

# ON STOLARSKY AND RELATED MEANS

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ABSTRACT. We give a simple proof of the Stolarsky means inequality as well as some related inequalities for similar means of Stolarsky type.

## 1. Introduction and Preliminaries

Let us consider the following means

$$\begin{aligned} E(x, y; r, s) &= \left\{ \frac{r(y^s - x^s)}{s(y^r - x^r)} \right\}^{\frac{1}{s-r}} \\ E(x, y; r, 0) &= E(0, r) = \left\{ \frac{y^r - x^r}{r(\ln y - \ln x)} \right\}^{1/r} \\ E(x, y; r, r) &= e^{-\frac{1}{r}} \left( \frac{x^{x^r}}{y^{y^r}} \right)^{1/(x^r - y^r)} \\ E(x, y; 0, 0) &= \sqrt{xy} \end{aligned}$$

where  $x$  and  $y$  are positive real numbers  $x \neq y$ ,  $r$  and  $s$  are any real numbers but 0.

These means, known in literature, are called Stolarsky means. Namely Stolarsky[1] in 1975 (see also [2, p.120]) introduced these means. Stolarsky proved that the function  $E(r, s)$  is increasing in both  $r$  and  $s$  i.e. for  $r \leq u$  and  $s \leq v$ , we have

$$(1) \quad E(x, y; r, s) \leq E(x, y; u, v).$$

In this paper, first we shall give a simple proof of inequality (1). Further we shall introduce two new classes of means of Stolarsky type.

## 2. A Simple Proof of Stolarsky Means Inequality

Note that  $E(r, s)$  is continuous, this means it is enough to prove (1) in the case where  $r, s, u, v \neq 0, r \neq s$  and  $u \neq v$ . We consider the following function

$$f(x) = p^2 \varphi_r(x) + 2pq \varphi_t(x) + q^2 \varphi_s(x) \quad \text{where } t = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R}, \text{ and}$$

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$$\varphi_r(x) = \begin{cases} x^r/r, & r \neq 0 \\ \ln x, & r = 0 \end{cases}$$

Now

$$\begin{aligned} f'(x) &= p^2 x^{r-1} + 2pqx^{t-1} + q^2 x^{s-1} \\ &= \left( px^{(r-1)/2} + qx^{(s-1)/2} \right)^2 \geq 0 \end{aligned}$$

This implies  $f$  is monotonically increasing. So for  $x \neq y$

$$\frac{f(x) - f(y)}{x - y} \geq 0$$

i.e.

$$p^2 \frac{\varphi_r(x) - \varphi_r(y)}{x - y} + 2pq \frac{\varphi_t(x) - \varphi_t(y)}{x - y} + q^2 \frac{\varphi_s(x) - \varphi_s(y)}{x - y} \geq 0.$$

Let

$$\phi(r) = \frac{\varphi_r(x) - \varphi_r(y)}{x - y}$$

then

$$p^2 \phi(r) + 2pq \phi(t) + q^2 \phi(s) \geq 0$$

i.e.

$$\phi^2(t) \leq \phi(r) \cdot \phi(s) \quad \text{where} \quad t = \frac{r+s}{2}.$$

This implies  $\phi$  is log-convex in Jensen sense.

Also  $\lim_{r \rightarrow 0} \phi(r) = \phi(0)$ , which implies  $\phi$  is continuous for all  $r \in \mathbb{R}$ . And therefore log-convex.

We need following lemma which proof can be found in [2].

**Lemma 2.1.** Let  $f$  be log-convex function and if,  $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$ , then the following inequality is valid:

$$(2) \quad \left( \frac{f(x_2)}{f(x_1)} \right)^{1/(x_2-x_1)} \leq \left( \frac{f(y_2)}{f(y_1)} \right)^{1/(y_2-y_1)}.$$

Applying Lemma 2.1 for  $f = \phi$ , (let  $r, s, u, v \neq 0$ ) we get an inequality

$$\left\{ \frac{r(y^s - x^s)}{s(y^r - x^r)} \right\}^{1/(s-r)} \leq \left\{ \frac{u(y^v - x^v)}{v(y^u - x^u)} \right\}^{1/(v-u)}.$$

Since  $E(r, s)$  is continuous, we have (1).

## CONCLUSION

In the literature, many researchers have published so many results on different major generalizations of convex function. Many authors today focus on interval-valued functions, which is known as the  $(h, m)$ -convex interval-valued function. Additionally, we give the rigorous proof of the famous Hermite-Hadamard type inequality for  $m$ -convex in intervalvalued.

## REFERENCES

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