Admission Test - 14 April 2013

1. The sets of integers A, B are defined by

$$A = \{13k - 5 \mid k = 1, 2, ..., 2013\},$$
$$B = \{17n - 10 \mid n = 1, 2, ..., 2013\}.$$

Compute the number of elements of $A \cup B$.

2. Show that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ and $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ are isomorphic as $\mathbb{Q}$–vector spaces, but they are not isomorphic fields.

3. a) Prove that the polynomial $P(X) = X^3 - X^2 - 15$ has no rational root.

b) Find an interval $(a, a+1)$ containing a root x_0 of $P(X)$.

c) Prove that $y_0 = x_0^2 + x_0 + 1$ is also a root of a polynomial with integer coefficients.

4. a) Draw the subsets X, Y of the plane $\mathbb{R}^2$:

$$X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (2y)^2 = 4\},$$
$$Y = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2} + \sqrt{(2y)^2} = 4\}.$$

b) Find the distance between the two sets $dist(X, Y)$.
(We remind that: $dist(X, Y) = \inf\{dist(x, y) \mid x \in X, y \in Y\}$.)

5. a) Compute the integral

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

b) Using the obtained result, deduce that

$$\frac{22}{7} - \frac{1}{4^4} < \pi < \frac{22}{7}.$$

Give a proof for all your statements.
Time allowed: 4 hours.
Each question is worth 20 points.
Do not use your calculator or mobile or other tables.
Organized by Abdus Salam School of Mathematical Sciences, GCU-Lahore