M.Phil Admission Test Spring 2010

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Mathematics , Dated: 2-2-2010,

Time: 90 minutes
Dated: 02-02-2010
Note: Section I is compulsory, Section II is for Applied Mathematics and Section III for Pure Mathematics candidates.
Use separate page for each question.

Section I

Q.1. Prove the Bernoulli's inequality
If $x > -1$, then $(1+x)^n \geq 1 + nx$, for all $n \in N$.

Q.2. Examine the convergence of the infinite series

$$1 + \frac{1}{(4)^{\frac{2}{3}}} + \frac{1}{(9)^{\frac{2}{3}}} + \frac{1}{(16)^{\frac{2}{3}}} + \dots$$

Q.3. Show that intersection of two normal subgroups is again a normal subgroup. Give an example to show that union of two subgroups need not to be a subgroup.

Q.4. (i) Prove that a metric space is a topological space.
(ii) Give an example of a topological space which is not a metric space.

Q.5. Find the general solution

$$y'' + e^{-x}y = 0.$$

Q.6. Find the dual basis of the following basis of $\mathbb{R}^3$: $\{(1,-2,3),(1,-1,1),(2,-4,7)\}$.

Q.7. Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation $\dot{x} = bx^{-3}$ where b is a positive constant. Find the force acting on the particle as a function of x.

Q.8. Show that the given function $u(x,y) = \frac{x}{x^2+y^2}, z \neq 0$ is harmonic. Find the corresponding conjugate harmonic function $v(x,y)$ and construct the analytic function $f(z) = u + iv$.

Q.9. (a) What is the rank of the tensors representing:
(i) A straight line (ii) A scalar (iii) A metric on a sphere having unit radius.
(b) Find the contracted tensor components of F_b^a, where $F_{db}^{ac} = A_d^a B_b^c$ and

$$A_d^a = \begin{bmatrix} 0 & 2 \\ 1 & -5 \end{bmatrix}, \quad B_b^c = \begin{bmatrix} 0 & -1 & 0 \\ -3 & 0 & 7 \end{bmatrix}$$

Q.10. Apply the Simpson's formula to compute the integral of $\sin x$ between 0 and $\frac{\Pi}{2}$ from the values provided in the following table:

x	0	$\frac{\Pi}{12}$	$\frac{\Pi}{6}$	$\frac{\Pi}{4}$	$\frac{\Pi}{3}$	$\frac{5\Pi}{12}$	$\frac{\Pi}{2}$
$sinx$	0.00000	0.25882	0.50000	0.70711	0.86603	0.96593	1.00000

Q.11. Consider the circular helix

$$\underline{\alpha}(s) = (r\cos\omega s, r\sin\omega s, h\omega s)$$

where $\omega = \frac{1}{\sqrt{r^2+h^2}}$. Is $\underline{\alpha}$ a unit speed curve? Compute the Frenet-Serret apparatus for the helix.

Q.12. Find the Laplace transform of $f(t) = \begin{cases} sint & 0 < t < \Pi \\ 0 & t > \Pi \end{cases}$.

Q.13. Show that a convergent sequence in a metric space is bounded.

Section II

Q.14. Use Simpson's $\frac{1}{3}$ rule to find the value of $\int_{0^\circ}^{360^\circ} \sin x \, dx$. Compare the result with exact solution.

Q.15. In a two dimensional incompressible flow the fluid velocity components are given by $u = x - 4y$, $v = -y - 4x$. Show that the flow satisfies the continuity equation and obtain the expression for the stream function. If the flow is of potential kind, obtain also the expression for the velocity potential.

Q.16. Solve the integral equation

$$u(x) = \sin x + 2\int_0^x \cos(x-t)u(t)dt.$$

Q.17. Prove that a displacement vector $\overline{u}$ can be written as

$$\overline{u} = \nabla\varphi + \nabla \times \overline{\Psi} \text{ where } \nabla \cdot \overline{\Psi} = 0.$$

Q.18. Write down the Maxwell's equations for homogenous isotropic and source-free medium. Also explain the terms their in. What are the constitutive relations.

Q.19. Using Dirac bra-ket algebra show that eigenstates corresponding to two distinct eigenvalues are orthogonal. Is the converse also true? Give an example.

Q.20. An initial frame F has coordinates (t, x, y, z) and frame F' has coordinates (t', x', y', z'), where F' is moving relative to F at speed $0.92c$ along the common $x - x'$ axis and the origins. Coinciding at $t = t' = 0$. If $x = 100m$, $y = 10m$, $z = 1m$ at $t = 5 \times 10^{-6}$secs, find the coordinates t', x', y', z' in F'. (Here c is the speed of light).

Section III

Q.21. If R is any commutative ring we define $GL(n, R)$ to the group of units of the matrix ring $M_n(R)$ in particular, $GL(n, \mathbb{Z})$ is the group of $n \times n$ matrices with integer coefficients whose inverses also have integer coefficients.

(i) Show that $GL(2, \mathbb{Z}) = \{A \in M_2(\mathbb{Z}) : \det(A) = \pm 1\}$.

(ii) Let A be an element of finite order in $GL(2, \mathbb{Z})$. Show that $ord(A)$ is $1, 2, 3, 4$ or 6.

Q.22. If the random variables X and Y have the joint probability density function given by $f_{X,Y}(x, y) = x + y \quad 0 < x < 1, \quad 0 < y < 1$ calculate the probability $P(X < Y)$.

Q.23. Let $T : H \to H$ and $W : H \to H$ be bounded linear operators on a complex Hilbert space H and $S = W^* \, T \, W$. Show that if T is self adjoint and positive, so is S.

Q.24. (a) Indicate true or false for a commutative ring R with identity.

i) Every ideal in R is a subring of R.

ii) Every maximal ideal is prime ideal in R.

iii) R/P is an integral domain $\Leftrightarrow$ P is prime ideal.

iv) $\mathbb{Z}$ is principal ideal domain.

v) $\mathbb{Z}_n$ is a finite field $\Leftrightarrow$ n is prime integer.

(b) Choose the correct one

	A	B	C
i) $1 + x + x^2$ has	roots in $\mathbb{Z}_2$	roots in $\mathbb{Z}_2$	neither
ii) $\mathbb{Z} \times \mathbb{Z}$ is	integral domain	field	neither
iii) $\mathbb{R} \times \mathbb{R}$ is	field	integral domain	neither
iv) $\mathbb{R}[X]$ is a	field	not a ring	neither
v) $X\mathbb{R}[X]$ is ...... ideal in $\mathbb{R}[X]$	Nil	trivial	neither.

Q.25. Find a relationship between $\ker \Pi$ and $\operatorname{Im} \phi$ for the sequences of homomorphisms of $\mathbb{Z}$-modules.

i) $0 \to 2\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\Pi} \mathbb{Z}/2\mathbb{Z} \to 0$

ii) $0 \to \mathbb{Z}^k \xrightarrow{\phi} \mathbb{Z}^n \xrightarrow{\Pi} \mathbb{Z}^{n-k} \to 0 \quad n > k$

iii) $0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Q} \xrightarrow{\Pi} \mathbb{Q} \to 0$

iv) $0 \to X\mathbb{R}[X] \xrightarrow{\phi} \mathbb{R}[X] \xrightarrow{\Pi} \mathbb{R} \to 0$

v) $0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \oplus \mathbb{Q} \xrightarrow{\Pi} \frac{\mathbb{Z} \oplus \mathbb{Q}}{\mathbb{Z}} \to 0$.

Q.26. Show that an abelian group A is divisible if and only if A is injective as $\mathbb{Z}$-module.

Q.27. Show that union and intersection of two measurable sets is a measurable set.