Maxima and Minima for Functions of Two Variable

Question

Test for maxima and minima

(i) $z = 1 - x^2 - y^2$ (ii) $z = x^2 + y^2$

(iii) $z = xy$ (iv) $z = x^3 - 3xy^2$

(v) $z = x^2 y^2$ (vi) $z = 4 - y^2$

Solution

(i) $z = 1 - x^2 - y^2$

$$\frac{\partial z}{\partial x} = -2x \ , \quad \frac{\partial z}{\partial y} = -2y$$

For critical points $\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$

$\Rightarrow\ x = 0,\ y = 0 \quad \Rightarrow\ (0,0)$ is the critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = -2, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 z}{\partial y^2} = -2$$

$B^2 - AC = 0 - 4 = -4 < 0$ and $A + C = -2 - 2 = -4 < 0$

$\Rightarrow$ $(0,0)$ is the point of maximum value

and maximum value of z at $(0,0)$ is 1.

(ii) *Do yourself as above*

(iii) $z = xy$

$$\frac{\partial z}{\partial x} = y \ , \quad \frac{\partial z}{\partial y} = x$$

For critical points $\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$

$\Rightarrow\ y = 0$ and $x = 0 \quad \Rightarrow\ (0,0)$ is the critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 0, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 1, \quad C = \frac{\partial^2 z}{\partial y^2} = 0$$

$B^2 - AC = (1)^2 - (0)(0) = 1 > 0$

Therefore $(0,0)$ is a saddle point.

(iv) $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 0 \quad \Rightarrow 3x^2 - 3y^2 = 0 \quad \Rightarrow\ x = -y \ \ \& \ \ x = y$$

$$\frac{\partial z}{\partial y} = 0 \quad \Rightarrow -6xy = 0 \quad \Rightarrow\ xy = 0$$

$\Rightarrow$ either $x = 0$ or $y = 0$ or *both are zero*

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 6x = 0 \quad \text{at } (0,0)$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -6y = 0 \quad \text{at } (0,0)$$

$$C = \frac{\partial^2 z}{\partial y^2} = -6x = 0 \quad \text{at } (0,0)$$

$\Rightarrow\ B^2 - AC = 0$ and $A + C = 0$

so we need further consideration for the nature of point.

$$\begin{aligned}\Delta z &= z(0+h, 0+k) - z(0,0)\\ &= z(h,k) - z(0,0)\\ &= z(h,k)\ = h^3 - 3hk\end{aligned}$$

For $h = k$ we have

$$\Delta z = h^3 - 3h^3 = -2h^3 \left| \begin{array}{l} > 0 \quad if \quad h < 0 \\ < 0 \quad if \quad h > 0 \end{array} \right.$$

$\Rightarrow$ $(0,0)$ is a saddle point.

(*v*) $z = f(x, y) = x^2 y^2$

$f_x = 0 \ \Rightarrow\ 2xy^2 = 0,\qquad f_y = 0 \ \Rightarrow\ 2x^2 y = 0$

$\Rightarrow$ $(0,0)$ is the critical point.

$A = f_{xx} = 2y^2 = 0$ at $(0,0)$

$B = f_{xy} = 4xy = 0$ at $(0,0)$

$C = f_{yy} = 2x^2 = 0$ at $(0,0)$

$\Rightarrow\ B^2 - AC = 0$ and $A + C = 0$

so we need further consideration

$$\begin{aligned}\Delta f &= f(x_0 + h, y_0 + h) - f(x_0, y_0)\\ &= f(h,k) - f(0,0) = h^2 k^2\end{aligned}$$

If $h = k$, we have

$$\Delta f = h^4 \geq 0 \quad \forall\ h$$

Thus $(0,0)$ is the point of minimum value.

Question

Find the critical points of the following functions and test for maxima and minima.

(*a*) $z = \sqrt{1 - x^2 - y^2}$

(*b*) $z = 2x^2 - xy - 3y^2 - 3x + 7y$

(*c*) $z = 1 + x^2 + y^2$

(*d*) $z = x^2 - 5xy - y^2$

(*e*) $z = x^2 - 2xy + y^2$

(*f*) $z = x^3 - 3xy^2 + y^3$

Solution

(*a*) $z = \sqrt{1 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(1 - x^2 - y^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{1 - x^2 - y^2}} = 0 \quad \Rightarrow\ x = 0$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}} = 0 \quad \Rightarrow\ y = 0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$\frac{\partial^2 z}{\partial x^2} = \frac{-\left[\sqrt{1 - x^2 - y^2} - x \cdot \left(\dfrac{-x}{\sqrt{1 - x^2 - y^2}}\right)\right]}{1 - x^2 - y^2}$$

$$= \frac{-\left[1-x^2-y^2+x^2\right]}{\left(1-x^2-y^2\right)^{3/2}} \quad = \frac{-1+y^2}{\left(1-x^2-y^2\right)^{3/2}}$$

$$\Rightarrow \; A = \frac{\partial^2 z}{\partial x^2} = -1 \qquad \text{at } (0,0)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)$$

$$= -y \cdot \left(-\frac{1}{2}\right)\left(1-x^2-y^2\right)^{-\frac{3}{2}}(-2x) \quad = \frac{-xy}{\left(1-x^2-y^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow \; B = \frac{\partial^2 z}{\partial x \partial y} = 0 \qquad \text{at } (0,0)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-\left[\left(1-x^2-y^2\right)^{\frac{1}{2}}(1) - y\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)\right]}{1-x^2-y^2} \quad = \frac{-1+x^2}{\left(1-x^2-y^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow \; C = \frac{\partial^2 z}{\partial y^2} = -1 \qquad \text{at } (0,0)$$

$$\Rightarrow \; B^2 - AC = 0 - (-1)(-1) = -1 < 0 \quad \text{and} \quad A + C = -1 - 1 = -2 < 0$$

$\Rightarrow$ z has a relative maxima at $(0,0)$.

(*b*) $z = 2x^2 - xy - 3y^2 - 3x + 7y$

$$\frac{\partial z}{\partial x} = 4x - y - 3, \qquad \frac{\partial z}{\partial y} = -x - 6y + 7$$

For critical points $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$

$\Rightarrow$ $4x - y - 3 = 0$ ……………… (*i*)

& $x + 6y - 7 = 0$ …………… (*ii*)

Multiplying equation (*i*) by 6 and adding in (*ii*)

$$\begin{array}{l} 24x - 6y - 18 = 0 \\ \;\;x + 6y - 7 \;\; = 0 \\ \hline 25x \qquad - 25 = 0 \end{array}$$

$\Rightarrow$ $x = 1$ $\Rightarrow$ $y = 1$

$\Rightarrow$ $(1,1)$ is the critical point

$$A = \frac{\partial^2 z}{\partial x^2} = 4, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -1, \quad C = \frac{\partial^2 z}{\partial y^2} = -6$$

$$B^2 - AC = (-1)^2 - (-4)(-6) = 25 > 0$$

$\Rightarrow$ There is a saddle point at $(1,1)$.

(*c*) $z = 1 + x^2 + y^2$

$$\frac{\partial z}{\partial x} = 2x, \qquad \frac{\partial z}{\partial y} = 2y$$

For critical points $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$ $\Rightarrow$ $(0,0)$ is the critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2,\quad B=\frac{\partial^2 z}{\partial x\partial y}=0,\quad C=\frac{\partial^2 z}{\partial y^2}=2$$

$\Rightarrow B^2-AC=(0)^2-(2)(2)=-4<0$ and $A+C=2+2=4>0$

$\Rightarrow$ The function has a relative minima at $(0,0)$.

(*d*) $z=x^2-5xy-y^2$

$$\frac{\partial z}{\partial x}=2x-5y,\qquad \frac{\partial z}{\partial y}=-5x-2y$$

$$\frac{\partial z}{\partial x}=0 \;\Rightarrow\; 2x-5y=0 \;\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y}=0 \;\Rightarrow\; -5x-2y=0 \;\ldots\ldots\ldots (ii)$$

(i) and (ii) gives $(0,0)$ is the critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2,\quad B=\frac{\partial^2 z}{\partial x\partial y}=-5,\quad C=\frac{\partial^2 z}{\partial y^2}=-2$$

$\Rightarrow B^2-AC=(-5)^2-(2)(-2)=25+4=29>0$

$\Rightarrow$ There is a saddle point at $(0,0)$.

(*e*) $z=x^2-2xy+y^2$

$$\frac{\partial z}{\partial x}=2x-2y,\qquad \frac{\partial z}{\partial y}=2y-2x$$

$$\frac{\partial z}{\partial x}=0,\ \frac{\partial z}{\partial y}=0 \;\Rightarrow\; x-y=0 \;\Rightarrow\; x=y$$

$\Rightarrow$ Every point on the line $y=x$ is a critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2,\quad B=\frac{\partial^2 z}{\partial x\partial y}=-2,\quad C=\frac{\partial^2 z}{\partial y^2}=2$$

$\Rightarrow B^2-AC=(-2)^2-(2)(2)=4-4=0$

Consider $\Delta z=z(x+h,y+k)-z(x,y)$

$\because x=y \quad \therefore \Delta z=z(x+h,x+k)-z(x,x)$

$$=(x+h)^2-2(x+h)(x+k)+(x+k)^2$$

$$=\left[(x+h)-(x+k)\right]^2\geq 0$$

$\Rightarrow$ Each point on the line $y=x$ gives a relative minimum.

(*f*) $z=x^3-3xy^2+y^3$

$$\frac{\partial z}{\partial x}=3x^2-3y^2,\qquad \frac{\partial z}{\partial y}=-6xy+3y^2$$

$$\frac{\partial z}{\partial x}=0 \;\Rightarrow\; 3x^2-3y^2=0 \;\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y}=0 \;\Rightarrow\; -6xy+3y^2=0 \;\ldots\ldots\ldots\ldots (ii)$$

From (i) and (ii), we have

$$3x^2-6xy=0 \;\Rightarrow\; x(x-2y)=0 \;\Rightarrow\; x=0,\ x=2y$$

Now $x=0 \;\Rightarrow\; y=0$

And $x=2y \;\Rightarrow\; (2y)^2-y^2=0 \;\Rightarrow\; y=0$

Hence (0,0) is the only critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=6x=0 \qquad \text{at } (0,0)$$

$$B=\frac{\partial^2 z}{\partial x\partial y}=-6y=0 \qquad \text{at } (0,0)$$

$$C=\frac{\partial^2 z}{\partial y^2}=-6x+6y=0 \qquad \text{at } (0,0)$$

$\Rightarrow\ B^2-AC=0$

Consider $\Delta z=z(h,k)-z(0,0)$

$$=h^3-3hk^2+k^3 \quad = h^3-3h^3+h^3 \qquad \text{when } h=k$$

$$=-h^3 \left| \begin{array}{l} <0 \quad when\ h>0 \\ >0 \quad when\ h<0 \end{array} \right.$$

$\Rightarrow$ There is a saddle point at $(0,0)$

Note : (i) If for a point $A=B=C=0$ and $\Delta z\ge 0$, then z is minimum at that point and if $\Delta z\le 0$, then z is maximum at that point.

(ii) If A, B, C are not zero and $B^2-AC=0$ then z is neither maximum nor minimum.

Question

Find the critical points of the following functions and test for maxima and minima.

(a) $z=x^3-2xy^2+y^3$

(b) $z=x^3+y^3-3x-12y+20$

(c) $z=x^3+y^3-63(x+y)+12xy$

(d) $z=xy(a-x-y)$

(e) $z=x^2-2xy+y^2+x^3-y^3+25$

(f) $z=x^2y^2-5x^2-8xy-5y^2$

(g) $z=2(x-y)^2-x^4-y^4$

(h) $z=2(x-y)^3-(x^4-y^4)$

(i) $z=x^2-5xy-y^3$

Solution

(***a***) $z=x^3-2xy^2+y^3$

$$\frac{\partial z}{\partial x}=3x^2-2y^2, \qquad \frac{\partial z}{\partial y}=-4xy+3y^2$$

$$\frac{\partial z}{\partial x}=0 \quad \Rightarrow\ 3x^2-2y^2=0 \ \ \ldots\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y}=0 \quad \Rightarrow\ -4xy+3y^2=0 \ \ \ldots\ldots\ldots\ldots (ii)$$

Adding (i) and (ii), we get

$$3x^2-4xy+y^2=0 \quad \Rightarrow\ 3x^2-3xy-xy+y^2=0$$

$$\Rightarrow\ 3x(x-y)-y(x-y)=0 \quad \Rightarrow\ (x-y)(3x-y)=0$$

If $x-y=0$, then $x=y$ in (i) gives

$$3x^2-2x^2=0 \quad \Rightarrow\ x=0 \quad \Rightarrow\ y=0.$$

And if $3x-y=0$, then $y=3x$ in (i) gives

$$3x^2-2(3x)^2=0 \quad \Rightarrow\ x=0 \quad \Rightarrow\ y=0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 6x = 0 \qquad \text{at } (0,0)$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -4y = 0 \qquad \text{at } (0,0)$$

$$C = \frac{\partial^2 z}{\partial y^2} = 6y - 4x = 0 \qquad \text{at } (0,0)$$

$\Rightarrow$ $A = B = C = 0$ at $(0,0)$ and hence $B^2 - AC = 0$

Now consider $\Delta z = z(h,k) - z(0,0)$

$$= h^3 - 2hk^2 + k^3$$

$$= h^3 - 2h^3 + h^3 = 0 \qquad \text{when } h = k$$

$\Rightarrow$ The nature of the point is undetermined.

(***b***) $z = x^3 + y^3 - 3x - 12y + 20$

$$\frac{\partial z}{\partial x} = 3x^2 - 3 \ , \qquad \frac{\partial z}{\partial y} = 3y^2 - 12$$

$$\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad x^2 - 1 = 0$$

$$\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad y^2 - 4 = 0$$

$\Rightarrow$ $x = \pm 1$, $y = \pm 2$, and the critical points are

$(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

$$A = \frac{\partial^2 z}{\partial x^2} = 6x, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 z}{\partial y^2} = 6y$$

$\Rightarrow$ $B^2 - AC = -36xy$

$$B^2 - AC = -36(1)(2) = -72 < 0 \qquad \text{at } (1,2)$$

$$B^2 - AC = -36(1)(-2) = 72 > 0 \qquad \text{at } (1,-2)$$

$$B^2 - AC = -36(-1)(2) = 72 > 0 \qquad \text{at } (-1,2)$$

$$B^2 - AC = -36(-1)(-2) = -72 < 0 \qquad \text{at } (-1,-2)$$

$\Rightarrow$ There is a saddle point at $(1,-2)$ and $(-1,2)$.

$B^2 - AC < 0$ while $A = 6 > 0$ at $(1,2)$

and $A = -6 < 0$ at $(-1,-2)$

$\Rightarrow$ z has relative minima at $(1,2)$ & relative maxima at $(-1,-2)$.

(***c***) $z = x^3 + y^3 - 63(x+y) + 12xy$

$$\frac{\partial z}{\partial x} = 3x^2 - 63 + 12y, \qquad \frac{\partial z}{\partial y} = 3y^2 - 63 + 12x$$

For critical points $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$.

$\Rightarrow$ $3x^2 + 12y - 63 = 0$ ……………… (i)

& $3y^2 + 12x - 63 = 0$ ……………….. (ii)

Subtracting (ii) from (i), we get

$$3x^2 - 3y^2 + 12y - 12x = 0$$

$$\Rightarrow \ x^2 - y^2 + 4(y - x) = 0$$

$$\Rightarrow \ (x-y)(x+y) - 4(x-y) = 0$$

$$\Rightarrow \ (x-y)(x+y-4) = 0$$

If $x-y=0$ then (i) gives $3x^2+12x-63=0$

$$\Rightarrow x^2+4x-21=0$$

$$\Rightarrow (x+7)(x-3)=0 \quad \Rightarrow x=-7,3$$

$\Rightarrow$ The critical points are $(-7,-7)$ & $(3,3)$.

If $x+y-4=0$ then $x=4-y$

Put this value of x in (ii), we have

$$3y^2+12(4-y)-63=0$$

$$\Rightarrow y^2+4(4-y)-21=0$$

$$\Rightarrow y^2-4y-5=0$$

$$\Rightarrow (y-5)(y+1)=0 \quad \Rightarrow y=5,\ -1$$

$y=5 \Rightarrow x=-1$ & $y=-1 \Rightarrow x=5$

$\Rightarrow (-1,5)$ and $(5,-1)$ are the other two critical points.

$$A=\frac{\partial^2 z}{\partial x^2}=6x, \quad B=\frac{\partial^2 z}{\partial x\partial y}=12, \quad C=\frac{\partial^2 z}{\partial y^2}=6y$$

$\Rightarrow B^2-AC=(12)^2-36xy=144-36xy$

At $(-7,-7)$, we have

$$B^2-AC=144-36(-7)(-7)<0 \quad \text{and} \quad A<0$$

$\Rightarrow (-7,7)$ is a point of relative maximum value.

At $(3,3)$, we have

$$B^2-AC=144-36(3)(3)=144-324<0 \quad \text{and} \quad A>0.$$

$\Rightarrow (3,3)$ is a point of relative minimum value.

At $(-1,5)$, we have

$$B^2-AC=144-36(-1)(5)>0$$

$\Rightarrow (-1,5)$ is a saddle point.

At $(-5,1)$, we have

$$B^2-AC=144-(-5)(1)>0$$

$\Rightarrow (-5,1)$ is also a saddle point.

(***d***) $z=xy(a-x-y)=axy-x^2y-xy^2$

$$\frac{\partial z}{\partial x}=ay-2xy-y^2$$

$$\frac{\partial z}{\partial y}=ax-x^2-2xy$$

$\frac{\partial z}{\partial x}=0 \Rightarrow ay-2xy-y^2=0$ .... (i)

$\frac{\partial z}{\partial y}=0 \Rightarrow ax-x^2-2xy=0$ .... (ii)

Subtracting (i) and (ii)

$$\begin{array}{l} ay-2xy-y^2=0 \\ \underline{\overset{-}{a}x\overset{+}{-}2xy\overset{+}{-}x^2=0} \\ ay-ax-y^2+x^2=0 \end{array}$$

$$\Rightarrow (x^2-y^2)-a(x-y)=0$$

$$\Rightarrow (x-y)(x+y)-a(x-y)=0$$

$$\Rightarrow (x-y)(x+y-a)=0$$

If $x-y=0 \Rightarrow x=y$ then (i) give

$$ax-2x^2-x^2=0 \quad \Rightarrow \quad ax-3x^2=0$$

$$\Rightarrow x(a-3x)=0 \Rightarrow x=0\,,\ a/3$$

$$\Rightarrow (0,0)\ \&\ \left(a/3\,,\,a/3\right) \text{ are the critical points.}$$

If $x+y-a=0$ then $y=a-x$ and (i) gives

$$a(a-x)-2x(a-x)-(a-x)^2=0$$

$$\Rightarrow a^2-ax-2ax+2x^2-a^2-x^2+2ax=0$$

$$\Rightarrow x^2-ax=0 \Rightarrow x(x-a)=0 \Rightarrow x=0,\ a$$

$\Rightarrow (0,a)\ \&\ (a,0)$ are the other two critical points.

$$A=\frac{\partial^2 z}{\partial x^2}=-2y\,,\quad B=\frac{\partial^2 z}{\partial x\partial y}=a-2x-2y\,,\quad C=\frac{\partial^2 z}{\partial x\partial y}=-2x$$

$$\Rightarrow B^2-AC=(a-2x-2y)^2-4xy$$

At $(0,0)$, we have $B^2-AC=a^2>0 \Rightarrow (0,0)$ is a saddle point.

At $\left(a/3\,,a/3\right)$, we have

$$B^2-AC=\left(a-2a/3-2a/3\right)^2-4\left(a/3\right)\left(a/3\right)$$

$$=\frac{a^2}{9}-\frac{4a^2}{9}<0 \quad \text{and} \quad A<0$$

$$\Rightarrow \left(\frac{a}{3},\frac{a}{3}\right) \text{ is a point of maximum value.}$$

At $(0,a)$, we have $B^2-AC=(a-2a)^2-4(0)(a)=a^2>0$

$\Rightarrow (0,a)$ is a saddle point.

At $(a,0)$, we have $B^2-AC=(a-2a)^2-4(a)(0)=a^2>0$

$\Rightarrow (a,0)$ is also a saddle point.

(e) $z=x^2-2xy+y^2+x^3-y^3+25$

$$\frac{\partial z}{\partial x}=2x-2y+3x^2$$

$$\frac{\partial z}{\partial y}=-2x+2y-3y^2$$

$$\frac{\partial z}{\partial x}=0 \Rightarrow 3x^2+2x-2y=0 \text{ .... } (i)$$

$$\frac{\partial z}{\partial y}=0 \Rightarrow 3y^2+2x-2y=0 \text{ .... } (ii)$$

Subtracting (i) and (ii), we have

$$3x^2-3y^2=0$$

$$\Rightarrow 3(x-y)(x+y)=0$$

$x-y=0 \Rightarrow x=y$, using in (i) we have

$$3x^2+2x-2x=0 \quad \Rightarrow \quad x=0$$

And $x+y=0 \Rightarrow x=-y$, using in (i) we have

$$3x^2+2x+2x=0$$

$$\Rightarrow 3x^2+4x=0 \quad \Rightarrow \quad x(3x+4)=0$$

$$\Rightarrow x=0,\quad x=-4/3$$

$x=0 \Rightarrow y=0$ and $x=-\frac{4}{3} \Rightarrow y=\frac{4}{3}$

$\Rightarrow$ The critical points are $(0,0)$ & $\left(-\frac{4}{3},\frac{4}{3}\right)$

$$A=\frac{\partial^2 z}{\partial x^2}=2+6x$$

$$B=\frac{\partial^2 z}{\partial x\partial y}=-2$$

$$C=\frac{\partial^2 z}{\partial y^2}=2-6y$$

$$B^2-AC=4-(2+6x)(2-6y)$$

At $(0,0)$, we have $B^2-AC=4-4=0 \Rightarrow$ Nature undetermined

At $\left(-\frac{4}{3},\frac{4}{3}\right)$, we have

$$B^2-AC=4-(2-8)(2-8)=4-(-6)(-6)<0 \quad \text{and} \quad A<0$$

$\therefore$ Relative maximum at $\left(-\frac{4}{3},\frac{4}{3}\right)$.

(*f*) $z=x^2y^2-5x^2-8xy-5y^2$

$$\frac{\partial z}{\partial x}=2xy^2-10x-8y$$

$$\frac{\partial z}{\partial y}=2x^2y-10y-8x$$

For critical points, we have

$$xy^2-5x-4y=0 \ \ldots\ldots\ldots\ldots (i)$$

$$x^2y-5y-4x=0 \ \ldots\ldots\ldots\ldots (ii)$$

Adding (*i*) and (*ii*), we have

$$xy^2+x^2y-9x-9y=0$$

$$\Rightarrow xy(y+x)-9(x+y)=0$$

$$\Rightarrow (x+y)(xy-9)=0$$

$x+y=0 \Rightarrow y=-x$ in (*i*) gives

$$x^3-5x+4x=0$$

$$\Rightarrow x^3-x=0 \quad \Rightarrow x(x-1)(x+1)=0$$

$$\Rightarrow x=0,1,-1$$

$x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=-1$

$x=-1 \Rightarrow y=1$

$\Rightarrow (0,0),\ (1,-1),\ (-1,1)$ are the critical points.

If $xy-9=0$, then $y=\frac{9}{x}$ in (*i*) gives $x^2-9=0 \Rightarrow x=\pm 3$

$x=3 \Rightarrow y=3$ and $x=-3 \Rightarrow y=-3$

$\Rightarrow (3,3)$ & $(-3,-3)$ are also the critical points.

$$A=\frac{\partial^2 z}{\partial x^2}=2y^2-10,\quad B=\frac{\partial^2 z}{\partial x\partial y}=4xy-8,\quad C=\frac{\partial^2 z}{\partial y^2}=2x^2-10$$

$$B^2-AC=(4xy-8)^2-(2y^2-10)(2x^2-10)$$

At $(0,0)$, we have

$$B^2 - AC = 64 - (-10)(-10) < 0 \quad \text{and} \quad A = -10 < 0$$

$\Rightarrow$ $(0,0)$ is the point of maximum value.

At $(1,-1)$, we have

$$B^2 - AC = (-4-8)^2 - (2-10)(2-10) = 144 - 64 > 0$$

$\Rightarrow$ $(1,-1)$ is a saddle point.

At $(-1,1)$, we have

$$B^2 - AC = (-4-8)^2 - (2-10)(2-10) = 144 - 64 > 0$$

$\Rightarrow$ $(-1,1)$ is a saddle point.

At $(3,3)$, we have

$$B^2 - AC = (36-8)^2 - (18-10)(18-10) = (24)^2 - 64 > 0$$

$\Rightarrow$ $(3,3)$ is a saddle point.

At $(-3,-3)$, we have

$$B^2 - AC = (36-8)^2 - (8)(8) > 0$$

$\Rightarrow$ $(-3,-3)$ is again a saddle point.

(***g***) $z = 2(x-y)^2 - x^4 - y^4$

$$\frac{\partial z}{\partial x} = 4(x-y) - 4x^3$$

$$\frac{\partial z}{\partial y} = -4(x-y) - 4y^3$$

For critical points

$$\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; x - y - x^3 = 0 \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; -x + y - y^3 = 0 \;\ldots\ldots\ldots\ldots\; (ii)$$

Addition of (i) and (ii) gives

$$x^3 + y^3 = 0$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$\Rightarrow$ $x + y = 0$ or $x^2 - xy + y^2 = 0$ which gives imaginary values.

$x + y = 0 \Rightarrow y = -x$ in (i) gives

$$x + x - x^3 = 0 \Rightarrow 2x - x^3 = 0$$

$$\Rightarrow x(2 - x^2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

$x = 0 \Rightarrow y = 0$

$x = \sqrt{2} \Rightarrow y = -\sqrt{2}$

$x = -\sqrt{2} \Rightarrow y = \sqrt{2}$

$\Rightarrow$ The critical points are $(0,0)$, $\left(\sqrt{2}, -\sqrt{2}\right)$, $\left(-\sqrt{2}, \sqrt{2}\right)$.

$$A = \frac{\partial^2 z}{\partial x^2} = 4 - 12x^2, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -4, \quad C = \frac{\partial^2 z}{\partial y^2} = 4 - 12y^2$$

$$B^2 - AC = 16 - (4 - 12x^2)(4 - 12y^2)$$

At $(0,0)$, we have $B^2 - AC = 0$

Consider $\Delta z = z(h,k) - z(0,0)$

$$= 2(h-k)^2 - h^4 - k^4 = -2h^4 \le 0 \quad \text{if } h = k$$

$\Rightarrow$ $(0,0)$ is the points of maximum value.

At $\left(\sqrt{2},-\sqrt{2}\right)$, we have

$$B^2 - AC = 16-(4-24)(4-24)$$
$$= 16-(-20)(-20) < 0 \quad \text{and} \quad A<0.$$

$\Rightarrow$ $\left(\sqrt{2},-\sqrt{2}\right)$ is a point of maximum value.

At $\left(-\sqrt{2},\sqrt{2}\right)$, we have

$$B^2 - AC = 16-(4-24)(4-24) < 0 \quad \text{and} \quad A<0.$$

$\Rightarrow$ $\left(-\sqrt{2},\sqrt{2}\right)$ is also a point of maximum vale.

(***h***) $z = 2(x-y)^3-(x^4-y^4)$

$$\frac{\partial z}{\partial x} = 6(x-y)^2 - 4x^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y} = -6(x-y)^2 + 4y^3 = 0 \quad \ldots\ldots\ldots\ldots (ii)$$

Adding (*i*) and (*ii*), we get

$$y^3 - x^3 = 0 \;\Rightarrow\; (y-x)(y^2+xy+x^2) = 0$$

$y-x=0 \;\Rightarrow\; y=x$ in (*i*) gives

$$4x^3 = 0 \;\Rightarrow\; x=0 \;\Rightarrow\; y=0$$

$x^2+xy+y^2=0$ gives imaginary values

$\Rightarrow$ $(0,0)$ is the only critical point

$$A = \frac{\partial^2 z}{\partial x^2} = 12(x-y) - 12x^2$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -12(x-y)$$

$$C = \frac{\partial^2 z}{\partial y^2} = 12(x-y) + 12y^2$$

at $(0,0)$, $A=B=C=0 \;\Rightarrow\; B^2-AC=0$

Consider $\Delta z = z(h,h) - z(0,0) = 0$

$\Rightarrow$ Nature undecided.

(***i***) $z = x^2 - 5xy - y^3$

$$\frac{\partial z}{\partial x} = 2x - 5y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial z}{\partial y} = -5x - 3y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From (*i*) $y = \frac{2x}{5}$

(*ii*) becomes $-5x - 3\left(\frac{4x^2}{25}\right) = 0$

$$\Rightarrow -125x - 12x^2 = 0$$

$$\Rightarrow 12x^2 + 125x = 0$$

$$\Rightarrow x(12x+125) = 0 \;\Rightarrow\; x = 0,\ -\frac{125}{12}$$

Remarks

$$x=0 \;\Rightarrow\; y=0 \quad \& \quad x=-\frac{125}{12} \;\Rightarrow\; y=\frac{2}{5}\left(-\frac{125}{12}\right)=-\frac{25}{6}$$

$\Rightarrow\; (0,0) \;\;\&\;\; \left(-\frac{125}{12},-\frac{25}{6}\right)$ are the critical points

$$A=\frac{\partial^2 z}{\partial x^2}=2, \quad B=\frac{\partial^2 z}{\partial x\partial y}=-5, \quad C=\frac{\partial^2 z}{\partial y^2}=-6y$$

$$B^2-AC=25+12y$$

At $(0,0)$, we have $B^2-AC=25>0 \;\Rightarrow\; (0,0)$ is a saddle point.

At $\left(-\frac{125}{12},-\frac{25}{6}\right)$, we have

$$B^2-AC=25+12\left(-\frac{25}{6}\right)=-25<0 \quad \text{and} \quad A=2>0$$

$\therefore\; \left(-\frac{125}{12},-\frac{25}{6}\right)$ is a point of maximum value.