Written Test Lecturer 1985

PUNJAB PUBLIC SERVICE COMMISSION, LAHORE.

Time Allowed: 2 hours **Maximum Marks: 100**

NOTE: Attempt any four questions. Some marks for each question are reserved for neatness any clarity of expression.

Q 1: (a) For any matrix A such that:

$$\left(I-A\right)^{-1} = I+A+A^2.$$

(b) Define an idempotent matrix. Show that if an idempotent matrix is non-singular then it must be the identity matrix.

(c) Express $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix}^2$ as a single determinant and evaluate.

Q 2: (a) If $A = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$, show that its characteristic roots are -1 and 5.

Verify that $\left(A-5I\right)\left(A+I\right)=0$. Explain.

(b) If the vectors $v_1, v_2, \ldots, v_n$ are linearly independent, show that

$v_1, v_1+v_2, v_1+v_2+v_3, \ldots, v_1+v_2+\ldots+v_n$ are also linearly independent.

(c) Which of the following transformations are linear.

(i) $T: C[0,1] \to C[0,1]$ given by $\left(Tf\right)(x) = f(x)+1,\ \ x\in[0,1],\ \ f(x)\in C[0,1]$

(ii) $T: C[0,1] \to \mathbb{R}$ given by $Tf = \int_0^1 f(x)\cdot g(x)dx$ where g is a fixed function.

Q 3: (a) Show that the union $H\cup K$ of two subgroups H and K of a group G is a subgroup if and only if either $H\subseteq K$ or $K\subseteq H$.

(b) Prove that every subgroup of a cyclic group is cyclic.

(c) Define the centre of a group. What are the centers of the following groups:

The group $\mathbb{Z}$ of integers under addition.

The group $\left\{\pm 1, \pm i, \pm j, \pm k\right\}$ of quaternions.

Q 4: (a) Define an ideal of a ring R. Show that the sum $I+J$ of two ideals I and J of a ring R is an ideal of R. Is $I\cap J$ also an ideal? Explain.

(b) Define a division ring. Is every field a division ring? What are the ideals of a field?

(c) Find all the sub-rings of the ring $\mathbb{Z}_{12}$ of integers module 12.

Q 5: (a) In a metric space $\left(\mathbf{X}, d\right)$, prove that

$\left|d(x,z)-d(y,z)\right| \le d(x,y)$ for all $x,y,z\in\mathbf{X}$.

(b) In a metric space $\left(\mathbf{X}, d\right)$, show that every open ball is open.

(c) Define the interior of a set in a metric space. If A° denotes the interior of a set A.
Show that $\left(A \cap B\right)^\circ = A^\circ \cap B^\circ$.

Q 6: (a) Let $\left(\mathbf{X}, d\right)$, $\left(\mathbf{Y}, d\right)$ be metric spaces. When is a function $f : \mathbf{X} \to \mathbf{Y}$ continuous?
Show that the sum $f + g$ of two single valued functions is continuous.
(b) Define uniform continuity. Is every uniformly continuous function continuous? Is the converse of this statement true? Justify.

Q 7: (a) Define a topological space. Show that for an infinite set $\mathbf{X}$, the collection consisting of θ and all subsets of $\mathbf{X}$ whose complements are finite is a topology on $\mathbf{X}$.
(b) Let $\mathbf{X} = \left\{a, b, c, d, e\right\}$. Determine whether or not each of the following classes of subsets of $\mathbf{X}$ is a topology on $\mathbf{X}$.

(i) $\tau_1 = \left\{\varphi, \mathbf{X}, \left\{a^2\right\}, \left\{a, b\right\}, \left\{a, c\right\}\right\}$

(ii) $\tau_2 = \left\{\varphi, \mathbf{X}, \left\{a\right\}, \left\{a, b\right\}, \left\{a, c, d\right\}, \left\{a, b, c, d\right\}\right\}$

Q 8: (a) If $f\left(t\right) = t\hat{i} - t^2\hat{j} + t^3\hat{k}$ and $f\left(t\right) = \sin t\,\hat{i} + \cos t\,\hat{j}$, then calculate $\dfrac{d}{dt}\left(f \cdot g\right)$ and $\dfrac{d}{dt}\left(f \times g\right)$.

(b) If the position vector of a moving point is given by $\vec{r} = t\,\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$. Find its velocity, speed and acceleration at time t and also at $t = \pi/2$.