h$$

Summary

Even Function of Period 2π. If f is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, \cdots$$

Odd Function of Period 2π. If f is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \cdots.$$

Sum and Scalar Multiple

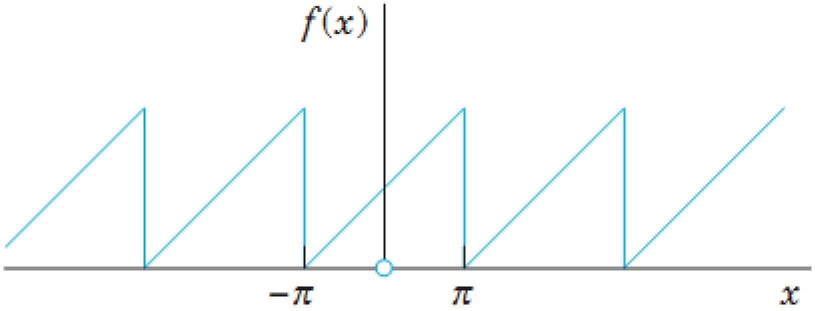
The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Example (Sawtooth Wave)

Find the Fourier series of the function

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$



Solution

We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$.

The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi}\left[\left.\frac{-x \cos nx}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right] = -\frac{2}{n}\cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -\frac{2}{2}$, $b_3 = \frac{2}{3}$, $b_4 = -\frac{2}{4}, \cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right).$$

Half Range Expansion of Fourier Series

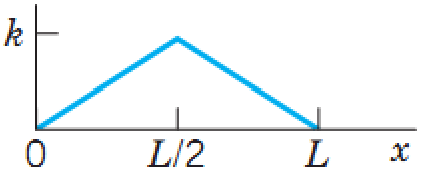
If a function is defined over half the range, say 0 to L, instead of the full range from −$-L$ to L, it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a **half range Fourier series**.

Conversely, the Fourier Series of an even or odd function can be analysed using the half range definition.

Example ("Triangle" and Its Half-Range Expansions)

Find the two half-range expansions of the function

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L - x) & \text{if} \quad \dfrac{L}{2} < x < L. \end{cases}$$



Solution

(a) ***Even periodic extension.***

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L - x)\,dx\right] = \frac{k}{2},$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_L^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L - x)\cos\frac{n\pi}{L}x\,dx\right].$$

We consider a_n. For the first integral we obtain by integration by parts

$$\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx = \frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\,\Bigg|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos \frac{n\pi}{2} - 1\right).$$

Similarly, for the second integral we obtain

$$\int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} x \, dx = \frac{L}{n\pi}(L - x) \sin \frac{n\pi}{L} x \bigg|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi}{L} x \, dx$$

$$= \left(0 - \frac{L}{n\pi}\left(L - \frac{L}{2}\right) \sin \frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos \frac{n\pi}{2}\right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2 \cos \frac{n\pi}{2} - \cos n\pi - 1\right).$$

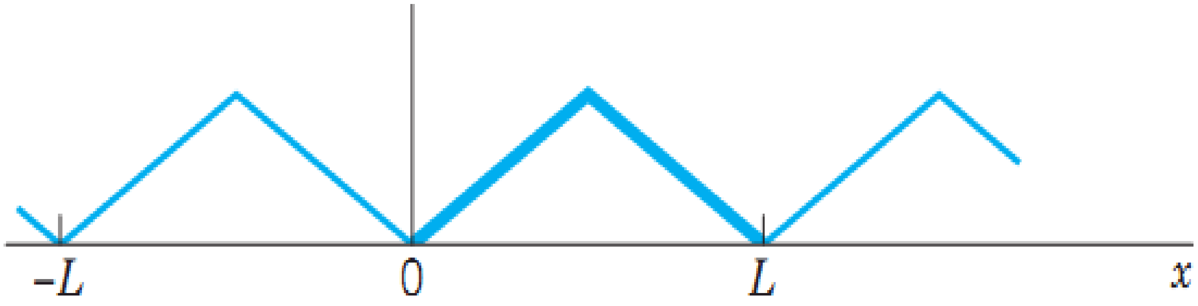
Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.



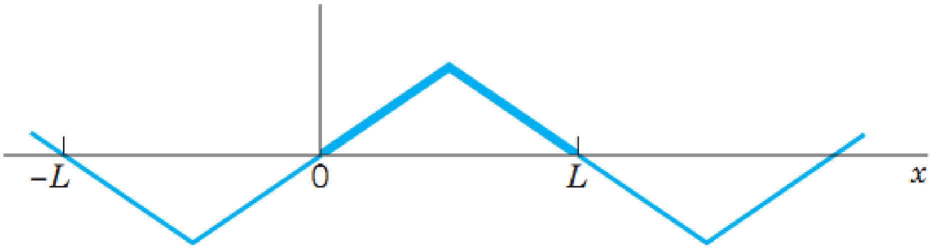
(b) ***Odd periodic extension.***

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Hence the other half-range expansion of $f(x)$ is

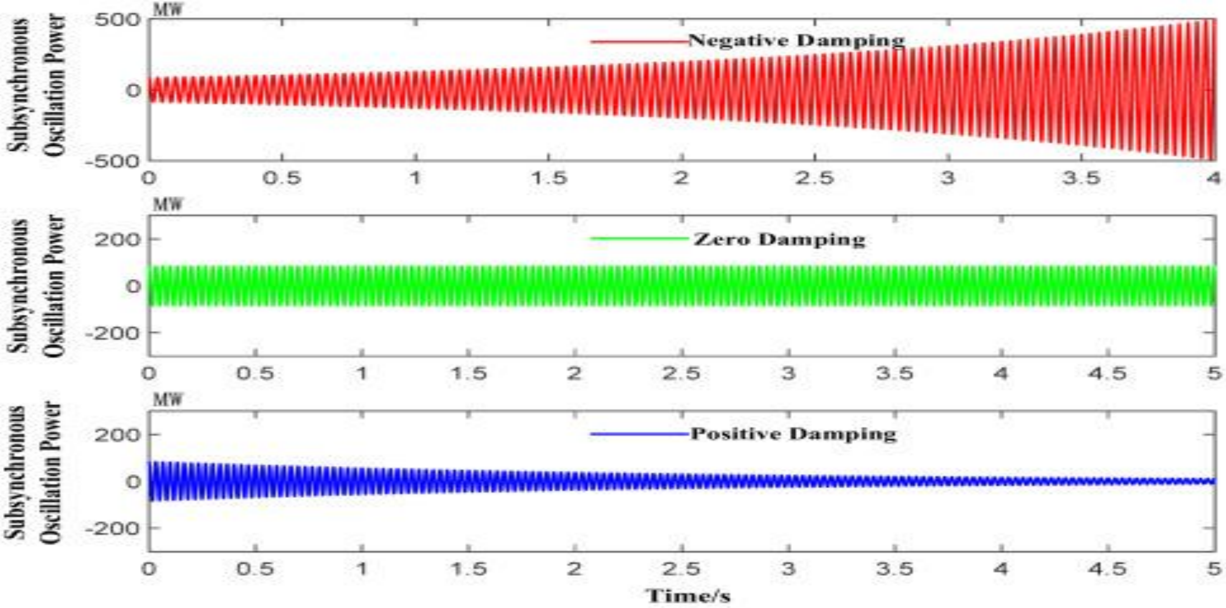
$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.



Forced Oscillations

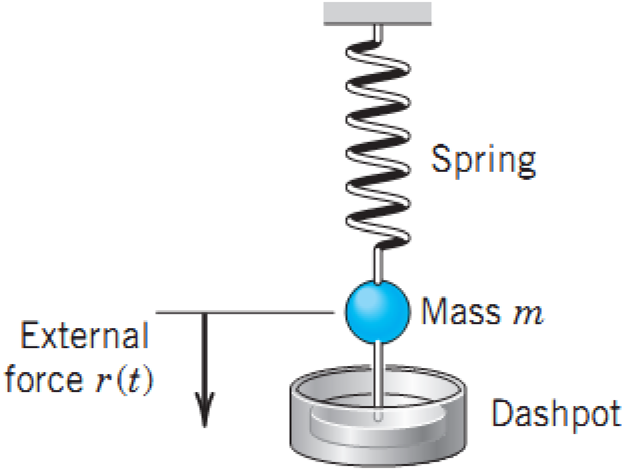
Forced oscillation, also known as a driven oscillation, is a type of oscillation where a system is subjected to an external periodic force. This external force causes the system to oscillate at the frequency of the driving force, rather than its natural frequency. Essentially, the system is "forced" to oscillate at a frequency different from its own, if not for the external force, its oscillations would eventually die out.



Forced oscillations of a body of mass m on a spring of modulus k are governed by the ODE

$$my'' + cy' + ky = r(t)$$

where $y = y(t)$ is the displacement from rest, c the damping constant, k the spring constant (spring modulus), and $r(t)$ the external force depending on time t.



Example (Forced Oscillations under a Nonsinusoidal Periodic Driving Force)

Evaluate

$$my'' + cy' + ky = r(t)$$

With

let $m = 1$ (g), $c = 0.05$ (g/sec), and $k = 25$ (g/sec^2)

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if} \quad -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if} \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

Solution

let $m = 1$ (g), $c = 0.05$ (g/sec), and $k = 25$ (g/sec^2)

$y'' + 0.05y' + 25y = r(t)$ where $r(t)$ is measured in g · cm/sec^2.

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if} \quad -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if} \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

We represent $r(t)$ by a Fourier series, finding

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right).$$

Then we consider the ODE

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, \cdots)$$

We know that

$$y_n = A_n \cos nt + B_n \sin nt.$$

By substituting this in previous we find that

$$A_n = \frac{4(25 - n^2)}{n^2\pi D_n}, \qquad B_n = \frac{0.2}{n\pi D_n}, \qquad \text{where} \qquad D_n = (25 - n^2)^2 + (0.05n)^2.$$

Since the ODE $y'' + 0.05y' + 25y = r(t)$ is linear, we may expect the steady-state solution to be

$$y = y_1 + y_3 + y_5 + \cdots$$

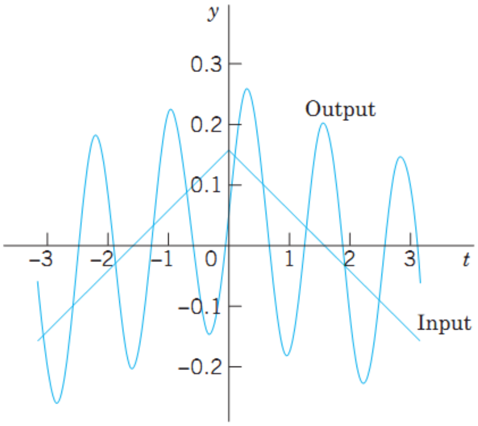
From A_n, B_nskipping D_n we find that the amplitude of y_n is

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}}.$$

Values of the first few amplitudes are

$$C_1 = 0.0531 \quad C_3 = 0.0088 \quad C_5 = 0.2037 \quad C_7 = 0.0011 \quad C_9 = 0.0003.$$

Figure shows the input (multiplied by 0.1) and the output. For $n = 5$ the quantity D_n is very small, the denominator of C_5 is small, and C_5 is so large that y_5 is the dominating term in y Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term y_1, whose amplitude is about 25% of that of y_5.



Approximation by Trigonometric Polynomials

Nth partial sum of the Fourier series

Let $f(x)$ be a function on the interval $-\pi \leq x \leq \pi$ that can be represented on this interval by a Fourier series. Then the Nth partial sum of the Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

is an approximation of the given $f(x)$.

Error/Square Error of the approximation

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx.$$

Square Error of the approximation with choice of the Coefficients

$$E^* = \int_{-\pi}^{\pi} f^2\,dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right].$$

Remember

Since the sum of squares of real numbers on the right cannot be negative,

$$E - E^* \geqq 0, \qquad \text{thus} \qquad E \geqq E^*$$

and $E = E^*$ if and only if $A_0 = a_0, \cdots, B_N = b_N.$

Minimum Square Error Theorem

The square error of F in a trigonometric polynomial of the same degree Nm (with fixed N) that is

$$F(x) = A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx) \qquad (N \text{ fixed}).$$

relative to f on the interval $-\pi \leq x \leq \pi$ is minimum if and only if the coefficients of F in above fromula are the Fourier coefficients of f. This minimum value E* is given by

$$E^* = \int_{-\pi}^{\pi} f^2\,dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right].$$

Bessel's Inequality

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx$$

Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \, dx.$$

Example (Minimum Square Error for the Sawtooth Wave)

Compute the minimum square error E^* of $F(x)$ with $N = 1, 2, \cdots, 10, 20, \cdots, 100$ and 1000 relative to

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

on the interval $-\pi \leqq x \leqq \pi$.

Solution

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi$$

Then its Fourier series is

$$F(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx\right)$$

And using

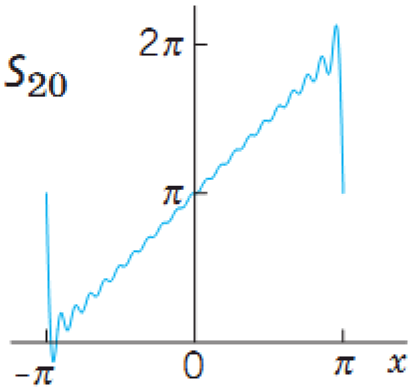
$$E^* = \int_{-\pi}^{\pi} f^2 \, dx - \pi\left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right].$$

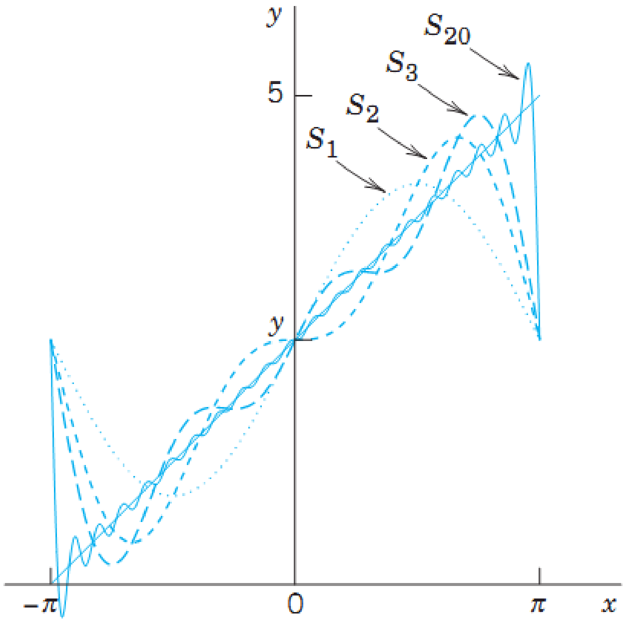
We have

$$E^* = \int_{-\pi}^{\pi} (x + \pi)^2 \, dx - \pi\left(2\pi^2 + 4\sum_{n=1}^{N} \frac{1}{n^2}\right)$$

Numeric values are:

N	E^*	N	E^*	N	E^*	N	E^*
1	8.1045	6	1.9295	20	0.6129	70	0.1782
2	4.9629	7	1.6730	30	0.4120	80	0.1561
3	3.5666	8	1.4767	40	0.3103	90	0.1389
4	2.7812	9	1.3216	50	0.2488	100	0.1250
5	2.2786	10	1.1959	60	0.2077	1000	0.0126





Although $|f(x) - F(x)|$ is large at $\pm\pi$ where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where "waves" remain owing to the "Gibbs phenomenon,"

Sturm–Liouville System (SL System)

The functions discussed in this chapter arise as solution of second order DE's which appear in special, rather than in general physical problems. So, these functions are usually known as "The Special Functions of Mathematical Physics"

Self Adjoint Operator:

An operatior 'A' defined over a linear space of functions is called Self Adjoint if $\langle u, Av\rangle = \langle v, Au\rangle$ which is equivalent to $\int_a^b u(x)[Av(x)]dx = \int_a^b [Au(x)]v(x)dx$ where the functions 'u' and 'v' are supposed to be real. In case of complex functions a slight modification is necessary.

Examples (just read):

- Sturm liouville Differential operator is Self Adjoint.
- The Hormonic oscillator equation is Self Adjoint.
- Legendre's equation is Self Adjoint.
- Laguerre's equation and Hermite equation are not Self adjoint but could be made using few conditions.

Sturm Liouville Equation (SL- Equation):

The SL equation is named after the German Mathematician John Sturm (1803 – 1855) and the French Mathematician Joseph Liouville (1809 - 1887), who did pioneering work on this DE and related problems.

Defination:

The Second order Ordinary Linear Homogeneous DE of the form

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$$

$\Rightarrow p(x)u'' + p'(x)u' + q(x)u + \lambda\, r(x)u = 0$ is called a Sturm Liouville equation OR briefly an SL equation. If $p(x), p'(x), q(x), r(x)$ are real and continuous over an interval [a,b].

Sturm Liouville Problem

The Second order Ordinary Linear Homogeneous DE of the form

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$$

$$\Rightarrow p(x)u'' + p'(x)u' + q(x)u + \lambda\, r(x)u = 0$$

Together with conditions

$k_1u + k_2u' = 0$ at $x = a$

$l_1u + l_2u' = 0$ at $x = b$

is called a Sturm Liouville problem.

Sturm Liouville (SL) Differential Operator:

A self adjoint operator of the form $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$ is called SL differential operator. This operator is a second order linear differential operator because it operate on everything to the right, not just by ordinary multiplication but also by the operation of differentiation.

Remark (Just Read):

- SL differential operator $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$ is called <u>normal operator</u> if $p(x) \neq 0$ in the range $x \in (-\infty, \infty)$
- In terms of SL differential operator SL equation can also be written as $L(u) + \lambda\, r(x)u = 0$

Singular Points: For SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ the points at which $p(x)\ and\ r(x)$ vanishes (i.e. become zero) over any interval [a,b] are called singular points.

Regular Points: For SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ the points at which $p(x)\ and\ r(x)$ do not vanishes (i.e. become zero) over any interval [a,b] are called regular points.

Weight Function: In SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ the continuous, non negative, real function $r(x)$ on any interval 'I' is called Weight Function.

Singular Sl Equation:

The SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ is called Singular SL equation in the interval [a,b] if the points $p(x)\ and\ r(x)$ vanishes (i.e. become zero) at any point the interval [a,b].

Examples:

- Legendre's DE $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$ with $p(x) = (1-x^2), q(x) = 0,\ r(x) = 1$ is singular at $x = \pm 1$.
- Bessele's DE is singular after few arrangements. At $x = 0$

Singular Sl System: A singular SL equation together with suitable linear homogeneous conditions on $u(x)$ leads to a singular SL system.

Regular Sl Equation:

The SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ is called regular SL equation in the interval [a,b] if the points $p(x)\ and\ r(x)$ do not vanishes (i.e. become zero) at any point the interval [a,b].

Examples:

- $u''(x) + \lambda u(x) = 0$ with $p(x) = 1 > 0, q(x) = 0, r(x) = 1 > 0$ is regular SL equation in every interval.
- Legendre's DE $\frac{d}{dx}\left\{(1 - x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$ with $p(x) = (1 - x^2), q(x) = 0,\ r(x) = 1$ is singular or is not regular at $x = \pm 1$.
- Bessele's DE is singular or is not regular. At $x = 0$

Regular Sl System: A Regular SL equation together with suitable end point conditions leads to a regular SL system.

Conditions are $\alpha u(a) + \alpha' u'(a) = 0$ and $\beta u(b) + \beta' u'(b) = 0$

Periodic SL Equation:

The SL equation $\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0$ is called periodic SL equation in the interval [a,b] if the points $p(x), q(x)\ and\ r(x)$ are periodic funcitons of period $b - a$.

Examples:

- $u'' + \lambda u = 0$ with $u(-\pi) = u(+\pi)\ \ and\, u'(-\pi) = u'(+\pi)$ is periodic SL equation.
- With period $2\,\pi$, The Mathieu DE $u'' + \lambda u + 16dCos2xu = 0$ with $u(-\pi) = u(+\pi)\ \ and\, u'(-\pi) = u'(+\pi)$ is periodic SL equation.

Periodic SL System: A Periodic SL equation together with suitable end ponint conditions leads to a Periodic SL system.

Conditions are $u(a) = u(b)\ \ and\, u'(a) = u'(b)$

BOUNDRY CONDITIONS ASSOCIATED WITH SL SYSTEM

- The boundry conditions $\alpha u(a) + \alpha' u'(a) = 0$ and $\beta u(b) + \beta' u'(b) = 0$ are called Separated Boundry Conditions are Unmixed Boundry Conditions.
- If the Separated Boundry conditions are of the form $u(a) = c_1\ \ and\, u(b) = c_2$ then they are called Drichlet BC's
- If the Separated Boundry conditions are of the form $u'(a) = c'_1\ \ and\, u'(b) = c'_2$ then they are called Neumann BC's
- If the Separated Boundry conditions are of the form $u(a) = u(b)\ \ \ and$ $u'(a) = u'(b)$ then they are called Periodic BC's

Eigenvalue Problems:
A non – zero solution of an SL sytem (Regular or Periodic) is said to be an eigensolution or eigenfunction corresponding to a value of the parameter λ in SL equation. The value of λ then called sn eigenvalue of the DE.
OR If an IVP or BVP contains a parameter λ in the DE and non – trivial solution(s) corresponding to certain values of λ can be found then the problem is called Eigenvalue Problem, and the corresponding values of λ are called Eigenvalues of the problem.

Example (Trigonometric Functions as Eigenfunctions. Vibrating String):
Find the eigenvalue and eigenfunctions (solutions) for the regular SL system $u'' + \lambda u = 0$ where λ is parameter and the boundry conditions are $u(0) = 0$ and $u(\pi) = 0$
Solution: (the end point conditions shows that the system is regular but not periodic) with $a = 0, b = \pi, p(x) = 1, q(x) = 0, r(x) = 1$
Now $u'' + \lambda u = 0 \Rightarrow D = \pm i\sqrt{\lambda}$
Then general solution becomes $u(x) = ACos\sqrt{\lambda}x + BSin\sqrt{\lambda}x \quad 0 \leq x \leq \pi$
Now using BC's $u(0) = 0 \Rightarrow A = 0 \quad \Rightarrow u(x) = BSin\sqrt{\lambda}x$
$u(\pi) = 0 \Rightarrow BSin\sqrt{\lambda}\pi = 0 \Rightarrow B \neq 0(gives\ trivial\ solution\ )\ Sin\sqrt{\lambda}\pi = 0$
$\Rightarrow \sqrt{\lambda}\pi = n\pi \Rightarrow \sqrt{\lambda} = n \quad ; n = \pm 1, \pm 2, \ldots\ldots.$
ommiting ,0, because gives trivial solution.
Hence $\lambda = \lambda_n = n^2 ; n = \pm 1, \pm 2, \ldots\ldots.$
are the eigenvalues for the non – trivial solution where the eigenfunctions are
$u_n(x) = b_n Sinnx \ ; n = \pm 1, \pm 2, \ldots\ldots$
whrer the constants b_n are in general different for each solution.

Example:
Show that if $u(x)$ and $v(x)$ are periodic solutions of the Mathieu equation with period π having the distinct eigenvalues then $\int_0^{\pi} u(x)v(x)dx = 0$
Solution:
we know that the Mathieu DE with period π is $u'' + \lambda u + 16dCos2xu = 0$ with end point conditions $u(0) = u(\pi)\ and\ u'(0) = u'(\pi)$
now if 'u' and 'v' are solutions of given equation corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively then
$u'' + \lambda_1 u + 16dCos2xu = 0$ ………...(i)
with end point conditions $u(0) = u(\pi)\ and\ u'(0) = u'(\pi)$
Similarly for 'v' we have
$v'' + \lambda_2 v + 16dCos2xv = 0$ …………..(ii)

with end point conditions $v(0) = v(\pi)\ and\ v'(0) = v'(\pi)$

Multiplying (i) with 'v' and (ii) with 'u' and subtracting we obtain

$(\lambda_1 - \lambda_2)uv = v''u - u''v$ ………….....(iii)

$\Rightarrow \int_0^\pi (\lambda_1 - \lambda_2)uvdx = \int_0^\pi v''u - u''vdx$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)dx = \int_0^\pi \frac{d}{dx}(v'u - u'v)dx$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)dx = |v'u - u'v|_0^\pi$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)dx = u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(0)v'(0) + u'(0)v(0)$

Now using end conditions.

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)dx = u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(\pi)v'(\pi) + u'(\pi)v(\pi)$

$\Rightarrow (\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)dx = 0 \Rightarrow (\lambda_1 - \lambda_2) \neq 0 \Rightarrow \int_0^\pi u(x)v(x)dx = 0$

as required.

Orthogonal Functions (Standard Form)

Functions $u(x)$ and $v(x)$ defined over [a,b] are said to be orthogonal w.r.to a weight function $w(x)$ if

$$\int_a^b w(x)u(x)v(x)dx = 0$$

Norm Form of $u(x)$

Functions $u(x)$ defined over [a,b] the norm form is given as

$$\|u(x)\| = \sqrt{(u,u)} = \sqrt{\int_a^b w(x)u^2(x)dx = 0}$$

Square Integrable Function: A function $f(x)$ is said to be square integrable with respect to a weight function $w(x) > 0$ over an interval [a,b] if $\int_a^b w(x)|f(x)|^2dx < \infty$

If $w(x) = 1$ then $\int_a^b |f(x)|^2dx < \infty$ in this case $f(x)$ is simply called square integrable.

Examples:

- Legendre's DE $\frac{d}{dx}\left\{(1 - x^2)\frac{du}{dx}\right\} + n(n + 1)u = 0$ is square integrable.
- Bessele's DE is square integrable.

Kronecker symbol

$$(u,v) = \int_a^b w(x)u(x)v(x)dx = \delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

Example (Orthogonal Functions. Orthonormal Functions. Notation)

The functions $y_m(x) = \sin mx$, $m = 1, 2, \cdots$ form an orthogonal set on the interval $-\pi \leqq x \leqq \pi$, because for $m \neq n$ we obtain by integration

$$(y_m, y_n) = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (m-n)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (m+n)x \, dx = 0, \quad (m \neq n).$$

The norm $\| y_m \| = \sqrt{(y_m, y_m)}$ equals $\sqrt{\pi}$ because

$$\|y_m\|^2 = (y_m, y_m) = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi \qquad (m = 1, 2, \cdots)$$

Hence the corresponding orthonormal set, obtained by division by the norm, is

$$\frac{\sin x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \frac{\sin 3x}{\sqrt{\pi}}, \quad \cdots.$$

Lagrange's Identity

Suppose $u(x)$ and $v(x)$ are two solutions of an SL equation, then the following identity must hold

$uL(v) - vL(u) = \frac{d}{dx}\{p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\}$

Which is called Differential form of Langrange's idenetity. While the integral form is given as follows

$\int_a^b [uL(v) - vL(u)]dx = \big|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\big|_a^b$

PROOF: Since $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$ therefore

$uL(v) - vL(u) = u\frac{d}{dx}\left\{p(x)\frac{dv}{dx}\right\} + q(x) - v\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} - q(x)$

$uL(v) - vL(u) = up(x)v'' + up'(x)v' - vp(x)u'' - vp'(x)u'$

$uL(v) - vL(u) = p(x)(uv'' - u''v) + p'(x)(uv' - u'v)$

$uL(v) - vL(u) = \frac{d}{dx}\{p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\}$

$\Rightarrow uL(v) - vL(u) = \frac{d}{dx}\{p(x)W(u,v)(x)\}$, W is called Wronskian of 'u' ,'v'

Taking integral from 'a' to 'b'

$\int_a^b [uL(v) - vL(u)]dx = \int_a^b \frac{d}{dx}\{p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\}dx$

$\int_a^b [uL(v) - vL(u)]dx = \big|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\big|_a^b$

IMPORTANCE: By using Lagrange's identity, we may prove reality, orthogonality and simplicity of eigenvalues of an SL system (regular or periodic)

REALITY OF EIGENVALUES

THEOREM – I : یہ یا اس سے اگلا رزلٹ، ان میں سے کوئی ایک یا دونوں کی اکٹھی سٹیٹمینٹ پوچھ لی جاتی ہے۔

The eigenvalues of an SL system (regular) are real.

PROOF: Let ‘u’ be eigenfunction corresponding to eigenvalue ‘λ’ then $u(x) \neq 0\ ;\forall x \epsilon (a,b)$

Now as $L(u) + \lambda\, r(x)u = 0 \Rightarrow L(u) = -\lambda\, r(x)u$ ……………..(i)

If possible let ‘λ’ be complex then $\bar{L}(u) = -\bar{\lambda}\, r(x)\bar{u}$

Now ‘p’,’q’,’r’ are real, therefore L is real hence

$\Rightarrow L(u) = -\bar{\lambda}\, r(x)\bar{u}$ ……………..(ii)

Now from Lagrange’s identity

$\int_a^b [uL(v) - vL(u)]dx = \left|p(x)\big(u(x)v'(x) - u'(x)v(x)\big)\right|_a^b$

Taking $v = \bar{u}$ then

$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = \left|p(x)\big(u(x)\bar{u}'(x) - u'(x)\bar{u}(x)\big)\right|_a^b$ …………….(A)

Now for a regular SL system $\alpha u(a) + \alpha' u'(a) = 0$ and $\beta u(b) + \beta' u'(b) = 0$

Similarly $\alpha \bar{u}(a) + \alpha' \bar{u}'(a) = 0$ and $\beta \bar{u}(b) + \beta' \bar{u}'(b) = 0$

If we substitute the values of $u(a), u(b), \bar{u}'(a), \bar{u}'(b)$ in R.H.S of (A) we find it will be zero. Hence $\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = 0$

Using (i) and (ii) $\int_a^b \big[u\big(-\bar{\lambda}\, r(x)\bar{u}\big) - \bar{u}(-\lambda\, r(x)u)\big]dx = 0$

$\Rightarrow \int_a^b \big[-\bar{\lambda}\, ru\bar{u} + \lambda\, ru\bar{u}\big]dx = 0 \Rightarrow \int_a^b \big(\lambda - \bar{\lambda}\big) ru\bar{u}dx = 0 \Rightarrow \int_a^b \big(\lambda - \bar{\lambda}\big) r|u|^2 dx = 0$

Now as $r(x) > 0$ also $|u|^2 > 0$ therefore $\int_a^b r|u|^2 dx > 0$

Then $\big(\lambda - \bar{\lambda}\big) = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real as required.

REALITY OF EIGENVALUES

THEOREM – II : یہ یا اس سے پچھلا رزلٹ، یا دونوں کی اکٹھی سٹیٹمینٹ پوچھ لی جاتی ہے۔

The eigenvalues of an SL system (periodic) are real.

PROOF: Let 'u' be eigenfunction corresponding to eigenvalue 'λ' then $u(x) \neq 0\ ; \forall x \epsilon (a, b)$

Now as $L(u) + \lambda\, r(x)u = 0 \Rightarrow L(u) = -\lambda\, r(x)u$ ……………..(i)

If possible let 'λ' be complex then $\bar{L}(u) = -\bar{\lambda}\, r(x)\bar{u}$

Now 'p','q','r' are real, therefore L is real hence

$\Rightarrow L(u) = -\bar{\lambda}\, r(x)\bar{u}$ ……………..(ii)

Now from Lagrange's identity

$\int_a^b [uL(v) - vL(u)]dx = \left| p(x)\big(u(x)v'(x) - u'(x)v(x)\big) \right|_a^b$

Taking $v = \bar{u}$ then

$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = \left| p(x)\big(u(x)\bar{u}'(x) - u'(x)\bar{u}(x)\big) \right|_a^b$ ……….(A)

Now for a periodic SL system $u(a) = u(b), u'(a) = u'(b), p(a) = p(b)$ and if B.C's are singular then $p(a) = p(b) = 0$ and R.H.S of (A) will be zero.

Hence $\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = 0$

Using (i) and (ii) $\int_a^b \left[u\big(-\bar{\lambda}\, r(x)\bar{u}\big) - \bar{u}(-\lambda\, r(x)u)\right]dx = 0$

$\Rightarrow \int_a^b \left[-\bar{\lambda}\, ru\bar{u} + \lambda\, ru\bar{u}\right]dx = 0 \Rightarrow \int_a^b \big(\lambda - \bar{\lambda}\big) ru\bar{u}dx = 0$

$\Rightarrow \int_a^b \big(\lambda - \bar{\lambda}\big) r|u|^2 dx = 0$

Now as $r(x) > 0$ also $|u|^2 > 0$ therefore $\int_a^b r|u|^2 dx > 0$

Then $\big(\lambda - \bar{\lambda}\big) = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real as required.

ORTHOGONALITY OF EIGENVALUES

Eigenfunctions of a regular or periodic SL system corresponding to distinct eigenvalues are orthogonal w.r.to weight function $r(x)$.

PROOF:

Let 'λ_m' and 'λ_n' be eigenvalues of an SL system with eigenfunctions $u_m(x)$ and $u_n(x)$ respectively then using Lagrange's identity

$L(u_m) = -\lambda_m\, r(x)u_m$ and $L(u_n) = -\lambda_n\, r(x)u_n$ with boundry conditions of the regular or periodic type.

Again using Lagrange's identity for $u_m(x)$ and $u_n(x)$

$\int_a^b [u_m L(u_n) - u_n L(u_m)]dx = \left|p(x)\big(u_m(x)u_n{}'(x) - u_m{}'(x)u_n(x)\big)\right|_a^b$

For a regular or periodic SL system, R.H.S = 0

Hence $\quad \int_a^b [u_m L(u_n) - u_n L(u_m)]dx = 0$

Using (i) and (ii) $\quad \int_a^b [u_m(-\lambda_n\, r(x)u_n) - u_n(-\lambda_m\, r(x)u_m)]dx = 0$

$\Rightarrow \int_a^b [-\lambda_n\, r u_m u_n + \lambda_m\, r u_m u_n]dx = 0 \Rightarrow \int_a^b (\lambda_m - \lambda_n) r u_m u_n dx = 0$

$\Rightarrow (\lambda_m - \lambda_n) \neq 0 \Rightarrow \int_a^b u_m u_n r dx = 0$

This shows that eigenvalues are orthogonal w.r.to weight function $r(x)$.

Example

Determine eigenvalues and eigenfuctions of the problem

$u'' + \lambda^2 u = 0\,; 0 < x < \pi$ with the boundry conditions

$u'(0) + 2u'(\pi) = 0$ and $u(\pi) = 0$

Solution

Given $u'' + \lambda^2 u = 0 \Rightarrow D = \pm i\lambda$

Then general solution becomes $u(x) = c_1 Cos\lambda x + c_2 Sin\lambda x$

$\Rightarrow u'(x) = -\lambda c_1 Sin\lambda x + \lambda c_2 Cos\lambda x$

Now using BC's $\quad u(\pi) = 0 \Rightarrow c_1 Cos\lambda\pi + c_2 Sin\lambda\pi = 0$ …………(i)

Now using BC's $\quad u'(0) + 2u'(\pi) = 0$

$-\lambda c_1 Sin\lambda 0 + \lambda c_2 Cos\lambda 0 + 2[-\lambda c_1 Sin\lambda\pi + \lambda c_2 Cos\lambda\pi] = 0$

$\Rightarrow \lambda c_2 - 2\lambda c_1 Sin\lambda\pi + 2\lambda c_2 Cos\lambda\pi = 0$

$\Rightarrow \lambda c_2 Cos\lambda\pi - \lambda c_1 2Sin\lambda\pi Cos\lambda\pi + 2\lambda c_2 Cos^2\lambda\pi = 0$

$\Rightarrow -\lambda c_1 Sin2\lambda\pi + \lambda c_2 Cos\lambda\pi(1 + 2Cos\lambda\pi) = 0$

$\Rightarrow c_1 Sin2\lambda\pi - c_2 Cos\lambda\pi(1 + 2Cos\lambda\pi) = 0$ …………(ii)

Substituiting c_2 from (i) into (ii) we get

$c_1(2 + Cos\lambda\pi) = 0 \Rightarrow c_1 \neq 0 \Rightarrow 2 + Cos\lambda\pi = 0 \Rightarrow Cos\lambda\pi = -2$

which cannot be satisfied for any real value of λ. Therefore the problem has only complex eigenvalues and complex eigenfunctions.

Example

Solve BVP defined by $u'' + \lambda^2 u = 0$ with $u(0) = 0, u(1) = 1; 0 < x < 1$

Solution: Given $u'' + \lambda^2 u = 0 \Rightarrow D = \pm i\lambda$

Then general solution becomes $u(x) = c_1 Cos\lambda x + c_2 Sin\lambda x$

Now using BC's $u(0) = 0 \Rightarrow c_1 = 0$

then given solution reduces to $u(x) = c_2 Sin\lambda x$

Now using BC's $u(1) = 1 \Rightarrow c_2 = \frac{1}{Sin\lambda}$

Hence the solution can be written as $u_\lambda(x) = \frac{Sin\lambda x}{Sin\lambda}$

Example

Express the function $f(x) = \begin{cases} 1 & ; 0 \le x \le \frac{1}{2} \\ 0 & ; \frac{1}{2} \le x \le 1 \end{cases}$ defined in the interval [0,1] in terms of eigenfunctions of the SL problem $y'' + \lambda^2 y = 0$ with the BC's $5y(1) + y'(1) = 0$ and $y(0) = 0$ where $0 < x < 1$

OR

Determine eigenvalues and eigenfuctions of the problem $u'' + \lambda^2 u = 0 ; 0 < x < 1$ with the boundry conditions $5u(1) + u'(1) = 0$ and $u(0) = 0$

Solution

Given $u'' + \lambda^2 u = 0 \Rightarrow D = \pm i\lambda$

Then general solution becomes $u(x) = c_1 Cos\lambda x + c_2 Sin\lambda x$

Now using BC's $u(0) = 0 \Rightarrow c_1 = 0$

Then $u(x) = c_2 Sin\lambda x$ and $u'(x) = \lambda c_2 Cos\lambda x$

Now using BC's $5u(1) + u'(1) = 0$

$5[c_2 Sin\lambda] + \lambda c_2 Cos\lambda = 0 \Rightarrow c_2[5Sin\lambda + \lambda Cos\lambda] = 0$

$\Rightarrow c_2 \neq 0 \Rightarrow [5Sin\lambda + \lambda Cos\lambda] = 0$

Given problem has infinite numbers of eigenvalues which satisfy the equation $tan\lambda_n = -\frac{\lambda_n}{5}$ where corresponding eigenfunctions are

$u_n = c_n Sin\lambda_n x$ n = 1,2,3,………

Example (Trigonometric Functions as Eigenfunctions. Vibrating String):

The ODE in Example $u'' + \lambda u = 0$ is a Sturm–Liouville equation with with $p = 1, q = 0, r = 1$. It follows that

the eigenfunctions $y_m = \sin mx$ ($m = 1, 2, \cdots$) are orthogonal on the interval $0 \leqq x \leqq \pi$.

Example (Trig Orthogonlity of the Legendre Polynomials)

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ may be written

$$[(1 - x^2)y']' + \lambda y = 0 \quad ; \quad \lambda = n(n + 1)$$

Hence, this is a Sturm–Liouville equation (1) with $p = 1 - x^2$, $q = 0$, and $r = 1$. Since $p(-1) = p(1) = 0$, we need no boundary conditions, but have a "***singular***" ***Sturm–Liouville problem*** on the interval $-1 \leqq x \leqq 1$. We know that for $n = 0, 1, \cdots$, hence $\lambda = 0, 1 \cdot 2, 2 \cdot 3, \cdots$, the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

SIMPLICITY OF EIGENVALUES

"The eigenvalues of a regular SL system are simple". i.e. to each eigenvalue there corresponds only one linearly independent eigenfunction.

In other words, if $u(x)$ and $v(x)$ are eigenfunctions corresponding to the same eigenvalue, then they must differ by a multiplicative constant.

PROOF: If possible let $u(x)$ and $v(x)$ be two linearly independent solutions corresponding to the same eigenvalue λ then using Lagrange's identity

$L(u) = -\lambda\, r(x)u$ and $L(v) = -\lambda\, r(x)v$

Then $uL(v) - vL(u) = -\lambda\, r(x)uv + \lambda\, r(x)uv = 0$ …………..(i)

But from Lagrange's identity, we have

$uL(v) - vL(u) = \frac{d}{dx}\{p(x)W(u,v)(x)\}$

Thus using (i) $\frac{d}{dx}\{p(x)W(u,v)(x)\} = 0 \quad ; \forall x \epsilon [a,b]$

Since in $[a,b]$, $\Rightarrow p(x) \neq 0 \Rightarrow W(u,v)(x) = 0$

It follows that $u(x)$ and $v(x)$ be linearly dependent solutions. i.e. to each eigenvalue there corresponds only one linearly independent eigenfunction.

ABEL'S FORMULA

If $u(x)$ and $v(x)$ are any two solutions of a regular or periodic SL equation, then $p(x)W(u,v)(x) = constant \;\; ; \forall x \epsilon [a,b]$

PROOF:

Since for a regular or periodic SL equation $uL(v) - vL(u) = 0$ for any pair of solutions. Hence from Lagrange's identity

$\frac{d}{dx}\{p(x)W(u,v)(x)\} = 0 \;\; ; \forall x \epsilon [a,b]$

$\Rightarrow p(x)W(u,v)(x) = constant \;\; ; \forall x \epsilon [a,b]$

THEOREM: Any eigenvalue 'λ' can be related to its eigenfunction $u(x)$ by

Rayleigh quotient $\quad \lambda = \frac{|-pu(x)u'(x)|_a^b + \int_a^b pu'^2(x)dx}{\int_a^b r(x)u^2 dx}$

This result cannot be used to determine eigenvalues, however, interesting and important results can be obtained from it.

EXAMPLE: Using Rayleigh quotient, discuss the signe of eigenvalue(s) of the SL system

$u'' + \lambda u = 0$ with $u(0) = 0, u(l) = 0, p(x) = 1, q(x) = 0, r(x) = 1$

Solution: here $a = 0, b = l, p(x) = 1, q(x) = 0, r(x) = 1$

Therefore using formula $\lambda = \frac{|-pu(x)u'(x)|_a^b + \int_a^b pu'^2(x)dx}{\int_a^b r(x)u^2 dx} = \frac{\int_0^l u'^2(x)dx}{\int_0^l u^2 dx}$

This result cannot be used to determine eigenvalues, however, interesting and important results about the eigenvalues can be obtained from it.

COMPLETENESS OF EIGENVALUES (just read)

"The eigenvalues of an SL system are complete"

OR" the set of eigenfunctions of an SL system are complete"

OR " Every function $u(x)\, ; x\epsilon[a,b]$ can be represented in terms of these eigenfunctions as $u(x) = \sum_{n=1}^{\infty} c_n u_n(x) \;\; ; x\epsilon[a,b]$

OR " A set of functions is said to be complete, if any function can be written as a linear combination of the function in the set, with constant coefficients." This is the generalization of the concept of the Fourier Series.

REMARKS:

- Legendre's polynomials are a complete set on $I = [-1,1]$
- Laguere polynomials are a complete set on $I = [0,\infty)$
- Hermite polynomials are a complete set on $I = (-\infty,\infty)$
- The eigenvalues of a Regular SL system are simple.i.e. Regular SL system have multiplicity 1.
- The eigenvalues of a Periodic SL system have multiplicity 2.

Orthogonal Series, Orthogonal Expansion, Or Generalized Fourier Series

Let $y_0, y_1, y_2, \cdots$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leqq x \leqq b$, and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

This is called an orthogonal series, orthogonal expansion, or generalized Fourier series.

Where Fourier coefficients are given as follows

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x)\, dx \qquad (n = 0, 1, \cdots).$$

Example (Fourier–Legendre Series)

A **Fourier–Legendre series** is an eigenfunction expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \cdots = a_0 + a_1 x + a_2 (\tfrac{3}{2} x^2 - \tfrac{1}{2}) + \cdots$$

in terms of Legendre polynomials

We have $r(x) = 1$ for Legendre's equation, and

Using

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x)\, dx \qquad (n = 0, 1, \cdots).$$

We have

$$a_m = \frac{2m + 1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx, \qquad m = 0, 1, \cdots$$

$$\|P_m\| = \sqrt{\int_{-1}^{1} P_m(x)^2\, dx} = \sqrt{\frac{2}{2m + 1}} \qquad (m = 0, 1, \cdots)$$

let $f(x) = \sin \pi x$. Then we obtain the coefficients

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} (\sin \pi x) P_m(x)\,dx, \qquad \text{thus} \qquad a_1 = \frac{3}{2}\int_{-1}^{1} x \sin \pi x \,dx = \frac{3}{\pi} = 0.95493, \qquad \text{etc.}$$

Hence the Fourier–Legendre series of $\sin \pi x$ is

$$\sin \pi x = 0.95493 P_1(x) - 1.15824 P_3(x) + 0.21929 P_5(x) - 0.01664 P_7(x) + 0.00068 P_9(x) - 0.00002 P_{11}(x) + \cdots .$$

The coefficient of P_{13} is about $3 \cdot 10^{-7}$. The sum of the first three nonzero terms gives a curve that practically coincides with the sine curve.

Example (Fourier–Bessel Series)

These series model vibrating membranes and other physical systems of circular symmetry. We derive these series in three steps.

***Step 1.* Bessel's equation as a Sturm–Liouville equation.** The Bessel function $J_n(x)$ with fixed integer $n \geqq 0$ satisfies Bessel's equation (Sec. 5.5)

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x}\dot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2) J_n(\tilde{x}) = 0$$

where $\dot{J}_n = dJ_n/d\tilde{x}$ and $\ddot{J}_n = d^2 J_n/d\tilde{x}^2$. We set $\tilde{x} = kx$. Then $x = \tilde{x}/k$ and by the chain rule, $\dot{J}_n = dJ_n/d\tilde{x} = (dJ_n/dx)/k$ and $\ddot{J}_n = J_n''/k^2$. In the first two terms of Bessel's equation, k^2 and k drop out and we obtain

$$x^2 J_n''(kx) + x J_n'(kx) + (k^2 x^2 - n^2) J_n(kx) = 0.$$

Dividing by x and using $(xJ_n'(kx))' = xJ_n''(kx) + J_n'(kx)$ gives the Sturm–Liouville equation

$$(5) \qquad [xJ_n'(kx)]' + \left(-\frac{n^2}{x} + \lambda x\right) J_n(kx) = 0 \qquad \lambda = k^2$$

with $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and parameter $\lambda = k^2$. Since $p(0) = 0$, Theorem 1 in Sec. 11.5 implies orthogonality on an interval $0 \leqq x \leqq R$ (R given, fixed) of those solutions $J_n(kx)$ that are zero at $x = R$, that is,

$$(6) \qquad J_n(kR) = 0 \qquad (n \text{ fixed}).$$

Note that $q(x) = -n^2/x$ is discontinuous at 0, but this does not affect the proof of Theorem 1.

***Step 2.* Orthogonality.** It can be shown (see Ref. [A13]) that $J_n(\tilde{x})$ has infinitely many zeros, say, $\tilde{x} = \alpha_{n,1} < \alpha_{n,2} < \cdots$ (see Fig. 110 in Sec. 5.4 for $n = 0$ and 1). Hence we must have

$$(7) \qquad kR = \alpha_{n,m} \qquad \text{thus} \qquad k_{n,m} = \alpha_{n,m}/R \qquad (m = 1, 2, \cdots).$$

This proves the following orthogonality property.

Orthogonality of Bessel Functions

For each fixed nonnegative integer n the sequence of Bessel functions of the first kind $J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$, $\cdots$ *with* $k_{n,m}$ *as in (7) forms an orthogonal set on the interval* $0 \leqq x \leqq R$ *with respect to the weight function* $r(x) = x$, *that is,*

$$(8) \qquad \int_0^R xJ_n(k_{n,m}x)J_n(k_{n,j}x)\,dx = 0 \qquad (j \neq m,\ n \text{ fixed}).$$

Hence we have obtained ***infinitely many orthogonal sets*** of Bessel functions, one for each of $J_0, J_1, J_2, \cdots$. Each set is orthogonal on an interval $0 \leqq x \leqq R$ with a fixed positive R of our choice and with respect to the weight x. The orthogonal set for J_n is $J_n(k_{n,1}x), J_n(k_{n,2}x), J_n(k_{n,3}x), \cdots$, where n is *fixed* and $k_{n,m}$ is given by (7).

***Step 3.* Fourier–Bessel series.** The Fourier–Bessel series corresponding to J_n (n fixed) is

$$(9) \qquad f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \cdots \qquad (n \text{ fixed}).$$

The coefficients are (with $\alpha_{n,m} = k_{n,m}R$)

$$\textbf{(10)} \qquad a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x\,f(x)\,J_n(k_{n,m}x)\,dx, \qquad m = 1, 2, \cdots$$

because the square of the norm is

$$(11) \qquad \|J_n(k_{n,m}x)\|^2 = \int_0^R xJ_n^2(k_{n,m}x)\,dx = \frac{R^2}{2} J_{n+1}^2(k_{n,m}R)$$

Example (Special Fourier–Bessel Series)

For instance, let us consider $f(x) = 1 - x^2$ and take $R = 1$ and $n = 0$ in the series

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \cdots \qquad (n \text{ fixed}).$$

simply writing λ for

$\alpha_{0,m}$. Then $k_{n,m} = \alpha_{0,m} = \lambda = 2.405, 5.520, 8.654, 11.792$, etc.

Next we calculate the coefficients a_m

$$a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx.$$

This can be integrated by a CAS or by formulas as follows. First use $[xJ_1(\lambda x)]' = \lambda x J_0(\lambda x)$ from Theorem 1 in Sec. 5.4 and then integration by parts,

$$a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx = \frac{2}{J_1^2(\lambda)} \left[\frac{1}{\lambda}(1 - x^2) x J_1(\lambda x) \Big|_0^1 - \frac{1}{\lambda} \int_0^1 x J_1(\lambda x)(-2x)\, dx \right].$$

The integral-free part is zero. The remaining integral can be evaluated by $[x^2 J_2(\lambda x)]' = \lambda x^2 J_1(\lambda x)$ from Theorem 1 in Sec. 5.4. This gives

$$a_m = \frac{4 J_2(\lambda)}{\lambda^2 J_1^2(\lambda)} \qquad (\lambda = \alpha_{0,m}).$$

Numeric values can be obtained from a CAS (or from the table on p. 409 of Ref. [GenRef1] in App. 1, together with the formula $J_2 = 2x^{-1}J_1 - J_0$ in Theorem 1 of Sec. 5.4). This gives the eigenfunction expansion of $1 - x^2$ in terms of Bessel functions J_0, that is,

$$1 - x^2 = 1.1081 J_0(2.405x) - 0.1398 J_0(5.520x) + 0.0455 J_0(8.654x) - 0.0210 J_0(11.792x) + \cdots.$$

A graph would show that the curve of $1 - x^2$ and that of the sum of first three terms practically coincide. ■

Example (Rectangular Wave)

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since f_L is even, $b_n = 0$ for all n.

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}.$$

This sequence of Fourier coefficients is called the **amplitude spectrum** of f_L because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$. Figure 280 shows this spectrum for the periods $2L = 4, 8, 16$. We see that for increasing L these amplitudes become more and more dense on the positive w_n-axis, where $w_n = n\pi/L$. Indeed, for $2L = 4, 8, 16$ we have 1, 3, 7 amplitudes per "half-wave" of the function $(2 \sin w_n)/(Lw_n)$ (dashed in the figure). Hence for $2L = 2^k$ we have $2^{k-1} - 1$ amplitudes per half-wave, so that these amplitudes will eventually be everywhere dense on the positive w_n-axis (and will decrease to zero).

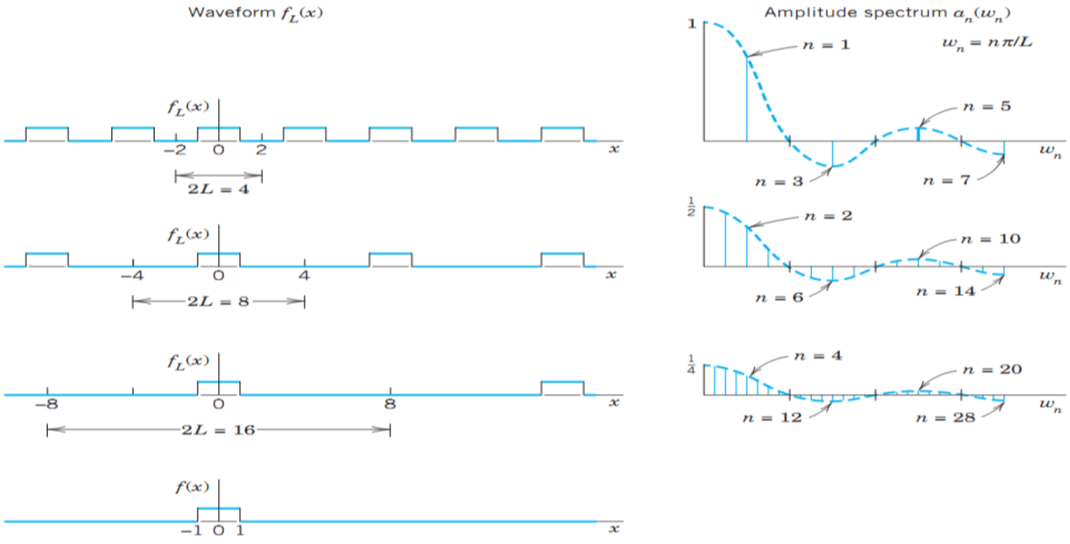


Fig. 280. Waveforms and amplitude spectra in Example 1

Fourier Integral

If $f\ (x)$ is a continuous, piecewise smooth, and absolutely integrable function, then the Fourier Integral of $f\ (x)$ is defined by

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw.$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

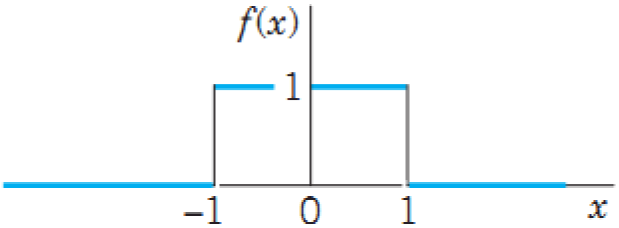
Applications of Fourier Integral

Example

(Single Pulse, Sine Integral. Dirichlet's Discontinuous Factor. Gibbs Phenomenon)

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$



Solution

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv = \frac{1}{\pi}\int_{-1}^{1} \cos wv\, dv = \left.\frac{\sin wv}{\pi w}\right|_{-1}^{1} = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} \sin wv\, dv = 0$$

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw.$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos wx \sin w}{w}\, dw.$$

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $(1 + 0)/2$, that is, $\frac{1}{2}$.

$$\int_0^{\infty} \frac{\cos wx \sin w}{w}\, dw = \begin{cases} \pi/2 & \text{if} \quad 0 \leqq x < 1, \\ \pi/4 & \text{if} \quad x = 1, \\ 0 & \text{if} \quad x > 1. \end{cases}$$

this integral is called Dirichlet's discontinous factor.

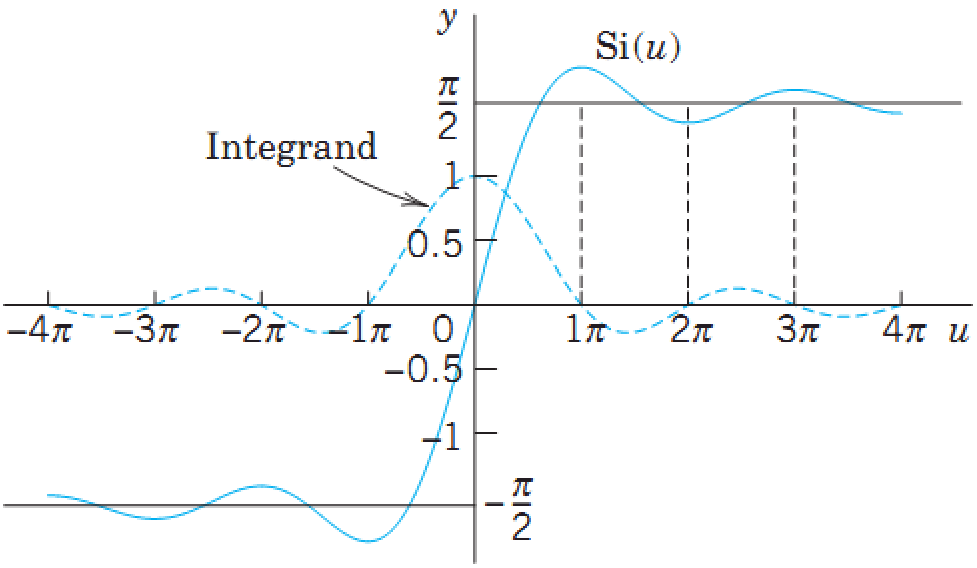
The case $x = 0$ is of particular interest. If $x = 0$, then

$$\int_0^\infty \frac{\sin w}{w}\,dw = \frac{\pi}{2}.$$

We see that this integral is the limit of the so-called sine integral

$$\operatorname{Si}(u) = \int_0^u \frac{\sin w}{w}\,dw$$

as $u \to \infty$. The graphs of $\operatorname{Si}(u)$ and of the integrand are shown in Fig.



Gibbs phenomenon for above example;

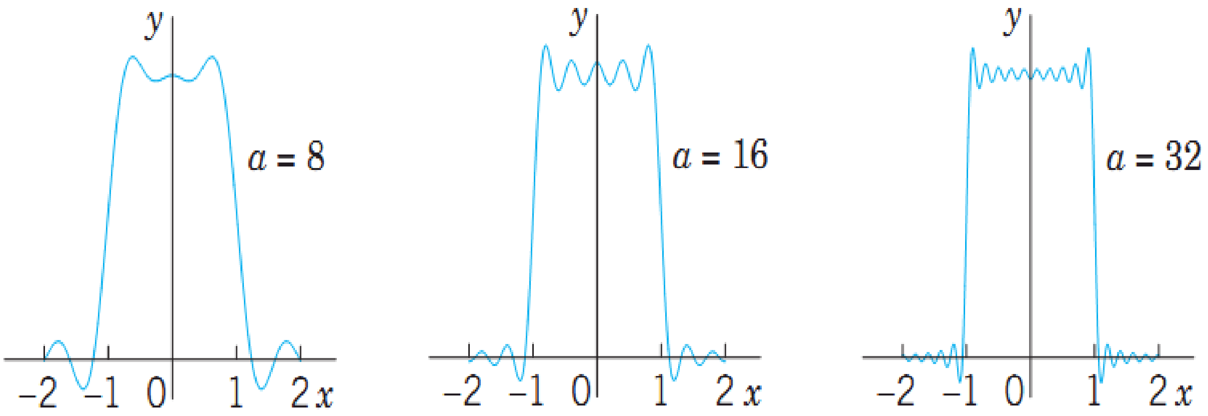


Figure 283 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as a approaches infinity. But this is not true; with increasing a, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series (see Sec. 11.2), is known as the **Gibbs phenomenon**. We can explain it by representing (9) in terms of sine integrals as follows. Using (11) in App. A3.1, we have

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin (w + wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin (w - wx)}{w}\,dw.$$

In the first integral on the right we set $w + wx = t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x + 1)a$. In the last integral we set $w - wx = -t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x - 1)a$. Since $\sin(-t) = -\sin t$, we thus obtain

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

From this and (8) we see that our integral (9) equals

$$\frac{1}{\pi}\operatorname{Si}(a[x + 1]) - \frac{1}{\pi}\operatorname{Si}(a[x - 1])$$

and the oscillations in Fig. 283 result from those in Fig. 282. The increase of a amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1. ■

Fourier cosine integral

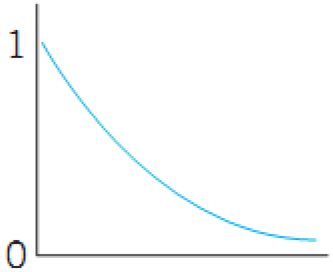
$$f(x) = \int_0^\infty A(w) \cos wx \, dw \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv.$$

Fourier sine integral

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \qquad \text{where} \qquad B(w) = \frac{2}{\pi}\int_0^\infty f(v) \sin wv \, dv.$$

Example (Laplace Integrals)

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ The result will be used to evaluate the so-called Laplace integrals.



Solution

(a) we have $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \cos wv \, dv$. Now, by integration by parts,

$$\int e^{-kv} \cos wv \, dv = -\frac{k}{k^2 + w^2} e^{-kv}\left(-\frac{w}{k} \sin wv + \cos wv\right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches zero because of the exponential factor. Thus $2/\pi$ times the integral from 0 to ∞ gives

$$A(w) = \frac{2k/\pi}{k^2 + w^2}.$$

the Fourier cosine integral representation

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2 + w^2}\,dw \qquad (x > 0, \quad k > 0).$$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2}\,dw = \frac{\pi}{2k}e^{-kx} \qquad (x > 0, \quad k > 0).$$

(b) Similarly, we have $B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\,dv$. By integration by parts,

$$\int e^{-kv}\sin wv\,dv = -\frac{w}{k^2 + w^2}e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

the Fourier sine integral representation

$$B(w) = \frac{2w/\pi}{k^2 + w^2}. \qquad f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\,dw.$$

From this representation we see that

$$\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\,dw = \frac{\pi}{2}e^{-kx} \qquad (x > 0, \quad k > 0).$$

The integrals are called the **Laplace integrals**.

Complex Form of Fourier Integral

The (real) Fourier integral is

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)[\cos wv\cos wx + \sin wv\sin wx]\, dv\, dw.$$

By the addition formula for the cosine the expression in the brackets […] equals cos (wv - wx) or, since the cosine is even, cos (wx - wv). We thus obtain

$$(1^*) \qquad f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right]dw.$$

The integral in brackets is an *even* function of w, call it $F(w)$, because $\cos(wx - wv)$ is an even function of w, the function f does not depend on w, and we integrate with respect to v (not w). Hence the integral of $F(w)$ from $w = 0$ to ∞ is $\frac{1}{2}$ times the integral of $F(w)$ from $-\infty$ to ∞. Thus (note the change of the integration limit!)

$$(1) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv\right]dw.$$

We claim that the integral of the form (1) with sin instead of cos is zero:

$$(2) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\, dv\right]dw = 0.$$

This is true since $\sin(wx - wv)$ is an odd function of w, which makes the integral in brackets an odd function of w, call it $G(w)$. Hence the integral of $G(w)$ from $-\infty$ to ∞ is zero, as claimed.

Using the Euler formula

$$e^{ix} = \cos x + i \sin x.$$

Taking $wx - wv$ instead of x and multiplying by $f(v)$ gives

$$f(v) \cos (wx - wv) + if(v) \sin (wx - wv) = f(v)e^{i(wx-wv)}.$$

Hence the result of adding (1) plus i times (2), called the **complex Fourier integral**, is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)e^{iw(x-v)}\, dv\, dw \qquad (i = \sqrt{-1}).$$

Fourier Transformation

If $f\ (x)$ is a continuous, piecewise smooth, and absolutely integrable function, then the Fourier transform of $f\ (x)$with respect to $x \in R$ is denoted by $F\ (k)$and is defined by

$$\mathcal{F}\ \{f\ (x)\} = \ F\ (k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\ f\ (x)\ \text{dx}$$

where k is called the Fourier transform variable and $exp\ (-ikx)$ is called the kernel of the transform.

Then, for all $x\ \in\ R$, the <u>INVERSE FOURIER TRANSFORM</u> of $F\ (k)$ is defined by

$$\mathcal{F}^{-1}\ \{F\ (k)\} = \ f\ (x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\text{ikx}}\ F\ (k)\ \text{dk}$$

Condition for Existence of Fourier Transformation

Fourier Transformation and Inverse Fourier Transformation exist if

(i) The function $f(x)$ or $F(k)$ is continuous or piecewise continuous over $(-\infty, \infty)$ and bounded.

(ii) The function $f(x)$ or $F(k)$ are absolutely integrable i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ or $\int_{-\infty}^{\infty}|F(k)|dk$ this condition is sufficient for existence of Fourier Transforamtion and Inverse Fourier Transformation.

Example:

Show that for a Guassian Function $\mathcal{F}\left\{Ne^{-\text{ax}^2}\right\} = \frac{N}{\sqrt{2\text{a}}}e^{\left(-\frac{k^2}{4a}\right)}$ $;a > 0\ ,N$ is constant.

Solution. We have, by definition

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\ f\ (x)\text{dx} = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}}\ .\ e^{-\text{ax}^2}\ \text{dx}$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\text{ikx}-\text{ax}^2}\ \text{dx} = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(\text{x}-\frac{\text{ik}}{2a}\right)^2+\frac{\text{k}}{4a^2}^2\right]}\ \text{dx}$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(\text{x}-\frac{\text{ik}}{2a}\right)^2}\ \text{dx}$$

Put $a\left(\text{x}-\frac{\text{ik}}{2a}\right)^2 = \text{P}^2 \Rightarrow \sqrt{a}\left(\text{x}-\frac{\text{ik}}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow \mathcal{F}\ \{f\ (x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(\text{x}-\frac{\text{ik}}{2a}\right)^2}\ \text{dx} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\text{P}^2}\ .\frac{dP}{\sqrt{a}}$$

$$\Rightarrow \mathcal{F}\ \{f\ (x)\} = \frac{Ne^{-\frac{k^2}{4a}}}{\sqrt{2\pi\text{a}}}.\sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-\text{P}^2}\ dP = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\ \{f\ (x)\} = \mathcal{F}\left\{Ne^{-\text{ax}^2}\right\} = \frac{N}{\sqrt{2\text{a}}}e^{\left(-\frac{k^2}{4a}\right)}$$

Consider $\mathbf{ikx - ax^2}$

$$= -\mathbf{a}\left(x^2 - \frac{\text{ikx}}{a}\right)$$

$$= -\mathbf{a}\begin{bmatrix}\left(x^2 - \frac{2\text{ikx}}{2a}\right) \\ + \left(\frac{\text{ik}}{2a}\right)^2 - \left(\frac{\text{ik}}{2a}\right)^2\end{bmatrix}$$

$$= -a\left[\left(\text{x} - \frac{\text{ik}}{2a}\right)^2 + \frac{\text{k}}{4a^2}^2\right]$$

Example: Find the Fourier transform of a box function

$$f(\mathrm{x}) = \begin{cases} 1\ , & |\mathrm{x}| < a \text{ or } -\mathrm{a} < x < a \\ 0 & ,\ |\mathrm{x}| > a \end{cases}$$

Solution. Let we have, by definition

$$\mathcal{F}\{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$$

$$\mathcal{F}\{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx}\ f\ (x)dx + \int_{-a}^{a} e^{ikx}\ f\ (x)dx + \int_{a}^{\infty} e^{ikx}\ f\ (x)dx\right]$$

$$\mathcal{F}\{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx}\ .0dx + \int_{-a}^{a} e^{ikx}\ .1dx + \int_{a}^{\infty} e^{ikx}\ .0dx\right]$$

$$\mathcal{F}\{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ikx}\ dx = \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ika} - e^{-ika}}{2i}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$$

Example:

Find the Fourier transform of $g(\mathrm{x}) = \frac{\mathrm{a}}{x^2+a^2}$

Solution. Let we have, by definition

$$\mathcal{F}\{g(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ g(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ \frac{\mathrm{a}}{x^2+a^2}\mathrm{dx}$$

$$\mathcal{F}\{g(z)\} = \frac{a}{\sqrt{2\pi}}\oint_{c}\frac{e^{\mathrm{ikz}}}{z^2+a^2}\mathrm{dz} \qquad \text{replacing 'x' with 'z'}$$

$\therefore e^{\mathrm{ikz}} = e^{\mathrm{ik}(x+iy)} = e^{\mathrm{ikx}}.e^{\mathrm{i^2ky}} = e^{\mathrm{ikx}}.e^{-\mathrm{ky}} \to 0 \quad as \quad y \to \infty \Rightarrow e^{\mathrm{ikz}} \to 0\ ; k > 0$

Similarly $e^{\mathrm{ikz}} \to 0\ ; k < 0$ when $y \to -\infty$

Let $g(z) = \frac{e^{\mathrm{ikz}}}{z^2+a^2} \Rightarrow z = \pm ai$ are the simple poles of $g(z)$

Now using $R(f,\alpha) = \lim_{s\to\alpha}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-\alpha)^n e^{st}F(s)]$

$$R(g,\alpha i) = R_1 = \lim_{z\to\alpha i}(z-\alpha i)\frac{e^{\mathrm{ikz}}}{(z-\alpha i)(z+\alpha i)} = \lim_{z\to\alpha i}\frac{e^{\mathrm{ikz}}}{(z+\alpha i)} = \frac{e^{\mathrm{ik}(\alpha i)}}{2\alpha i} = \frac{e^{-\alpha\mathrm{k}}}{2\alpha i}$$

Similarly

$$R(g,-\alpha i) = R_2 = \lim_{z\to-\alpha i}(z+\alpha i)\frac{e^{\mathrm{ikz}}}{(z-\alpha i)(z+\alpha i)} = \lim_{z\to\alpha i}\frac{e^{\mathrm{ikz}}}{(z-\alpha i)} = \frac{e^{\mathrm{ik}(-\alpha i)}}{-2\alpha i} = \frac{e^{\alpha\mathrm{k}}}{-2\alpha i}$$

Now $\Rightarrow \mathcal{F}\{g(x)\} = \frac{a}{\sqrt{2\pi}}\oint_{c}\frac{e^{\mathrm{ikz}}}{z^2+a^2}\mathrm{dz} = \frac{a}{\sqrt{2\pi}}.2\pi i\sum_j R_j = \frac{a}{\sqrt{2\pi}}.2\pi i[R_1+R_2]$

Now we use $2\pi i$ for the contour as a semi circle in upper half plane and $-2\pi i$ for the contour as a semi circle in lower half plane

$$\Rightarrow \mathcal{F}\{g(x)\} = \frac{a}{\sqrt{2\pi}}.[(2\pi i)R_1 + (2\pi i)R_2] = \frac{a}{\sqrt{2\pi}}.2\pi i[R_1 - R_2]$$

$$\Rightarrow \mathcal{F}\{g(x)\} = \frac{a}{\sqrt{2\pi}}.2\pi i\left[\frac{e^{-\alpha\mathrm{k}}}{2\alpha i} + \frac{e^{\alpha\mathrm{k}}}{2\alpha i}\right] = \sqrt{\frac{\pi}{2}}\left[e^{-\alpha\mathrm{k}} + e^{\alpha\mathrm{k}}\right]$$

$$\Rightarrow \mathcal{F}\{g(x)\} = \sqrt{\frac{\pi}{2}}\left[e^{\alpha|\mathrm{k}|} + e^{\alpha|\mathrm{k}|}\right] \qquad \therefore k > 0, k < 0 \Rightarrow |\mathrm{k}| = \pm k$$

$$\Rightarrow \mathcal{F}\{g(x)\} = \sqrt{\frac{\pi}{2}}.2e^{\alpha|\mathrm{k}|} = \sqrt{2\pi}e^{\alpha|\mathrm{k}|}$$

Example: Show that $\mathcal{F}\left\{e^{-\mathrm{ax}^2}\right\} = \frac{1}{\sqrt{2\mathrm{a}}}e^{\left(-\frac{k^2}{4a}\right)}$ $\quad ; a > 0$

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}} f(x)\mathrm{dx} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,.\,e^{-\mathrm{ax}^2}\,\mathrm{dx}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}-\mathrm{ax}^2}\,\mathrm{dx} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2+\frac{\mathrm{k}}{4a^2}^{2}\right]}\,\mathrm{dx}$$

$$\mathcal{F}\{f(x)\} == \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2}\,\mathrm{dx}$$

Put $a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2 = \mathrm{P}^2 \Rightarrow \sqrt{a}\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2}\,\mathrm{dx} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{P}^2}\,.\frac{dP}{\sqrt{a}}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\,\pi\,\mathrm{a}}}.\sqrt{\pi} \qquad \therefore \int_{-\infty}^{\infty} e^{-\mathrm{P}^2}\,dP = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \mathcal{F}\left\{e^{-\mathrm{ax}^2}\right\} = \frac{1}{\sqrt{2\mathrm{a}}}e^{\left(-\frac{k^2}{4a}\right)}$$

Consider $\mathbf{ikx - ax^2}$

$$= -\mathbf{a}\left(x^2 - \frac{\mathbf{ikx}}{a}\right)$$

$$= -\mathbf{a}\begin{bmatrix}\left(x^2 - \frac{2\mathbf{ikx}}{2a}\right) \\ +\left(\frac{\mathbf{ik}}{2a}\right)^2 - \left(\frac{\mathbf{ik}}{2a}\right)^2\end{bmatrix}$$

$$= -\boldsymbol{a}\left[\left(\mathbf{x} - \frac{\mathbf{ik}}{2a}\right)^2 + \frac{\mathbf{k}}{4a^2}^{2}\right]$$

Example: Show that $\mathcal{F}\left\{e^{-\mathrm{a}|\mathrm{x}|}\right\} = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}$ $\quad ; a > 0$

Solution. We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}} f(x)\mathrm{dx} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,.\,e^{-\mathrm{a}|\mathrm{x}|}\,\mathrm{dx}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}-\mathrm{a}|\mathrm{x}|}\,\mathrm{dx} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{0} e^{(\mathrm{a}+\mathrm{ik})\mathrm{x}}\,dx + \frac{1}{\sqrt{2\,\pi}}\int_{0}^{\infty} e^{-(\mathrm{a}-\mathrm{ik})\mathrm{x}}\,dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\,\pi}}\left[\frac{1}{\mathrm{a}+\mathrm{ik}} + \frac{1}{\mathrm{a}-\mathrm{ik}}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}$$

Example: Show that $\mathcal{F}\left\{\mathcal{X}_{[-a,a]}(x)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$

where $\mathcal{X}_{[-a,a]}(x) = \mathrm{H}(\mathrm{a}-|\mathrm{x}|) = \begin{cases} 1\ , & |\mathrm{x}| < a \ \text{ or } -\mathrm{a} < x < a \\ 0 & ,\ |\mathrm{x}| > a \end{cases}$

Solution. Let us consider $f(x) = \mathcal{X}_{[-a,a]}(x)$ then We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}} f(x)\mathrm{dx}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx} f(x)dx + \int_{-a}^{a} e^{ikx} f(x)dx + \int_{a}^{\infty} e^{ikx} f(x)dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ikx}\,.0dx + \int_{-a}^{a} e^{ikx}\,.1dx + \int_{a}^{\infty} e^{ikx}\,.0dx\right]$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ikx}\,dx = \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ika} - e^{-ika}}{2i}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{Sinak}{k}\right)$$

Example: Find the Fourier transformation of the function $f(x) = \frac{1}{\sqrt{|x|}}$

Solution

We have, by definition

$$\mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\; f\,(x)dx$$

$$\mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\;\frac{1}{\sqrt{|x|}}dx \qquad \ldots\ldots\ldots(i)$$

$$As\;|x| = \begin{cases} x & ; x > 0 \quad : (0,\infty) \\ -x & ; x < 0 \quad : (-\infty, 0) \end{cases}$$

$$(i) \Rightarrow \mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ikx}\;\frac{1}{\sqrt{|x|}}dx + \int_{0}^{\infty} e^{ikx}\;\frac{1}{\sqrt{|x|}}dx\right]$$

$$\Rightarrow \mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ikx}\;\frac{1}{\sqrt{-x}}dx + \int_{0}^{\infty} e^{ikx}\;\frac{1}{\sqrt{x}}dx\right]$$

for first integral let $x' = -x$ *then* $dx' = -dx$ *with if* $x \to 0, -\infty$ *then* $x' \to 0, \infty$

for second integral let $x = x'$ *then* $dx = dx'$ *with if* $x \to 0, \infty$ *then* $x' \to 0, \infty$

$$\Rightarrow \mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{-ikx'}\;\frac{1}{\sqrt{x'}}(-dx') + \int_{0}^{\infty} e^{ikx'}\;\frac{1}{\sqrt{x'}}dx'\right]$$

$$\Rightarrow \mathcal{F}\,\{f\,(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{-ikx'}\;\frac{1}{\sqrt{x'}}dx' + \int_{0}^{\infty} e^{ikx'}\;\frac{1}{\sqrt{x'}}dx'\right]$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty}\frac{1}{\sqrt{x'}}\left(e^{ikx'} + e^{-ikx'}\right)dx'\right]$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \frac{2}{\sqrt{2\pi}}\left[\int_{0}^{\infty}\frac{1}{\sqrt{x'}}\left(\frac{e^{ikx'} + e^{-ikx'}}{2}\right)dx'\right]$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx'}{\sqrt{x'}}dx'$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \sqrt{\frac{2}{\pi}}Re\int_{0}^{\infty}\frac{e^{-ikx'}}{\sqrt{x'}}dx' \qquad \because e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{ikx'} = \cos kx' + i\sin kx'$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \sqrt{\frac{2}{\pi}}Re\int_{0}^{\infty}\frac{e^{-t}}{\sqrt{\frac{t}{ik}}}\frac{dt}{ik} \qquad \because t = ikx', x' = \frac{t}{ik}$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \sqrt{\frac{2}{\pi}}.\frac{1}{\sqrt{ik}}Re\int_{0}^{\infty}\frac{e^{-t}}{\sqrt{t}}dt = \sqrt{\frac{2}{\pi}}.\frac{1}{\sqrt{ik}}Re\int_{0}^{\infty} e^{-t}\,t^{-\frac{1}{2}}\,dt$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \sqrt{\frac{2}{\pi}}.\frac{1}{\sqrt{ik}}.\sqrt{\pi}$$

$$\Rightarrow \mathcal{F}\,\{f(x)\} = \sqrt{\frac{2}{ik}}$$

By gamma function

$$\because \sqrt{x} = \int_{0}^{\infty} e^{-u}\,u^{x-1}\,du$$

$$\because \Gamma\left(\frac{1}{2}\right) = \int_{0}^{\infty} e^{-u}\,u^{\frac{1}{2}-1}\,du$$

$$\because \sqrt{\pi} = \int_{0}^{\infty} e^{-u}\,u^{-\frac{1}{2}}\,du$$

Example: Find the Fourier transformation of the function $f(x) = \frac{1}{x}$

Solution

We have, by definition

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\ f(x)dx$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\ \frac{1}{x}dx \quad \ldots\ldots\ldots(i)$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ikx}\ \frac{1}{x}dx + \int_{0}^{\infty} e^{ikx}\ \frac{1}{x}dx\right]$$

for first integral let $x = -x'$ *then* $dx = -dx'$ *with if* $x \to 0, -\infty$ *then* $x' \to 0, \infty$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{-ikx'}\ \frac{1}{-x'}(-dx') + \int_{0}^{\infty} e^{ikx}\ \frac{1}{x}dx\right]$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[-\int_{0}^{\infty} e^{-ikx'}\ \frac{1}{x'}dx' + \int_{0}^{\infty} e^{ikx}\ \frac{1}{x}dx\right]$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[-\int_{0}^{\infty} e^{-ikx}\ \frac{1}{x}dx + \int_{0}^{\infty} e^{ikx}\ \frac{1}{x}dx\right]$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left(e^{ikx} - e^{-ikx}\right)\frac{1}{x}dx$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\frac{e^{ikx} - e^{-ikx}}{2i}\right)\frac{1}{x}dx$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{sinkx}{x}dx$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{sinz}{\frac{z}{s}}\frac{dz}{s} \qquad \textit{putting } kx = z$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{sinz}{z}dz$$

$$\Rightarrow \mathcal{F}\{f(x)\} = \frac{2i}{\sqrt{2\pi}}.\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}\,i$$

Example

Find the Fourier transform of $f\ (x) = 1$ if $|x| < 0$ and $f\ (x) = 0$ otherwise

Solution

Let we have, by definition

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} e^{ikx}\ f\ (x)dx + \int_{-1}^{1} e^{ikx}\ f\ (x)dx + \int_{1}^{\infty} e^{ikx}\ f\ (x)dx\right]$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{ikx}\ .1dx$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{ikx}\ dx = \frac{1}{\sqrt{2\pi}}\left|\frac{e^{ikx}}{ik}\right|_{-1}^{1} = \frac{1}{ik\sqrt{2\pi}}\left(e^{ik} - e^{-ik}\right)$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{ik\sqrt{2\pi}}(2iSink) = \frac{2}{k\sqrt{2\pi}}(Sink) = \sqrt{\frac{2}{\pi}}\left(\frac{Sink}{k}\right)$$

Example

Find $\mathcal{F}\ \{e^{-\mathrm{ax}}\}$ of $f\ (x) = e^{-\mathrm{ax}}$ if $x > 0$ and $f\ (x) = 0$ if $x < 0$. Here $a > 0$

Solution

We have, by definition

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,e^{-\mathrm{ax}}\ \mathrm{dx}$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-(\mathrm{a-ik})\mathrm{x}}\ \mathrm{dx} = \frac{1}{\sqrt{2\,\pi}}\int_{0}^{\infty} e^{-(\mathrm{a-ik})\mathrm{x}}\ \mathrm{dx}$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{-(\mathrm{a-ik})\sqrt{2\,\pi}}\left|e^{-(\mathrm{a-ik})\mathrm{x}}\right|_{0}^{\infty} = \frac{1}{-(\mathrm{a-ik})\sqrt{2\,\pi}}\left(e^{-(\mathrm{a-ik})\infty} - e^{-(\mathrm{a-ik})0}\right)$$

$$\mathcal{F}\ \{f\ (x)\} = \frac{1}{-(\mathrm{a-ik})\sqrt{2\,\pi}}$$

Physical Interpretation of Fourier Transform

The Fourier transform provides a way to decompose complex signals or functions into a sum of simpler sinusoidal components, revealing their frequency content. This is particularly useful in physics for understanding waves, where it helps analyze energy distribution across different frequencies. Essentially, the Fourier transform maps a function from one domain (e.g., time or space) to another (e.g., frequency or momentum).

PROPERTIES OF FOURIER TRANSFORMS

Linearity Property: The Fourier Transformation $\mathcal{F}$ Is Linear.

Proof. Let $u\,(x) = af(x) + bg(x)$ where a and b are constants.

We have, by definition

$$\mathcal{F}\,\{u\,(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ \mathrm{u\,(x)dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ [af(x) + bg(x)]\mathrm{dx}$$

$$\mathcal{F}\,\{u\,(x)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f(x)\mathrm{dx} + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ g(x)\mathrm{dx}$$

$$\mathcal{F}\,\{u\,(x)\} = \mathrm{a}\mathcal{F}\,\{f(x)\} + b\mathcal{F}\,\{g(x)\}$$

$\mathcal{F}\,\{af(x) + bg(x)\} = \mathrm{a}\mathcal{F}\,\{f(x)\} + b\mathcal{F}\,\{g(x)\}$ hence proved.

Linearity Property: The Inverse Fourier Transformation $\mathcal{F}^{-1}$ Is Linear.

Proof. Let $U\,(k) = aF(k) + bG(k)$ where a and b are constants.

We have, by definition

$$\mathcal{F}^{-1}\,\{U\,(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ \mathrm{U(k)dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ [aF(k) + bG(k)]\mathrm{dk}$$

$$\mathcal{F}^{-1}\,\{U\,(k)\} = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ F(k)\mathrm{dk} + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ G(k)\mathrm{dk}$$

$\mathcal{F}^{-1}\,\{aF(k) + bG(k)\} = \mathrm{a}\mathcal{F}^{-1}\,\{F(k)\} + \mathrm{b}\mathcal{F}^{-1}\,\{G(k)\}$ hence proved.

Shifting Property: Let $\mathcal{F}\,\{f\ (x)\}$ be a Fourier transform of $f(x)$. Then

- **(i)** $\mathcal{F}\,[f\ (x\ -\ a)] = e^{\mathrm{ika}}\,F(k)$ where 'a' is a real constant.

Proof. From the definition, we have, for a > 0,

$$\mathcal{F}\,[f\ (x\ -\ a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x\ -\ a)\mathrm{dx}$$

Put $x\ - a = x' \Rightarrow dx = dx'$ also as $x \to \pm\infty$ then $x' \to \pm\infty$

$$\mathcal{F}\,[f\ (x\ -\ a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}(x'+a)}\ f\ (x')\mathrm{dx}' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}'}\,.\,e^{\mathrm{ika}}\ f\ (x')\mathrm{dx}'$$

$$\mathcal{F}\,[f\ (x\ - a)] = e^{\mathrm{ika}}\,.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}'}\ f\ (x')\mathrm{dx}' = e^{\mathrm{ika}}\,\mathcal{F}\,\{f\ (x)\} = e^{\mathrm{ika}}\,F(k)$$

- **(ii)** $\mathcal{F}\left[e^{\mathrm{iax}}f\ (x)\right] = F(k + a)$ where 'a' is a real constant.

Proof. From the definition, we have, for a > 0,

$$\mathcal{F}\left[e^{\mathrm{iax}}f\ (x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,e^{\mathrm{iax}}\,f\ (x)\ \mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k+a)\mathrm{x}}\,f\ (x)\ \mathrm{dx}$$

$$\mathcal{F}\left[e^{\mathrm{iax}}f\ (x)\right] = F(k + a)$$

Scaling Property: If $\mathcal{F}$ is the Fourier transform of f, then

$\mathcal{F}\ [f\ (cx)]\ =\ (\frac{1}{|c|})\ F\ (\frac{k}{c})$ where c is a real nonzero constant.

Proof. For $c\ \neq\ 0$ we have $\mathcal{F}\ [f\ (cx)]\ =\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (cx)\mathrm{dx}$

$$\mathcal{F}\ [f\ (cx)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}\left(\frac{x'}{c}\right)}\ f\ (x')\frac{dx'}{c} = \frac{1}{c}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}\left(\frac{x'}{c}\right)}\ f\ (x')dx' = \frac{1}{c}F\ (\frac{k}{c})$$

Since $c\ \neq\ 0$ then either $c <\ 0$ or $c >\ 0$

If $c >\ 0$ then $\mathcal{F}\ [f\ (cx)] = \frac{1}{+c}F\ (\frac{k}{c})$ If $c <\ 0$ then $\mathcal{F}\ [f\ (cx)] = \frac{1}{-c}F\ (\frac{k}{c})$

Hence $\mathcal{F}\ [f\ (cx)]\ =\ (\frac{1}{|c|})\ F\ (\frac{k}{c})$

Conjugation Property: Let f is real then $F\ (-k) = \overline{F\ (k)}$

Proof. Since f is real therefore $f\ (x) = \overline{f\ (x)}$ then by defination

$$F\ (k) = \mathcal{F}\ [f(x)]\ =\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$$

$$\overline{F\ (k)} = \mathcal{F}\ \left[\overline{f\ (x)}\right] =\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ \overline{f\ (x)}\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(-k)\mathrm{x}}\ f\ (x)\mathrm{dx} = F\ (-k)$$

Hence $F\ (-k) = \overline{F\ (k)}$

Attenuation Property:

For a function $f\ (x)$ the result will be , $\mathcal{F}\ [e^{ax}f\ (x)]\ = F\ (k - ai)$

Proof. By definition $F\ (k) = \mathcal{F}\ [f(x)]\ =\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$

Then $\mathcal{F}\ [e^{ax}f\ (x)]\ =\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ e^{ax}f\ (x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ e^{-i^2ax}f\ (x)\mathrm{dx}$

$\mathcal{F}\ [e^{ax}f\ (x)]\ = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k-ai)\mathrm{x}}\ f\ (x)\mathrm{dx}$ ……….(i)

Also $F\ (k - ai) = \mathcal{F}\ [f(x)]\ =\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k-ai)\mathrm{x}}\ f\ (x)\mathrm{dx}$ ……….(ii)

Thus from (i) and (ii) $\mathcal{F}\ [e^{ax}f\ (x)]\ = F\ (k - ai)$

Modulation Property(i):

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2}[F\,(k+a) + F\,(k-a)]$$

Proof. By definition $\mathcal{F}\,[Cosaxf\,(x)] = \mathcal{F}\,[\left(\frac{e^{iax}+e^{-\mathrm{i}ax}}{2}\right)f\,(x)]$

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2}\left[\mathcal{F}\left\{e^{iax}f\,(x)\right\} + \mathcal{F}\left\{e^{-iax}f\,(x)\right\}\right]$$

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2}[F\,(k+a) + F\,(k-a)]$$

Modulation Property (ii):

$$\mathcal{F}\,[Sinaxf\,(x)] = \frac{1}{2i}[F\,(k+a) - F\,(k-a)]$$

Proof. By definition $\mathcal{F}\,[Sinaxf\,(x)] = \mathcal{F}\,[\left(\frac{e^{iax}-e^{-\mathrm{i}ax}}{2i}\right)f\,(x)]$

$$\mathcal{F}\,[Cosaxf\,(x)] = \frac{1}{2i}\left[\mathcal{F}\left\{e^{iax}f\,(x)\right\} - \mathcal{F}\left\{e^{-iax}f\,(x)\right\}\right]$$

$$\mathcal{F}\,[Sinaxf\,(x)] = \frac{1}{2i}[F\,(k+a) - F\,(k-a)]$$

Property: if $f\,(x)$ is real and even then $F\,(k)$ is real.

Proof. Since f is real therefore $f\,(x) = \overline{f\,(x)}$ ....(i) and $f\,(-x) = f\,(x)$ ……(ii) then by defination

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f\,(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f\,(-x)\mathrm{dx}$$

$$F\,(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\mathrm{ikx}'}\,f\,(x')(-dx') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}'}\,\overline{f\,(x')}dx'$$

Hence $F\,(k) = \overline{F\,(k)}$ then $F\,(k)$ is real.

Property: if $f\,(x)$ is real and odd then $F\,(k)$ is pure imaginary.

Proof. Since f is real therefore $f\,(x) = \overline{f\,(x)}$ ………..(i)
and is odd $f\,(-x) = -f\,(x)$ ………(ii) then by defination

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f\,(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(-f\,(-x))\mathrm{dx}$$

$$F\,(k) = \mathcal{F}\,[f(x)] = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f\,(-x)\mathrm{dx}$$

$$F\,(k) = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\mathrm{ikx}'}\,f\,(x')(-dx') = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}'}\,f\,(x')dx'$$

Hence $F\,(k) = -\overline{F\,(k)}$ or $\overline{F\,(k)} = -F\,(k)$ then $F\,(k)$ is pure imaginary.

Property: if $f\ (x)$ is complex then $\mathcal{F}\ \left[\overline{f(-x)}\right] = \overline{F\ (k)}$

Proof. by definition

$$\mathcal{F}\ \left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ \overline{f(-x)}\mathrm{dx} = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{\mathrm{ik}(-x')}\ \overline{f(x')}(-dx')$$

$$\mathcal{F}\ \left[\overline{f(-x)}\right] = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\mathrm{ik}x'}\ \overline{f(x')}dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x'}\ \overline{f(x')}dx'$$

$$\mathcal{F}\ \left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x}\ \overline{f(x)}dx$$ replacing x' with x

$$\mathcal{F}\ \left[\overline{f(-x)}\right] = \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\imath kx}\ f(x)dx} = \overline{F\ (k)}$$

$$\mathcal{F}\ \left[\overline{f(-x)}\right] = \overline{F\ (k)}$$ as required.

DIFFERENTIATION PROPERTY (Higher Derivative Theorem):

Let f be continuous and piecewise smooth in $(-\infty, \infty)$. Let $f(x)$ approach zero as $|x| \to \infty$. If f and f' are absolutely integrable, then

$$\mathcal{F}\ [f'\ (x)]\ =\ (-ik)\mathcal{F}\ [f\ (x)]\ =\ (-ik)F\ (k)$$

Proof.

$$\mathcal{F}\ [f'\ (x)]\ = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f'\ (x)\mathrm{dx}$$

$$\mathcal{F}\ [f'(x)] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{\mathrm{ikx}} f\ (x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(ik)\,f\ (x)\mathrm{dx}\right]$$

$$\mathcal{F}\ [f'(x)] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}\right]$$

$$\mathcal{F}\ [f'\ (x)]\ =\ (-ik)\mathcal{F}\ [f\ (x)]\ =\ (-ik)F\ (k)$$

For n = 2

$$\mathcal{F}\ [f''\ (x)]\ = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f''\ (x)\mathrm{dx}$$

$$\mathcal{F}\ [f''(x)] = \frac{1}{\sqrt{2\pi}}\left[\left|e^{\mathrm{ikx}} f'(x)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,(ik)\,f'(x)\mathrm{dx}\right]$$

$$\mathcal{F}\ [f''(x)] = \frac{1}{\sqrt{2\pi}}\left[0 + (-ik)\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f'(x)\mathrm{dx}\right]$$

$$\mathcal{F}\ [f''(x)] = (-ik)\mathcal{F}\ [f'(x)] = (-ik)(-ik)F\ (k) = (-ik)^2\ F\ (k)$$

This result can be easily extended. If f and its first (n − 1) derivatives are continuous, and if its <u>nth derivative</u> is piecewise continuous, then

$$\mathcal{F}\ [f^n(x)]\ =\ (-ik)^n\mathcal{F}\ [f\ (x)]\ =\ (-ik)^n\ F\ (k)\quad n =\ 0, 1, 2, \ldots\ldots\ldots\ldots\ldots$$

provided f and its derivatives are absolutely integrable. In addition, we assume that f and its first (n − 1) derivatives tend to zero as |x| tends to infinity.

Example

Find the Fourier transform of xe^{-x^2}

Solution

$$\begin{aligned}\mathcal{F}(xe^{-x^2}) &= \mathcal{F}\{-\tfrac{1}{2}(e^{-x^2})'\}\\ &= -\tfrac{1}{2}\mathcal{F}\{(e^{-x^2})'\}\\ &= -\tfrac{1}{2}iw\mathcal{F}(e^{-x^2})\\ &= -\frac{1}{2}iw\frac{1}{\sqrt{2}}e^{-w^2/4}\\ &= -\frac{iw}{2\sqrt{2}}e^{-w^2/4}.\end{aligned}$$

Convolution Function / Faultung Function

The function $(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x - \xi)\, g(\xi)\, d\xi$

is called the convolution of the functions f and g over the interval (−∞,∞)

Note

The convolution satisfies the following properties:

1. $f * g = g * f$ (commutative)
2. $f * (g * h) = (f * g) * h$ (associative)
3. $* (ag + bh) = a(f * g) + b(f * h)$, (distributive)

where a and b are constants.

Property: $\boldsymbol{f * g = g * f}$

Proof: since by definition $(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x - \xi)\, g(\xi)\, d\xi$

Put $x - \xi = \propto \Rightarrow d\xi = -d\propto also\ \xi = x - \propto$ and if $\xi \to \pm\infty$ then $\propto \to \mp\infty$ then

$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(\propto) g(x - \propto)(-d\propto) = g * f$

$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x - \propto) f(\propto)\ (d\propto) = g * f$

Hence $f * g = g * f$

Convolution Theorem

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g).$$

Proof:

By the definition,

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\,g(x-p)\,dp\; e^{-iwx}\,dx.$$

An interchange of the order of integration gives

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\,g(x-p)\,e^{-iwx}\,dx\,dp.$$

Instead of x we now take $x - p = q$ as a new variable of integration. Then $x = p + q$ and

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\,g(q)\,e^{-iw(p+q)}\,dq\,dp.$$

This double integral can be written as a product of two integrals and gives the desired result

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-iwp}\,dp \int_{-\infty}^{\infty} g(q)\,e^{-iwq}\,dq$$

$$= \frac{1}{\sqrt{2\pi}}[\sqrt{2\pi}\,\mathcal{F}(f)][\sqrt{2\pi}\,\mathcal{F}(g)] = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g).$$

Convolution / Faultung Theorem

If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution $(f\ *\ g)$ is the product $F\ (k)G(k)$. That is, $\quad \mathcal{F}\ \{f\ (x) *\ g\ (x)\} =\ F\ (k)G(k)$

Or, equivalently, $\quad \mathcal{F}^{-1}\ \{F\ (k)G(k)\}\ =\ f\ (x)\ *\ g\ (x)$

Or

$$\mathcal{F}^{-1}\ \{F\ (k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ F\ (k)G(k)\mathrm{dk} = (f\ *\ g)\ (x)\ =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\ (x-\xi)\ g\ (\xi)\ d\xi$$

Or

$$\mathcal{F}(f * g) = \sqrt{2\pi}\ \mathcal{F}(f)\,\mathcal{F}(g).$$

Proof: By definition, we have

$$\mathcal{F}^{-1}\ \{F\ (k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ F\ (k)G(k)\mathrm{dk}$$

$$\mathcal{F}^{-1}\ \{F\ (k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\ F(k)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx'}}\ g\ (x')\mathrm{dx'}\right\}\mathrm{dk}$$

By changing the order of integration

$$\mathcal{F}^{-1}\ \{F\ (k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik(x-x')}}\ F(k)\mathrm{dk}\right] g\ (x')\mathrm{dx'}$$

$$\mathcal{F}^{-1}\ \{F\ (k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\ (x-x')g\ (x')dx'$$

$$\mathcal{F}^{-1}\ \{F\ (k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\ (x-\xi)\ g(\xi)\ d\xi = (f * g)(x)$$

Where we replace ξ with x'

Hence $\mathcal{F}^{-1}\ \{F\ (k)G(k)\}\ =\ f\ (x)\ *\ g\ (x)$

Or $\quad \mathcal{F}\ \{f\ (x) *\ g\ (x)\} =\ F\ (k)G(k)$

Or

$$\mathcal{F}(f * g) = \sqrt{2\pi}\ \mathcal{F}(f)\,\mathcal{F}(g).$$

Parseval's Formula of 1st And 2nd Kind

Theorem given by Marc Anotoine des Chenes Parseval (1755 – 1836)

1ST KIND: According to this formula $\int_{-\infty}^{\infty}|f(\mathrm{x})|^2\,dx = \int_{-\infty}^{\infty}|F(\mathrm{k})|^2\,dk$

Proof: The convolution formula gives

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\,F\,(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\,(\xi)g\,(x-\xi)d\xi$

$\int_{-\infty}^{\infty} f\,(\xi)g\,(x-\xi)\,d\xi = \int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\,F\,(k)G(k)\mathrm{dk}$

which is, by putting $x = 0$ $\quad \int_{-\infty}^{\infty} f\,(\xi)g\,(-\xi)\,d\xi = \int_{-\infty}^{\infty} F\,(k)G(k)\mathrm{dk}$

$\int_{-\infty}^{\infty} f\,(x)g\,(-x)\,dx = \int_{-\infty}^{\infty} F\,(k)G(k)\mathrm{dk}$

Putting $g\,(-x) = \overline{f\,(x)}$ then $g\,(x) = \overline{f\,(-x)} \Rightarrow \mathcal{F}\,\{g\,(x)\} = \mathcal{F}\,\{\overline{f\,(-x)}\}$

$\Rightarrow G(k) = \overline{F\,(k)}$ $\quad \therefore \mathcal{F}\,\{\overline{f\,(-x)}\} = \overline{F\,(k)}$ for complex f.

$\int_{-\infty}^{\infty} f\,(x)\overline{f\,(x)}\,dx = \int_{-\infty}^{\infty} F\,(k)\overline{F\,(k)}\mathrm{dk}$

where the bar denotes the complex conjugate.

$\Rightarrow \int_{-\infty}^{\infty}|f(\mathrm{x})|^2\,dx = \int_{-\infty}^{\infty}|F(\mathrm{k})|^2\,dk$

In terms of the notation of the norm, this is $\|f\| = \|F\|$

2ND KIND: According to this formula $\int_{-\infty}^{\infty} F\,(k)G(k)\,dk = \int_{-\infty}^{\infty} f\,(u)g\,(-u)\,du$

PROOF: The convolution formula gives

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\,F\,(k)G(k)\mathrm{dk} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\,(u)g\,(x-u)du$

by putting $x = 0$ we get $\int_{-\infty}^{\infty} F\,(k)G(k)\,\mathrm{dk} = \int_{-\infty}^{\infty} f\,(u)g\,(-u)du$

Boundedness and Continuity of Fourier Transformation

If $f(x)$ is piecewise smooth and absolutely integrable function on the interval $(-\infty,\infty)$ then its fourier transformation $F(k)$ is bounded and continuous.

Proof: given that $f(x)$ is piecewise smooth and absolutely integrable function i.e.

$J = \int_{-\infty}^{\infty}|f(x)|dx$

now by definition $\quad F\,(k) = \mathcal{F}\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f\,(x)\mathrm{dx}$

For boundedness taking mod on both sides

$\Rightarrow |F\,(k)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\,f\,(x)\mathrm{dx}\right| \le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|e^{\mathrm{ikx}}\right||f\,(x)|\mathrm{dx}$

$\Rightarrow |F\,(k)| \le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f\,(x)|\,\mathrm{dx}$ $\quad$ since $\left|e^{\mathrm{ikx}}\right| = 1$

$\Rightarrow |F\,(k)| \le \frac{1}{\sqrt{2\pi}}.J$ $\quad$ since $J = \int_{-\infty}^{\infty}|f(x)|dx$

$\Rightarrow |F\,(k)| \le \lambda$ $\quad$ where $\lambda = \frac{1}{\sqrt{2\pi}}.J \in R$

$\Rightarrow F\ (k)$ is bounded.

Now for continuity of $F\ (k)$ we have

$F\ (k+h)-F\ (k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(k+h)\mathrm{x}}\ f\ (x)\mathrm{dx}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$

$F\ (k+h)-F\ (k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\left(e^{\mathrm{i}h\mathrm{x}}-1\right)f(x)\mathrm{dx}=\mathrm{I}(k,h)$ say

$\lim_{h\to 0}[F\ (k+h)-F\ (k)]=\lim_{h\to 0}\mathrm{I}(k,h)$ ……….(i)

Now $\lim_{h\to 0}\mathrm{I}(k,h)$ exists if $\mathrm{I}(k,h)$ is uniformly convergent.

For this consider

$\Rightarrow |\mathrm{I}(k,h)|=\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\left(e^{\mathrm{i}h\mathrm{x}}-1\right)f(x)\mathrm{dx}\right|$

$\Rightarrow |\mathrm{I}(k,h)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|e^{\mathrm{ikx}}\right|\left|e^{\mathrm{i}h\mathrm{x}}-1\right||f\ (x)|\mathrm{dx}$

$\Rightarrow |\mathrm{I}(k,h)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1)\,|Coshx+iSinhx-1||f\ (x)|\mathrm{dx}$

$\Rightarrow |\mathrm{I}(k,h)|\le\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|(Coshx-1)+iSinhx||f\ (x)|\,\mathrm{dx}$

$\Rightarrow |\mathrm{I}(k,h)|\le\frac{1}{\sqrt{2\pi}}.\sqrt{2}\int_{-\infty}^{\infty}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}$

$\Rightarrow |\mathrm{I}(k,h)|\le\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}$

$\Rightarrow \lim_{h\to 0}|\mathrm{I}(k,h)|\le\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{h\to 0}\sqrt{1-Coshx}|f\ (x)|\,\mathrm{dx}\quad\to 0$

$\Rightarrow \lim_{h\to 0}|\mathrm{I}(k,h)|\le 0\Rightarrow\lim_{h\to 0}\mathrm{I}(k,h)=0$

$(i)\Rightarrow\lim_{h\to 0}[F\ (k+h)-F\ (k)]=0$

$\Rightarrow\lim_{h\to 0}F\ (k+h)=F\ (k)\Rightarrow F\ (k)$ is continuous.

Hence If $f(x)$ is piecewise smooth and absolutely integrable function on the interval $(-\infty,\infty)$ then its fourier transformation $F(k)$ is bounded and continuous.

Riemann Lebesque Theorem

If $f(x)$ is piecewise smooth and absolutely integrable function then

$\lim_{|\mathrm{k}|\to\infty}F\ (k)=0$

Proof: given that $f(x)$ is piecewise smooth and absolutely integrable function i.e.

$J=\int_{-\infty}^{\infty}|f(x)|dx$

now by definition $F\ (k)=\mathcal{F}\ [f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\ f\ (x)\mathrm{dx}$

$F\ (k)=\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]$

$\Rightarrow |F\ (k)|=\left|\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]\right|$

$\Rightarrow |F\ (k)|=\left|\frac{1}{\sqrt{2\pi}}\left[\left|f(x)\frac{e^{\mathrm{ikx}}}{ik}\right|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{e^{\mathrm{ikx}}}{ik}f'(x)\mathrm{dx}\right]\right|$

$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left|\left.|f(x)|\frac{|e^{\text{ikx}}|}{|ik|}\right|_{-\infty}^{\infty} + \left|-\int_{-\infty}^{\infty}\frac{e^{\text{ikx}}}{ik}f'(x)\text{dx}\right|\right.$

$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right] + \int_{-\infty}^{\infty}\frac{|e^{\text{ikx}}|}{|ik|}|f'(x)|\text{dx}$

$\Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}\left[\lim_{x\to\infty}\frac{|f(x)|}{|k|} - \lim_{x\to-\infty}\frac{|f(x)|}{|k|}\right] + \int_{-\infty}^{\infty}\frac{1}{|k|}|f'(x)|\text{dx}$ ……..(i)

Since $f(x)$ is absolutely integrable function then $\lim_{x\to\pm\infty}|f(x)| = 0$

$(i) \Rightarrow |F\ (k)| \leq \frac{1}{\sqrt{2\pi}}.\frac{1}{|k|}\int_{-\infty}^{\infty}|f'(x)|\ \text{dx}$ ……..(ii)

Since $f(x)$ is piecewise smooth then $f'(x)$ will be piecewise continuous and therefore $\int_{-\infty}^{\infty}|f'(x)|\,dx = I$

$(ii) \Rightarrow \lim_{|\text{k}|\to\infty}|F\ (k)| \leq \lim_{|\text{k}|\to\infty}\frac{1}{\sqrt{2\pi}}.\frac{1}{|k|}.I = 0 \Rightarrow \lim_{|\text{k}|\to\infty}|F\ (k)| = 0$

Fourier Transform of the Function of the Form $[x^n f(x)]$

Let f be piecewise continuous on the interval $[-l, l]$ for every positive 'l' and $\int_{-\infty}^{\infty}|x^n f(x)|$ converges then

$\mathcal{F}\ [x^n f(x)] = \frac{1}{i^n}F^n(k) = i^{-n}F^n(k)$; $n = 0, 1, 2, \ldots$..............

Proof. By definition $\mathcal{F}\ [f\ (x)] = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\text{ikx}}\ f(x)\text{dx}$

$\Rightarrow F'(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\text{ikx}}\ (ix)f(x)\text{dx}$ diff. w.r.to 'k'

$\Rightarrow i^{-1}F'(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\text{ikx}}\ (x)f(x)\text{dx} = \mathcal{F}\ [xf(x)] = i^{-1}F^1(k)$

$\Rightarrow F''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\text{ikx}}\ (ix)^2f(x)\text{dx}$ again diff. w.r.to 'k'

$\Rightarrow i^{-2}F''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\text{ikx}}\ (x^2)f(x)\text{dx} = \mathcal{F}\ [x^2f(x)] = i^{-2}F^2(k)$

Continuing in this manner we can get the required result as follows;

$\mathcal{F}\ [x^n f(x)] = i^{-n}F^n(k) = \frac{1}{i^n}F^n(k)$; $n = 0, 1, 2, \ldots$..............

$\mathcal{F}\ [x^n f(x)] = (-i)^n\frac{d^n}{dk^n}F(k)$; $n = 0, 1, 2, \ldots$..............

Where we use the result $i^{-n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i}\times\frac{i}{i}\right)^n = \left(\frac{i}{i^2}\right)^n = (-i)^n$

Fourier Transform of an Integral

Let f be piecewise continuous on the interval $(-\infty,\infty)$ and that $\int_{-\infty}^{\infty}|f(x)| < \infty$ also $F(0) = 0$ with $\mathcal{F}\,[f\,(x)] = F(k)$ then

$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,\mathrm{dx}'\right\} = \frac{1}{-ik}F(k) = \frac{i}{k}F(k)$

Proof. Let $g(x) = \int_{-\infty}^{x} f(x')\,\mathrm{dx}'$ ……………..(i)

Given that $\mathcal{F}\,[f\,(x)] = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, f(x)\mathrm{dx}$

$\Rightarrow F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ putting k = 0 also $e^{0} = 1$

$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{dx} = 0$ since $F(0) = 0$

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\mathrm{dx} = 0 \Rightarrow \lim_{x\to\infty}\int_{-\infty}^{x} f(x')\,\mathrm{dx}' = 0 \Rightarrow \lim_{x\to\infty} g(x) = 0$

Now from (i) we get by using Leibniz Rule

$g'(x) = f(x') \Rightarrow \mathcal{F}\,\{g'(x)\} = \mathcal{F}\,\{f(x')\} \Rightarrow (-ik)\mathcal{F}\,\{g(x)\} = F(k)$

$\Rightarrow \mathcal{F}\,\{g(x)\} = \frac{1}{-i\mathrm{k}}F(k)$

$\Rightarrow \mathcal{F}\,\{g(x)\} = \mathcal{F}\left\{\int_{-\infty}^{x} f(x')\,\mathrm{dx}'\right\} = \frac{1}{-ik}F(k) = \frac{i}{\mathrm{k}}F(k)$

Fourier Integral Theorem

If $f\,(x)$ is real valued function over $(-\infty,+\infty)$ and the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ is absolutely convergent then $f\,(x) = \frac{1}{\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

PROOF: Since $\int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$ is absolutely convergent then F.T and I.F.T of function exists.

$f\,(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}$ since $\mathcal{F}^{-1}\,\{F\,(k)\} = f\,(x)$

$f\,(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk} + \int_{0}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}\right]$ …………(i)

Put in 1st term $-k = k' \Rightarrow dk = -dk'$ also if $k \to -\infty, 0$ then $k' \to \infty, 0$

$(i) \Rightarrow f\,(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{\mathrm{i}k'\mathrm{x}}\, F(-k')(-dk') + \int_{0}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}\right]$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{\mathrm{i}k'\mathrm{x}}\, F(-k')dk' + \int_{0}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}\right]$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{\mathrm{ikx}}\, F(-k)dk + \int_{0}^{\infty} e^{-\mathrm{ikx}}\, F(k)\mathrm{dk}\right]$ replacing k' with k

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left[e^{\mathrm{ikx}}\,\overline{F(k)} + e^{-\mathrm{ikx}}\,F(k)\right]\mathrm{dk}$ …………(ii) $\therefore F(-k) = \overline{F(k)}$

Consider $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ik}x'}\, f(x')\mathrm{dx}'$

$\Rightarrow \overline{F(k)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ik}x'}\,\overline{f(x')}\mathrm{dx}'$ taking conjugate

Then $e^{-\mathrm{ikx}}F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\mathrm{ik}(x-x')}\,f(x')\mathrm{dx}'$

Also $e^{\mathrm{ikx}}\overline{F(k)}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mathrm{ik}(x-x')}\,\overline{f(x')}\mathrm{dx}'$

Since $f(x)$ is real therefore $\overline{f(x')}=f(x')$

Now $e^{\mathrm{ikx}}\overline{F(k)}+e^{-\mathrm{ikx}}F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{\mathrm{ik}(x-x')}+e^{-\mathrm{ik}(x-x')}\right]f(x')\mathrm{dx}'$

$e^{\mathrm{ikx}}\overline{F(k)}+e^{-\mathrm{ikx}}F(k)=\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}\left[e^{\mathrm{ik}(x-x')}+e^{-\mathrm{ik}(x-x')}\right]f(x')\mathrm{dx}'$

$e^{\mathrm{ikx}}\overline{F(k)}+e^{-\mathrm{ikx}}F(k)=\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}Cos\mathrm{k}(x-x')\,f(x')\mathrm{dx}'$

$(ii)\Rightarrow f(x)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}Cos\mathrm{k}(x-x')\,f(x')\mathrm{dx}'\,\mathrm{dk}$

$f\,(x)=\frac{2}{2\pi}\int_{0}^{\infty}dk\int_{-\infty}^{\infty}Cosk(x-x')f(x')dx'$

$f\,(x)=\frac{1}{\pi}\int_{0}^{\infty}dk\int_{-\infty}^{\infty}Cosk(x-x')f(x')dx'$ as required.

The Fourier Transforms of Step and Impulse Functions

The Heaviside unit step function is defined by

$H\,(x\,-\,a)\,=\begin{cases}0 & \mathrm{x}<\mathrm{a}\\ 1 & \mathrm{x}\geq\mathrm{a}\end{cases}$ where a ≥ 0

The Fourier transform of the Heaviside unit step function can be easily determined. We consider first

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mathrm{ikx}}\,H\,(x\,-\,a)\mathrm{dx}$

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{\mathrm{ikx}}\,H\,(x\,-\,a)\mathrm{dx}+\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{\mathrm{ikx}}\,H\,(x\,-\,a)\mathrm{dx}$

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{\mathrm{ikx}}\,.0\mathrm{dx}+\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{\mathrm{ikx}}\,.1\mathrm{dx}=\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{\mathrm{ikx}}\,\mathrm{dx}$

This integral does not exist. However, we can prove the existence of this integral by defining a new function

$$H\,(x\,-\,a)e^{-\alpha\mathrm{x}}\;=\begin{cases}0 & \mathrm{x}<\mathrm{a}\\ e^{-\alpha\mathrm{x}} & \mathrm{x}\geq\mathrm{a}\end{cases}$$

This is evidently the unit step function as $\alpha\;\rightarrow\;0$. Thus, we find the Fourier transform of the unit step function as

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\lim_{\alpha\rightarrow0}\mathcal{F}\,[H\,(x\,-\,a)e^{-\alpha\mathrm{x}}]$

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\lim_{\alpha\rightarrow0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mathrm{ikx}}\,H\,(x\,-\,a)e^{-\alpha x}\mathrm{dx}$

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\lim_{\alpha\rightarrow0}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{\mathrm{ikx}}\,e^{-\alpha x}\mathrm{dx}=\lim_{\alpha\rightarrow0}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{\mathrm{i(k-\alpha)x}}\,\mathrm{dx}$

$\mathcal{F}\,[H\,(x\,-\,a)]\,=\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{\mathrm{ikx}}\,\mathrm{dx}=\frac{e^{\mathrm{ika}}}{\sqrt{2\pi}ik}$ For a = 0 ⇒ $\mathcal{F}\,[H\,(x\,)]\,=\frac{1}{\sqrt{2\pi}ik}$

An impulse function is defined by

$$p\,(x\,) = \begin{cases} \mathrm{h} & \mathrm{a} - \varepsilon < \mathrm{x} < \mathrm{a} + \varepsilon \\ 0 & \mathrm{x} \leq \mathrm{a} - \varepsilon \text{ or } \mathrm{x} \geq \mathrm{a} + \varepsilon \end{cases}$$

where h is large and positive, $a > 0$, and ε is a small positive constant, This type of function appears in practical applications; for instance, a force of large magnitude may act over a very short period of time.

The Fourier transform of the impulse function is

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{ikx}}\, p\,(x\,)\mathrm{dx}$$

$$\mathcal{F}\,[p\,(x\,)\,] =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\mathrm{a}-\varepsilon} e^{\mathrm{ikx}}\, p\,(x\,)\mathrm{dx} + \frac{1}{\sqrt{2\pi}}\int_{\mathrm{a}-\varepsilon}^{\mathrm{a}+\varepsilon} e^{\mathrm{ikx}}\, p\,(x\,)\mathrm{dx} + \frac{1}{\sqrt{2\pi}}\int_{\mathrm{a}+\varepsilon}^{\infty} e^{\mathrm{ikx}}\, p\,(x\,)\mathrm{dx}$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{1}{\sqrt{2\pi}}\int_{\mathrm{a}-\varepsilon}^{\mathrm{a}+\varepsilon} h e^{\mathrm{ikx}}\,\mathrm{dx} = \frac{h}{\sqrt{2\pi}}\left|\frac{e^{\mathrm{ikx}}}{ik}\right|_{\mathrm{a}-\varepsilon}^{\mathrm{a}-\varepsilon}$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{h}{\sqrt{2\pi}}\cdot\frac{1}{ik}\left(e^{\mathrm{ik(a+\varepsilon)}} - e^{\mathrm{ik(a-\varepsilon)}}\right)$$

$$\mathcal{F}\,[p\,(x\,)\,] = \frac{h}{\sqrt{2\pi}}\cdot\frac{e^{\mathrm{ika}}}{ik}\left(e^{\mathrm{ik\varepsilon}} - e^{-\mathrm{ik\varepsilon}}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{\mathrm{ika}}\left(\frac{e^{\mathrm{ik\varepsilon}} - e^{-\mathrm{ik\varepsilon}}}{2ik\varepsilon}\right) = \frac{2h\varepsilon}{\sqrt{2\pi}}e^{\mathrm{ika}}\left(\frac{Sink\varepsilon}{k\varepsilon}\right)$$

Now if we choose the value of $h = (\frac{1}{2\varepsilon})$ then the impulse defined by

$$I\,(\varepsilon) = \int_{-\infty}^{\infty} p\,(x\,)\mathrm{dx} = \int_{\mathrm{a}-\varepsilon}^{\mathrm{a}+\varepsilon}\frac{1}{2\varepsilon}\mathrm{dx} = 1$$

which is a constant independent of ε. In the limit as $\varepsilon \to 0$, this particular function $p_{\varepsilon}\,(x\,)$ with h = (1/2ε) satisfies $\lim_{\varepsilon \to 0} p_{\varepsilon}\,(x\,) = 0\ ; x \neq 0$ and $\lim_{\varepsilon \to 0} I\,(\varepsilon) = 1$

Thus, we arrive at the result $\delta\,(x - a) = 0, x \neq a$, and $\int_{-\infty}^{\infty}\delta(x - a)\mathrm{dx} = 1$

This is the Dirac delta function

We now define the Fourier transform of $\delta(x)$ as the limit of the transform of $p_{\varepsilon}\,(x\,)$. We then consider

$$\mathcal{F}\,[\delta\,(x - a)\,] = \lim_{\varepsilon \to 0}\mathcal{F}\,[p_{\varepsilon}\,(x\,)\,] = \lim_{\varepsilon \to 0}\frac{e^{\mathrm{ika}}}{\sqrt{2\pi}}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) = \frac{e^{\mathrm{ika}}}{\sqrt{2\pi}}$$

in which we note that, by L'Hospital's rule, $\lim_{\varepsilon \to 0}\left(\frac{Sink\varepsilon}{k\varepsilon}\right) = 1$

When a = 0, we obtain $\mathcal{F}\,[\delta\,(x)\,] = \frac{1}{\sqrt{2\pi}}$

Error Function:

The error function is defined by $erf\ (x)\ = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2}\ d\eta$

This is a widely used and tabulated function.

Example: Slowing-down of Neutrons Consider the following physical problem

$u_t = u_{xx} + \delta\ (x)\ \delta\ (t)$

$u\ (x, 0)\ =\ \delta\ (x)\ ;\ lim_{|x|\to\infty}\ u\ (x, t)\ =\ 0$

This is the problem of an infinite medium which slows neutrons, in which a source of neutrons is located. Here u (x, t) represents the number of neutrons per unit volume per unit time and $\delta(x)\ \delta(t)$ represents the source function.

Solution: The Fourier transformation of equation yields

$U_t + k^2U = \frac{1}{\sqrt{2\pi}} \delta\ (t)$ $\quad\therefore \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ or $\mathcal{F}\ \{\delta(x)\} = \frac{1}{\sqrt{2\pi}}$

The solution of this, after applying the condition

$U(k, 0) = \frac{1}{\sqrt{2\pi}}$ is $U(k, t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}$

Hence, the inverse Fourier transform gives the solution of the problem

$u\ (x, t)\ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\ e^{-k^2 t - \mathrm{ikx}} dk = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$

Fourier Cosine Transformation and Inverse

Let $f\ (x)$ be defined for $0\ \le\ x\ <\ \infty$, and extended as an even function in $(-\infty, \infty)$ satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the Fourier cosine transform of $f(x)$ and its inverse transform are defined by

$$\mathcal{F}_C\ \{f\ (x)\} =\ F_c\ (k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f\ (x)\ Coskxdx$$

$$\mathcal{F}^{-1}{}_C\ \{F_c\ (k)\} = f\ (x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c\ (k)\ Coskxdk$$

Fourier Sine Transformation and Inverse

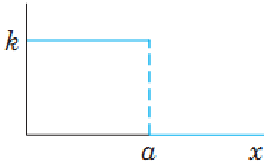
Let $f\ (x)$ be defined for $0\ \le\ x\ <\ \infty$, and extended as an odd function in $(-\infty, \infty)$ satisfying the conditions of Fourier Integral formula. Then, at the points of continuity, the Fourier sine transform of $f(x)$ and its inverse transform are defined by

$$\mathcal{F}_s\ \{f\ (x)\} =\ F_s\ (k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f\ (x)\ Sinkxdx$$

Example:

Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$$



Solution:

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin wx\,dx,$$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\,k\int_0^{a}\cos wx\,dx = \sqrt{\frac{2}{\pi}}\,k\left(\frac{\sin aw}{w}\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\,k\int_0^{a}\sin wx\,dx = \sqrt{\frac{2}{\pi}}\,k\left(\frac{1-\cos aw}{w}\right).$$

Example:

Show that $\mathcal{F}_C\,\{e^{-x}\} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+k^2}\right)$

Solution: We have, by definition

$$\mathcal{F}_C\,\{f\,(x)\} = F_c\,(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f\,(x)\,Coskxdx$$

$$\mathcal{F}_C\,\{e^{-x}\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[e^{-(-ik)x} + e^{-(ik)x}\right]dx$$

$$\mathcal{F}_C\,\{e^{-x}\} = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{1-ik} + \frac{1}{1+ik}\right]dx$$

$$\mathcal{F}_C\,\{e^{-x}\} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+k^2}\right)$$

Example:

Show that $\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,Coskxdx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-(a-ik)x} + e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \frac{1}{2}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} + \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right) \quad ; a > 0$$

Example:

Show that $\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,Sinkxdx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)dx = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\int_0^\infty\left[e^{-(a-ik)x} - e^{-(a+ik)x}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \frac{1}{2i}.\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} - \frac{1}{a+ik}\right]dx$$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{k}{a^2+k^2}\right) \quad ; a > 0$$

Example:

Show that $\mathcal{F}_s^{\,-1}\left\{\frac{1}{k}e^{-sk}\right\} = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{x}{s}\right)$

Solution: To prove this we use the standard definite integral

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\,Sinkxdk$$

$$\mathcal{F}_s^{\,-1}\left\{\frac{1}{k}e^{-sk}\right\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}e^{-sk}\,Sinkxdk$$

$$f'(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-sk}}{k}kcoskxdk = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-sk}\,coskxdk$$

Now using formula $\int e^{ax}Cosbxdx = \frac{e^{ax}}{a^2+b^2}[aCosbx + bSinbx]$

$$f'(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-sk}\,coskxdk = \left|\frac{e^{-sk}}{s^2+x^2}[-sCoskx + xSinkx]\right|_0^\infty = \frac{s}{s^2+x^2}$$

Then $\mathcal{F}_s{}^{-1}\{e^{-sk}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k} e^{-sk}\, Sinkxdk = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+x^2}$

Integrating both sides we have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k} e^{-sk}\, Sinkxdk \;=\; \sqrt{\frac{2}{\pi}}\int \frac{sds}{s^2+x^2} + C = \sqrt{\frac{2}{\pi}}\left|s.\frac{1}{s}tan^{-1}\left(\frac{x}{s}\right)\right| + C$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k} e^{-sk}\, Sinkxdk \;=\; \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{x}{s}\right) + C$$

Using $f(0) = 0$ we have $C = 0$

Consequently

$$\mathcal{F}_s{}^{-1}\left\{\frac{1}{k}e^{-sk}\right\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-sk}}{k} Sinkxdk = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{x}{s}\right)$$

Example:

Show that $\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(a^2+k^2)^2}$; $a > 0$

Solution: We have, by definition

$$\mathcal{F}_C\{f\,(x)\} = F_c\,(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,Coskxdx$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\,Coskxdx$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\left|x\left(\int e^{-ax}Coskxdx\right)\right|_0^\infty - \int_0^\infty\left(\int e^{-ax}Coskxdx\right)dx\right] \text{ ……..(i)}$$

Now using formula $\int e^{ax}Cosbxdx = \frac{e^{ax}}{a^2+b^2}[aCosbx + bSinbx]$ one becomes

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\Bigg[\left|x.\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty - \int_0^\infty\Bigg(\frac{e^{-ax}}{a^2+k^2}[-aCoskx +$$

$$kSinkx]\Bigg)dx\Bigg]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[(0-0) + \frac{a}{a^2+k^2}\int_0^\infty e^{-ax}Coskxdx - \frac{k}{a^2+k^2}\int_0^\infty e^{-ax}Sinkxdx\right]$$

$$= \sqrt{\frac{2}{\pi}}\Bigg[\frac{a}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty - \frac{k}{a^2+k^2}\Bigg|\frac{e^{-ax}}{a^2+k^2}[-aSinkx -$$

$$kCoskx]\Bigg|_0^\infty\Bigg]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left\{0-\left(\frac{-a}{a^2+k^2}\right)\right\} - \frac{k}{a^2+k^2}\left\{0-\left(\frac{-k}{a^2+k^2}\right)\right\}\right]$$

$$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2}\right]$$

$\mathcal{F}_C\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-k^2}{(a^2+k^2)^2}\right]$ as required.

Example:

Show that $\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2ak}{(a^2+k^2)^2}$ $\quad ;\ a > 0$

Solution: We have, by definition

$$\mathcal{F}_s\{f\,(x)\} = F_s\,(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,Sinkxdx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\,Sinkxdx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\left|x\left(\int e^{-ax}Sinkxdx\right)\right|_0^\infty - \int_0^\infty\left(\int e^{-ax}Sinkxdx\right)dx\right] \text{........(i)}$$

Now using formula $\int e^{ax}Sinbxdx = \frac{e^{ax}}{a^2+b^2}[aSinbx - bCosbx]$ one becomes

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\left|x.\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty - \int_0^\infty\left(\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right)dx\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[(0-0) + \frac{a}{a^2+k^2}\int_0^\infty e^{-ax}Sinkxdx + \frac{k}{a^2+k^2}\int_0^\infty e^{-ax}Coskxdx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aSinkx - kCoskx]\right|_0^\infty + \frac{k}{a^2+k^2}\left|\frac{e^{-ax}}{a^2+k^2}[-aCoskx + kSinkx]\right|_0^\infty\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\left\{0 - \left(\frac{-k}{a^2+k^2}\right)\right\} + \frac{k}{a^2+k^2}\left\{0 - \left(\frac{-a}{a^2+k^2}\right)\right\}\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{ak}{(a^2+k^2)^2} + \frac{ak}{(a^2+k^2)^2}\right]$$

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2ak}{(a^2+k^2)^2} \quad ;\ a > 0$$

as required.

Example:

Calculate Fourier Sine Transform of the function $f\,(x) = e^{-x}Cosx$

Solution: We have, by definition

$$\mathcal{F}_s\{f\,(x)\} = F_s\,(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f\,(x)\,Sinkxdx$$

$$\mathcal{F}_s\{e^{-x}Cosx\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}Cosx\,Sinkxdx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\,(2SinkxCosx)dx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\,[Sin(kx+x) + Sin(kx-x)]dx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\,Sin(k+1)xdx + \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\,Sin(k-1)xdx$$

$$\mathcal{F}_s\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}I_1 + \frac{1}{\sqrt{2\pi}}I_2 \qquad \text{................(i)}$$

Now using formula $\int e^{ax} Sinbxdx = \frac{e^{ax}}{a^2+b^2}[aSinbx - bCosbx]$

$$I_1 = \int_0^\infty e^{-x}\, Sin(k+1)xdx = \left|\frac{e^{-x}}{(-1)^2+(k+1)^2}[(-1)Sin(k+1)x - (k+1)Cos(k+1)x]\right|_0^\infty$$

$$I_1 = \left[0 - \frac{e^0}{1+(k+1)^2}\{-0-(k+1)(1)\}\right] = \frac{1}{1+k^2+2k+1}(k+1) = \frac{(k+1)}{k^2+2k+2}$$

Similarly

$$I_2 = \int_0^\infty e^{-x}\, Sin(k-1)xdx = \left|\frac{e^{-x}}{(-1)^2+(k-1)^2}[(-1)Sin(k-1)x - (k-1)Cos(k-1)x]\right|_0^\infty$$

$$I_2 = \left[0 - \frac{e^0}{1+(k-1)^2}\{-0-(k-1)(1)\}\right] = \frac{1}{1+k^2-2k+1}(k-1) = \frac{(k-1)}{k^2-2k+2}$$

$$(i) \Rightarrow \mathcal{F}_s\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}}\left[\frac{(k+1)}{k^2+2k+2} + \frac{(k-1)}{k^2-2k+2}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2k^3}{k^4+4}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{k^3}{k^4+4}\right]$$

Example:

Calculate Fourier Sine Transform of the function $f(x) = \begin{cases} Sinx & 0 \le x < \pi \\ 0 & x > \pi \end{cases}$

Solution: We have, by definition

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, Sinkxdx$$

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\pi Sinx\, Sinkxdx + \sqrt{\frac{2}{\pi}}\int_\pi^\infty 0.\, Sinkxdx$$

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\pi Sinx\, Sinkxdx = \sqrt{\frac{2}{\pi}}\left(-\frac{1}{2}\right)\int_0^\pi(-2SinxSinkx)\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\int_0^\pi[Cos(kx+x) - Cos(kx-x)]\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\int_0^\pi Cos(k+1)x\,dx + \frac{1}{\sqrt{2\pi}}\int_0^\pi Cos(k-1)x\,dx$$

$$\mathcal{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}}\left|\frac{Sin(k+1)x}{k+1}\right|_0^\pi + \frac{1}{\sqrt{2\pi}}\left|\frac{Sin(k-1)x}{k-1}\right|_0^\pi$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left|\frac{Sin(kx-x)}{k-1} - \frac{Sin(kx+x)}{k+1}\right|_0^\pi = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{Sin(k\pi-\pi)}{k-1} - \frac{Sin(k\pi+\pi)}{k+1}\right) - 0\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{Sink\pi Cos\pi - Cosk\pi Sin\pi}{k-1} - \frac{Sink\pi Cos\pi + Cosk\pi Sin\pi}{k+1}\right]$$

$$\mathcal{F}_s\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\frac{-Sink\pi}{k-1} + \frac{Sink\pi}{k+1}\right] \qquad \text{since } Cos\pi = -1$$

$$\mathcal{F}_s\{f(x)\} = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{1}{k+1} - \frac{1}{k-1}\right] = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{k-1-k-1}{(k+1)(k-1)}\right] = \frac{Sink\pi}{\sqrt{2\pi}}\left[\frac{-2}{k^2-1}\right]$$

$$\mathcal{F}_s\{f(x)\} = -\sqrt{\frac{2}{\pi}}\left[\frac{Sink\pi}{k^2-1}\right]$$

Linearity Property: The Fourier Cosine Transform $\mathcal{F}_C$ Is Linear.

Proof:

$$\mathcal{F}_C\,\{af+bg\}=\sqrt{\frac{2}{\pi}}\int_0^\infty [af+bg]\,\text{coskxdx}$$

$$\mathcal{F}_C\,\{af+bg\}=a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\text{coskxdx}+\text{b}\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\,\text{coskxdx}$$

$$\mathcal{F}_C\,\{af(x)+bg(x)\}=\text{a}\mathcal{F}_C\,\{f(x)\}+b\mathcal{F}_C\,\{g(x)\}$$

Linearity Property: The Fourier Sine Transform $\mathcal{F}_S$ Is Linear.

Proof:

$$\mathcal{F}_S\,\{af+bg\}=\sqrt{\frac{2}{\pi}}\int_0^\infty [af+bg]\,\text{sinkxdx}$$

$$\mathcal{F}_S\,\{af+bg\}=a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\text{sinkxdx}+\text{b}\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\,\text{sinkxdx}$$

$$\mathcal{F}_S\,\{af(x)+bg(x)\}=\text{a}\mathcal{F}_S\,\{f(x)\}+b\mathcal{F}_S\,\{g(x)\}$$

Theorem

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

$$\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$$

Proof:

$$\begin{aligned}
\mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx \\
&= \sqrt{\frac{2}{\pi}}\left[f(x)\cos wx\Big|_0^\infty + w\int_0^\infty f(x)\sin wx\,dx\right] \\
&= -\sqrt{\frac{2}{\pi}}f(0) + w\mathcal{F}_s\{f(x)\};
\end{aligned}$$

Theorem

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

$$\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$$

Proof:

$$\begin{aligned}\mathcal{F}_s\{f'(x)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx \\ &= \sqrt{\frac{2}{\pi}}\left[f(x)\sin wx\Big|_0^\infty - w\int_0^\infty f(x)\cos wx\,dx\right] \\ &= 0 - w\mathcal{F}_c\{f(x)\}.\end{aligned}$$

Theorem
Let f (x) and its first derivative vanish as x → ∞. If $F_c\ (k)$ is the Fourier cosine transform, then $\mathcal{F}_C\ \{f''\ (x)\} = -k^2 F_c\ (k) - \sqrt{\frac{2}{\pi}} f'\ (0)$

Proof: Consider $f\ (x)$ is real and $\lim_{x\to\infty}|f\ (x)| = 0$ then

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''\ (x)\ Coskxdx$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \left[|Coskxf'\ (x)|_0^\infty - \int_0^\infty f'\ (x)\ (-kSinkx)dx \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \left[\lim_{x\to\infty}|Coskxf'\ (x)| - \lim_{x\to 0}|Coskxf'\ (x)| + k \int_0^\infty f'\ (x)\ Sinkxdx \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} \left[0 - f'\ (0) + k \int_0^\infty f'\ (x)\ Sinkxdx \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\ (0) + k \left\{ \sqrt{\frac{2}{\pi}} |Sinkxf\ (x)|_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty f\ (x)\ (kCoskx)dx \right\} \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\ (0) + k \left\{ \sqrt{\frac{2}{\pi}} |Sinkxf\ (x)|_0^\infty - k\sqrt{\frac{2}{\pi}} \int_0^\infty f\ (x)\ (Coskx)dx \right\} \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = \left[-\sqrt{\frac{2}{\pi}} f'\ (0) + k \left\{ \sqrt{\frac{2}{\pi}} \left(\lim_{x\to\infty}|Sinkxf\ (x)| - \lim_{x\to 0}|Sinkxf\ (x)| \right) - kF_c\ (k) \right\} \right]$$

$$\mathcal{F}_C\ \{f''\ (x)\} = -k^2 F_c\ (k) - \sqrt{\frac{2}{\pi}} f'\ (0)$$

In a similar manner, the Fourier cosine transforms of higher-order derivatives of f (x) can be obtained.

Theorem

Let f (x) and its first derivative vanish as x → ∞. If $F_s\ (k)$ is the Fourier cosine transform, then $\mathcal{F}_s\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}} kf\ (0) - k^2 F_s\ (k)$

Proof: Consider $f\ (x)$ is real and $\lim_{x\to\infty}|f\ (x)| = 0$ then

$$\mathcal{F}_s\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''\ (x)\, Sinkxdx$$

$$\mathcal{F}_s\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}}\left[|Sinkxf'\ (x)|_0^\infty - \int_0^\infty f'\ (x)\,(kCosx)dx\right]$$

$$\mathcal{F}_s\ \{f''\ (x)\} =$$

$$\sqrt{\frac{2}{\pi}}\left[\lim_{x\to\infty}|Sinkxf'\ (x)| - \lim_{x\to 0}|Sinkxf'(x)| - k\int_0^\infty f'\ (x)\, Coskxdx\right]$$

$$\mathcal{F}_s\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}}\left[0 - 0 - k\int_0^\infty f'\ (x)\, Coskxdx\right]$$

$$\mathcal{F}_s\ \{f''\ (x)\} = -k\left[\sqrt{\frac{2}{\pi}}|Coskxf\ (x)|_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty f\ (x)\,(-kSinkx)dx\right]$$

$$\mathcal{F}_s\ \{f''\ (x)\} = -k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf\ (x)| - \lim_{x\to 0}|Coskxf\ (x)|) + k\sqrt{\frac{2}{\pi}}\int_0^\infty f\ (x)\,(Sinkx)dx\right]$$

$$\mathcal{F}_s\ \{f''\ (x)\} = -k\left[\sqrt{\frac{2}{\pi}}(\lim_{x\to\infty}|Coskxf\ (x)| - \lim_{x\to 0}|Coskxf\ (x)|) + kF_s\ (k)\right]$$

$$\mathcal{F}_s\ \{f''\ (x)\} = \sqrt{\frac{2}{\pi}}kf\ (0) - k^2 F_s\ (k)$$

In a similar manner, the Fourier sine transforms of higher-order derivatives of $f\ (x)$ can be obtained.

REMARK:

- $\mathcal{F}\ [f^n(x)] = (-ik)^n \mathcal{F}\ [f\ (x)] = (-ik)^n\ F\ (k)\ \ n = 0, 1, 2, \ldots\ldots\ldots$
- If $\mathcal{F}\ \{u_t\} = \mathcal{F}\ \{u_x\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}\ \{u\ (x,t)\} = (-ik)\ \mathcal{F}\ \{u\ (x,t)\}$ when ‘x’ varies not ‘t’
- When range of spatial variable is infinite then Fourier transform is used rather than the sine or cosine.
- If boundry conditions are of the form $u(0,t) = value$ then use Sine transform, while conditions are of the form $u_x(0,t) = value$ then use Cosine transform,

EXAMPLE: Solve the potential equation for the potential $u(x,y)$ in the semi infinite strip $0 < x < c;\ y > 0$ that satisfies the following conditions;
$u(0,y) = 0\ ;\quad u_y(x,0) = 0\ ;\ u_x(c,y) = f(y)$

Solution: the potential equation is given as $u_{xx} + u_{yy} = 0\ ;\ 0 < x < c;\ y > 0$
Since the BC's are in the form $u_y(x,0) = constant$ therefor we use fourier cosine transform w.r.to 'y'

$$\mathcal{F}_C\,\{u_{xx}\} + \mathcal{F}_C\,\{u_{yy}\} = 0 \Rightarrow \frac{d^2}{dx^2}\mathcal{F}_C\,\{u(x,y)\} + \mathcal{F}_C\,\{u_{yy}\} = 0$$

$$\Rightarrow \frac{d^2}{dx^2}U_C\,(x,k) + \left[-k^2U_c\,(x,k) - \sqrt{\frac{2}{\pi}}u_y(x,0)\right] = 0$$

$$\Rightarrow \frac{d^2}{dx^2}U_C\,(x,k) - k^2U_c\,(x,k) = 0$$

Then general solution will be $U_c(x,k) = c_1e^{kx} + c_2e^{-kx}$ …………(i)

Now using BC's $u\,(0,y) = 0 \Rightarrow \mathcal{F}_C\{u\,(0,y)\} = 0 \Rightarrow U_c(0,k) = 0$

$(i) \Rightarrow U_c(0,k) = 0 = c_1e^0 + c_2e^0 \Rightarrow c_1 = -c_2$

Now $\frac{d}{dx}U_C\,(x,k) = c_1ke^{kx} - c_2ke^{-kx}$ …………(ii)

using BC's $u_x(c,y) = f(y) \Rightarrow \mathcal{F}_C\{u_x(c,y)\} = f(y) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k)$

$$(ii) \Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = c_1ke^{kc} - c_2ke^{-kc}$$

$$\Rightarrow \frac{d}{dx}U_c(c,k) = F_c(k) = -c_2ke^{kc} - c_2ke^{-kc} \qquad \text{since } c_1 = -c_2$$

$$\Rightarrow F_c(k) = -c_2k(e^{kc} + e^{-kc}) \Rightarrow c_2 = -\frac{F_c(k)}{2k\left(\frac{e^{kc}+e^{-kc}}{2}\right)} = -\frac{F_c(k)}{2kCoshkc}$$

$$\Rightarrow c_2 = -\frac{F_c(k)}{2kCoshkc} \Rightarrow c_1 = \frac{F_c(k)}{2kCoshkc} \qquad \text{since } c_1 = -c_2$$

Then $(i) \Rightarrow U_c(x,k) = \frac{F_c(k)}{2kCoshkc}e^{kx} - \frac{F_c(k)}{2kCoshkc}e^{-kx}$

$$U_c(x,k) = \frac{F_c(k)}{kCoshkc}\left(\frac{e^{kx}-e^{-kx}}{2}\right) = \frac{F_c(k)}{kCoshkc}Sinhkx$$

$$\Rightarrow \mathcal{F}_C{}^{-1}\{U_c(x,k)\} = \mathcal{F}_C{}^{-1}\left\{\frac{F_c(k)}{kCoshkc}Sinhkx\right\}$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{F_c(k)}{kCoshkc}Sinhkx\,Coskxdk = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}F_c(k)dk$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{SinhkxCoskx}{kCoshkc}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f\,(y')\,Cosky'dy'\right]dk$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{SinhkxCoskxCosky\prime}{kCoshkc}f\,(y')dy'dk$$

EXAMPLE: Solve the problem using Fourier Transformation method $u_t = u_{xx}$ with $u(0,t) = u_0\,;\quad u(x,0) = 0\ ; x > 0, t > 0, u_0 > 0$

Solution: BC's suggest that we should use fourier sine transform w.r.to 'x'

$\mathcal{F}_s\,\{u_t\} = \mathcal{F}_s\,\{u_{xx}\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F}_s\,\{u(x,t)\} = \mathcal{F}_s\,\{u_{xx}\}$

$\Rightarrow \frac{d}{dt}U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku(0,t) - k^2U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku_0 - k^2U_s\,(k,t)$

$\Rightarrow \frac{d}{dt}U_s\,(k,t) + k^2U_s\,(k,t) = \sqrt{\frac{2}{\pi}}ku_0$ …………….(i)

This is 1st order, linear, non – homogeneous ODE

Therefore I.F. = $e^{\int k^2dt} = e^{k^2t}$

$(i) \Rightarrow e^{k^2t}\frac{\partial}{\partial t}U_s\,(k,t) + k^2U_s\,(k,t)e^{k^2t} = \sqrt{\frac{2}{\pi}}ku_0e^{k^2t}$

$\Rightarrow \int\frac{d}{dt}e^{k^2t}\,U_sdt = \int\sqrt{\frac{2}{\pi}}ku_0e^{k^2t}dt$ + Cosntant

$\Rightarrow e^{k^2t}U_s = \sqrt{\frac{2}{\pi}}ku_0\frac{e^{k^2t}}{k^2} + c \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} + ce^{-k^2t}$ …………….(ii)

Now using IC's $u(x,0) = 0 \Rightarrow \mathcal{F}_s\{u\,(x,0)\} = 0 \Rightarrow U_s(k,0) = 0$

$(ii) \Rightarrow U_s(k,0) = 0 = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} + ce^0 \Rightarrow c = -\sqrt{\frac{2}{\pi}}\frac{u_0}{k}$

Thus $(ii) \Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{k} - \sqrt{\frac{2}{\pi}}\frac{u_0}{k}e^{-k^2t} = \sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)$

$\Rightarrow \mathcal{F}_s^{-1}\{U_s(k,t)\} = \mathcal{F}_s^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)\right\}$

$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{u_0}{k}\left(1 - e^{-k^2t}\right)Sinkxdk = \frac{u_0}{k}\frac{2}{\pi}\int_0^\infty\left(1 - e^{-k^2t}\right)Sinkxdk$

EXAMPLE: Solve the problem using Fourier Transformation method $u_t = u_{xx}$ with $u_x(0,t) = 0\,,\ u(x,0) = f(x);\ \ 0 < x < \infty\ , t > 0$

Solution: BC's suggest that we should use fourier cosine transform w.r.to 'x'

$\mathcal{F}_C\,\{u_t\} = \mathcal{F}_C\,\{u_{xx}\} \Rightarrow \frac{d}{dt}\mathcal{F}_C\,\{u(x,y)\} = \mathcal{F}_C\,\{u_{xx}\}$

$\Rightarrow \frac{d}{dt}U_C\,(k,t) = \left[-k^2U_c\,(k,t) - \sqrt{\frac{2}{\pi}}u_x(0,t)\right] = -k^2U_c\,(k,t) - 0$

$\Rightarrow \frac{d}{dt}U_C\,(k,t) + k^2U_c\,(k,t) = 0$ …………….(i)

This is 1st order, linear, homogeneous ODE

Then general solution will be $U_c(k,t) = Ae^{-k^2t}$ ………….(ii)

Now using IC's

$u(x,0) = f(x) \Rightarrow \mathcal{F}_c\{u\,(x,0)\} = \mathcal{F}_c\{f(x)\} \Rightarrow U_c(k,0) = F_c(k)$

Thus $(i) \Rightarrow U_c(k,0) = F_c(k) = Ae^0 \Rightarrow A = F_c(k)$

$(i) \Rightarrow U_c(k,t) = F_c(k)e^{-k^2t}$

$\Rightarrow {\mathcal{F}_C}^{-1}\{U_c(k,t)\} = {\mathcal{F}_C}^{-1}\{F_c(k)e^{-k^2t}\}$

$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)e^{-k^2t}\, Coskxdk$

$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x')\, Coskx'dx'\right]e^{-k^2t}\, Coskxdk$

$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(x')\, Coskx'dx'\right]e^{-k^2t}\, Coskxdk$

Example:

Solve the problem using Fourier Transformation method

$u_{xx} = u_t\,;\ \ 0 < x < \infty\ ,t \geq 0$

with $u(x,0) = e^{-ax^2}$; $u(x), u'(x) \to 0\ \ as\ \ x \to \pm\infty$

Solution: since $x \to \pm\infty$ therefore we should use fourier transform w.r.to 'x'

$\mathcal{F}\,\{u_{xx}\} = \mathcal{F}\,\{u_t\}$

$\Rightarrow (-ik)^2\mathcal{F}\,\{u(x,t)\} = \frac{d}{dt}\mathcal{F}\,\{u(x,t)\} \Rightarrow -k^2U(k,t) = \frac{d}{dt}U(k,t)$

$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -k^2 \Rightarrow \int\frac{dU}{U} = -k^2\int dt \Rightarrow lnU = -k^2t + A$

$\Rightarrow U(k,t) = e^{-k^2t+A} \Rightarrow U(k,t) = ce^{-k^2}$ ……………(i) where $e^A = c$

Now using IC's

$u(x,0) = e^{-ax^2} \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{e^{-ax^2}\}$

$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}.e^{-ax^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ax^2}dx$

$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2+{\frac{\mathrm{k}}{4a^2}}^2\right]}\ \mathrm{dx}$

$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2}\ \mathrm{dx}$

Consider ikx − ax²

$= -\mathbf{a}\left(x^2 - \frac{\mathrm{ikx}}{a}\right)$

$= -\mathbf{a}\left[\begin{matrix}\left(x^2 - \frac{2\mathrm{ikx}}{2a}\right)\\ +\left(\frac{\mathrm{ik}}{2a}\right)^2 - \left(\frac{\mathrm{ik}}{2a}\right)^2\end{matrix}\right]$

$= -a\left[\left(\mathrm{x} - \frac{\mathrm{ik}}{2a}\right)^2 + {\frac{\mathrm{k}}{4a^2}}^2\right]$

Put $a\left(\mathrm{x} - \frac{\mathrm{ik}}{2a}\right)^2 = \mathrm{P}^2 \Rightarrow \sqrt{a}\left(\mathrm{x} - \frac{\mathrm{ik}}{2a}\right) = P \Rightarrow \sqrt{a}dx = dP \Rightarrow dx = \frac{dP}{\sqrt{a}}$

$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(\mathrm{x}-\frac{\mathrm{ik}}{2a}\right)^2}\ \mathrm{dx} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{P}^2}\ .\frac{dP}{\sqrt{a}}$

$\Rightarrow U(k,0) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi a}}.\sqrt{\pi}\quad \therefore \int_{-\infty}^{\infty} e^{-\mathrm{P}^2}\ dP = \sqrt{\pi}$

$\Rightarrow U(k,0) = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$ ……………(ii)

$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}$

Thus $\Rightarrow U(k,t) = \frac{1}{\sqrt{2a}}e^{\left(-\frac{k^2}{4a}\right)}e^{-k^2} = \frac{1}{\sqrt{2a}}e^{-k^2\left(\mathrm{t}+\frac{1}{4a}\right)}$

$$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\text{a}}}e^{-k^2\left(\text{t}+\frac{1}{4a}\right)}\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\text{a}}}.\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,.\,e^{-k^2\left(\text{t}+\frac{1}{4a}\right)}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\text{a}\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(\text{t}+\frac{1}{4a}\right)\left\{k^2 + \frac{ikx}{\left(\text{t}+\frac{1}{4a}\right)}\right\}\right]dk \;\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Since $k^2 + \frac{ikx}{\left(\text{t}+\frac{1}{4a}\right)} = k^2 + 2(k)\left(\frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right) + \left(\frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right)^2 - \left(\frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right)^2$

$$k^2 - \frac{ikx}{\left(\text{t}+\frac{1}{4a}\right)} = \left(k + \frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right)^2 + \frac{x^2}{4\left(\text{t}+\frac{1}{4a}\right)^2}$$

$$(iii) \Rightarrow u(x,t) = \frac{1}{\sqrt{4\text{a}\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(\text{t}+\frac{1}{4a}\right)\left(k + \frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right)^2\right]e^{\left(\frac{x^2}{4\left(\text{t}+\frac{1}{4a}\right)^2}\right)}dk$$

$$\Rightarrow u(x,t) = \frac{e^{\left(\frac{x^2}{4\left(\text{t}+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4\text{a}\pi}}\int_{-\infty}^{\infty} Exp\left[-\left(\text{t}+\frac{1}{4a}\right)\left(k + \frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right)^2\right]dk \;\ldots\ldots\ldots\ldots\text{(iv)}$$

Now put $\left(\text{t}+\frac{1}{4a}\right)\left(k + \frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right)^2 = \text{m}^2 \Rightarrow \sqrt{\left(\text{t}+\frac{1}{4a}\right)}\left(k + \frac{ix}{2\left(\text{t}+\frac{1}{4a}\right)}\right) = m$

$$\Rightarrow \sqrt{\left(\text{t}+\frac{1}{4a}\right)}\,dk = dm \Rightarrow dk = \frac{1}{\sqrt{\left(\text{t}+\frac{1}{4a}\right)}}dm$$

$$(iv) \Rightarrow u(x,t) = \frac{e^{\left(\frac{x^2}{4\left(\text{t}+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4\text{a}\pi}.\sqrt{\left(\text{t}+\frac{1}{4a}\right)}}\int_{-\infty}^{\infty} e^{-\text{m}^2}\;dm = \frac{e^{\left(\frac{x^2}{4\left(\text{t}+\frac{1}{4a}\right)^2}\right)}}{\sqrt{4\text{a}\pi}.\frac{1}{\sqrt{4\text{a}}}\sqrt{4\text{at}+1}}.\sqrt{\pi}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\text{at}+1}}e^{\left(\frac{ax^2}{4\text{at}+1}\right)}$$

Example: Solve the problem using Fourier Transformation method
$u_t(x,t) = \propto^2 u_{xx}(x,t)\,;\ -\infty < x < \infty\ ,t > 0$
with $u_x(x,0) = f(x);\ |u(x,0)| < \infty$
Solution: since $x \to \pm\infty$ therefore we should use fourier transform w.r.to 'x'
$\mathcal{F}\,\{u_t\} = \propto^2 \mathcal{F}\,\{u_{xx}\}$
$\Rightarrow \frac{d}{dt}\mathcal{F}\,\{u(x,t)\} = \propto^2 (-ik)^2\mathcal{F}\,\{u(x,t)\} \Rightarrow \frac{d}{dt}U(k,t) = -\propto^2 k^2 U(k,t)$
$\Rightarrow \frac{1}{U}\frac{dU}{dt} = -\propto^2 k^2 \Rightarrow \int\frac{dU}{U} = -\propto^2 k^2\int dt \Rightarrow lnU = -\propto^2 k^2 t + A$
$\Rightarrow U(k,t) = e^{-\propto^2 k^2 t + A} \Rightarrow U(k,t) = ce^{-\propto^2 k^2}$ ……………(i) where $e^A = c$
Now using IC's
$u_x(x,0) = f(x)\ and\ |u(x,0)| < \infty \Rightarrow u(x,0) = f(x)$
$\Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} \quad \Rightarrow U(k,0) = F(k)$
$(i) \Rightarrow U(k,0) = ce^0 \Rightarrow c = F(k)$
Thus $(i) \Rightarrow U(k,t) = F(k)e^{-\propto^2 k^2 t}$
$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{F(k)e^{-\propto^2 k^2}\right\}$
$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,.\,F(k)e^{-\propto^2 k^2 t}dk$
$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx'\right]e^{-\propto^2 k^2 t}dk$
$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t}dk\right]f(x')dx'$ ……………(iii)
Now consider $I = \int_{-\infty}^{\infty} e^{-ik(x-x')-\propto^2 k^2 t}dk$
$I = \int_{-\infty}^{\infty} e^{-iku-\beta k^2}dk$ put $x - x' = u$ and $\propto^2 t = \beta$
$I = \int_{-\infty}^{\infty} e^{-\beta\left(k^2+\frac{ik}{\beta}\right)}dk$
$I = \int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{iu}{2\beta}\right)^2}.\,e^{-\frac{u^2}{4\beta}}dk$
$I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-\beta\left(k+\frac{iu}{2\beta}\right)^2}dk$ ……….(iv)
Put $\beta\left(k + \frac{iu}{2\beta}\right)^2 = P^2 \Rightarrow \sqrt{\beta}\left(k + \frac{iu}{2\beta}\right) = P \Rightarrow \sqrt{\beta}dk = dP \Rightarrow dk = \frac{dP}{\sqrt{\beta}}$
$(iv) \Rightarrow I = e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-P^2}\,.\frac{dP}{\sqrt{\beta}} = \frac{dP}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}\int_{-\infty}^{\infty} e^{-P^2}\ dp = \frac{1}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}.\sqrt{\pi}$
$(iii) \Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\beta}}e^{-\frac{u^2}{4\beta}}f(x')dx'$
$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi}.\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\propto^2 t}}e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$
$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi\propto^2 t}.}\int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4(\propto^2 t)}}f(x')dx'$

Consider $\boldsymbol{k^2 + \frac{ik}{\beta}u}$

$= \boldsymbol{k^2 + 2k\left(\frac{iu}{2\beta}\right) + \left(\frac{iu}{2\beta}\right)^2 - \left(\frac{iu}{2\beta}\right)^2}$

$= \boldsymbol{\left(k + \frac{iu}{2\beta}\right)^2 - \left(\frac{iu}{2\beta}\right)^2}$

$= \boldsymbol{\left(k + \frac{iu}{2\beta}\right)^2 + \frac{u^2}{4\beta^2}}$

Example:

Solve the problem using Fourier Transformation method $u_{xxxx} = \frac{1}{a^2} u_{tt}$
with $u(x,0) = f(x); u_t(x,0) = ag'(x)$ and $g, u, u_x, u_{xx}, u_{xxx} \to 0 \; as \;\; x \to \pm\infty$

Solution: since $x \to \pm\infty$ therefore we should use fourier transform w.r.to 'x'

$\mathcal{F}\,\{u_{xxxx}\} = \frac{1}{a^2}\,\mathcal{F}\,\{u_{tt}\}$

$\Rightarrow (-ik)^4 \mathcal{F}\,\{u(x,t)\} = \frac{1}{a^2}\frac{d^2}{dt^2}\mathcal{F}\,\{u(x,t)\} \Rightarrow a^2k^4U(k,t) = \frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U - a^2k^4U = 0$

$\Rightarrow U(k,t) = Ae^{ak^2t} + Be^{-ak^2t}$ ……………(i)

$\Rightarrow \frac{d}{dt}U(k,t) = Aak^2e^{ak^2t} - Bak^2e^{-ak^2t}$ ……………(ii)

Now using IC's $u(x,0) = f(x) \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{f(x)\} \Rightarrow U(k,0) = F(k)$

Then $(i) \Rightarrow U(k,0) = Ae^0 + Be^0 \Rightarrow A + B = F(k)$ ……………(iii)

Also $u_t(x,0) = ag'(x) \Rightarrow \mathcal{F}\{u_t\,(x,0)\} = \mathcal{F}\{ag'(x)\}$

$\Rightarrow \frac{d}{dt}U(k,0) = a(-ik)^1\mathcal{F}\,\{g'(x)\} \Rightarrow \frac{d}{dt}U(k,0) = -iakG(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aak^2e^0 - Bak^2e^0 \Rightarrow -iakG(k) = Aak^2 - Bak^2$

$\Rightarrow -iG(k) = (A-B)k \Rightarrow A - B = -\frac{i}{k}G(k)$ ……………(iv)

Adding (iii) and (iv) $\quad A = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]$

Subtracting (iii) and (iv) $\quad B = \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]$

Then (i) becomes

$\Rightarrow U(k,t) = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]e^{ak^2t} + \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]e^{-ak^2t}$

$\Rightarrow U(k,t) = F(k)\left[\frac{e^{ak^2t} + e^{-ak^2t}}{2}\right] - \frac{i}{k}G(k)\left[\frac{e^{ak^2t} - e^{-ak^2t}}{2}\right]$

$\Rightarrow U(k,t) = F(k)Coshak^2t - \frac{i}{k}G(k)Sinhak^2t$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k)Coshak^2t\} - \mathcal{F}^{-1}\left\{\frac{i}{k}G(k)Sinhak^2t\right\}$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-ikx}\,F(k)Coshak^2tdk - \int_{-\infty}^{\infty} e^{-ikx}\,\frac{i}{k}G(k)Sinhak^2tdk\right]$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,U(k,t)dk$ is our required solution.

Example:

Solve the problem using Fourier Transformation method $u_{xx} = \frac{1}{c^2} u_{tt}$

with $u(x,0) = p(x); u_t(x,0) = q(x)$ and $u, u_x \to 0 \; as \;\; x \to \pm\infty$

Solution: since $x \to \pm\infty$ therefore we should use fourier transform w.r.to ‘x’

$\mathcal{F}\,\{u_{xx}\} = \frac{1}{c^2}\,\mathcal{F}\,\{u_{tt}\}$

$\Rightarrow (-ik)^2 \mathcal{F}\,\{u(x,t)\} = \frac{1}{c^2}\frac{d^2}{dt^2}\mathcal{F}\,\{u(x,t)\} \Rightarrow -c^2k^2U(k,t) = \frac{d^2}{dt^2}U(k,t)$

$\Rightarrow \frac{d^2}{dt^2}U + c^2k^2U = 0 \Rightarrow U(k,t) = c_1Cosxkt + c_2Sinckt$

$\Rightarrow U(k,t) = c_1\left(\frac{e^{ickt}+e^{-ickt}}{2}\right) + c_2\left(\frac{e^{ickt}-e^{-ickt}}{2}\right)$

$\Rightarrow U(k,t) = \left(\frac{c_1+c_1}{2}\right)e^{ickt} + \left(\frac{c_1-c_1}{2}\right)e^{-ickt}$

$\Rightarrow U(k,t) = Ae^{ickt} + Be^{-ickt}$ ……………(i)

$\Rightarrow \frac{d}{dt}U(k,t) = Aice^{ickt} - Bice^{-ickt}$ ……………(ii)

Now using IC’s $u(x,0) = p(x) \Rightarrow \mathcal{F}\{u\,(x,0)\} = \mathcal{F}\{p(x)\} \Rightarrow U(k,0) = P(k)$

Then $(i) \Rightarrow U(k,0) = Ae^0 + Be^0 \Rightarrow A + B = P(k)$ ……………(iii)

Also $u_t(x,0) = q(x) \Rightarrow \mathcal{F}\{u_t\,(x,0)\} = \mathcal{F}\{q(x)\} \Rightarrow \frac{d}{dt}U(k,0) = Q(k)$

Then $(ii) \Rightarrow \frac{d}{dt}U(k,0) = Aicke^0 - Bicke^0$

$\Rightarrow Q(k) = ick(A-B)k \Rightarrow A - B = \frac{1}{ick}Q(k)$ ……………(iv)

Adding (iii) and (iv) $\qquad A = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]$

Subtracting (iii) and (iv) $\quad B = \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]$

Then (i) becomes

$\Rightarrow U(k,t) = \frac{1}{2}\left[P(k) + \frac{1}{ick}Q(k)\right]e^{ickt} + \frac{1}{2}\left[P(k) - \frac{1}{ick}Q(k)\right]e^{-ickt}$

$\Rightarrow U(k,t) = P(k)\left[\frac{e^{ickt}+e^{-ickt}}{2}\right] + \frac{1}{ick}Q(k)\left[\frac{e^{ickt}-e^{-ickt}}{2}\right]$

$\Rightarrow \mathcal{F}^{-1}\{U(k,t)\} =$

$\frac{1}{2}\left[\mathcal{F}^{-1}\{P(k)e^{ickt}\} + \mathcal{F}^{-1}\{P(k)e^{-ickt}\}\right] + \frac{1}{2ick}\mathcal{F}^{-1}\{Q(k)(e^{ickt} - e^{-ickt})\}$ ….(A)

$\mathcal{F}^{-1}\{P(k)e^{ickt}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,P(k)e^{ickt}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(x-ct)k}\,P(k)dk$

$\mathcal{F}^{-1}\{P(k)e^{ickt}\} = P(x-ct)$

Similarly $\quad \mathcal{F}^{-1}\{P(k)e^{-ickt}\} = P(x+ct)$

And consider $\qquad q(x) = \mathcal{F}^{-1}\{Q(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,Q(k)dk$

$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{x-ct}^{x+ct}\int_{-\infty}^{\infty} e^{-ikx}\,Q(k)dkdx$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{x-ct}^{x+ct} e^{-ikx'}dx'Q(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{e^{-ikx'}}{-ik}\right|_{x-ct}^{x+ct} Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-ik}\left[e^{-ik(x+ct)} - e^{-ik(x-ct)}\right]Q(k)dk$$

$$\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{ik}\left[e^{-ik(x-ct)} - e^{-ik(x+ct)}\right]Q(k)dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[e^{ickt} - e^{-ickt}\right]\frac{Q(k)}{k}dk$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} q(x)dx = \frac{1}{2ic}\mathcal{F}^{-1}\left\{\left(e^{ickt} - e^{-ickt}\right)\frac{Q(k)}{k}dk\right\}$$

$$(A) \Rightarrow u(x,t) = \frac{1}{2}\left[P(x+ct) + P(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} q(x')dx'$$

Fourier Inversion Formula:

The proper inversion formula is given as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, F(k)\ \mathrm{dk}$$

The formula nearly states that f is the fourier transform of $F(k)$

where $F(k) = \mathcal{F}\,\{f(x)\}$

PROOF:

by Fourier integral theorem $f\,(x) = \frac{1}{\pi}\int_0^{\infty} dk\int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$

$$\Rightarrow f\,(x) = \frac{1}{\pi}\int_0^{\infty} dk\int_{-\infty}^{\infty} Cosk(x-x')f(x')dx'$$

$$\Rightarrow f\,(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')dx'\int_0^{\infty} Cosk(x-x')dk \qquad \text{changing the order}$$

$$\Rightarrow f\,(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')dx'.\lim_{m\to\infty}\int_0^{m} Cosk(x-x')dk \quad \ldots\ldots\ldots(\mathrm{i})$$

Since $\int_{-m}^{m} Cosk(x'-x)dk = 2\int_0^{m} Cosk(x-x')dk$ ………(ii)

Also $\int_{-m}^{m} Sink(x'-x)dk = 0 \Rightarrow i\int_{-m}^{m} Sink(x-x')dk = 0$ ………(iii)

On subtraction from (ii) and (iii) we have

$$\int_{-m}^{m}[Cosk(x-x') - iSink(x-x')]dk = 2\int_0^{m} Cosk(x-x')dk$$

$$\Rightarrow \int_{-m}^{m} e^{-ik(x-x')}dk = 2\int_0^{m} Cosk(x-x')dk$$

$$\Rightarrow \int_0^{m} Cosk(x-x')dk = \frac{1}{2}\int_{-m}^{m} e^{-ik(x-x')}dk \quad \ldots\ldots\ldots(\mathrm{iv})$$

Hence from (i) and (iv)

$$\Rightarrow f\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx'.\lim_{m\to\infty}\int_{-m}^{m} e^{-ik(x-x')}dk$$

$$\Rightarrow f\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx'\int_{-\infty}^{\infty} e^{-ik(x-x')}dk$$

$$\Rightarrow f\,(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,dk.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}f(x')dx'$$

$$\Rightarrow f\,(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{ikx}}\, F(k)\ \mathrm{dk} \qquad \text{as required.}$$

SEQUENCE AND SERIES

Sequence

An arrangement of numbers in any specific order is called sequence or progression.
Or A sequence is a function whose domain is a subset of the set of natural numbers.
A sequence is a special type of a function from a subset of N to R or C.
Sometimes, the domain of a sequence is taken to be a subset of the set {0, 1, 2, 3,...}, i.e., the set of non-negative integers. If all members of a sequence are real numbers, then it is called a real sequence.
Sequences are usually named with letters a, b, c etc., and n is used instead of x as a variable. If a natural number n belongs to the domain of a sequence a, the corresponding element in its range is denoted by a_n. For convenience, a special notation a_n is adopted for a(n)and the symbol $\{a_n\}$ or $a_1, a_2, a_3,....,a_n$,...is used to represent the sequence a. The elements in the range of the sequence $\{a_n\}$ are called its **terms**; that is, a_1 is the first term, a_2 the second term and a_n the nth term or the general term.

Types of Sequence

- Arithmetic Sequence (Progression)
- Geometric Sequence (Progression)
- Harmonic Sequence (Progression)

Finite and Infinite Sequence

If the domain of a sequence is a finite set, then the sequence is called a finite sequence otherwise, an infinite sequence.
An infinite sequence has no last term.
Some examples of sequences are;
i) 1, 4, 9,...,121
ii) 1, 3, 5, 7, 9,...,21
iii) 1, 2, 4,...
iv) 1, 3, 7, 15, 31,...
v) 1, 6, 20, 56,...
The sequences (i) and (ii) are finite whereas the sequences (iii) to (v) are infinite.

Real Sequence

A real sequence is one whose terms are real.

Convergence

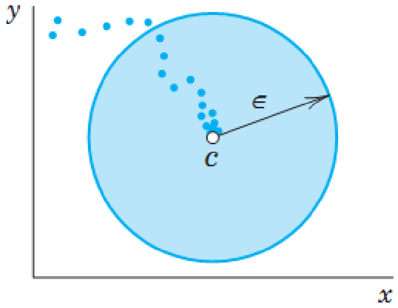
A **convergent sequence** $z_1, z_2, \cdots$ is one that has a limit c, written

$$\lim_{n\to\infty} z_n = c \qquad \text{or simply} \qquad z_n \to c.$$

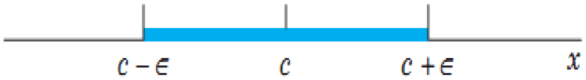
By definition of **limit** this means that for every $\epsilon > 0$ we can find an N such that

$$|z_n - c| < \epsilon \qquad \text{for all } n > N;$$

geometrically, all terms z_n with $n > N$ lie in the open disk of radius ϵ and center c and only finitely many terms do not lie in that disk.



Convergent complex sequence



Convergent real sequence

Divergent Sequence

A divergent sequence is one that does not converge.

Convergent and Divergent Sequences

The sequence $\{i^n/n\} = \{i, -\frac{1}{2}, -i/3, \frac{1}{4}, \cdots\}$ is convergent with limit 0.

The sequence $\{i^n\} = \{i, -1, -i, 1, \cdots\}$ is divergent, and so is $\{z_n\}$ with $z_n = (1+i)^n$.

Sequences of the Real and the Imaginary Parts

The sequence $\{z_n\}$ with $z_n = x_n + iy_n = 1 - 1/n^2 + i(2 + 4/n)$ is $6i, \frac{3}{4} + 4i, \frac{8}{9} + 10i/3, \frac{15}{16} + 3i, \cdots$. (Sketch it.) It converges with the limit $c = 1 + 2i$. Observe that $\{x_n\}$ has the limit $1 = \operatorname{Re} c$ and $\{y_n\}$ has the limit $2 = \operatorname{Im} c$. This is typical. It illustrates the following theorem by which the convergence of a *complex* sequence can be referred back to that of the two *real* sequences of the real parts and the imaginary parts.

Theorem: (Criteria for Convergence)

Sequences of the Real and the Imaginary Parts

A sequence $z_1, z_2, \cdots, z_n, \cdots$ *of complex numbers* $z_n = x_n + iy_n$ *(where* $n = 1, 2, \cdots$*) converges to* $c = a + ib$ *if and only if the sequence of the real parts* $x_1, x_2, \cdots$ *converges to* a *and the sequence of the imaginary parts* $y_1, y_2, \cdots$ *converges to* b.

Example:

Show that the sequence $z_n = -1 + i\frac{(-1)^n}{n^2}; n = 1,2,3, \ldots.$ Converges to -1

Solution:

$$lim_{n\to\infty}\left[-1 + i\frac{(-1)^n}{n^2}\right] = lim_{n\to\infty}[-1] + i\, lim_{n\to\infty}\left[\frac{(-1)^n}{n^2}\right] = -1 + i.0 = -1$$

Another way:

$$|z_n - (-1)| = \left|-1 + i\frac{(-1)^n}{n^2} - (-1)\right| = \left|i\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2} < \varepsilon \text{ whenever } n > \frac{1}{\sqrt{\varepsilon}}$$

Example:

Show that the sequence $z_n = \frac{3+ni}{n+2ni}$ Converges to $\frac{2}{5} + \frac{1}{5}i$

Solution:

$$lim_{n\to\infty}\left[\frac{3+ni}{n+2ni}\right] = lim_{n\to\infty}\left[\frac{n\left(\frac{3}{n}+i\right)}{n(1+2i)}\right] = lim_{n\to\infty}\left[\frac{\left(\frac{3}{n}+i\right)}{(1+2i)}\right] = \frac{2}{5} + \frac{1}{5}i$$

Series

The sum of an indicated number of terms in a sequence is called a series.

Given a sequence $z_1, z_2, \cdots, z_m, \cdots$, we may form the sequence of the sums

$$s_1 = z_1, \qquad s_2 = z_1 + z_2, \qquad s_3 = z_1 + z_2 + z_3, \quad \cdots$$

and in general

$$s_n = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \cdots).$$

Here s_n is called the ***n*th partial sum** of the *infinite series* or **series**

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

The $z_1, z_2, \cdots$ are called the **terms** of the series.

For example, the sum of the first seven terms of the sequence $\{n^2\}$ is the series, $1 + 4 + 9 + 16 + 25 + 36 + 49$.

The above series is also named as the 7th partial sum of the sequence $\{n^2\}$. If the number of terms in a series is finite, then the series is called a finite series, while a series consisting of an unlimited number of terms is termed as an infinite series.

Convergent Series

A **convergent series** is one whose sequence of partial sums converges, say,

$$\lim_{n\to\infty} s_n = s. \qquad \text{Then we write} \qquad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

and call s the **sum** or *value* of the series. A series that is not convergent is called a **divergent series**.

Note

If we omit s_n the terms of from the nth partial sum of the series

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

there remains

$$R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots$$

This is called the remainder of the series.

Theorem: Real and Imaginary Parts of Partial Sum

A series $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$

with $z_m = x_m + iy_m$ *converges and has the sum* $s = u + iv$ *if and only if* $x_1 + x_2 + \cdots$ *converges and has the sum* u *and* $y_1 + y_2 + \cdots$ *converges and has the sum* v.

Divergence Test

If a series $z_1 + z_2 + \cdots$ *converges, then* $\lim_{m\to\infty} z_m = 0$. *Hence if this does not hold, the series diverges.*

Proof

If $z_1 + z_2 + \cdots$ converges, with the sum s, then, since $z_m = s_m - s_{m-1}$,

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty}(s_m - s_{m-1}) = \lim_{m\to\infty} s_m - \lim_{m\to\infty} s_{m-1} = s - s = 0.$$

CAUTION! $z_m \to 0$ is *necessary* for convergence but *not sufficient*, as we see from the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which satisfies this condition but diverges

Cauchy's Convergence Principle for Series

A series $z_1 + z_2 + \cdots$ *is convergent if and only if for every given* $\epsilon > 0$ *(no matter how small) we can find an* N *(which depends on* ϵ*, in general) such that*

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots$$

Absolute Convergence

A series $z_1 + z_2 + \cdots$ is called **absolutely convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$$ is convergent.

Conditionally Convergence

If $z_1 + z_2 + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \cdots$ is called, more precisely, **conditionally convergent**.

Example (A Conditionally Convergent Series)

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots$ converges, but only conditionally since the harmonic series diverges,

Note

If a series is absolutely convergent, it is convergent.

Comparison Test

If a series $z_1 + z_2 + \cdots$ is given and we can find a convergent series $b_1 + b_2 + \cdots$ with nonnegative real terms such that $|z_1| \leqq b_1, |z_2| \leqq b_2, \cdots$, then the given series converges, even absolutely.

Proof

By Cauchy's principle, since $b_1 + b_2 + \cdots$ converges, for any given $\epsilon > 0$ we can find an N such that

$$b_{n+1} + \cdots + b_{n+p} < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots.$$

From this and $|z_1| \leqq b_1, |z_2| \leqq b_2, \cdots$ we conclude that for those n and p,

$$|z_{n+1}| + \cdots + |z_{n+p}| \leqq b_{n+1} + \cdots + b_{n+p} < \epsilon.$$

Hence, again by Cauchy's principle, $|z_1| + |z_2| + \cdots$ converges, so that $z_1 + z_2 + \cdots$ is absolutely convergent.

GEOMETRIC SERIES

Any series of the form $\sum_{n=1}^{\infty} az^{n-1} = a + az + az^2 + az^3 + \ldots\ldots\ldots$ is called Geometric Series. It is Convergent Series and its sum is $\sum_{n=1}^{\infty} az^{n-1} = \frac{a}{1-z}$

OSCILLATORY SERIES

Any series is said to be Oscillatory if neither the partial sum tends to finite and definite limit nor tends to $+\infty$ or $-\infty$ rather oscillate between two numbers.

POWER SERIES

Any series of the form
$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \ldots\ldots\ldots$ is called Power Series.

DE – ALEMBERT OR RATIO TEST

Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series such that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$ then

i. If $L < 1$ then series will be absolutely convergent.
ii. If $L > 1\ \ or\ L = \infty$ then series will be divergent.
iii. If $L = 1$ then test fail.

ROOT TEST

Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series such that $\lim_{n\to\infty} |z_n|^{1/n} = L$ then

i. If $L < 1$ then series will be absolutely convergent.
ii. If $L > 1$ then series will be divergent.
iii. If $L = 1$ then test fail

Example (Ratio Test)

Is the following series convergent or divergent? (First guess, then calculate.)

$$\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!} = 1 + (100+75i) + \frac{1}{2!}(100+75i)^2 + \cdots$$

Solution. By **Ratio Test**, the series is convergent, since

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+75i|^{n+1}/(n+1)!}{|100+75i|^n/n!} = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \longrightarrow L = 0.$$

Example (more general than Ratio Test)

Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$. Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

Solution

The ratios of the absolute values of successive terms are

$\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$

Hence convergence follows from Ratio Test - I. Since the sequence of these ratios has no limit, Ratio Test – II is not applicable.

Convergence Behavior of Power Series

Power series have variable terms (functions of z), but if we fix z, then all the concepts for series with constant terms in the last section apply. Usually a series with variable terms will converge for some z and diverge for others. For a power series the situation is simple. The series may converge in a disk with center or in the whole z-plane or only at z_0.

Example (Convergence in a Disk. Geometric Series)

The *geometric series* $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$

converges absolutely if $|z| < 1$ and diverges if $|z| \geqq 1$

Example (Convergence for Every z)

The power series (which will be the Maclaurin series of e^z in Sec. 15.4)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is absolutely convergent for every z. In fact, by the ratio test, for any fixed z,

$$\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Example (Convergence Only at the Center. (Useless Series))

The following power series converges only at $z = 0$, but diverges for every $z \neq 0$, as we shall show.

$$\sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^3 + \cdots$$

In fact, from the ratio test we have

$$\left|\frac{(n+1)! z^{n+1}}{n! z^n}\right| = (n+1)|z| \to \infty \quad \text{as} \quad n \to \infty \quad (z \text{ fixed and } \neq 0).$$

Theorem (Convergence of a Power Series)

(a) *Every power series* (1) *converges at the center* z_0.

(b) *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* z *closer to* z_0 *than* z_1, *that is,* $|z - z_0| < |z_1 - z_0|$. See Fig. 365.

(c) *If* (1) *diverges at* $z = z_2$, *it diverges for every* z *farther away from* z_0 *than* z_2. See Fig. 365.

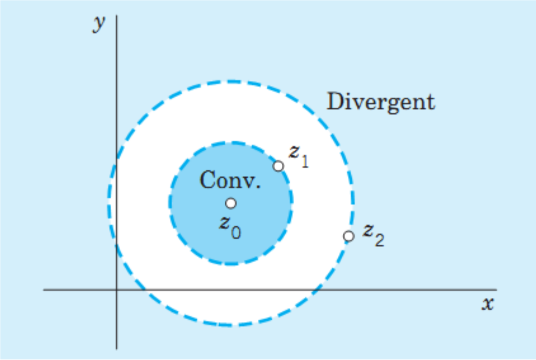


Fig. 365. Theroem 1

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

RADIUS OF CONVERGENCE AND DISC OR CIRCLE OF CONVERGENCE

A circle centered at z_0 having radius $R > 0$ for which the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at every point within the circle $|z-z_0| = R$ then R is called Radius of convergence and Region or Domain of convergenc is defined as $|z-z_0| < R$

HOW TO FIND RADIUS OF CONVERGENCE?

Suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series.

If $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$ or $\lim_{n\to\infty}|z_n|^{1/n} = L$ then

Radius of Convergence $= R = \frac{1}{L}$

Example: If $\sum_{n=0}^{\infty}\left(1+\frac{1}{n}\right)^{n^n} z^n$ then find circle of convergence.

Solution: Since $\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty}\left(1+\frac{1}{n}\right)^{n^n}(z-0)^n$

$\Rightarrow a_n = \left(1+\frac{1}{n}\right)^{n^n}$ & $z_0 = 0$

By Root test; $\lim_{n\to\infty}|z_n|^{1/n} = L$

$\Rightarrow L = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^{n^n}\right|^{1/n} = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^{n}\right| = e$ use $a_n = z_n$

Then Radius of convergence $= R = \frac{1}{L} = \frac{1}{e}$

Circle of Convergence is $|z-z_0| = R \Rightarrow |z-0| = \frac{1}{e} \Rightarrow |z| = \frac{1}{e}$

Example:

If $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$ then find disk and region of convergence.

Solution: Since $\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$

$\Rightarrow a_n = \frac{(-1)^n}{n}$ & $z_0 = 2i$

By Ratio test; $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$

$\Rightarrow L = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n}+1}{n+1}}{\frac{(-1)^n}{n}}\right|$ we may use $a_n = z_n$

$\Rightarrow L = 1$

Then Radius of convergence $= R = \frac{1}{L} = 1$

Circle of Convergence is $|z-z_0| = R \Rightarrow |z-2i| = 1$

Region (domain) of Convergence is $|z-z_0| < R \Rightarrow |z-2i| < 1$

Remark:

The power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ at $z_0 = 0$ In the complex plane.

- Either converges for all values of z
- Or converges only for z = 0
- Or converges for z in some region.

Example(Behavior on the Circle of Convergence)

On the circle of convergence (radius $R = 1$ in all three series),

$\Sigma z^n/n^2$ converges everywhere since $\Sigma 1/n^2$ converges,

$\Sigma z^n/n$ converges at -1 (by Leibniz's test) but diverges at 1,

Σz^n diverges everywhere.

Cauchy–Hadamard formula

Radius of Convergence R

Suppose that the sequence $|a_{n+1}/a_n|, n = 1, 2, \cdots,$ *converges with limit* L^*. *If* $L^* = 0$, *then* $R = \infty$; *that is, the power series* (1) *converges for all* z. *If* $L^* \neq 0$ (*hence* $L^* > 0$), *then*

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad \textbf{(Cauchy–Hadamard formula}^{1}\textbf{)}.$$

If $|a_{n+1}/a_n| \to \infty$, *then* $R = 0$ (*convergence only at the center* z_0).

CAUCHY's HADAMARD THEOREM

For every series $\sum_{n=0}^{\infty} a_n z^n$ *there exsists a number* R *such that* $0 < R < \infty$ *called radius of convergence then the series converges absolutely for every* $|z| < R$

Proof:

For series $\sum_{n=0}^{\infty} a_n z^n$ there exsists a number $R = \frac{1}{L}$ where $L = \lim_{n\to\infty} |a_n|^{1/n}$

$\Rightarrow L = \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$ and $|z| < R$ then there exists a number ρ such that

$|z| < \rho < R$

$\Rightarrow \rho < R \Rightarrow \frac{1}{R} < \frac{1}{\rho} \Rightarrow |a_n|^{1/n} < \frac{1}{\rho} \Rightarrow |a_n| < \frac{1}{\rho^n} \Rightarrow |a_n||z|^n < \frac{|z|^n}{\rho^n}$

$\sum_{n=0}^{\infty} |a_n z^n| < \sum_{n=0}^{\infty} \frac{|z|^n}{\rho^n}$

Since $\sum_{n=0}^{\infty} \frac{|z|^n}{\rho^n} = 1 + \frac{|z|^1}{\rho^1} + \frac{|z|^2}{\rho^2} + \ldots\ldots\ldots\ldots$ is geometric series which is convergent under the condition $|z| < \rho \Rightarrow \frac{|z|}{\rho} < 1$

therefore by comparison test

$\sum_{n=0}^{\infty} |a_n z^n|$ is convergent series.

$\Rightarrow \sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent series for every $|z| < R$

Example(Radius of Convergence)

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n \text{ is}$$

$$R = \lim_{n\to\infty}\left[\frac{(2n!)}{(n!)^2}\Big/\frac{(2n+2)!}{((n+1)!)^2}\right] = \lim_{n\to\infty}\left[\frac{(2n!)}{(2n+2)!}\cdot\frac{((n+1)!)^2}{(n!)^2}\right] = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$$

The series converges in the open disk $|z-3i| < \frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$.

Example (Cauchy Hadamard is not helpful, its extention)

Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4+\cdots.$$

Solution. The sequence of the ratios $\frac{1}{6}$, $2(2+\frac{1}{4})$, $1/(8(2+\frac{1}{4}))$, $\cdots$ does not converge, so that Theorem is of no help. It can be shown that

$$(6^*) \qquad R = 1/\tilde{L}, \qquad \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

This still does not help here, since $(\sqrt[n]{|a_n|})$ does not converge because $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = \frac{1}{2}$ for odd n, whereas for even n we have

$$\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1 \qquad \text{as} \qquad n\to\infty,$$

so that $\sqrt[n]{|a_n|}$ has the two limit points $\frac{1}{2}$ and 1. It can further be shown that

$$(6^{**}) \qquad R = 1/\tilde{l}, \qquad \tilde{l} \text{ the greatest limit point of the sequence } \left\{\sqrt[n]{|a_n|}\right\}.$$

Here $\tilde{l} = 1$, so that $R = 1$. *Answer.* The series converges for $|z| < 1$. ■

Functions Given by Power Series

Theorem (Continuity of the Sum of a Power Series)

If a function f(z) can be represented by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R).$$

with radius of convergence R > 0, then f(z) is continuous at z = 0.

Identity Theorem for Power Series. Uniqueness

Let the power series $a_0 + a_1 z + a_2 z^2 + \cdots$ *and* $b_0 + b_1 z + b_2 z^2 + \cdots$ *both be convergent for* $|z| < R$, *where* R *is positive, and let them both have the same sum for all these* z. *Then the series are identical, that is,* $a_0 = b_0, a_1 = b_1, a_2 = b_2, \cdots$.

Hence if a function $f(z)$ *can be represented by a power series with any center* z_0, *this representation is* ***unique***.

Termwise Differentiation of a Power Series

The derived series of a power series has the same radius of convergence as the original series.

Termwise Integration of Power Series

The power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \cdots$ *term by term has the same radius of convergence as the original series.*

TAYLOR SERIES:

Suppose a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a function within the circle of convergence $|z-z_0| = R$ then following series is known as Taylor series in complex analysis;

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$$

SPECIAL CASE: When $z_0 = 0$ then Taylor Series becomes Maclaurin Series i.e.

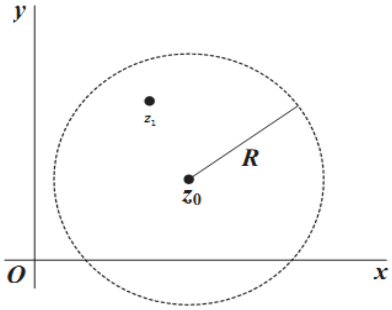
$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(z)^n$$

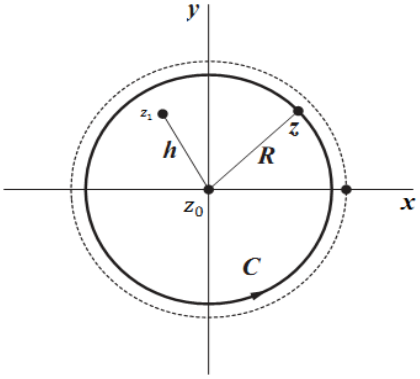
TAYLOR SERIES THEOREM

Suppose that a function f´ is analytic throughout a disk $|z-z_0| < R$, *centered at* z_0 *and with radius R (Fig.). Then f´(z) has the power series representation*

$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$ *with* $a_n = \frac{f^n(z_0)}{n!}$

That is, series $f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$ *converges to f´(z) when z lies in the stated open disk.*





Proof:

Let C be a circle $|z-z_0| < R$ centered at z_0 having radius R. also condiser $z_1 = z_0 + h$ be another point inside the circle C. also function is analytic in domain D. then by using Cauchy Integral Formula

$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)}dz$$

$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)-h}dz$$

$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)\left[1-\frac{h}{z-z_0}\right]}dz$$

$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}\left[1-\frac{h}{z-z_0}\right]^{-1} dz \quad \text{........(i)}$$

Consider

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}+\ \ldots\ldots\ldots$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}\left[1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots\right]\ldots\ldots\ldots$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}\left[1-\frac{h}{z-z_0}\right]^{-1}$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}\left[\frac{z-z_0-h}{z-z_0}\right]^{-1}$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}(z-z_0)}{(z-z_0)^{n+1}(z-z_0-h)}$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^n(z-z_0-h)}\quad\ldots\ldots\text{(ii)}$$

$$(i)\Rightarrow f(z_0+h)=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)}\left[1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^n(z-z_0-h)}\right]dz$$

$$\Rightarrow f(z_0+h)=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)}dz+\frac{h}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}dz+\frac{h^2}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^3}dz+\cdots+\frac{h^n}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz+\frac{h^{n+1}}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}(z-z_0-h)}dz$$

$$f(z_0+h)=f(z_0)+\frac{h}{1!}f'(z_0)+\frac{h^2}{2!}f''(z_0)+\cdots+\frac{h^n}{n!}f^n(z_0)+R_n\quad\ldots\text{(iii)}$$

Where $\qquad R_n=\frac{h^{n+1}}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}(z-z_0-h)}dz$

Now we will prove $R_n\to 0$ as $n\to\infty$

$$\Rightarrow |R_n|=\left|\frac{h^{n+1}}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}(z-z_0-h)}dz\right|$$

$$\Rightarrow |R_n|\le\frac{h^{n+1}}{2\pi|i|}\int_C\left|\frac{f(z)}{(z-z_0-h)}\right|\frac{|dz|}{|z-z_0|^{n+1}}<\frac{h^{n+1}}{2\pi|i|}\frac{M}{R^{n+1}}\int_C|dz|$$

$$\Rightarrow |R_n|<\frac{h^{n+1}}{2\pi|i|}\frac{M}{R^{n+1}}(2\pi R)=Mh\left(\frac{h}{R}\right)^n$$

$$\Rightarrow |R_n|<Mh\left(\frac{h}{R}\right)^n\to 0\text{ as }n\to\infty$$

Then equation (iii) becomes

$$f(z_0+h)=f(z_0)+\frac{h}{1!}f'(z_0)+\frac{h^2}{2!}f''(z_0)+\cdots+\frac{h^n}{n!}f^n(z_0)$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n\qquad with\quad z-z_0=h$$

Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if *f* is analytic at z_0, it is analytic throughout some neighborhood $|z - z_0| < \varepsilon$ of that point; and ε may serve as the value of R_0 in the statement of Taylor's theorem. Also, if *f* is entire, *R* can be chosen arbitrarily large ; and the condition of validity becomes $|z - z_0| < \infty$. The series then converges to *f (z)* at each point *z* in the finite plane.
When it is known that *f* is analytic everywhere inside a circle centered at z_0, convergence of its Taylor series about z_0 to *f* (z) for each point *z* within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to *f* (z) within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 at which *f* *fails* to be analytic.

Example
Expand $f(z) = e^z$ using Maclaurin series in the form of infinite series. Also find the region of convergence when $z_0 = 0$
Solution:
Given $f(z) = e^z \Rightarrow f(0) = 1$
$f'(z) = e^z \Rightarrow f'(0) = 1$
$f''(z) = e^z \Rightarrow f''(0) = 1$
Continuing in this manner $f^n(z) = e^z \Rightarrow f^n(0) = 1$
Using Maclaurin series $f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(z)^n$
$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots\ldots\ldots\ldots$ Required series.
Now consider $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
$\Rightarrow a_n = \frac{1}{n!}, a_{n+1} = \frac{1}{(n+1)!}$ & $z_0 = 0$
By Ratio test; $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$
$\Rightarrow L = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right|$ we may use $a_n = z_n$
$\Rightarrow L = \frac{1}{\infty} = 0$
Then Radius of convergence $= R = \frac{1}{0} = \infty$
Region (domain) of Convergence is $|z - z_0| < R \Rightarrow |z| < \infty$

Example: Expand $f(z) = Sinz$ using Taylor's series when $z_0 = \frac{\pi}{4}$

Solution: Given $f(z) = Sinz \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(z) = Cosz \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(z) = -Sinz \Rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -Cosz \Rightarrow f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Continuing in this ………………

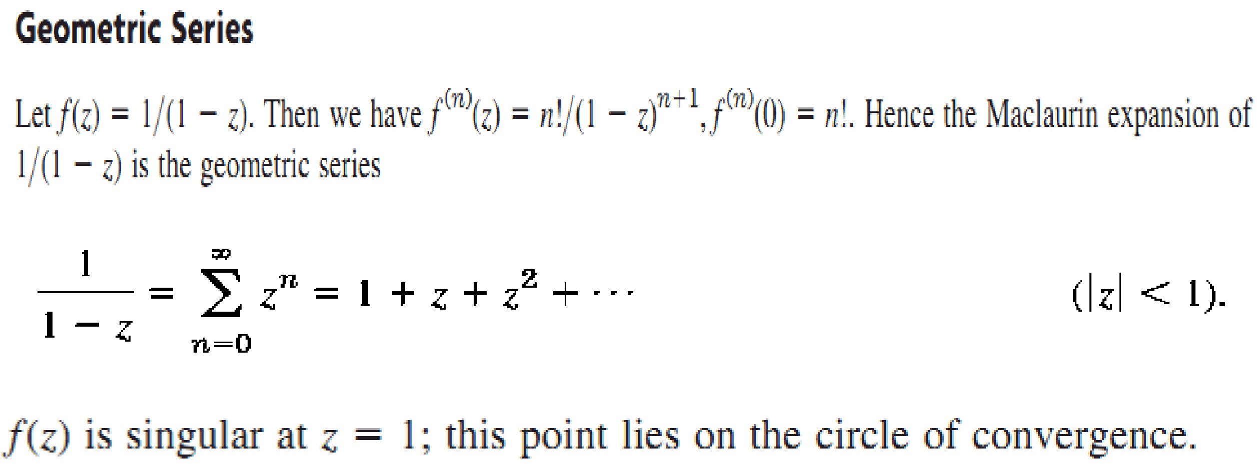
Using Taylor's series $f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{4}\right)}{n!}\left(z - \frac{\pi}{4}\right)^n$

$f(z) = Sinz = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z - \frac{\pi}{4}\right)^2$ ……………. Required series.

Power Series as Taylor Series

A power series with a nonzero radius of convergence is the Taylor series of its sum.

Important Special Taylor Series



Geometric Series

Let $f(z) = 1/(1 - z)$. Then we have $f^{(n)}(z) = n!/(1 - z)^{n+1}, f^{(n)}(0) = n!$. Hence the Maclaurin expansion of $1/(1 - z)$ is the geometric series

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1).$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence.

Exponential Function

We know that the exponential function e^z (Sec. 13.5) is analytic for all z, and $(e^z)' = e^z$. Hence from (1) with $z_0 = 0$ we obtain the Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots .$$

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} .$$

Euler formula

$$e^{iy} = \cos y + i \sin y.$$

Trigonometric and Hyperbolic Functions

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Logarithm

$$\operatorname{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots \qquad (|z| < 1).$$

Replacing z by $-z$ and multiplying both sides by -1, we get

$$-\operatorname{Ln}(1-z) = \operatorname{Ln}\frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (|z| < 1).$$

By adding both series we obtain

$$\operatorname{Ln}\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1).$$

Example

Find the Maclaurin series of $f(z) = 1/(1+z^2)$.

Solution

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \quad (|z| < 1).$$

Example

Find the Maclaurin series of $f(z) = \arctan z$.

Solution

$f'(z) = 1/(1+z^2)$.

Integrating the following and using $f(0) = 0$

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \quad (|z| < 1).$$

$$\arctan z = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad (|z| < 1);$$

this series represents the principal value of $w = u + iv = \arctan z$ defined as that value for which $|u| < \frac{\pi}{2}$

Example

Develop $1/(c - z)$ in powers of $z - z_0$, where $c - z_0 \neq 0$.

Solution

$$\frac{1}{c-z} = \frac{1}{c - z_0 - (z - z_0)} = \frac{1}{(c - z_0)\left(1 - \dfrac{z - z_0}{c - z_0}\right)} = \frac{1}{c - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n$$

$$= \frac{1}{c - z_0}\left(1 + \frac{z - z_0}{c - z_0} + \left(\frac{z - z_0}{c - z_0}\right)^2 + \cdots\right).$$

This series converges for

$$\left|\frac{z - z_0}{c - z_0}\right| < 1, \qquad \text{that is,} \qquad |z - z_0| < |c - z_0|.$$

Example (Binomial Series, Reduction by Partial Fractions)

Find the Taylor series of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Solution

We develop $f(z)$ in partial fractions and the first fraction in a **binomial series**

$$\frac{1}{(1 + z)^m} = (1 + z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$

$$= 1 - mz + \frac{m(m + 1)}{2!} z^2 - \frac{m(m + 1)(m + 2)}{3!} z^3 + \cdots$$

With m = 2 and the second fraction in a geometric series, and then add the two series term by term. This gives

$$f(z) = \frac{1}{(z + 2)^2} + \frac{2}{z - 3} = \frac{1}{[3 + (z - 1)]^2} - \frac{2}{2 - (z - 1)} = \frac{1}{9}\left(\frac{1}{[1 + \frac{1}{3}(z - 1)]^2}\right) - \frac{1}{1 - \frac{1}{2}(z - 1)}$$

$$= \frac{1}{9} \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{z - 1}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z - 1}{2}\right)^n = \sum_{n=0}^{\infty} \left[\frac{(-1)^n (n + 1)}{3^{n+2}} - \frac{1}{2^n}\right] (z - 1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z - 1) - \frac{23}{108}(z - 1)^2 - \frac{275}{1944}(z - 1)^3 - \cdots.$$

the whole series converges for $|z - 1| < 2$.

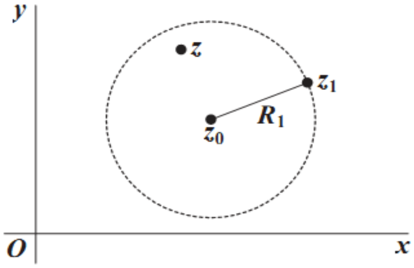
ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series.

We recall that a series of complex numbers converges *absolutely* if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

Theorem:

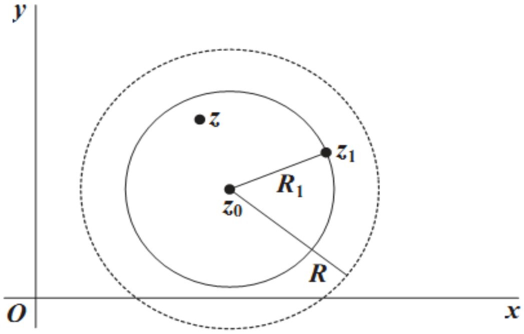
If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \ldots..$ *converges when* $z = z_1$ $(z_1 \neq z_0)$*, then it is absolutely convergent at each point* z *in the open disk* $|z - z_0| < R_1$ *where* $R_1 = |z_1 - z_0|$



The theorem tells us that the set of all points inside some circle centered at z_0 is a region of convergence for the power series , provided it converges at some point other than z_0. The greatest circle centered at z_0 such that series converges at each point inside is called the *circle of convergence* of series. The series cannot converge at any point z_2 outside that circle, according to the theorem ; for if it did, it would converge everywhere inside the circle centered at z_0 and passing through z_2. The first circle could not, then, be the circle of convergence.

Theorem:

If z_1 *is a point inside the circle of convergence* $|z - z_0| = R$ *of a power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *then that series must be uniformly convergent in the closed disk* $|z - z_0| \leq R_1$*, where* $R_1 = |z_1 - z_0|$



LAURENT SERIES

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $(z - z_0)$.We now present the theory of such representations, and we begin with ***Laurent's theorem.***

Theorem:

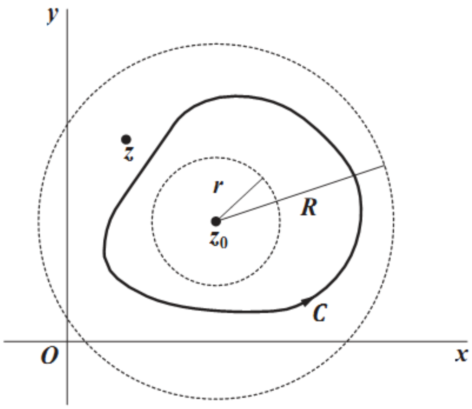
Suppose that a function f is analytic throughout an annular domain $r < |z - z_0| < R$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain (Fig). *Then, at each point in the domain, $f(z)$ has the series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$ *and* $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

∞ *For $n = 0,1,2,3,\ldots\ldots\ldots\ldots$* ∞

$$b_n = \frac{1}{2\pi i}\oint_C (z^* - z_0)^{n-1} f(z^*)\, dz^*$$



PROOF:

Let $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ be two concentric circles forming the annular domain D such that $r < |\boldsymbol{z} - \boldsymbol{z_0}| < R$ then suppose that $\boldsymbol{z} = \boldsymbol{z_0} + \boldsymbol{h}$ is a point in this annular domain D so f is analytic in this annular domain. So by using C. I. Formula

$$f(z_0 + h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)} dz = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-(z_0+h)} dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-(z_0+h)} dz$$

$$f(z_0 + h) = I_1 + I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Now let $I_1 = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-z_0)-h} dz$

$$I_1 = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-z_0)\left[1-\frac{h}{z-z_0}\right]} dz$$

$$I_1 = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-z_0)}\left[1 - \frac{h}{z-z_0}\right]^{-1} dz \qquad \ldots\ldots\text{(i)}$$

Consider

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}+\ldots\ldots\ldots$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}\left[1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots\right]\ldots\ldots\ldots$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}\left[1-\frac{h}{z-z_0}\right]^{-1}$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^{n+1}}\left[\frac{z-z_0-h}{z-z_0}\right]^{-1}$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}(z-z_0)}{(z-z_0)^{n+1}(z-z_0-h)}$$

$$\left[1-\frac{h}{z-z_0}\right]^{-1}=1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^n(z-z_0-h)}\quad\ldots\ldots\text{(ii)}$$

$$(i)\Rightarrow I_1=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)}\left[1+\frac{h}{z-z_0}+\frac{h^2}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^n}+\frac{h^{n+1}}{(z-z_0)^n(z-z_0-h)}\right]dz$$

$$\Rightarrow I_1=\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)}dz+\frac{h}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^2}dz+\frac{h^2}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^3}dz+\cdots+\frac{h^n}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^{n+1}}dz+\frac{h^{n+1}}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^{n+1}(z-z_0-h)}dz$$

$$I_1=f(z_0)+\frac{h}{1!}f'(z_0)+\frac{h^2}{2!}f''(z_0)+\cdots+\frac{h^n}{n!}f^n(z_0)+R_n\quad\ldots\ldots\ldots\ldots\text{(iii)}$$

Where $R_n=\frac{h^{n+1}}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^{n+1}(z-z_0-h)}dz$

Now we will prove $R_n\to 0$ as $n\to\infty$

$$\Rightarrow |R_n|=\left|\frac{h^{n+1}}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^{n+1}(z-z_0-h)}dz\right|$$

$$\Rightarrow |R_n|\le\frac{h^{n+1}}{2\pi|i|}\int_{C_2}\left|\frac{f(z)}{(z-z_0-h)}\right|\frac{|dz|}{|z-z_0|^{n+1}}<\frac{h^{n+1}}{2\pi|i|}\frac{M}{R^{n+1}}\int_{C_2}|dz|$$

$$\Rightarrow |R_n|<\frac{h^{n+1}}{2\pi|i|}\frac{M}{R^{n+1}}(2\pi R)=Mh\left(\frac{h}{R}\right)^n\ \Rightarrow |R_n|<Mh\left(\frac{h}{R}\right)^n\to 0\text{ as }n\to\infty$$

Then equation (iii) becomes

$$I_1=f(z_0)+\frac{h}{1!}f'(z_0)+\frac{h^2}{2!}f''(z_0)+\cdots\ldots\ldots+\frac{h^n}{n!}f^n(z_0)$$

$$I_1=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n\quad \textit{with}\quad z-z_0=h$$

$$I_1=\sum_{n=0}^{\infty}a_n(z-z_0)^n\quad \textit{with}\quad a_n=\frac{f^n(z_0)}{n!}$$

Now consider $I_2=\frac{-1}{2\pi i}\int_{C_1}\frac{f(z)}{z-(z_0+h)}dz=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z_0-z+h}dz$

$$I_2=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{h\left[1-\frac{z-z_0}{h}\right]}dz$$

$$I_2=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{h}\left[1-\frac{z-z_0}{h}\right]^{-1}dz\quad\ldots\ldots\ldots\text{(iv)}$$

Consider

$$\left[1-\frac{z-z_0}{h}\right]^{-1}=1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots+\frac{(z-z_0)^n}{h^n}+\frac{(z-z_0)^{n+1}}{h^{n+1}}+\ldots\ldots\ldots$$

$$\left[1-\frac{z-z_0}{h}\right]^{-1}=1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots+\frac{(z-z_0)^n}{h^n}+\frac{(z-z_0)^{n+1}}{h^{n+1}}\left[1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots\right]\ldots\ldots\ldots$$

$$\left[1-\frac{z-z_0}{h}\right]^{-1}=1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots+\frac{(z-z_0)^n}{h^n}+\frac{(z-z_0)^{n+1}}{h^{n+1}}\left[1-\frac{z-z_0}{h}\right]^{-1}$$

$$\left[1-\frac{z-z_0}{h}\right]^{-1}=1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots+\frac{(z-z_0)^n}{h^n}+\frac{(z-z_0)^{n+1}}{h^{n+1}}\left[\frac{h-z+z_0}{h}\right]^{-1}$$

$$\left[1-\frac{z-z_0}{h}\right]^{-1}=1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots+\frac{(z-z_0)^n}{h^n}+\frac{(z-z_0)^{n+1}}{h^n(h-z+z_0)}\quad\ldots\ldots\ldots..(\text{v})$$

$$(iv)\Rightarrow I_1=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{h}\left[1+\frac{z-z_0}{h}+\frac{(z-z_0)^2}{h^2}+\cdots+\frac{(z-z_0)^n}{h^n}+\frac{(z-z_0)^{n+1}}{h^n(h-z+z_0)}\right]dz$$

$$\Rightarrow I_2=$$

$$\frac{1}{2\pi ih}\int_{C_1}f(z)dz+\frac{1}{2\pi ih^2}\int_{C_1}f(z)(z-z_0)^2dz+\frac{1}{2\pi ih^3}\int_{C_1}f(z)(z-z_0)^3dz+\cdots+\frac{1}{2\pi ih^n}\int_{C_1}f(z)(z-z_0)^ndz+\frac{1}{2\pi ih^{n+1}}\int_{C_1}\frac{f(z)(z-z_0)^{n+1}}{(h-z+z_0)}dz$$

$$\Rightarrow I_2=$$

$$\frac{1}{h}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^{-1+1}}dz\right]+\frac{1}{h^2}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^{-2+1}}\right]dz+\frac{1}{h^3}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^{-3+1}}\right]+\cdots+\frac{1}{h^n}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^{-n+1}}\right]+\frac{-1}{2\pi ih^{n+1}}\int_{C_1}\frac{f(z)(z-z_0)^{n+1}}{z-(z_0+h)}dz$$

$$I_2=\frac{1}{h}b_1+\frac{1}{h^2}b_2+\frac{1}{h^3}b_3+\cdots+\frac{1}{h^n}b_n+R_n\quad\ldots\ldots\ldots..(\text{vi})$$

Where $\qquad R_n=\frac{-1}{2\pi ih^{n+1}}\int_{C_1}\frac{f(z)(z-z_0)^{n+1}}{z-(z_0+h)}dz$

Now we will prove $R_n\to 0$ as $n\to\infty$

$$\Rightarrow |R_n|=\left|\frac{-1}{2\pi ih^{n+1}}\int_{C_1}\frac{f(z)(z-z_0)^{n+1}}{z-(z_0+h)}dz\right|$$

$$\Rightarrow |R_n|\le\frac{1}{2\pi|i|h^{n+1}}\int_{C_1}\left|\frac{f(z)}{z-(z_0+h)}\right||z-z_0|^{n+1}|dz|<\frac{1}{2\pi h^{n+1}}Mr^{n+1}\int_{C_1}|dz|$$

$$\Rightarrow |R_n|<\frac{1}{2\pi h^{n+1}}Mr^{n+1}(2\pi r)=Mr\left(\frac{r}{h}\right)^n$$

$$\Rightarrow |R_n|<Mr\left(\frac{r}{h}\right)^n\to 0\text{ as }n\to\infty$$

Then equation (vi) becomes

$$I_2=\frac{1}{h}b_1+\frac{1}{h^2}b_2+\frac{1}{h^3}b_3+\cdots+\frac{1}{h^n}b_n$$

$$I_2=\sum_{n=1}^{\infty}\frac{1}{h^n}b_n\Rightarrow I_1=\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

Using both values in (A) we get the required

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

Remark:

- **Uniqueness property of Laurent's Theorem**:
  suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$; $r\ |z-z_0|< \ < R$ then the series is necessarily identical with the Laurent's Series for $f(z)$
- **Uniqueness property of Taylor's Theorem**:
  suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$; $r < |z-z_0| < R$ then the series is necessarily identical with the Taylor's Series for $f(z)$
- Suppose that $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ be two Laurent's Series expansions which converges in the same annulus then their Product $f(z).g(z)$ also converges and represents the Laurent's Series expansion.
- If z replace by $\frac{1}{z}$ in a given function $f(z)$ then $f(z)$ does not change. Then we have $a_n = b_n = a_{-n}$
- Finding Laurent's Series if condition appears in the form $|z-z_0| < R$ then take constant as common and no need to take variable in most cases while if condition appears in the form $|z-z_0| > R$ then take variable as common necessarily from denominator.

Example:

Show that $f(z) = \frac{1}{4z-z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$ when $0 < |z| < 4$

Solution:

Given that $f(z) = \frac{1}{4z-z^2} = \frac{1}{4z\left[1-\frac{z}{4}\right]} = \frac{1}{4z}\left[1-\frac{z}{4}\right]^{-1}$

$$f(z) = \frac{1}{4z-z^2} = \frac{1}{4z}\left[1+\frac{z}{4}+\frac{z^2}{4^2}+\frac{z^3}{4^3}+\cdots\right]$$

$$f(z) = \frac{1}{4z-z^2} = \left[\frac{1}{4z}+\frac{1}{4z}.\frac{z}{4}+\frac{1}{4z}.\frac{z^2}{4^2}+\frac{1}{4z}.\frac{z^3}{4^3}+\cdots\right]$$

$$f(z) = \frac{1}{4z-z^2} = \frac{1}{4z}+\frac{1}{4^2}+\frac{z^1}{4^3}+\frac{z^2}{4^4}+\cdots$$

$$f(z) = \frac{1}{4z-z^2} = \frac{z^{-1}}{4}+\frac{z^0}{4^2}+\frac{z^1}{4^3}+\frac{z^2}{4^4}+\cdots$$

$$f(z) = \frac{1}{4z-z^2} = \frac{z^{0-1}}{4^{0+1}}+\frac{z^{-1+1}}{4^{1+1}}+\frac{z^{2-1}}{4^{2+1}}+\frac{z^{3-1}}{4^{3+1}}+\cdots$$

Thus $f(z) = \frac{1}{4z-z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$ when $0 < |z| < 4$

Example:

For the function $f(z) = \frac{Z+1}{Z-1}$ find the Laurent's series representation in the punctured disk $0 < |z| < \infty$

Solution:

Given function $f(z) = \frac{Z+1}{Z-1}$ has the singular point z = 1 and analytic in the given domain and The representation of $f(z)$ in the unbounded domain $0 < |z| < \infty$ is a Laurent Series and the fact that $\left|\frac{1}{z}\right| < 1$ when 'z' is a point in given domain then replacing 'z' with '1/z'

$$f(z) = \frac{Z+1}{Z-1} = -\frac{1+\frac{1}{z}}{1-\frac{1}{z}} = -\left(1+\frac{1}{z}\right)\frac{1}{1-\frac{1}{z}} = -\left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{1}{z^n}$$

$$f(z) = \left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \quad \text{since } a_n = b_n = a_{-n}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=1}^{\infty}\frac{1}{z^n} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} \quad \text{for 'n – 1' in place of 'n'}$$

Example (Use of Maclaurin Series)

Find the Laurent series of $z^{-5}\sin z$ with center 0.

Solution

$$z^{-5}\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040}z^2 + - \cdots \qquad (|z| > 0).$$

Here the "annulus" of convergence is the whole complex plane without the origin and the principal part of the series at 0 is $z^{-4} - \frac{1}{6}z^{-2}$.

Example (Substitution)

Find the Laurent series of $z^2e^{1/z}$ with center 0.

Solution

$$z^2e^{1/z} = z^2\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots \qquad (|z| > 0)$$

Example (Development of 1/(1-z))

Develop $1/(1-z)$ **(a)** in nonnegative powers of z, **(b)** in negative powers of z.

Solution.

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1\text{).}$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1\text{).}$$

Example (Laurent Expansions in Different Concentric Annuli)

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

Solution

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1\text{).}$$

Multiplying with 1/z

(I) $$\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1),$$

(II) $$\frac{1}{z^3 - z^4} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \qquad (|z| > 1).$$

Example (Use of Partial Fractions)

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z + 3}{z^2 - 3z + 2}$ with center 0.

Solution. In terms of partial fractions,

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}.$$

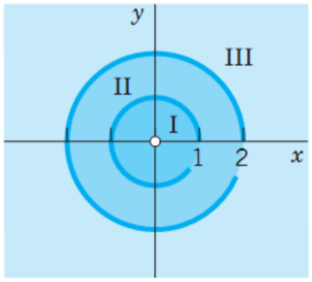
(a) and (b) in Example 3 take care of the first fraction. For the second fraction,

(c) $$-\frac{1}{z-2} = \frac{1}{2\left(1 - \frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

(d) $$-\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$

(I) From (a) and (c), valid for $|z| < 1$ (see Fig. 371),

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots .$$



(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots .$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots .$$

Example:

Show that $f(z) = Cos\left(z + \frac{1}{z}\right)$ can be expand as a Laurent's Series

$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$ where $a_n = \frac{1}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)\, Cosn\theta d\theta$

Solution: Given that $f(z) = Cos\left(z + \frac{1}{z}\right)$ and f is non – analytic at z = 0

Put $z = \frac{1}{z}$ in given function then $f(z) = Cos\left(\frac{1}{z} + z\right) = Cos\left(z + \frac{1}{z}\right)$

$f(z) = f\left(\frac{1}{z}\right) \Rightarrow a_n = b_n$ since $b_n = a_{-n}$

then Laurent's Series will be expand as

$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$

$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for $z_0 = 0$

$f(z) = a_0 + a_1z^1 + a_{-1}z^{-1} + a_2z^2 + a_{-2}z^{-2} + \ldots\ldots\ldots\ldots\ldots\ldots$

$f(z) = a_0 + a_1z^1 + \frac{a_{-1}}{z^1} + a_2z^2 + \frac{a_{-2}}{z^2} + \ldots\ldots\ldots\ldots\ldots\ldots$

$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \frac{a_{-n}}{z^n} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \frac{a_n}{z^n}$ $\quad\therefore a_n = b_n = a_{-n}$

$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$

Now we will find a_n

Since we know that $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

$\Rightarrow a_n = \frac{1}{2\pi i}\int_C \frac{Cos\left(z+\frac{1}{z}\right)}{(z-0)^{n+1}}dz \Rightarrow a_n = \frac{1}{2\pi i}\int_C \frac{Cos\left(z+\frac{1}{z}\right)}{z^{n+1}}dz$ ………(i)

Put

$z = e^{i\theta}, \frac{1}{z} = e^{-i\theta} \Rightarrow z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2Cos\theta, dz = ie^{i\theta}d\theta; 0 < \theta < 2\pi$

$$(i) \Rightarrow a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{Cos(2Cos\theta)}{(e^{i\theta})^{n+1}}\left(ie^{i\theta}d\theta\right) = \frac{1}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)e^{-ni\theta}d\theta$$

$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)(Cosn\theta - iSinn\theta)d\theta$

$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)Cosn\theta d\theta - \frac{i}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)Sinn\theta d\theta$

$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)Cosn\theta d\theta - \frac{i}{2\pi}I$ ………(ii)

To get required value of a_n we just show I = 0

Let $I = \int_0^{2\pi} Cos(2Cos\theta)Sinn\theta d\theta$

$I = \int_0^{2\pi} Cos[2Cos(2\pi - \theta)]Sinn(2\pi - \theta)d\theta \quad \therefore \int_0^a f(z)dz = \int_0^a f(a - z)dz$

$I = \int_0^{2\pi} Cos[2Cos(2\pi - \theta)]Sin(2n\pi - n\theta)d\theta$

$I = \int_0^{2\pi} Cos(2Cos\theta)(-Sinn\theta)d\theta = -\int_0^{2\pi} Cos(2Cos\theta)Sinn\theta d\theta = -I$

$I + I = 2I = 0 \Rightarrow I = 0$

$(ii) \Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} Cos(2Cos\theta)Cosn\theta d\theta$ which is our required.

RESIDUE INTEGRATION

ISOLATED SINGULAR POINTS

Recall that a point z_0 is called a **singular point** of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.

A singular point z_0 is said to be ***isolated*** if, there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic.

Example: The function $f(z) = \frac{z-1}{z^5(z^2+9)}$ has the three isolated singular points $z = 0$ and $z = \pm 3i$.

Example: The function $f(z) = \frac{1}{Sin\left(\frac{\pi}{z}\right)}$ has the singular points $z = 0$ and $z = 1/n$ $(n = 1, 2, \ldots)$. Each singular point except $z = 0$ is isolated. The singular point $z = 0$ is not isolated because every deleted ε neighborhood of the origin contains other singular points of the function.

ZERO OF A FUNCTION:

A number $\boldsymbol{z = z_0}$ is called zero of a function $\boldsymbol{f(z)}$ if $\boldsymbol{(z_0) = 0}$. further we can say that an analytic function $\boldsymbol{f(z)}$ has a zero of order 'n' at point $\boldsymbol{z = z_0}$ if $\boldsymbol{f(z_0) = 0, f'(z_0) = 0, f''(z_0) = 0}, \ldots \ldots, \boldsymbol{f^{n-1}(z_0) = 0}$ but $\boldsymbol{f^n(z_0) \neq 0}$

Remark:

i. A function $\boldsymbol{f(z)}$ is analytic in some disk $\boldsymbol{|z - z_0| < R}$ has a zero of order 'n' at $\boldsymbol{z = z_0}$ iff $\boldsymbol{f(z)}$ can be written as $\boldsymbol{f(z) = (z - z_0)^n \varphi(z)}$ where $\boldsymbol{\varphi(z)}$ is analytic at $\boldsymbol{z_0}$ and $\boldsymbol{\varphi(z_0) \neq 0}$

ii. A zero of order one is called simple zero.

iii. A zero of order 'n' is called a zero of multiplicity 'n'

Example:

i. For $\boldsymbol{(z) = z - 3i}$; $\boldsymbol{z_0 = 3i}$ we have $\boldsymbol{f(z_0) = 0}$ but $\boldsymbol{f'(z_0) \neq 0}$ then $\boldsymbol{z_0 = 3i}$ is a simple zero of given function.

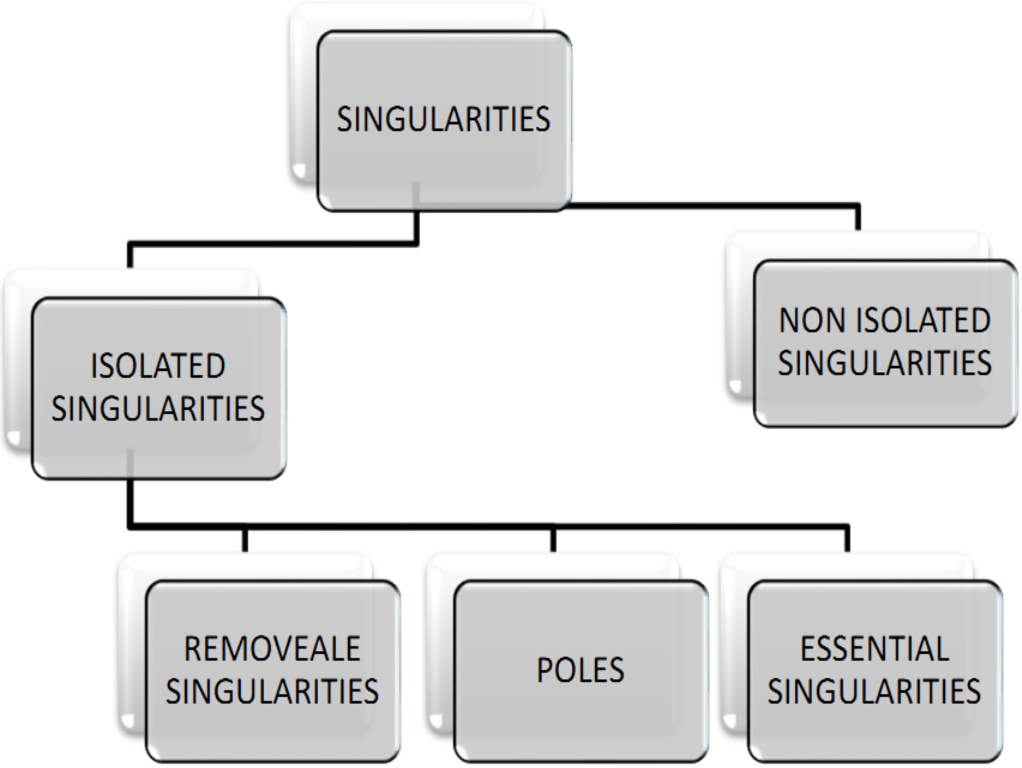
ii. For $\boldsymbol{(z) = (z - i)^3}$; $\boldsymbol{z_0 = i}$
we have $\boldsymbol{f(z_0) = 0, f'(z_0) = 0, f''(z_0) = 0}$ but $\boldsymbol{f'''(z_0) \neq 0}$ then $\boldsymbol{z_0 = i}$ is a zero of order 3 or zero of multiplicity 3.

SINGULARITY:
If a complex valued function $f(z)$ failed to be analytic at point $z = z_0$ then this point is said to be singular point or singularity.

Example:

i. Function $f(z) = \frac{e^z}{z-1}$ is non – analytic at $z_0 = 1$ so $z_0 = 1$ is a singularity of $f(z)$
ii. Function $f(z) = \frac{Sinz}{z}$ is non – analytic at $z_0 = 0$ so $z_0 = 0$ is a singularity of $f(z)$
iii. Function $f(z) = e^{\frac{1}{z-2}}$ is non – analytic at $z_0 = 2$ so $z_0 = 2$ is a singularity of $f(z)$

TYPES OF SINGULARITES



ISOLATED SINGULARITIES:

A point $\boldsymbol{z = z_0}$ is said to be an isolated singularity of a function $\boldsymbol{f(z)}$ if $\boldsymbol{f(z)}$ is analytic at each point in the neighborhood of $\boldsymbol{z = z_0}$ except at $\boldsymbol{z = z_0}$

Example:

i. Function $\boldsymbol{f(z) = \frac{e^z}{z-1}}$ is non – analytic at $\boldsymbol{z_0 = 1}$ so $\boldsymbol{z_0 = 1}$ is an isolated singularity of $\boldsymbol{f(z)}$

ii. Function $\boldsymbol{f(z) = \frac{1}{Sin\pi z}}$ is non – analytic at $\boldsymbol{z_0 = \frac{1}{k}}$; $\boldsymbol{k = \pm 1, \pm 2, \dots\dots}$ so $\boldsymbol{z_0 = \frac{1}{k}}$ is an isolated singularity of $\boldsymbol{f(z)}$

NON – ISOLATED SINGULARITIES:

A point $\boldsymbol{z = z_0}$ is said to be a non – isolated singularity of a function $\boldsymbol{f(z)}$ if in the neighborhood of $\boldsymbol{z = z_0}$ there exists other points where $\boldsymbol{f(z)}$ is not analytic.

Example:

i. Function $\boldsymbol{f(z) = logz}$ is non – analytic at $\boldsymbol{z_0 = 0}$ so $\boldsymbol{z_0 = 0}$ is a non – isolated singularity of $\boldsymbol{f(z)}$ there exist also other points where the function will be non – analytic.

ii. Function $\boldsymbol{f(z) = \frac{1}{Sin\pi z}}$ is non – analytic at $\boldsymbol{z_0 = 0}$ so $\boldsymbol{z_0 = 0}$ is a non – isolated singularity of $\boldsymbol{f(z)}$ there exist also other points where the function will be non – analytic.

REMOVEABLE (ARTIFICIAL) SINGULARITIES:

A point $\boldsymbol{z = z_0}$ is said to be a removable singularity of a function $\boldsymbol{f(z)}$ if the principle part of Laurent's Series expansion contains no term.

i.e. for Laurent's Series $\boldsymbol{f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}$

we have the form as follows;

$$\boldsymbol{f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n}$$

Example:

i. Function $\boldsymbol{f(z) = \frac{Sinz}{z}}$ has removable singularity at $\boldsymbol{z_0 = 0}$

Since $\boldsymbol{f(z) = \frac{Sinz}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}$ contains no Principle Part.

ii. Function $\boldsymbol{f(z) = \frac{1-Cosz}{z^2}}$ has removeable singularaity at $\boldsymbol{z_0 = 0}$

Since $\boldsymbol{f(z) = \frac{1-Cosz}{z^2} = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots}$ contains no Principle Part.

POLE

A point $z = z_0$ is called pole of order 'n' of a function $f(z)$ if the principle part of Laurent's Series expansion contains finite numbers of terms.

i.e. For Laurent's Series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

we have the form as follows;

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{k} \frac{b_n}{(z-z_0)^n}$$

Example:

i. Function $f(z) = \frac{Sinz}{z^4}$ has a pole of order '3' at $z_0 = 0$

Since $f(z) = \frac{Sinz}{z} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right] = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$ contains finite terms.

ii. Function $f(z) = \frac{1}{z-3}$ has a pole of order '1' at $z_0 = 3$

ESSENTIAL SINGULARITIES:

A point $z = z_0$ is called pole of order 'n' of a function $f(z)$ if the principle part of Laurent's Series expansion contains infinite numbers of terms.

i.e. for Laurent's Series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

we have the form as follows;

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Example:

i. Function $f(z) = Sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots$ has an essential singularity at $z_0 = 0$

ii. Function $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ has an essential singularity at $z_0 = 0$

Remark:

i. A function $f(z)$ is analytic in a domain $0 < |z - z_0| < R$ has a pole of order 'n' at $z = z_0$ iff $f(z)$ can be written as $f(z) = \frac{\varphi(z)}{(z-z_0)^n}$ where $\varphi(z)$ is analytic at z_0 and $\varphi(z_0) \neq 0$

ii. If the functions $g \& h$ are analytic at $z = z_0$ have a zero of order 'n' at $z = z_0$ and $g(z_0) \neq 0$ then the function $f(z) = \frac{g(z)}{h(z)}$ has a pole of order 'n' at $z = z_0$

Remark:

If want to check the behavior of function at infinity then

i. Make substitution $z = \frac{1}{w}$

ii. Investigate the behavior of new function at $w = 0$ actually that is $z = \infty$

Example:

Find the nature of singularity of the function $\boldsymbol{f(z) = e^{2z}}$ at $\boldsymbol{z = \infty}$

Solution:

Given that $\boldsymbol{f(z) = e^{2z}}$

Make substitution $z = \frac{1}{w}$ $\Rightarrow f\left(\frac{1}{w}\right) = e^{2/w}$ then $w = 0$ is singularity.

Let $g(w) = e^{2/w} = 1 + \frac{2}{w} + \frac{\left(\frac{2}{w}\right)^2}{2!} + \frac{\left(\frac{2}{w}\right)^3}{3!} + \cdots$

$\Rightarrow g(w) = e^{2/w} = 1 + \dfrac{2}{w} + \dfrac{2}{2w^2} + \dfrac{2}{6w^3} + \cdots$

$\Rightarrow g(w)$ has an essential singularity at $w = 0$

$\Rightarrow \boldsymbol{f(z)}$ has an essential singularity at $z = \infty$

Example:

Find the nature of singularity of the function $\boldsymbol{f(z) = z^2(z+1)}$ at $\boldsymbol{z = \infty}$

Solution:

Given that $\boldsymbol{f(z) = z^2(z+1)}$

Make substitution $z = \frac{1}{w}$ $\Rightarrow f\left(\frac{1}{w}\right) = \frac{1}{w^2}\left(\frac{1}{w} + 1\right)$ then $w = 0$ is singularity.

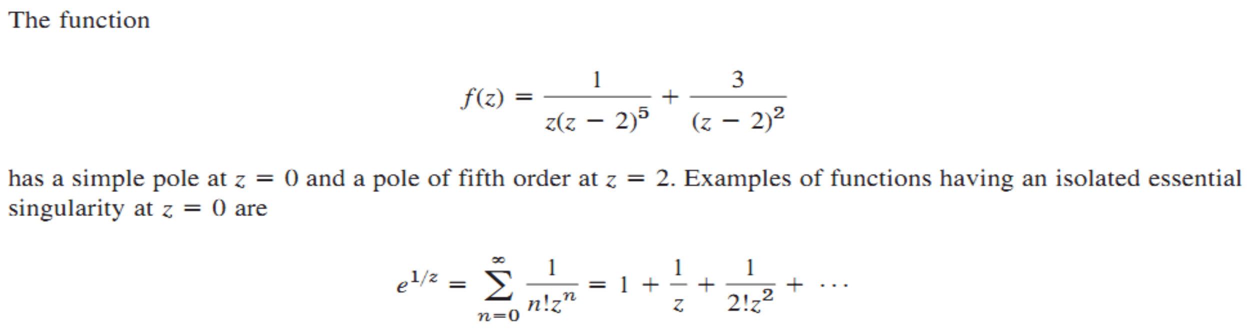
Let $g(w) = \frac{1}{w^2}\left(\frac{1}{w} + 1\right) = \frac{1}{w^2}\left(\frac{1+w}{w}\right) = \frac{1+w}{w^3}$

$\Rightarrow g(w) = \dfrac{1+w}{(w-0)^3} = \dfrac{\varphi(w)}{(w-0)^3}\,(say)$

$\Rightarrow g(w)$ has a pole of order 3 at $w = 0$

$\Rightarrow f(z)$ has a pole of order 3 at $z = \infty$

Example (Poles. Essential Singularities)



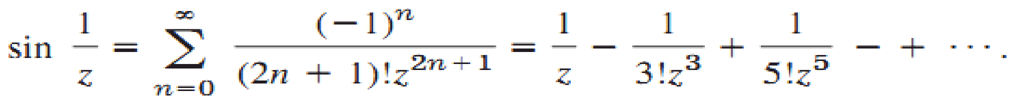
The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

And



$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots.$$

Theorem

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

Example (Behavior Near a Pole)

$f(z) = 1/z^2$ has a pole at $z = 0$, and $|f(z)| \to \infty$ as $z \to 0$ in any manner.

Picard's Theorem

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ϵ-neighborhood of z_0.

Example

The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Relationship between Poles and Zeros

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. Then $1/f(z)$ has a pole of nth order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.

Residue

DEFINATION: If a function f has an isolated singularity at a point $z = z_0$ then f has Laurent's expansion as follows;
$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ then coefficient b_1 of $\frac{1}{(z-z_0)}$ is called Rasidue of a function $f(z)$ at $z = z_0$ and it is denoted by
$b_1 = (f, z_0)$ or $b_1 = Res_{z=z_0} f(z)$

Example: Find the residue of $f(z) = \frac{1}{(z-1)^2(z-3)}$ at z = 1.

Solution: Given $f(z) = \frac{1}{(z-1)^2(z-3)}$

$$f(z) = \frac{1}{(z-1)^2(z-2-1)} = \frac{1}{(z-1)^2(2-(z-1))} = \frac{-1}{2(z-1)^2\left(1-\frac{(z-1)}{2}\right)}$$

$$f(z) = \frac{-1}{2(z-1)^2}\left(1-\frac{(z-1)}{2}\right)^{-1}$$

$$f(z) = \frac{-1}{2(z-1)^2}\left(1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right)$$

$$f(z) = \frac{-1}{2(z-1)^2}+\frac{-1}{4(z-1)}-\frac{1}{8}$$

$$f(z) = \frac{-\frac{1}{2}}{(z-1)^2}+\frac{-\frac{1}{4}}{(z-1)}-\frac{1}{8}$$

This is Laurent Series expansion in $\left|\frac{z-1}{2}\right| < 1$ or $0 < |z-1| < 2$ and by

definition of residue $Res(f,1) = -\frac{1}{4}$

Example: Find the residue of $f(z) = \frac{1}{Sinhz}$ at z = 0.

Solution: Given $f(z) = \frac{1}{Sinhz}$

$$f(z) = \frac{1}{z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots} = \frac{1}{z\left[1+\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)\right]}$$

$$f(z) = \frac{1}{z}\left[1+\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)\right]^{-1}$$

$$f(z) = \frac{1}{z}\left[1-\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)+\cdots\right]$$

$$f(z) = \frac{1}{z}+\frac{-z}{3!}+\frac{-z^3}{5!}+\cdots$$

This is Laurent's Series expansion at z = 0

Then by definition of residue $b_1 = (f,0) = 1$

Keep in mind: Pole is a finite order singularity. We will discuss it later.

THEOREM (Residue of a function at a pole of order 'n')

If $\boldsymbol{f(z)}$ has a pole of order 'n' at $\boldsymbol{z = z_0}$ then

$$\boldsymbol{Res(f, z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)]}$$

Proof:

If $\boldsymbol{f(z)}$ has a pole of order 'n' at $\boldsymbol{z = z_0}$ then the Laurent's Series expansion will be as follows;

$$\boldsymbol{f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{n} \frac{b_k}{(z-z_0)^k}}$$

$$\boldsymbol{f(z) = a_0(z-z_0)^0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}}$$

Multiplying both sides by $\boldsymbol{(z-z_0)^n}$

$$\boldsymbol{(z-z_0)^n f(z) = a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + a_2(z-z_0)^{n+2} + \cdots + b_1(z-z_0)^{n-1} + b_2(z-z_0)^{n-2} + \cdots + b_n} \quad \text{................(i)}$$

Now differentiating (i) w.r.to 'z' $\boldsymbol{(n - 1\ time)}$

$$\boldsymbol{\frac{d}{dz}[(z-z_0)^n f(z)] = na_0(z-z_0)^{n-1} + (n+1)a_1(z-z_0)^n + (n+2)a_2(z-z_0)^{n+1} + \cdots + (n-1)b_1(z-z_0)^{n-2} + (n-2)b_2(z-z_0)^{n-3} + \cdots + b_{n-1} + 0}$$

$$\boldsymbol{\frac{d^2}{dz^2}[(z-z_0)^n f(z)] = n(n-1)a_0(z-z_0)^{n-2} + n(n+1)a_1(z-z_0)^{n-1} + (n+1)(n+2)a_2(z-z_0)^n + \cdots + (n-1)(n-2)b_1(z-z_0)^{n-3} + (n-2)(n-3)b_2(z-z_0)^{n-4} + \cdots + b_{n-2} + 0 + 0}$$

Continuing in this manner we get

$$\boldsymbol{\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = [n(n-1)\ldots\ldots 3.2.1]a_0(z-z_0)^{n-(n-1)} + \cdots + [(n-1)(n-2)\ldots\ldots 3.2.1]b_1(z-z_0)^{n-n}}$$

$$\boldsymbol{\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = n!\,a_0(z-z_0) + (n-1)!\,b_1}$$

Applying $\boldsymbol{\lim\limits_{z\to z_0}}$ on both sides

$$\boldsymbol{\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = \lim_{z\to z_0}[n!\,a_0(z-z_0) + (n-1)!\,b_1]}$$

$$\boldsymbol{\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = 0 + (n-1)!\,b_1}$$

$$\boldsymbol{\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = (n-1)!\,b_1}$$

$$\boldsymbol{\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = b_1 = Res(f, z_0)}$$

Hence we get the result. $\boldsymbol{Res(f, z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)]}$

Remark:

i. If function has a simple pole at $\boldsymbol{z = z_0}$ then
$\boldsymbol{Res(f, z_0) = \lim_{z \to z_0}[(z - z_0)f(z)]}$

ii. For a quotient function $\boldsymbol{f(z) = \frac{g(z)}{h(z)}}$ where 'g' and 'h' are analytic at $\boldsymbol{z = z_0}$ then $\boldsymbol{Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}}$

iii. A function which has more than one poles is called **Meromorphic function**.

Example: Find the residue of $f(z) = \frac{z}{z^2+16}$ at all poles.

Solution: Given $f(z) = \frac{z}{z^2+16} = \frac{z}{(z+4i)(z-4i)}$ is a meromorphic function and

has two poles of order 1 (simple poles) at $z = \pm 4i$

Then by using Residue formula for Simple pole at $z = z_0$

$$Res(f, z_0) = \lim_{z \to z_0}[(z - z_0)f(z)]$$

$$Res(f, -4i) = \lim_{z \to -4i}\left[(z + 4i)\frac{z}{(z+4i)(z-4i)}\right] = \frac{1}{2}$$

$$Res(f, 4i) = \lim_{z \to 4i}\left[(z - 4i)\frac{z}{(z+4i)(z-4i)}\right] = \frac{1}{2}$$

Example: Find the residue of $f(z) = \frac{Cosz}{z^2(z-\pi)}$ at $z = \pi$

Solution: Given $f(z) = \frac{Cosz}{z^2(z-\pi)}$ has a simple pole at $z = \pi$

Then by using Residue formula for Simple pole at $z = z_0$

$$Res(f, z_0) = \lim_{z \to z_0}[(z - z_0)f(z)]$$

$$Res(f, \pi) = \lim_{z \to \pi}[(z - \pi)f(z)]$$

$$Res(f, \pi) = \lim_{z \to \pi}\left[(z - \pi)\frac{Cosz}{z^2(z-\pi)}\right] = \frac{Cos\pi}{\pi^2}$$

$$Res(f, \pi) = \frac{-1}{\pi^2}$$

Example:

Find the residue of $f(z)=\frac{e^z}{z^2(z-\pi i)^4}$ at $z=0,\pi i$

Solution:

Given $f(z)=\frac{e^z}{z^2(z-\pi i)^4}$ is a meromorphic function and has two poles at $z=0$ and $z=\pi i$

$z=0$ is a pole of order 2

And

$z=\pi i$ is a pole of order 4

By using formula $\boldsymbol{Res(f,z_0)=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)]}$

$$Res(f,0)=\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}[(z-0)^2 f(z)]$$

$$Res(f,0)=\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2\frac{e^z}{z^2(z-\pi i)^4}\right]=\lim_{z\to 0}\frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^4}\right]$$

$Res(f,0)=\lim_{z\to 0}\left[\frac{e^z(z-\pi i-4)}{(z-\pi i)^5}\right]=\frac{\pi-4i}{\pi^5}$ after solving

Also $Res(f,\pi i)=\frac{1}{(4-1)!}\lim_{z\to\pi i}\frac{d^{4-1}}{dz^{4-1}}[(z-\pi i)^4 f(z)]$

$$Res(f,\pi i)=\frac{1}{3!}\lim_{z\to\pi i}\frac{d^3}{dz^3}\left[(z-\pi i)^4\frac{e^z}{z^2(z-\pi i)^4}\right]$$

$$Res(f,\pi i)=\frac{1}{6}\lim_{z\to\pi i}\frac{d^3}{dz^3}\left[\frac{e^z}{z^2}\right]$$

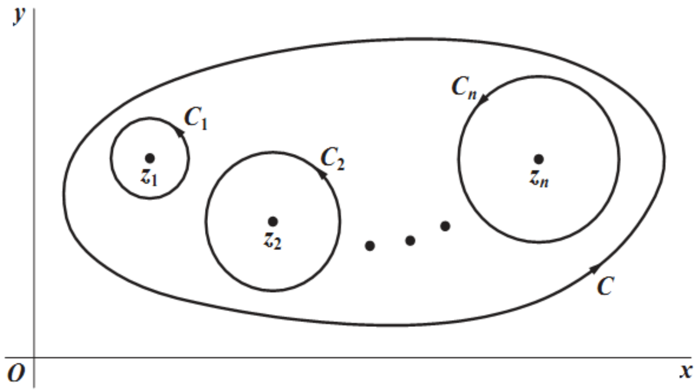
$Res(f,\pi i)=\frac{(\pi^3-18\pi)+(6\pi^2-24)i}{6\pi^5}$ after solving

CAUCHY's RESIDUE THEOREM

If, except for a *finite* number of singular points, a function f is analytic inside a simple closed contour C, those singular points must be isolated. The following theorem, which is known as *Cauchy's residue theorem,* is a precise statement of the fact that if f is also analytic on C and if C is positively oriented, then the value of the integral of f around C is $2\pi i$ times the *sum* of the residues of f at the singular points inside C.

Theorem. *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k ($k = 1, 2, \ldots, n$) inside C* (Fig)*, then*

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$



Proof:

Let $C_1, C_2, C_3, \ldots\ldots\ldots\ldots\ldots, C_n$ are the circles with center $z_1, z_2, z_3, \ldots\ldots, z_n$ and radius of each is 'r' as 'r' is so small that these circles do not overlap and lies inside C.

Now $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots\ldots + \int_{C_n} f(z)dz$ …….(i)

Suppose $f(z)$ has a pole (finite order singularity) of order 'm' at $z = z_1$ then the Laurent's Series expansion will be

$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n}$

Let $f(z) = [\varphi(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n] + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$

$\int_{C_1} f(z)dz = \int_{C_1} \varphi(z)dz + \int_{C_1} \frac{b_1}{z-z_0}dz + \int_{C_1} \frac{b_2}{(z-z_0)^2}dz + \cdots + \int_{C_1} \frac{b_m}{(z-z_0)^m}dz$

By using the following two results;

$\int_{C_1} \varphi(z)dz = 0$ and $\int_{C_1} \frac{1}{(z-z_0)^m}dz = \begin{cases} 2\pi i & if\ m = 1 \\ 0 & if\ m \neq 1 \end{cases}$

Then $\int_{C_1} f(z)dz = 0 + b_1(2\pi i) + b_2(0) + \cdots + b_m(0)$

$\int_{C_1} f(z)dz = b_1(2\pi i)$

$\int_{C_1} f(z)dz = 2\pi i Res(f, z_1)$

Similarly $\int_{C_2} f(z)dz = 2\pi i Res(f, z_2)$

And continuing in this manner we get $\int_{C_n} f(z)dz = 2\pi i Res(f, z_n)$

Using all values in (i)

$\int_C f(z)dz = 2\pi i Res(f, z_1) + 2\pi i Res(f, z_2) + \ldots\ldots\ldots + 2\pi i Res(f, z_n)$

$\int_C f(z)dz = 2\pi i[Res(f, z_1) + Res(f, z_2) + \cdots + Res(f, z_n)]$

$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$ required result.

Example: Evaluate the integral $\int_C \frac{1}{(z-1)(z+2)^2} dz$ with $C: |z| = 3$

Solution: Given $f(z) = \frac{1}{(z-1)(z+2)^2}$ is a meromorphic function and has two poles at $z = 1$ and $z = -2$

$z = 1$ is simple pole And $z = -2$ is a pole of order 2

For simple poles using $Res(f, z_0) = \lim_{z\to z_0}[(z - z_0)f(z)]$

$Res(f, 1) = \lim_{z\to 1}\left[(z - 1)\frac{1}{(z-1)(z+2)^2}\right] = \frac{1}{9}$

By using formula $Res(f, z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)]$

$$Res(f, -2) = \frac{1}{(2-1)!}\lim_{z\to -2}\frac{d^{2-1}}{dz^{2-1}}[(z+2)^2 f(z)]$$

$$Res(f, -2) = \frac{1}{1!}\lim_{z\to -2}\frac{d}{dz}\left[(z+2)^2\frac{1}{(z-1)(z+2)^2}\right] = \lim_{z\to -2}\frac{d}{dz}\left[\frac{1}{(z-1)}\right]$$

$Res(f, -2) = \lim_{z\to -2}\left[\frac{-1}{(z-1)^2}\right] = -\frac{1}{9}$ after solving

Now $\int_C \frac{1}{(z-1)(z+2)^2} dz = 2\pi i \sum_{k=1}^{2} Res(f, z_k)$

$\int_C \frac{1}{(z-1)(z+2)^2} dz = 2\pi i[Res(f, 1) + Res(f, -2)] = 0$

$\int_C \frac{1}{(z-1)(z+2)^2} dz = 0$

Example: Evaluate the integral $\int_C \frac{z^2-z+1}{(z-1)(z-4)(z+3)} dz$ with $C: |z| = 5$

Solution: Given $f(z) = \frac{z^2-z+1}{(z-1)(z-4)(z+3)}$ has three poles at $z = 1, z = 4$ and $z = -3$ of order 1 (Simple poles)

For simple poles using $Res(f, z_0) = \lim_{z\to z_0}[(z - z_0)f(z)]$

$$Res(f, 1) = \lim_{z\to 1}\left[(z-1)\frac{z^2-z+1}{(z-1)(z-4)(z+3)}\right] = \lim_{z\to 1}\left[\frac{z^2-z+1}{(z-4)(z+3)}\right] = -\frac{1}{12}$$

$$Res(f, 4) = \lim_{z\to 4}\left[(z-4)\frac{z^2-z+1}{(z-1)(z-4)(z+3)}\right] = \lim_{z\to 4}\left[\frac{z^2-z+1}{(z-1)(z+3)}\right] = \frac{13}{21}$$

$$Res(f, -3) = \lim_{z\to -3}\left[(z+3)\frac{z^2-z+1}{(z-1)(z-4)(z+3)}\right] = \lim_{z\to -3}\left[\frac{z^2-z+1}{(z-1)(z-4)}\right] = \frac{13}{28}$$

Now $\int_C \frac{z^2-z+1}{(z-1)(z-4)(z+3)} dz = 2\pi i \sum_{k=1}^{3} Res(f, z_k)$

$$\int_C \frac{z^2-z+1}{(z-1)(z-4)(z+3)} dz = 2\pi i[Res(f, 1) + Res(f, 4) + Res(f, -3)] = 1$$

$$\int_C \frac{z^2-z+1}{(z-1)(z-4)(z+3)} dz = 1$$

Example (Evaluation of an Integral by Means of a Residue)

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

Solution

We obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!} - \frac{z^3}{7!} + - \cdots$$

which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -\frac{1}{3}!$. From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4} dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

Example (Use the Right Laurent Series!)

Integrate $f(z) = 1/(z^3 - z^4)$ clockwise around the circle C: $|z| = \frac{1}{2}$.

Solution

$z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$. Now $z = 1$ lies outside C. Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$.

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1).$$

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

Example (Residue at a Simple Pole)

$f(z) = (9z + i)/(z^3 + z)$ has a simple pole at i because $z^2 + 1 = (z + i)(z - i)$,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z \to i} (z - i) \frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

Example (Residue at a Pole of Higher Order)

$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$ has a pole of second order at $z = 1$ because the denominator equals $(z + 4)(z - 1)^2$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{d}{dz}[(z - 1)^2 f(z)] = \lim_{z \to 1} \frac{d}{dz}\left(\frac{50z}{z + 4}\right) = \frac{200}{5^2} = 8.$$

Example (Integration by the Residue Theorem. Several Contours)

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4 - 3z}{z^2 - z}\, dz$$

Solution

The integrand has simple poles at 0 and 1, with residues

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

Answer: (a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0.

Example

Integrate $(\tan z)/(z^2-1)$ counterclockwise around the circle C: $|z| = \frac{3}{2}$.

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C. Because of the denominator $z^2 - 1 = (z-1)(z+1)$ the given function has simple poles at ± 1. We thus obtain from (4) and the residue theorem

$$\oint_C \frac{\tan z}{z^2-1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=1} \frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1} \frac{\tan z}{z^2-1}\right)$$

$$= 2\pi i\left(\frac{\tan z}{2z}\bigg|_{z=1} + \frac{\tan z}{2z}\bigg|_{z=-1}\right)$$

$$= 2\pi i \tan 1 = 9.7855i.$$

■

Example (Poles and Essential Singularities)

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz.$$

Solution. Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4); note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16},$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2}\cdot\frac{1}{z} + \cdots \qquad (|z| > 0).$$

Answer: $2\pi i(-\frac{1}{16} - \frac{1}{16} + \frac{1}{2}\pi^2) = \pi(\pi^2 - \frac{1}{4})i = 30.221i$ by the residue theorem.

■

Residue Integration of Real Integrals

TYPE – I:

If we have an integral of the form $\int_0^{2\pi} F(Sin\theta, Cos\theta)\, d\theta$ where F is a rational function of $Sin\theta$ and $Cos\theta$ then we can solve it using following procedure.

- Put $z = e^{i\theta} \;\; ; 0 \le \theta \le 2\pi \Rightarrow \frac{dz}{d\theta} = ie^{i\theta} = iz \;\Rightarrow d\theta = \frac{dz}{iz}$
- Put $Sin\theta = \frac{z - z^{-1}}{2i}$ and $Cos\theta = \frac{z + z^{-1}}{2}$
- Rewrite the integral in the form $\int_0^{2\pi} F(\theta)\, d\theta = \int_C F(z) dz$ where C is positively oriented unit circle $|z| = 1$
- Calculate the poles of $F(z)$. Say at $z_0, z_1, z_2, \ldots\ldots\ldots$ select those poles which lie in the unit circle $|z| = 1$. Then find the residues at the selected poles $R_1(F, z_0), R_2(F, z_1)$ etc.
- Using Cauchy Residue Formula make the form as follows;

$$\int_0^{2\pi} F(Sin\theta, Cos\theta)\, d\theta = \int_C F(z) dz = 2\pi i \sum_{j=1}^{n} R_j$$

Example

Show by the present method that $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi.$

Solution. We use $\cos\theta = \frac{1}{2}(z + 1/z)$ and $d\theta = dz/iz$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$

$$= -\frac{2}{i} \oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

We see that the integrand has a simple pole at $z_1 = \sqrt{2} + 1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2 = \sqrt{2} - 1$ (where $z - \sqrt{2} + 1 = 0$) inside C with residue [by (3), Sec. 16.3]

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left[\frac{1}{z - \sqrt{2} - 1}\right]_{z=\sqrt{2}-1}$$

$$= -\frac{1}{2}.$$

Answer: $2\pi i(-2/i)(-\frac{1}{2}) = 2\pi$. (Here $-2/i$ is the factor in front of the last integral.) ■

Example: Prove that $\int_0^{2\pi}\frac{d\theta}{a+bCos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$ where $a>b>0$

Solution:

- Put $z=e^{i\theta}\ \ ;0\le\theta\le2\pi\Rightarrow\frac{dz}{d\theta}=ie^{i\theta}=iz\ \Rightarrow d\theta=\frac{dz}{iz}$
- $Cos\theta=\frac{z+z^{-1}}{2}$
- Rewrite the integral in the form $\int_0^{2\pi}\frac{d\theta}{a+bCos\theta}=\int_C\frac{\frac{dz}{iz}}{a+b\left(\frac{z+z^{-1}}{2}\right)}$ where C is positively oriented unit circle $|z|=1$

  $\Rightarrow\int_0^{2\pi}\frac{d\theta}{a+bCos\theta}=\frac{2}{i}\int_C\frac{dz}{bz^2+2az+b}=\frac{2}{i}\int_C F(z)dz$ with $F(z)=\frac{1}{bz^2+2az+b}$
- To Calculate the poles of $F(z)$

  Firstly we will find the roots of $bz^2+2az+b$ that will be as follows;

  $\frac{-a\pm\sqrt{a^2-b^2}}{b}$ we may write these as $\propto=\frac{-a+\sqrt{a^2-b^2}}{b},\beta=\frac{-a-\sqrt{a^2-b^2}}{b}$

  Since $a>b>0\Rightarrow|\beta|>1,\propto\beta=1$ as $|\propto|<1$

  Here $z=\propto$ is the only simple pole which lie inside the C i.e. $|z|=1$

  $\Rightarrow Res(F,\propto)=\lim_{z\to\propto}[(z-\propto)F(z)]=\lim_{z\to\propto}\left[(z-\propto)\frac{1}{bz^2+2az+b}\right]$

  $\Rightarrow Res(F,\propto)=\lim_{z\to\propto}\left[(z-\propto)\frac{1}{b(z-\propto)(z-\beta)}\right]=\lim_{z\to\propto}\left[\frac{1}{b(z-\beta)}\right]=\frac{1}{b(\propto-\beta)}$

  $\Rightarrow Res(F,\propto)=\frac{1}{2b\frac{\sqrt{a^2-b^2}}{b}}=\frac{1}{2\sqrt{a^2-b^2}}$
- Using Cauchy Residue Formula make the form as follows;

  $\Rightarrow\int_0^{2\pi}\frac{d\theta}{a+bCos\theta}=\frac{2}{i}\int_C F(z)dz=\frac{2}{i}\times2\pi i\times Res(F,\propto)=\frac{2}{i}\times2\pi i\times\frac{1}{2\sqrt{a^2-b^2}}$

  $\Rightarrow\int_0^{2\pi}\frac{d\theta}{a+bCos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$

Likewise: we can Prove that $\int_0^{2\pi}\frac{d\theta}{a+bSin\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$ where $a>b>0$

Example: Prove that $\int_0^{2\pi}\frac{d\theta}{(a+bCos\theta)^2}=\frac{2\pi a}{(a^2-b^2)^{3/2}}$

Solution:

Since we know that $\int_0^{2\pi}\frac{d\theta}{a+bCos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$

$\Rightarrow\int_0^{2\pi}-\frac{d\theta}{(a+bCos\theta)^2}=-\frac{1}{2}.\frac{2\pi a}{\left(a^2-b^2\right)^{\frac{3}{2}}}.2a$ taking derivative w.r.to ‘a’

$$\Rightarrow\int_0^{2\pi}\frac{d\theta}{(a+bCos\theta)^2}=\frac{2\pi a}{\left(a^2-b^2\right)^{3/2}}$$

TYPE – II:

If we have an integral of the form $\int_0^\infty f(x)\,dx$ or $\int_{-\infty}^{\infty} f(x)\,dx$ then we can solve it using following procedure.

- Replace 'x' by 'z' in the integrand and test whether $zf(z) \to 0$ as $|z| \to \infty$
- Find the poles of $f(z)$, locate those poles which lie in the upper half plane. Find the residue at the located poles.
- Use formula $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$ or $\int_0^\infty f(x)\,dx = \pi i \sum R^+$ where $\sum R^+$ denotes sum of residues at poles lying in the upper half plane.

Remark:

- No poles lies on the real axis.
- Let $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials such that $Q(z) = 0$ has no real roots. And the degree of $P(z)$ is at least 2 less than that of $Q(z)$ so that $zf(z) \to 0$ as $|z| \to \infty$

Example: Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{3\pi}{8}$

Solution:

- Replace 'x' by 'z' in the integrand and test whether $zf(z) \to 0$ as $|z| \to \infty$

Given $f(x) = \frac{1}{(x^2+1)^2} \Rightarrow f(z) = \frac{1}{(z^2+1)^2} \Rightarrow zf(z) = z\frac{1}{(z^2+1)^2}$

Clearly $zf(z) \to 0$ as $|z| \to \infty$

- Find the poles of $f(z)$, locate those poles which lie in the upper half plane.

  The poles of $f(z)$ are at $z = \pm i$ of order 3. The only pole which lies in the upper half plane is $z = i$ of order 3.
- Find the residue at the located poles. i.e. $z = i$ of order 3

  $$Res(f,i) = \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z-i)^3\frac{1}{(z-i)^3(z+i)^3}\right]$$

  $$Res(f,i) = \frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right] = \frac{6}{32i}$$
- Use formula $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$

  $\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^2} = 2\pi i \times \frac{6}{32i} = \frac{3\pi}{8}$ as required.

Example: Prove that $\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^4}$ where a > 0

Solution:

- Replace 'x' by 'z' in the integrand and test whether $zf(z) \to 0$ as $|z| \to \infty$

Given $f(x) = \frac{dx}{x^4+a^4} \Rightarrow f(z) = \frac{1}{z^4+a^4} \Rightarrow zf(z) = z\left(\frac{1}{z^4+a^4}\right)$

Clearly $zf(z) \to 0$ as $|z| \to \infty$

- Find the poles of $f(z)$, locate those poles which lie in the upper half plane.

The poles of $f(z)$ are the roots of $z^4 + a^4 = 0$

$\Rightarrow z^4 = -a^4 \Rightarrow z^4 = a^4 e^{(2n+1)\pi i} \Rightarrow z = ae^{\frac{(2n+1)\pi i}{4}}$; n = 0,1,2,3

The poles are at $ae^{\frac{\pi i}{4}}, ae^{\frac{3\pi i}{4}}, ae^{\frac{5\pi i}{4}}, ae^{\frac{7\pi i}{4}}$. The only pole which lies in the upper half plane are $ae^{\frac{\pi i}{4}}, ae^{\frac{3\pi i}{4}}$

- Find the residue at the located poles

Let $z = \beta$ denote any one of these poles. Such that in $z^4 + a^4 = 0$ we have $\beta^4 = -a^4$

$$Res(f,\beta) = \lim_{z\to\beta}\left[(z-\beta)\frac{1}{z^4+a^4}\right] \quad ; \frac{0}{0}\ form$$

$$Res(f,\beta) = \lim_{z\to\beta}\left[\frac{1}{4z^3}\right] = \frac{1}{4\beta^3} = \frac{\beta}{4\beta^4} = \frac{\beta}{-4a^4} \quad \therefore \beta^4 = -a^4$$

$$Sum\ of\ Residues = \sum R^+ = -\frac{1}{4a^4}[\beta] = -\frac{1}{4a^4}\left[e^{\frac{\pi i}{4}} + e^{\frac{3\pi i}{4}}\right]$$

$$Sum\ of\ Residues = \sum R^+ = -\frac{1}{4a^4}\left[\frac{e^{\frac{\pi i}{4}}+e^{-\frac{\pi i}{4}}}{2i}\right] \times 2i$$

$$Sum\ of\ Residues = \sum R^+ = -\frac{2i}{4a^4} Sin\frac{\pi}{4} = -\frac{2i}{4a^4}\frac{1}{\sqrt{2}} = -\frac{i}{2\sqrt{2}a^4}$$

- Use formula $\int_0^\infty f(x)\,dx = \pi i \sum R^+$

$\Rightarrow \int_0^\infty \frac{dx}{x^4+a^4} = \pi i \times \frac{-i}{2\sqrt{2}a^4} = \frac{\pi}{2\sqrt{2}a^4}$ as required.

Example

show that $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

Solution

Solution. Indeed, $f(z) = 1/(1+z^4)$ has four simple poles at the points (make a sketch)

$$z_1 = e^{\pi i/4}, \qquad z_2 = e^{3\pi i/4}, \qquad z_3 = e^{-3\pi i/4}, \qquad z_4 = e^{-\pi i/4}.$$

The first two of these poles lie in the upper half-plane (Fig. 375). From (4) in the last section we find the residues

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4}.$$

(Here we used $e^{\pi i} = -1$ and $e^{-2\pi i} = 1$.) By (1) in Sec. 13.6 and (7) in this section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = -\frac{2\pi i}{4}\left(e^{\pi i/4} - e^{-\pi i/4}\right) = -\frac{2\pi i}{4}\cdot 2i\sin\frac{\pi}{4} = \pi\sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since $1/(1+x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Fourier Integrals

TYPE – III:

If we have an integral of the form $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}Cosmx\,dx$ or $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}Sinmx\,dx$ or $\int_{0}^{\infty}\frac{P(x)}{Q(x)}Cosmx\,dx$ or $\int_{0}^{\infty}\frac{P(x)}{Q(x)}Sinmx\,dx$ where then we can solve it using following procedure.

- Replace ‘x’ by ‘z’ in the integrand $f(x)=\frac{P(x)}{Q(x)}$ and $Sinmx$ or $Cosmx$ by e^{imz}
- Find the poles of $f(z)e^{imz}$, locate those poles which lie in the upper half plane. Find the residue at the located poles.
- Then by Cauchy Residue theorem use the following formulae
  $\int_{-\infty}^{\infty} f(x)Sinmx\,dx = Im(2\pi i\sum R)$
  $\int_{-\infty}^{\infty} f(x)Cosmx\,dx = Re(2\pi i\sum R)$
  $\int_{0}^{\infty} f(x)Sinmx\,dx = Im(\pi i\sum R)$
  $\int_{0}^{\infty} f(x)Cosmx\,dx = Re(\pi i\sum R)$

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx = -2\pi\sum \text{Im Res}\,[f(z)e^{isz}],$$

$$\int_{-\infty}^{\infty} f(x)\sin sx\,dx = 2\pi\sum \text{Re Res}\,[f(z)e^{isz}]. \qquad (s>0)$$

Remark:

- $P(x)$ and $Q(x)$ are polynomials such that $Q(x)=0$ has no real roots. And the degree of $Q(x)$ exceeds the degree of $P(x)$
- **Jordan’s Inequality:**
  $\frac{2\theta}{\pi} \le Sin\theta \le \theta$ where $0 \le \theta \le \frac{\pi}{2}$
  Or $\int_0^{\pi} e^{-RSin\theta}\,d\theta < \frac{\pi}{R}$; $R>0$
  Or $\int_0^{\frac{\pi}{2}} e^{-RSin\theta}\,d\theta \le \frac{\pi}{2R}$; $R>0$

- **Jordan’s Lemma:**
  if $f(z)$ is complex valued function such that $f(z)\to 0$ as $z\to\infty$ and $f(z)$is the meromorphic in the upper half plane then
  $\boldsymbol{\lim_{R\to\infty}\int_C f(z)e^{imz}dz = 0}$
  where C denotes the semi-circle $|z|=R; Im(z)>0$

  Or if a function $f(z)$ is analytic at all points in the upper half plane that are exterior to the semi circle $|z|=R_0$. and if C_R denotes a semi circle $z=Re^{i\theta}$ $(0\le\theta\le\pi)$ where $R>R_0$ then for all points z on C_R there will be a positive constant M_R such that
  $|f(z)|\le M_R$ and $\lim_{R\to\infty} M_R = 0$

Example: Evaluate $\int_0^\infty \frac{Cosmxdx}{a^2+x^2}$ and deduce the value of $\int_0^\infty \frac{xSinmxdx}{a^2+x^2}$ where 'a' and 'm' are constants.

Solution:

- The given integral $\int_0^\infty \frac{Cosmxdx}{a^2+x^2}$ becomes $\int_C \frac{e^{imz}dz}{z^2+a^2}$
- The poles of $f(z)e^{imz} = \frac{e^{imz}}{z^2+a^2}$ are the zeros of $z^2 + a^2 = 0$

$\Rightarrow z^2 = -a^2 \Rightarrow z = \pm ai$. The only pole which lies in the upper half plane is $z = ai$ of order 1.

$$Res(f, ai) = \lim_{z\to ai}\left[(z - ai)\frac{e^{imz}}{z^2+a^2}\right] = \frac{e^{-ma}}{2ai}$$

- Then by Cauchy Residue theorem use the following formulae

$$\int_0^\infty f(x)Cosmx\,dx = Re(\pi i \textstyle\sum R)$$

$$\int_0^\infty \frac{Cosmxdx}{a^2+x^2} = Re\left(\pi i \times \frac{e^{-ma}}{2ai}\right) = \frac{\pi}{2a}e^{-ma}$$

Diff. w.r.to 'm' we get $\int_0^\infty \frac{-Sinmx.xdx}{a^2+x^2} = \frac{\pi}{2a}e^{-ma}(-a)$

$$\int_0^\infty \frac{xSinmxdx}{a^2+x^2} = \frac{\pi}{2}e^{-ma}$$

Example: Prove that $\int_{-\infty}^\infty \frac{Cosxdx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$; $a > b > 0$

Solution:

- The given integral $\int_{-\infty}^\infty \frac{Cosxdx}{(x^2+a^2)(x^2+b^2)}$ becomes $\int_C \frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)}$
- The poles of $f(z)e^{imz} = \frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)}$ are $z = \pm ai$, $z = \pm bi$

The only pole which lies in the upper half plane are $z = ai, bi$ of order 1.

$$Res(f, ai) = \lim_{z\to ai}\left[(z - ai)\frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)}\right]$$

$$Res(f, ai) = \lim_{z\to ai}\left[(z - ai)\frac{e^{iz}dz}{(z-ai)(z+ai)(z^2+b^2)}\right]$$

$$Res(f, ai) = \lim_{z\to ai}\left[\frac{e^{iz}dz}{(z+ai)(z^2+b^2)}\right] = \frac{e^{-a}}{2ai(b^2-a^2)}$$

$$Res(f, bi) = \lim_{z\to bi}\left[(z - bi)\frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)}\right]$$

$$Res(f, bi) = \lim_{z\to bi}\left[(z - bi)\frac{e^{iz}dz}{(z-bi)(z+bi)(z^2+a^2)}\right]$$

$$Res(f, bi) = \lim_{z\to bi}\left[\frac{e^{iz}dz}{(z+bi)(z^2+a^2)}\right] = \frac{e^{-b}}{2bi(a^2-b^2)}$$

- Then by Cauchy Residue theorem use the following formulae

$$\int_{-\infty}^\infty f(x)Cosmx\,dx = Re(2\pi i \textstyle\sum R)$$

$$\int_{-\infty}^\infty \frac{Cosxdx}{(x^2+a^2)(x^2+b^2)} = Re\left[2\pi i \times \left(\frac{e^{-a}}{2ai(b^2-a^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}\right)\right] = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Example

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2}\, dx = \frac{\pi}{k} e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2}\, dx = 0 \qquad (s > 0, k > 0).$

Solution. In fact, $e^{isz}/(k^2 + z^2)$ has only one pole in the upper half-plane, namely, a simple pole at $z = ik$, and from (4) in Sec. 16.3 we obtain

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \frac{e^{-ks}}{2ik}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2 + x^2}\, dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{-ks}.$$

Since $e^{isx} = \cos sx + i \sin sx$, this yields the above results [see also (15) in Sec. 11.7.] ■

Another Kind of Improper Integral

We consider an improper integral

$$\int_A^B f(x)\, dx$$

whose integrand becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty.$$

Then we may write

$$\int_A^B f(x)\, dx = \lim_{\epsilon \to 0} \int_A^{a-\epsilon} f(x)\, dx + \lim_{\eta \to 0} \int_{a+\eta}^B f(x)\, dx$$

$$\int_A^B f(x)\, dx = \lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\, dx + \int_{a+\epsilon}^B f(x)\, dx\right]$$

Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then

$$\lim_{r\to 0} \int_{C_2} f(z)\, dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

Principal Value of Function

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

Example

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} .$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1}$$

$$= -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2}$$

$$= \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i}$$

$$= \frac{1}{6+2i} = \frac{3-i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here.

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi}{10}.$$

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خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (آمین)

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