Maxima and Minima for Functions with side conditions. Lagrange's Multiplier.

Remarks

Question

Find the critical points of $w = xyz$ subject to the condition $x^2 + y^2 + z^2 = 1$.

Solution

We form the function

$$j = f + l\,g = xyz + l\,(x^2 + y^2 + z^2 - 1)$$

and obtain four equations

$$\frac{\partial j}{\partial x} = yz + 2l\,x = 0$$

$$\frac{\partial j}{\partial y} = xz + 2l\,y = 0$$

$$\frac{\partial j}{\partial z} = xy + 2l\,z = 0$$

& $\quad x^2 + y^2 + z^2 - 1 = 0$

Multiplying the first three equations by x, y, z respectively, adding and using in fourth equation we find $l = -\dfrac{3xyz}{2}$.

Using this relation we have $(0,0,\pm 1)$, $(0,\pm 1,0)$, $(\pm 1,0,0)$, and $\left(\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}\right)$ as the critical points.

Question

Find the critical points of the function $z = x^2 + 24xy + 8y^2$ where $x^2 + y^2 = 25$. Test for maxima & minima.

Solution

$$F(x, y, l) = x^2 + 24xy + 8y^2 + l\,(x^2 + y^2 - 25)$$

$F_x = 2x + 24y + 2l\,x = 0$ ……………. (i)

$F_y = 24x + 16y + 2l\,y = 0$ …………… (ii)

& $\quad x^2 + y^2 - 25 = 0$ ………………….. (iii)

$(i) \Rightarrow (1 + l)x + 12y = 0$ …………..…. (iv)

$(ii) \Rightarrow 12x + (8 + l)y = 0$ …………..…. (v)

Multiplying equation (iv) by 12, (v) by $(1 + l)$ and adding

$$12(1 + l)x + 144y = 0$$

$$\underset{-}{}12(1 + l)x \underset{-}{+} (1 + l)(8 + l)y = 0$$

$$144y - (1 + l)(8 + l)y = 0$$

$\Rightarrow y = 0 \quad$ or $\quad l^2 + 9l - 136 = 0$

$\Rightarrow y = 0, \quad l = 8, \ -17$

From (ii), $\quad y = 0 \quad \Rightarrow \quad x = 0$

$\because$ $(0,0)$ does not satisfy (iii) $\quad \therefore$ It is not a critical point.

$l = 8 \quad \Rightarrow \quad x = -\dfrac{4y}{3} \qquad$ form (iv)

Put this value of x in (iii)

$\Rightarrow \dfrac{16y^2}{9} + y^2 = 25 \quad \Rightarrow \quad y = \pm 3$

$\Rightarrow (-4,3) \quad \& \quad (4,-3)$ are the critical points.

Similarly when $l = -17$, we have $x = \dfrac{3y}{4}$ from (iv)

And putting the value of x in (iii) we get $y = \pm 4$

$\Rightarrow (\pm 3, \pm 4)$ are the other two critical points.

$A = F_{xx} = 2 + 2l$

$B = F_{xy} = 24$

$C = F_{yy} = 16 + 2l$

When $l = 8$

$A = 2 + 16 = 18$, $\quad B = 24$, $\quad C = 16 + 16 = 32$

and so $\quad B^2 - AC = 576 - 576 = 0$

$F(x, y, l) = x^2 + 24xy + 8y^2 + 8(x^2 + y^2 - 25)$ when $l = 8$

$\Rightarrow F(x, y) = 9x^2 + 24xy + 16y^2 - 200$

At $(-4,3)$

$$\begin{aligned}\Delta F &= F(-4+h, 3+h) - F(-4,3)\\ &= 9(-4+h)^2 + 24(-4+h)(3+h) + 16(3+h)^2 - 200\\ &\qquad -9(-4)^2 + 24(-4)(-3) - 16(3)^2 + 200\\ &= 9(16 - 8h + h^2) + 24(h^2 - h - 12) + 16(9 + 6h + h^2)\\ &\qquad -144 + 288 - 144\\ &= 144 - 72h + 9h^2 + 24h^2 - 24h - 288 + 144 + 96h + 26h^2\\ &\qquad -144 + 288 - 144\\ &= 49h^2 \geq 0\end{aligned}$$

$\Rightarrow (-4,3)$ is the point of minimum value.

Similarly $(4,-3)$ gives a point of minimum value.

And when $l = -17$, $(\pm 3, \pm 4)$ are the point of maximum value.

Question

Find the critical points of $w = x + z$, where $x^2 + y^2 + z^2 = 1$. Test for a maxima and minima.

Solution

Consider the function

$F(x, y, z) = x + z + l(x^2 + y^2 + z^2 - 1)$

$F_x = 1 + 2lx$, $\quad F_y = 2ly$, $\quad F_z = 1 + 2lz$

For critical points, we have

$1 + 2lx = 0$ …………..…. (i)

$2ly = 0$ ………………..…. (ii)

$1 + 2lz = 0$ …………….. (iii)

and $\quad x^2 + y^2 + z^2 = 1$ ………… (iv)

Solving these equations we have $l = \pm \dfrac{1}{\sqrt{2}}$

$l = \dfrac{1}{\sqrt{2}}$ gives $\left(\dfrac{-1}{\sqrt{2}}, 0, \dfrac{-1}{\sqrt{2}}\right)$ as the critical point.

$l = -\dfrac{1}{\sqrt{2}}$ gives $\left(\dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}}\right)$ as the critical point.

$A = F_{xx} = 2l$, $B = F_{xy} = 0$, $C = F_{yy} = 2l$

i) $l = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $A = \sqrt{2}$, $B = 0$, $C = \sqrt{2}$

so $B^2 - AC = 0 - 2 < 0$ and $A > 0$

$\Rightarrow \left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$ is a point of relative minimum value.

ii) $l = -\frac{1}{\sqrt{2}}$ $\Rightarrow$ $A = -\sqrt{2}$, $B = 0$, $C = -\sqrt{2}$

so $B^2 - AC = 0 - 2 < 0$ and $A < 0$

$\Rightarrow \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ is a point of relative maximum value.

Question

Find the critical points of $w = xyz$ where $x^2 + y^2 = 1$ & $x - z = 0$. Test for the maxima and minima.

Solution

Consider $F = xyz + l_1(x^2 + y^2 - 1) + l_2(x - z)$

For critical points

$F_x = yz + 2l_1x + l_2 = 0$ …………. (i)

$F_y = xz + 2l_1y = 0$ …………….. (ii)

$F_z = xy - l_2 = 0$ ……………….. (iii)

& $x^2 + y^2 = 1$ …………..……. (iv)

$x - z = 0$ …………….……. (v)

From (iii) $l_2 = xy$ and from (ii) $l_1 = -\frac{xz}{2y}$

Putting in (i), we get

$$yz - \frac{x^2 z}{y} + xy = 0$$

$$\Rightarrow y^2 z - x^2 z + xy^2 = 0$$

$\because$ $x = z$ form (iv) $\therefore$ $y^2 x - x^3 + xy^2 = 0$

$$\Rightarrow 2xy^2 - x^3 = 0$$

But from (iv), $y^2 = 1 - x^2$ $\Rightarrow$ $2x(1 - x^2) - x^3 = 0$

$$\Rightarrow 3x^3 - 2x = 0 \Rightarrow x(3x^2 - 2) = 0 \Rightarrow x = 0,\ \pm\sqrt{\frac{2}{3}}$$

$\Rightarrow$ The critical points are $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$, $\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$, $(0,1,0)$ and $(0,-1,0)$.

$A = F_{xx} = 2l_1$, $B = F_{xy} = z$, $C = F_{yy} = 2l_1$

$B^2 - AC = z^2 - 4l_1^2$

From (ii) $l_1^2 = \frac{x^2 z^2}{4y^2}$ $\Rightarrow$ $B^2 - AC = z^2 - \frac{z^2 x^2}{y^2} = \frac{z^2(y^2 - x^2)}{y^2}$

i) At $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$, we have $B^2 - AC = \frac{\frac{2}{3}\left(\frac{1}{3} - \frac{2}{3}\right)}{1/3} < 0$

And $A = F_{xx} = 2l_1 = -\frac{xz}{y} = \frac{-2/3}{1/\sqrt{3}} < 0$

$\Rightarrow$ Function is maximum at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$.

Similarly we can show that w is maximum at $(0,-1,0)$ and minimum at $\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$ & $(0,1,0)$.

Question

Find the point to the curves $x^2 - xy + y^2 - z^2 = 1$, $x^2 + y^2 = 1$ nearest to the origin $(0,0,0,)$.

Solution

Let (x,y,z) be a point on the curve. Then its distance from the origin is given by $\sqrt{x^2 + y^2 + z^2}$

We are to minimize $f = d^2 = x^2 + y^2 + z^2$

subject to the conditions $x^2 - xy + y^2 - z^2 = 1$, $x^2 + y^2 = 1$

Consider

$$F = x^2 + y^2 + z^2 + l_1\left(x^2 - xy + y^2 - z^2 - 1\right) + l_2\left(x^2 + y^2 - 1\right)$$
$$F_x = 2x + (2x - y)\,l_1 + 2\,l_2 x$$
$$F_y = 2y + (2y - x)\,l_1 + 2\,l_2 y$$
$$F_z = 2z + l_1(-2z)$$

For critical points, we have

$$2x\left(1 + l_1 + l_2\right) - l_1 y = 0 \quad \ldots\ldots\ldots\ldots \ (i)$$
$$2y(1 + l_1 + l_2) - l_1 x = 0 \quad \ldots\ldots\ldots\ldots\ (ii)$$
$$2z(1 - l_1) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (iii)$$
$$x^2 - xy + y^2 - z^2 - 1 = 0 \quad \ldots\ldots\ldots\ldots \ (iv)$$
$$x^2 + y^2 - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (v)$$

From (iii), we have $z = 0$ & $l_1 = 1$.

$z = 0$ in (iv) gives $x^2 - xy + y^2 - 1 = 0 \quad \Rightarrow \quad xy = x^2 + y^2 - 1$

But $x^2 + y^2 - 1 = 0 \quad \Rightarrow \quad xy = 0$

$\Rightarrow$ $x = 0$ or $y = 0$ or *both are zero.*

We can not take $x = 0$, $y = 0$ at a same time because it gives $(0,0,0)$ which is origin itself.

$z = 0$, $x = 0$ in (v) $\Rightarrow$ $y^2 = 1$ $\Rightarrow$ $y = \pm 1$

$\Rightarrow$ $(0,\pm 1,0)$ are the critical points

& $z = 0$, $y = 0$ in (v) $\Rightarrow$ $x^2 = 1$ $\Rightarrow$ $x = \pm 1$

$\Rightarrow$ $\left(\pm 1,0,0\right)$ are the other critical points.

$\because$ $f = d^2 = 1$ at these four points

$\therefore$ These are the required points at which function is nearest to origin.

Remarks

Question

Find the shortest distance from the origin to the curve

$$x^2 + 8xy + 7y^2 = 225$$

Solution

We are to find the minimum value of $f = d^2 = x^2 + y^2$

subject to the condition $x^2 + 8xy + 7y^2 = 225$.

Consider $F = x^2 + y^2 + l\,(x^2 + 8xy + 7y^2 - 225)$

$$F_x = 2x + l\,(2x + 8y)$$

$$F_y = 2y + l\,(8x + 14y)$$

For critical points

$$x + l\,(x + 4y) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$y + l\,(4x + 7y) = 0 \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

$$x^2 + 8xy + 7y^2 - 225 = 0 \;\ldots\ldots\; (iii)$$

$$(i) \;\Rightarrow\; (1 + l)x + 4l\,y = 0 \quad\Rightarrow\; \frac{x}{y} = -\frac{4l}{1+l}$$

$$(ii) \;\Rightarrow\; 4l\,x + (1 + 7l)y = 0 \quad\Rightarrow\; \frac{x}{y} = -\frac{1+7l}{4l}$$

$$\Rightarrow\; \frac{x}{y} = -\frac{4l}{1+l} = -\frac{1+7l}{4l} \qquad\Rightarrow\; 16l^2 = (1+l)(1+7l)$$

$$\Rightarrow\; 16l^2 = 1 + l + 7l + 7l^2 \qquad\Rightarrow\; 9l^2 - 8l - 1 = 0$$

$$\Rightarrow\; (l - 1)(9l + 1) = 0 \qquad\qquad\Rightarrow\; l = 1,\; -\frac{1}{9}$$

$$l = 1 \;\Rightarrow\; \frac{x}{y} = -2 \;\Rightarrow\; x = -2y$$

Putting this value of x in equation (iii) we have

$$(-2y)^2 + 8(-2y)\,y + 7y^2 = 225$$

$$\Rightarrow\; 4y^2 - 16y^2 + 7y^2 = 225 \quad\Rightarrow\; -5y^2 = 225$$

which gives imaginary values of y.

$$l = -\frac{1}{9} \quad\Rightarrow\; \frac{x}{y} = -\frac{4\left(-\tfrac{1}{9}\right)}{1 - \tfrac{1}{9}} = \frac{\tfrac{4}{9}}{\tfrac{8}{9}} = \frac{1}{2} \quad\Rightarrow\; y = 2x$$

Putting in (iii), we have

$$x^2 + 8x(2x) + 7(2x)^2 = 225$$

$$\Rightarrow\; x^2 + 16x^2 + 28x^2 = 225$$

$$\Rightarrow\; 45x^2 = 225 \;\Rightarrow\; x^2 = 5 \;\Rightarrow\; x = \pm\sqrt{5}$$

$$x = \sqrt{5} \quad\Rightarrow\; y = 2\sqrt{5}$$

& $\quad x = -\sqrt{5} \;\Rightarrow\; y = -2\sqrt{5}$

$\therefore$ The critical points are $\left(\sqrt{5}, 2\sqrt{5}\right)$ & $\left(-\sqrt{5}, -2\sqrt{5}\right)$.

$$d^2_{(\pm\sqrt{5},\pm 2\sqrt{5})} = 25$$

$\Rightarrow$ Shortest distance $=$ $d = 5$

Remarks

Question

Find a point (x,y,z) on the sphere $x^2+y^2+z^2=1$ which is farthest from the point $(1,2,3)$.

Solution

We are to maximize

$$f(x,y,z)=(x-1)^2+(y-2)^2+(z-3)^2$$

subject to the condition $x^2+y^2+z^2=1$

Let $F=(x-1)^2+(y-2)^2+(z-3)^2+l(x^2+y^2+z^2-1)$

For critical points

$$F_x=2(x-1)+2lx=0$$

$$F_y=2(y-2)+2ly=0$$

$$F_z=2(z-3)+2lz=0 \quad \text{and} \quad x^2+y^2+z^2=1$$

$$\Rightarrow\ x-1+lx=0 \ \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$y-2+ly=0 \ \ldots\ldots\ldots\ldots\ (ii)$$

$$z-3+lz=0 \ \ldots\ldots\ldots\ldots\ (iii)$$

$$x^2+y^2+z^2=1 \ \ldots\ldots\ldots\ldots\ (iv)$$

$$\Rightarrow\ x=\frac{1}{1+l},\quad y=\frac{2}{1+l},\quad z=\frac{3}{1+l}$$

Putting in (iv)

$$\left(\frac{1}{1+l}\right)^2+\left(\frac{2}{1+l}\right)^2+\left(\frac{3}{1+l}\right)^2=1$$

$$\Rightarrow\ 14=(1+l)^2$$

$$\Rightarrow\ l+1=\pm\sqrt{14} \quad \Rightarrow\ l=-1\pm\sqrt{14}$$

$$\Rightarrow\ x=\frac{1}{\pm\sqrt{14}},\quad y=\frac{2}{\pm\sqrt{14}},\quad z=\frac{3}{\pm\sqrt{14}}$$

Clearly $\left(\frac{-1}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-3}{\sqrt{14}}\right)$ is the point which is farthest from $(1,2,3)$.

Question

Find the extreme values of $z=6-4x-3y$, provided x & y satisfy $x^2+y^2=1$.

Solution

Define $F=6-4x-3y+l\left(x^2+y^2-1\right)$

For critical points, we have

$$F_x=-4+2lx=0 \ \ldots\ldots\ldots\ldots\ (i)$$

$$F_y=-3+2ly=0 \ \ldots\ldots\ldots\ldots\ (ii)$$

and $x^2+y^2=1 \ \ldots\ldots\ldots\ldots\ (iii)$

From (i) and (ii) we have $x=\frac{2}{l}$, $y=\frac{3}{2l}$

Putting these values in (iii) we get $l=\pm\frac{5}{2}$

$$l=\frac{5}{2} \quad\Rightarrow\ x=\frac{2}{5/2}=\frac{4}{5} \quad \& \quad y=\frac{3}{2\cdot 5/2}=\frac{3}{5}$$

$$l=-\frac{5}{2} \quad \Rightarrow \quad x=\frac{2}{-5/2}=-\frac{4}{5} \quad \& \quad y=\frac{3}{2\cdot(-5/2)}=-\frac{3}{5}$$

$\Rightarrow \left(\frac{4}{5},\frac{3}{5}\right)$ & $\left(-\frac{4}{5},-\frac{3}{5}\right)$ are the critical points.

$A=F_{xx}=2l\,,\quad B=F_{xy}=0\,,\quad C=F_{yy}=2l$

$$\Rightarrow\ B^2-AC=0-4l^2=-4\left(\pm\frac{5}{2}\right)^2=-25<0$$

$\Rightarrow$ F is maximum or minimum at the critical points.

Now at $\left(\frac{4}{5},\frac{3}{5}\right)$, we have $A=5>0$

And at $\left(-\frac{4}{5},-\frac{3}{5}\right)$, we have $A=-5<0$

$\Rightarrow$ The function is min. at $\left(\frac{4}{5},\frac{3}{5}\right)$ and max. at $\left(-\frac{4}{5},-\frac{3}{5}\right)$.

Question

Find the critical point of $f(x,y)=x^2+2y^2+2xy+2x+3y$. Where $x^2-y=1$. Test for maxima and minima.

Solution

Define $F=x^2+2y^2+2xy+2x+3y+l(x^2-y-1)$

For critical points, we have

$F_x=2x+2y+2+2lx=0$ ………. (i)

$F_y=4y+2x+3-l=0$ ………… (ii)

and $x^2-y-1=0$ ………………. (iii)

From (i) $\quad l=\dfrac{-x-y-1}{x}$

From (ii) $\quad l=2x+4y+3$

$$\Rightarrow\ \frac{-x-y-1}{x}=2x+4y+3$$

$$\Rightarrow\ -x-y-1=2x^2+4xy+3x$$

$$\Rightarrow\ 2x^2+4x+4xy+y+1=0$$

But from (iii) $\quad x^2=1+y$

$$\Rightarrow\ 2(1+y)+4x+4xy+y+1=0$$

$$\Rightarrow\ 4x+4xy+3y+3=0$$

$$\Rightarrow\ 4x(1+y)+3(y+1)=0$$

$$\Rightarrow\ (y+1)(4x+3)=0$$

$\Rightarrow$ Either $y=-1$ or $x=-\frac{3}{4}$

If $y=-1$, we get $x^2=0$ from (iii)

$\Rightarrow$ $(0,-1)$ is a critical point and $l=-1$ in this case.

If $x=-\frac{3}{4}$, we get $\frac{9}{16}-1=y$ i.e. $y=-\frac{7}{16}$

$\Rightarrow \left(-\frac{3}{4},-\frac{7}{16}\right)$ is the other critical point and $l=-\frac{1}{4}$ in this. case.

Remarks

Now $A = F_{xx} = 2 + 2l$, $B = F_{xy} = 2$, $C = F_{yy} = 4$

$\Rightarrow B^2 - AC = 4 - 4(2 + 2l) = -4 - 8l$

$l = -1 \Rightarrow B^2 - AC = 4 > 0 \Rightarrow f$ is neither maximum nor minimum at $(0,1)$.

$l = -\frac{1}{4} \Rightarrow B^2 - AC = -4 - 8\left(-\frac{1}{4}\right) = -4 + 2 = -2 < 0$

and $A = 2 + 2l = 2 + 2\left(-\frac{1}{4}\right) = 2 - \frac{1}{2} > 0$

$\Rightarrow \left(-\frac{3}{4}, -\frac{7}{16}\right)$ is the point of minimum value.

Question

Find the critical points of $z = x^2 + y^2$ when $x^3 + y^3 = 6xy$, Also test for maxima and minima.

Solution

Define $F = x^2 + y^2 + l(x^3 + y^3 - 6xy)$

For critical points we have

$F_x = 2x + 3l x^2 - 6l y = 0$ …..……. (*i*)

$F_y = 2y + 3l y^2 - 6l x = 0$ ……….. (*ii*)

and $x^3 + y^3 - 6xy = 0$ …………… (*iii*)

from (*i*) $l = \dfrac{-2x}{3x^2 - 6y}$

from (*ii*) $l = \dfrac{-2y}{3y^2 - 6x}$

$\Rightarrow \dfrac{-2x}{3x^2 - 6y} = \dfrac{-2y}{3y^2 - 6x}$

$\Rightarrow x(3y^2 - 6x) = y(3x^2 - 6y)$

$\Rightarrow x(y^2 - 2x) = y(x^2 - 2y)$

$\Rightarrow xy^2 - 2x^2 = x^2y - 2y^2$

$\Rightarrow x^2y - xy^2 + 2x^2 - 2y^2 = 0$

$\Rightarrow xy(x - y) + 2(x - y)(x + y) = 0$

$\Rightarrow (x - y)(2x + 2y + xy) = 0$

$\Rightarrow$ Either $x - y = 0$ or $2x + 2y + xy = 0$

If $x - y = 0$ then (*iii*) becomes $x^3 + x^3 - 6x^2 = 0$

$\Rightarrow 2x^3 - 6x^2 = 0 \Rightarrow x^2(x - 3) = 0$

$\Rightarrow x = 0,\ 3$

$\Rightarrow x = 0,\ y = 0$ & $x = 3,\ y = 3$

$\Rightarrow (0,0)$ & $(3,3)$ are the critical points.

At $(0,0)$, $l = \dfrac{-2x}{3x^2 - 6y} = \dfrac{-2x}{3x^2 - 6x}$ $\because x - y = 0 \Rightarrow x = y$

$= \dfrac{-2}{3x - 6} = \dfrac{-2}{3(0) - 6} = \dfrac{1}{3}$

And at $(3,3)$, $l = -\dfrac{2}{3}$

$A = F_{xx} = 2 + 6 l x$

$B = F_{xy} = -6 l$

$C = F_{yy} = 2 + 6 l y$

At $(0,0)$, we have $A = 2$, $B = -2$, $C = 2$

And $\therefore\ B^2 - AC = 0$

Consider $\Delta z = z(h,h) - z(0,0) = h^2 + h^2 = 2h^2 \geq 0$

$\Rightarrow$ $(0,0)$ is the point of minimum value.

At $(3,3)$, we have $A = 2 + 6\left(-\dfrac{2}{3}\right)(3) = -10$

$B = -6\left(-\dfrac{2}{3}\right) = 4$

$C = 2 + 6\left(-\dfrac{2}{3}\right)(3) = -10$

and $\therefore\ B^2 - AC = 16 - 100 < 0$ and $A = -10 < 0$

$\Rightarrow$ $(3,3)$ is a point of maximum value.

Question

Find the points in the plane $2x + 3y - z = 5$ nearest to the origin.

Solution

We are to minimize $f = d^2 = x^2 + y^2 + z^2$

subject to $2x + 3y - z - 5 = 0$.

Define $F = x^2 + y^2 + z^2 + l\left(2x + 3y - z - 5\right)$

$F_x = 2x + 2l = 0$ ……………..... (i)

$F_y = 2y + 3l = 0$ …………….... (ii)

$F_z = 2z - l = 0$ ………………..(iii)

and $2x + 3y - z - 5 = 0$ …..….. (iv)

$x = -l$, $y = \dfrac{-3l}{2}$, $z = \dfrac{l}{2}$ from (i), (ii) & (iii) resp.

(iv) becomes $-2l - \dfrac{9l}{2} - \dfrac{l}{2} - 5 = 0$

$\Rightarrow$ $4l + 9l + l = -10$

$\Rightarrow$ $l = -\dfrac{10}{14} = -\dfrac{5}{7}$

$\Rightarrow$ $x = \dfrac{5}{7}$, $y = \dfrac{15}{14}$, $z = -\dfrac{5}{14}$

$\Rightarrow$ $\left(\dfrac{5}{7}, \dfrac{15}{14}, \dfrac{-5}{14}\right)$ is the critical point.

$A = F_{xx} = 2$, $B = F_{xy} = 0$, $C = F_{yy} = 2$

$B^2 - AC = 0 - 4 < 0$ and $A = 2 > 0$

$\Rightarrow$ F is relative minimum at $\left(\dfrac{5}{7}, \dfrac{15}{14}, \dfrac{-5}{14}\right)$ so this is the required point.