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Admission Test Ph.D. 2007

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Instructions

Maximum time allowed: 3½ hours.
Each problem worth 20 points.
Write only solutions of the problems, each problem on separate sheet(s).
Give rigorous proofs for all your answers.

PROBLEMS

Problem 1: For every positive integer n, consider the open intervals $\Delta_{n,k} = \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, where $k = 0,1,\ldots,2^n - 1$. Define the function

$$r_n(x) = \begin{cases} (-1)^k & if \quad x \in \Delta_{n,k} \\ 0 & if \quad x = \frac{k}{2^n} \end{cases}$$

Give positive integers $n_1 \le n_2 \le \ldots \le n_p$, Compute the integral

$$I\left(n_1,\ldots,n_p\right) = \int_0^1 r_{n_1}(x)\, r_{n_2}(x) \ldots r_{n_p}(x)\, dx$$

Problem 2: *a*) Show that the additive group $\mathbb{Z} \times \mathbb{Z}$ is not cyclic and find the minimal set of generators for it.
b) Let $n > 0$ be an integer, $\mathbb{R}$ be the additive group of real numbers and $\mathbf{U}$ be the multiplicative group of complex numbers of modulo 1. Show that the group $\mathbb{R}/n\mathbb{Z}$ and $\mathbf{U}$ are isomorphic.

Problem 3: A circle with center at O is divided into n equal arcs, $n \ge 2$, by the points $A_1, A_2, \ldots, A_n$. Find the sum of radius vectors $\overrightarrow{OA_i}$, $i = 1,2,\ldots,n$.

Problem 4: Let V be the real vector space of the polynomials $P(X) \in \mathbb{R}[X]$ of degree at most 2 and $f : V \to V$ the function defined by $f\left(P(X)\right) = P\left(X-1\right)$, for all $P(X) \in V$.

a) Show that f is an isomorphism of vector spaces.
b) Find the polynomial $P(X)$ such that $f\left(P(X)\right) = X^2 + X + 1$.
c) Find the matrix A of f with respect to the basis $\left\{1, X, X^2\right\}$.
d) Find the invariant subspaces of f.
e) Compute A^n, for all $n \ge 2$.

Problem 5: For any positive integer n let $C^n = [0,1]^n$ be the n-dimensional unite cube. Define

$$I_n = \int_{C^n} \min\{x_1, x_2, \ldots, x_n\} dx_1 dx_2 \ldots dx_n .$$

a) Compute I_2 and I_3.

b) Compute I_n for all $n \geq 2$.