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Ph.D Admission Test – April 2009

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Instructions

a) Maximum time allowed: 3 hours.
b) Each problem is worth 20 points.
c) Write only solutions of the problems, solution of each problem on separate sheet(s).
d) Give rigorous proofs for all your answers.

Problem 1. Show that in any triangle $\triangle ABC$ the following inequality holds:

$$\cos A + \cos B + \cos C \leq \frac{3}{2}$$. What is the minimal value of the sum?

Problem 2. A 3×3 real matrix is called *magic* if there is a real number S such that in each row, column, and diagonal the sum of the elements equals S.

a) Show that the set of magic matrices is a real vector space and find its dimension.
b) Can you find a basis of this vector space with integral matrices?

Problem 3. A continuous function $f : D \to \mathbb{R}$ is defined on the set $D = \left\{ z \in \mathbb{C} : 1 \leq z \leq 2 \right\}$.

a) Show that for any $r \in \left[1,2\right]$ there is an element $z_r \in D$, $\left|z_r\right| = r$ such that $f(z_r) = \sup_{|z|=r} f(z)$. Also show that, for any $\alpha \in [0,2\pi]$, there is an element $z_\alpha \in D$, $\arg z_\alpha = \alpha$, such that $f(z_\alpha) = \inf_{\arg z=\alpha} f(z)$.

b) Let $M_r = \sup_{|z|=r} f(z)$ and $A_\alpha = \inf_{\arg z=\alpha} f(z)$. Put in the increasing order the two numbers

$$M = \inf_{r\in[1,2]} M_r \quad \text{and} \quad A = \sup_{\alpha\in[0,2\pi]} A_\alpha .$$

Problem 4.

a) Show that any finitely generated subgroup of the group $(\mathbb{Q}+)$ is cyclic, but the group itself is not cyclic.
b) Can you find a surjective function $f : \mathbb{N} \to \mathbb{Q} \setminus \{0\}$ such that:
$$f(xy) = f(x)f(y)?$$

Problem 5. For a continuous function $f : [0,1] \to \mathbb{R}$, compute

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)dx$$ (particular case: $f(x)$ is a polynomial function).

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