Written Test Lecturer 1996

PUNJAB PUBLIC SERVICE COMMISSION, LAHORE.

Time Allowed: 2 hours **Maximum Marks: 100**

NOTE: Attempt any FOUR questions only.

I. (a) Express $\cos^4\theta \sin^3\theta$ in a series of multiples of θ. 12

(b) Separate into real and imaginary parts $\tan^{-1}(x+iy)$. 13

II. (a) If $u = \sin nx + \cos nx$, show that $u_r = n^r\{1+(-1)^r \sin 2nx\}^{1/2}$, where u_r denotes the rth derivative of u with respect to x. 12

(b) Evaluate $\int_0^1 \arctan\left(\dfrac{2x-1}{1+x-x^2}\right)dx$. 13

III. (a) Obtain a reduction formula for $\int \sin^m x \cos^n x\, dx$, where m and n are positive integers. Hence evaluate $\int_0^{\Pi/2} \sin^m x \cos^n x\, dx$, where m and n are both even positive integers. 12

(b) Evaluate $\lim_{x\to 1}\left|\dfrac{1}{\pi^2(x-1)^2} - \dfrac{1}{\sin^2\pi x}\right|$

IV. (a) Show that in a conic, the semi-latus rectum is the harmonic mean between the segments of a focal chord. 12

(b) Find the length of the cardioid $r = a(1+\sin\theta)$. 13

V. (a) Applying elementary row operations, find the inverse of the matrix 12

$$\begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$$

(b) If w is the cube-root of unity, then without expanding either side, prove that the determinant 13

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

VI. (a) Define a plane and prove that any equation of the first degree in x, y and z is the equation of a plane. 12

(b) The points A (3, 2, -4), B (-1, 1, 2), C (-2, 3, 3) and D (-3, -2, 1) are the corners of a tetrahedron. Find (i) volume of the tetrahedron and (ii) the shortest distance between AC and BD. 13

VII. Solve the differential equations:

(a) $d^2y/dx^2 + y = \operatorname{cosec} x$. 12

(b) $x^4\, d^2y/dx^2 + 3x^3\, dy/dx - 8x^2y = x^4 - 1$, given that when

$x = 1$, $dy/dx = 1/9$ and $y = 1/18$ 1

VIII. (a) If $\underline{a}, \underline{b}, \underline{c}$ are non-coplanar vectors, prove that 1

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$$

(b) Let (X, d) be a metric space and let $d' : X \times X \longrightarrow R$ and $d'' : X \times X \longrightarrow R$ be given by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} \text{ and } d''(x_1, x_2) = \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)}.$$

Prove that d' is a metric but d'' is not a metric on X. 13