M.Phil Admission Test Fall 2007

Quaid-e-Azam University, Islamabad.

Mathematics , Dated: 16-8-2007, http://www.mathcity.org

Time: 9:00 a.m. to 10:30 a.m.
Dated: 16-08-2007.
Note: Section I is compulsory, section II is for Applied Mathematics and section III is for Pure Mathematics.

Section I

Q.1 **Point out the true or false statement(s).**

(a) Every straight line is a regular curve.
(b) Torsion at all points of a straight line is not zero
(c) The torsion ay any point of a plane curve is zero.
(d) The normal curvature of a sphere is zero.
(e) There are infinitely many involutes to the space curve.
(f) First fundamental form determines the arc length of the shortest possible path between two points on a surface.
(g) The tangent vectors at a point on a surface in the direction of two parametric curves is dependent vectors.
(h) The radius of curvature of a straight line is zero.
(i) The radius of curvature of a circle is equal to its radius.
(j) The magnitude of a tangent vector of a curve is equal to one if it's parametric representation is natural.

Q.2 Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } x^4+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

Does the function $f(x,y)$ is continuous at the origin?

Q.3 Test the convergence of

$$\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$$

Q.4 Solve

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0<x<1,\ t>0$$

$$u(0,t)=0, \quad u(1,t)=0, \quad t>0$$

$$u(x,0)=x, \quad 0<x<1.$$

Q.5 Solve the following differential Equation

$$y'' + 2y(y')^3 = 0$$

Q.6 Define the christoffel symbol. Show that the christoffel symbol is not a tensor

Q.7 For a given matrix $A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$. Use method of triangularisation to find A^{-1}.

Q.8 If H is a cyclic normal subgroup of a group G then every subgroup of H is normal in G.

Q.9 Let $(N, \|\cdot\|)$ be a normed space, $a, b \in N$, $r > 0$. Show that $A = \{x \in N : \|x-a\| + \|x-b\| < r\}$ is a convex subset of N.

Q.10 a) Sketch the closure of the set $Re\left(\frac{1}{z}\right) \leq 1/2$

b) Does the analytic function $f(z) = u(x,y) + i\,v(x,y)$ exist for each $v(x,y) = x^3 + iy^3$? why?

Q.11 I. Define topology and metric on a non-empty set.

II. verify that a metric space is a topological space.

III Let R be the set of real numbers and

$$R[x] = \left\{ f(x) = \sum_{i=0}^{n} r_i x^i : r_i\text{'s} \in R \text{ and } n \in N \cup \{0\} \right\}$$

be the set of all polynomials in one indeterminate x with coefficients in R.

a) Is $d : R[x] \times R[x] \longrightarrow R$, defined by

$$d\left(\sum_{i=0}^{n} r_i x^i, \sum_{i=0}^{m} s_i x^i\right) = |r_0 - s_0|$$ a metric on $R[x]$?

b) Is $d' : R[x] \times R[x] \longrightarrow R$, defined by

$$d'\left(\sum_{i=0}^{n} r_i x^i, \sum_{i=0}^{m} s_i x^i\right) = \sum_{i=0}^{\max(n,m)} (r_i - s_i)$$

a metric on $R[x]$?

Q.12 Define the term Moment of inertia.
Obtain the Moment of inertia of a uniform rod of mass m and length '2a' about the perpendicular bisector

Q.13 Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\{x_1, x_2, x_3\}$ is a basis for a subspace W of the vector space R^4 over the field R. Construct the orthonormal basis for W.

Section II Applied Mathematics

Q.14 Use the method of successive approximations to solve the Volterra integral equation

$$U(x) = x - \int_0^x (x-t)\, u(t)\, dt$$

Q.15 Verify that the following stress fields satisfy the equilibrium equations in the absence of body forces

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0$$

$$\tau_{xz} = -A\, y\, (x^2+y^2)^{-1}$$

$$\tau_{yz} = A\, x\, (x^2+y^2)^{-1}$$

Q.16 What is a Photon Gas. Derive the equation of state

Q.17 Under what condition does the

a) velocity field $V = (a_1x + b_1y + c_1z)\,i + (a_2x + b_2y + c_2z)\,j + (a_3x + b_3y + c_3z)\,k$ where $a_1, b_1, \ldots$ etc = constant, represent an incompressible flow

b) For the velocity field $u = a(x^2 - x^2)$, $v = -2axy$ If the flow is irrotational find velocity potential

Q.18 Solve the Schrödinger equation for one of the following cases.

(i) Step Potential (ii) Harmonic Oscillator (iii) Hydrogenic Atom (iv) Free Particle

Q.19 Derive an explicit scheme to solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 \le x \le l$ Under the boundary conditions $u(x,t) = 0$ at $x = 0$ and l for $t > 0$ using the finite differences.

Q-20 Obtain Hamilton's equations of motion for a particle executing simple harmonic oscillation.

Section III Pure Mathematics

Q.21 Let R be a commutative ring with identity.
(i) Define ideal, prime ideal, maximal ideal and primary ideal of R
(ii) Prove that every prime ideal is primary ideal
(iii) Define $\sqrt{I}$, the radical of an ideal I of R. Moreover prove that $I \subseteq \sqrt{I}$
(iv) Define nil radical of R (i.e. $Nil(R)$) and Jacobson radical of R (i.e. $J(R)$)
(v) when $\mathcal{N}(R) = J(R)$?

Q.22 Show that an abelian group D is divisible iff D is injective as $\mathbb{Z}$-module.

Q.23 Show that measure of an interval is its length

Q.24 Let R be a commutative ring with unity and I be an ideal of R. Show that I is a maximal ideal of R iff R/I is a field.

Q.25 Let P_1 and P_2 be projections on a Hilbert space H. Show that the sum $P = P_1 + P_2$ is a projection on H iff $P_1(H)$ and $P_2(H)$ are orthogonal.

Q.26 Show that $Cov(X, Y) = E(XY) - E(X)E(Y)$

Q.27 Prove that there is one-one correspondence between actions of the group G on the set Ω and representation of G by permutations of Ω.