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## Ph.D Admission Test – April 2009

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## Instructions

- a) Maximum time allowed: 3 hours.
- b) Each problem is worth 20 points.
- c) Write only solutions of the problems, solution of each problem on separate sheet(s).
- d) Give rigorous proofs for all your answers.

**Problem 1.** Show that in any triangle  $\triangle ABC$  the following inequality holds:

 $\cos A + \cos B + \cos C \le \frac{3}{2}$ . What is the minimal value of the sum?

**Problem 2.** A  $3 \times 3$  real matrix is called *magic* if there is a real number S such that in each row, column, and diagonal the sum of the elements equals S.

- a) Show that the set of magic matrices is a real vector space and find its dimension.
- b) Can you find a basis of this vector space with integral matrices?

**Problem 3.** A continuous function  $f: D \to \mathbb{R}$  is defined on the set  $D = \{z \in \mathbb{C} : 1 \le z \le 2\}$ .

a) Show that for any  $r \in [1,2]$  there is an element  $z_r \in D$ ,  $|z_r| = r$  such that

 $f(z_r) = \sup_{|z|=r} f(z)$ . Also show that, for any  $\alpha \in [0, 2\pi]$ , there is an element

$$z_{_{\alpha}}\in D,\,\arg z_{_{\alpha}}=\alpha$$
 , such that  $\,f(z_{_{\alpha}})=\inf_{\arg z=\alpha}\,f(z)$  .

b) Let  $M_r = \sup_{|z|=r} f(z)$  and  $A_{\alpha} = \inf_{\arg z=\alpha} f(z)$ . Put in the increasing order the two numbers

$$M=\inf_{\boldsymbol{r}\in[1,2]}M_{\boldsymbol{r}}\quad\text{and}\quad A=\sup_{\boldsymbol{\alpha}\in[0,2\pi]}A_{\boldsymbol{\alpha}}\,.$$

## Problem 4.

- a) Show that any finitely generated subgroup of the group (Q+) is cyclic, but the group itself is not cyclic.
- b) Can you find a surjective function  $f : \mathbb{N} \to \mathbb{Q} \setminus \{0\}$  such that: f(xy) = f(x)f(y)?

**Problem 5.** For a continuous function  $f:[0,1] \to \mathbb{R}$ , compute

 $\lim_{n \to \infty} n \int_0^1 x^n f(x) dx \quad \text{(particular case: } f(x) \text{ is a polynomial function).}$ 

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