We discussed Riemann-Stieltjes’s integrals of the form \( \int_{a}^{b} f \, d\alpha \) under the restrictions that both \( f \) and \( \alpha \) are defined and bounded on a finite interval \([a,b]\).

To extend the concept, we shall relax these restrictions on \( f \) and \( \alpha \).

**Definition**

The integral \( \int_{a}^{b} f \, d\alpha \) is called an improper integral of first kind if \( a = -\infty \) or \( b = +\infty \) or both i.e. one or both integration limits is infinite.

**Definition**

The integral \( \int_{a}^{b} f \, d\alpha \) is called an improper integral of second kind if \( f(x) \) is unbounded at one or more points of \( a \leq x \leq b \). Such points are called singularities of \( f(x) \).

**Notations**

We shall denote the set of all functions \( f \) such that \( f \in R(\alpha) \) on \([a,b]\) by \( R(\alpha;a,b) \). When \( \alpha(x) = x \), we shall simply write \( R(\alpha,b) \) for this set. The notation \( \alpha \uparrow \) on \([a,\infty)\) will mean that \( \alpha \) is monotonically increasing on \([a,\infty)\).

**Definition**

Assume that \( f \in R(\alpha;a,b) \) for every \( b \geq a \). Keep \( a, \alpha \) and \( f \) fixed and define a function \( I \) on \([a,\infty)\) as follows:

\[
I(b) = \int_{a}^{b} f(x) \, d\alpha(x) \quad \text{if} \quad b \geq a \quad \ldots \ldots \quad (i)
\]

The function \( I \) so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol \( \int_{a}^{\infty} f(x) \, d\alpha(x) \) or by \( \int_{a}^{\infty} f \, d\alpha \).

The integral \( \int_{a}^{\infty} f \, d\alpha \) is said to converge if the limit

\[
\lim_{b \to \infty} I(b) \quad \ldots \ldots \quad (ii)
\]

exists (finite). Otherwise, \( \int_{a}^{\infty} f \, d\alpha \) is said to diverge.

If the limit in (ii) exists and equals \( A \), the number \( A \) is called the value of the integral and we write \( \int_{a}^{\infty} f \, d\alpha = A \).

**Example**

Consider \( \int_{1}^{b} x^{-p} \, dx \).

\[
\int_{1}^{b} x^{-p} \, dx = \frac{(1 - b^{-p})}{p - 1} \quad \text{if} \quad p \neq 1, \quad \text{the integral} \quad \int_{1}^{\infty} x^{-p} \, dx \quad \text{diverges if} \quad p < 1. \quad \text{When} \quad p > 1, \quad \text{it converges and has the value} \quad \frac{1}{p - 1}.
\]

If \( p = 1 \), we get \( \int_{1}^{b} x^{-1} \, dx = \log b \to \infty \) as \( b \to \infty \). \( \Rightarrow \int_{1}^{\infty} x^{-1} \, dx \) diverges.
**Example**

Consider $\int_0^b \sin 2\pi x \, dx$

$\therefore \int_0^b \sin 2\pi x \, dx = \frac{(1 - \cos 2\pi b)}{2\pi} \to \infty \quad \text{as} \quad b \to \infty$.

$\therefore$ the integral $\int_0^\infty \sin 2\pi x \, dx$ diverges.

**Note**

If $\int_a^\infty f \, d\alpha$ and $\int_{-\infty}^a f \, d\alpha$ are both convergent for some value of $a$, we say that the integral $\int_{-\infty}^\infty f \, d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^\infty f \, d\alpha = \int_{-\infty}^a f \, d\alpha + \int_a^\infty f \, d\alpha$$

The choice of the point $a$ is clearly immaterial.

If the integral $\int_{-\infty}^\infty f \, d\alpha$ converges, its value is equal to the limit: $\lim_{b \to +\infty} \int_a^b f \, d\alpha$.

**Theorem**

Assume that $\alpha \uparrow$ on $[a, +\infty)$ and suppose that $f \in R(\alpha; a, b)$ for every $b \geq a$.

Assume that $f(x) \geq 0$ for each $x \geq a$. Then $\int_a^\infty f \, d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^b f \, d\alpha \leq M \quad \text{for every} \quad b \geq a.$$  

**Proof**

We have $I(b) = \int_a^b f(x) \, d\alpha(x), \quad b \geq a$

$\Rightarrow I \uparrow$ on $[a, +\infty)$

Then $\lim_{b \to +\infty} I(b) = \sup \{ I(b) | b \geq a \} = M > 0$ and the theorem follows

$\Rightarrow \int_a^b f \, d\alpha \leq M$ for every $b \geq a$ whenever the integral converges.
**Theorem: (Comparison Test)**

Assume that \( \alpha \uparrow \) on \([a, +\infty)\). If \( f \in R(\alpha; a, b) \) for every \( b \geq a \), if

\[ 0 \leq f(x) \leq g(x) \quad \text{for every} \quad x \geq a, \]

and if \( \int_a^\infty g \, d\alpha \) converges, then \( \int_a^\infty f \, d\alpha \) converges and we have

\[ \int_a^\infty f \, d\alpha \leq \int_a^\infty g \, d\alpha. \]

**Proof**

Let \( I_1(b) = \int_a^b f \, d\alpha \) and \( I_2(b) = \int_a^b g \, d\alpha \), \( b \geq a \)

\[ \therefore \quad 0 \leq f(x) \leq g(x) \quad \text{for every} \quad x \geq a \]

\[ \therefore \quad I_1(b) \leq I_2(b) \quad \text{.................. (i)} \]

\[ \therefore \quad \int_a^\infty g \, d\alpha \text{ converges} \quad \exists \text{ a constant } M > 0 \text{ such that} \]

\[ \int_a^\infty g \, d\alpha \leq M, \quad b \geq a \quad \text{...............(ii)} \]

From (i) and (ii) we have \( I_1(b) \leq M, \quad b \geq a \)

\[ \Rightarrow \quad \lim_{b \to \infty} I_1(b) \quad \text{exists and is finite.} \]

\[ \Rightarrow \quad \int_a^\infty f \, d\alpha \quad \text{converges.} \]

Also \( \lim_{b \to \infty} I_1(b) \leq \lim_{b \to \infty} I_2(b) = M \)

\[ \Rightarrow \quad \int_a^\infty f \, d\alpha \leq \int_a^\infty g \, d\alpha. \]

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**Theorem (Limit Comparison Test)**

Assume that \( \alpha \uparrow \) on \([a, +\infty)\). Suppose that \( f \in R(\alpha; a, b) \) and that \( g \in R(\alpha; a, b) \) for every \( b \geq a \), where \( f(x) \geq 0 \) and \( g(x) \geq 0 \) if \( x \geq a \). If

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \]

then \( \int_a^\infty f \, d\alpha \) and \( \int_a^\infty g \, d\alpha \) both converge or both diverge.

**Proof**

For all \( b \geq a \), we can find some \( N > 0 \) such that

\[ \left| \frac{f(x)}{g(x)} - 1 \right| < \varepsilon \quad \forall \quad x \geq N \quad \text{for every} \quad \varepsilon > 0. \]

\[ \Rightarrow \quad 1 - \varepsilon < \frac{f(x)}{g(x)} < 1 + \varepsilon \]

Let \( \varepsilon = \frac{1}{2} \), then we have

\[ \frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2} \]

\[ \Rightarrow \quad g(x) < 2f(x) \quad \text{.........(i)} \quad \text{and} \quad 2f(x) < 3g(x) \quad \text{.........(ii)} \]
From (i) \[ \int_a^\infty g \, d\alpha < 2 \int_a^\infty f \, d\alpha \]
\[ \Rightarrow \int_a^\infty g \, d\alpha \text{ converges if } \int_a^\infty f \, d\alpha \text{ converges and } \int_a^\infty f \, d\alpha \text{ diverges if } \int_a^\infty f \, d\alpha \text{ diverges.} \]

From (ii) \[ 2 \int_a^\infty f \, d\alpha < 3 \int_a^\infty g \, d\alpha \]
\[ \Rightarrow \int_a^\infty f \, d\alpha \text{ converges if } \int_a^\infty g \, d\alpha \text{ converges and } \int_a^\infty g \, d\alpha \text{ diverges if } \int_a^\infty f \, d\alpha \text{ diverges.} \]
\[ \Rightarrow \text{The integrals } \int_a^\infty f \, d\alpha \text{ and } \int_a^\infty g \, d\alpha \text{ converge or diverge together.} \]

\[ \textbf{Note} \]

The above theorem also holds if \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = c \), provided that \( c \neq 0 \). If \( c = 0 \), we can only conclude that convergence of \( \int_a^\infty g \, d\alpha \) implies convergence of \( \int_a^\infty f \, d\alpha \).

\[ \textbf{Example} \]

For every real \( p \), the integral \( \int_1^\infty e^{-x^p} \, dx \) converges.

This can be seen by comparison of this integral with \( \int_1^\infty \frac{1}{x^2} \, dx \).

Since \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{e^{-x^p}}{\frac{1}{x^2}} \), where \( f(x) = e^{-x^p} \) and \( g(x) = \frac{1}{x^2} \).

\[ \Rightarrow \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} e^{-x^p} \cdot \frac{x^{p+2}}{e^x} = 0 \]
and \( \therefore \int_1^\infty \frac{1}{x^2} \, dx \) is convergent

\( \therefore \) the given integral \( \int_1^\infty e^{-x^p} \, dx \) is also convergent.

\[ \textbf{Theorem} \]

Assume \( \alpha \uparrow \) on \([a, +\infty)\). If \( f \in \text{R}(\alpha; a, b) \) for every \( b \geq a \) and if \( \int_a^\infty |f| \, d\alpha \)
converges, then \( \int_a^\infty f \, d\alpha \) also converges.

Or: An absolutely convergent integral is convergent.

\[ \textbf{Proof} \]

If \( x \geq a \), \( \pm f(x) \leq |f(x)| \)
\[ \Rightarrow |f(x)| - f(x) \geq 0 \]
\[ \Rightarrow 0 \leq |f(x)| - f(x) \leq 2|f(x)| \]
\[ \Rightarrow \int_a^\infty (|f| - f) \, d\alpha \text{ converges.} \]

Subtracting from \( \int_a^\infty f \, d\alpha \) we find that \( \int_a^\infty f \, d\alpha \) converges.

( \because \text{ Difference of two convergent integrals is convergent } )

\( \Rightarrow \) \textbf{Note}

\( \int_a^\infty f \, d\alpha \) is said to converge absolutely if \( \int_a^\infty |f| \, d\alpha \) converges. It is said to be convergent conditionally if \( \int_a^\infty f \, d\alpha \) converges but \( \int_a^\infty |f| \, d\alpha \) diverges.

\( \Rightarrow \) \textbf{Remark}

Every absolutely convergent integral is convergent.

\( \Rightarrow \) \textbf{Theorem}

Let \( f \) be a positive decreasing function defined on \([a, +\infty)\) such that \( f(x) \to 0 \) as \( x \to +\infty \). Let \( \alpha \) be bounded on \([a, +\infty)\) and assume that \( f \in R(\alpha; a, b) \) for every \( b \geq a \). Then the integral \( \int_a^\infty f \, d\alpha \) is convergent.

\textbf{Proof}

Integration by parts gives

\[
\int_a^b f \, d\alpha = \left[ f(x) \cdot \alpha(x) \right]_a^b - \int_a^b \alpha(x) \, df
\]

\[= f(b) \cdot \alpha(b) - f(a) \cdot \alpha(a) + \int_a^b \alpha \, d(-f)
\]

It is obvious that \( f(b)\alpha(b) \to 0 \) as \( b \to +\infty \)

(\because \alpha \text{ is bounded and } f(x) \to 0 \text{ as } x \to +\infty)

and \( f(a)\alpha(a) \) is finite.

\( \therefore \) the convergence of \( \int_a^b f \, d\alpha \) depends upon the convergence of \( \int_a^b \alpha \, d(-f) \).

Actually, this integral converges absolutely. To see this, suppose \( |\alpha(x)| \leq M \) for all \( x \geq a \) (\because \alpha(x) \text{ is given to be bounded})

\[\Rightarrow \int_a^b |\alpha(x)| \, d(-f) \leq \int_a^b M \, d(-f)
\]

But \( \int_a^b M \, d(-f) = M \int_a^b -f \, df = M \int_a^b -f \, d(-f) \to M \int_a^b f \, d\alpha \) as \( b \to \infty \).

\( \Rightarrow \int_a^\infty M \, d(-f) \) is convergent.

\( \therefore \int_a^\infty f \, d\alpha \) is convergent.

\( \therefore \int_a^\infty |f| \, d\alpha \) is convergent. (Comparison Test)

\[\Rightarrow \int_a^\infty f \, d\alpha \text{ is convergent.}
\]
Theorem (Cauchy condition for infinite integrals)

Assume that \( f \in R(\alpha; a, b) \) for every \( b \geq a \). Then the integral \( \int_{a}^{\infty} f \, d\alpha \) converges if, and only if, for every \( \varepsilon > 0 \) there exists a \( B > 0 \) such that \( c > b > B \) implies

\[
\left| \int_{b}^{c} f(x) \, d\alpha(x) \right| < \varepsilon
\]

**Proof**

Let \( \int_{a}^{\infty} f \, d\alpha \) be convergent. Then \( \exists \ B > 0 \) such that

\[
\left| \int_{a}^{b} f \, d\alpha - \int_{a}^{\infty} f \, d\alpha \right| < \frac{\varepsilon}{2} \quad \text{for every} \quad b \geq B \quad \ldots \ldots \ldots \ldots \ldots (i)
\]

Also for \( c > b > B \),

\[
\left| \int_{a}^{c} f \, d\alpha - \int_{a}^{b} f \, d\alpha \right| < \frac{\varepsilon}{2} \quad \ldots \ldots \ldots \ldots \ldots (ii)
\]

Consider

\[
\left| \int_{b}^{c} f \, d\alpha \right| = \left| \int_{a}^{c} f \, d\alpha - \int_{a}^{b} f \, d\alpha \right|
\]

\[
= \left| \int_{a}^{c} f \, d\alpha - \int_{a}^{\infty} f \, d\alpha + \int_{a}^{\infty} f \, d\alpha - \int_{a}^{b} f \, d\alpha \right|
\]

\[
\leq \left| \int_{a}^{c} f \, d\alpha - \int_{a}^{\infty} f \, d\alpha \right| + \left| \int_{a}^{\infty} f \, d\alpha - \int_{a}^{b} f \, d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

\[
\Rightarrow \left| \int_{b}^{c} f \, d\alpha \right| < \varepsilon \quad \text{when} \quad c > b > B.
\]

Conversely, assume that the Cauchy condition holds.

Define \( a_{n} = \int_{a}^{a+n} f \, d\alpha \) if \( n = 1, 2, \ldots \).

The sequence \( \{a_{n}\} \) is a Cauchy sequence \( \Rightarrow \) it converges.

Let \( \lim_{n \to \infty} a_{n} = A \)

Given \( \varepsilon > 0 \), choose \( B > 0 \) such that \( \left| \int_{b}^{c} f \, d\alpha \right| < \frac{\varepsilon}{2} \) if \( c > b > B \).

and also that \( \left| a_{n} - A \right| < \frac{\varepsilon}{2} \) whenever \( a + n \geq B \).

Choose an integer \( N \) such that \( a + N > B \) i.e. \( N > B - a \)

Then, if \( b > a + N \), we have

\[
\left| \int_{a}^{b} f \, d\alpha - A \right| = \left| a_{N} - A \right| + \left| \int_{a+N}^{b} f \, d\alpha \right|
\]

\[
\leq \left| a_{N} - A \right| + \left| \int_{a+N}^{b} f \, d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

\[
\Rightarrow \int_{a}^{\infty} f \, d\alpha = A
\]

This completes the proof.
Remarks
It follows from the above theorem that convergence of \( \int_{a}^{\infty} f \, d\alpha \) implies
\[
\lim_{b \to \infty} \int_{a}^{b} f \, d\alpha = 0 \quad \text{for every fixed } \varepsilon > 0.
\]
However, this does not imply that \( f(x) \to 0 \) as \( x \to \infty \).

Theorem
Every convergent infinite integral \( \int_{a}^{\infty} f(x) \, dx \) can be written as a convergent infinite series. In fact, we have
\[
\int_{a}^{\infty} f(x) \, dx = \sum_{k=1}^{\infty} a_k \quad \text{where} \quad a_k = \int_{a+k-1}^{a+k} f(x) \, dx \quad \text{……… (1)}
\]

Proof
\[\because \int_{a}^{\infty} f \, d\alpha \quad \text{converges}, \quad \text{the sequence} \quad \left\{ \int_{a}^{a+n} f \, d\alpha \right\} \quad \text{also converges.}
\]
But \[\int_{a}^{a+n} f \, d\alpha = \sum_{k=1}^{n} a_k \]. Hence the series \( \sum_{k=1}^{\infty} a_k \) converges and equals \( \int_{a}^{\infty} f \, d\alpha \).

Remarks
It is to be noted that the convergence of the series in (1) does not always imply convergence of the integral. For example, suppose \( a_k = \int_{k}^{k+1} \sin 2\pi x \, dx \). Then each \( a_k = 0 \) and \( \sum a_k \) converges.

However, the integral \[\int_{0}^{\infty} \sin 2\pi x \, dx = \lim_{b \to \infty} \int_{0}^{b} \sin 2\pi x \, dx = \lim_{b \to \infty} \frac{1 - \cos 2\pi b}{2\pi} \] diverges.

Improper Integral of the Second Kind

Definition
Let \( f \) be defined on the half open interval \((a, b]\) and assume that \( f \in R(\alpha; x, b) \) for every \( x \in (a, b] \). Define a function \( I \) on \((a, b]\) as follows:
\[
I(x) = \int_{a}^{x} f \, d\alpha \quad \text{if} \quad x \in (a, b] \quad \text{……… (i)}
\]

The function \( I \) so defined is called an improper integral of the second kind and is denoted by the symbol \( \int_{a+}^{b} f(t) \, d\alpha(t) \) or \( \int_{a+}^{b} f \, d\alpha \).

The integral \( \int_{a+}^{b} f \, d\alpha \) is said to converge if the limit
\[\lim_{x \to a+} I(x) \quad \text{……… (ii)} \quad \text{exists (finite)}.
\]
Otherwise, \( \int_{a+}^{b} f \, d\alpha \) is said to diverge. If the limit in (ii) exists and equals \( A \), the number \( A \) is called the value of the integral and we write \( \int_{a+}^{b} f \, d\alpha = A \).
Similarly, if \( f \) is defined on \([a,b]\) and \( f \in R(\alpha;a,x) \ \forall \ x \in [a,b] \) then

\[
I(x) = \int_{a}^{b} f \, dx \quad \text{if} \quad x \in [a,b]
\]

is also an improper integral of the second kind and is denoted as \( \int_{a}^{b} f \, dx \) and is convergent if \( \lim_{x \to b^-} I(x) \) exists (finite).

**Example**

\( f(x) = x^{-p} \) is defined on \((0,b]\) and \( f \in R(\alpha;b,\infty) \) for every \( x \in (0,b] \).

\[
I(x) = \int_{x}^{b} x^{-p} \, dx \quad \text{if} \quad x \in (0,b]
\]

\[
= \int_{0^+}^{b} x^{-p} \, dx = \lim_{\varepsilon \to 0^+} \int_{x}^{b} x^{-p} \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \left[ \frac{x^{-p}}{1-p} \right]_{x}^{b} = \lim_{\varepsilon \to 0^+} \frac{b^{1-p} - x^{1-p}}{1-p}, \quad (p \neq 1)
\]

\[
= \begin{cases} 
\text{finite}, & p < 1 \\
\text{infinite}, & p > 1 
\end{cases}
\]

When \( p = 1 \), we get

\[
\int_{0^+}^{b} \frac{1}{x} \, dx = \log b - \log \varepsilon \to \infty \quad \text{as} \quad \varepsilon \to 0.
\]

\( \Rightarrow \int_{0^+}^{b} x^{-1} \, dx \) also diverges.

Hence the integral converges when \( p < 1 \) and diverges when \( p \geq 1 \).

**Note**

If the two integrals \( \int_{a}^{c} f \, dx \) and \( \int_{c}^{b} f \, dx \) both converge, we write

\[
\int_{a}^{b} f \, dx = \int_{a}^{c} f \, dx + \int_{c}^{b} f \, dx
\]

The definition can be extended to cover the case of any finite number of sums.

We can also consider mixed combinations such as

\[
\int_{a}^{b} f \, dx \bigg|_{c}^{d} + \int_{d}^{f} f \, dx \quad \text{which can be written as} \quad \int_{a}^{f} f \, dx.
\]

**Example**

Consider \( \int_{0^+}^{\infty} x^{p-1} \, dx \), \( (p > 0) \)

This integral must be interpreted as a sum as

\[
\int_{0^+}^{\infty} x^{p-1} \, dx = \int_{0^+}^{1} x^{p-1} \, dx + \int_{1}^{\infty} x^{p-1} \, dx
\]

\[
= I_1 + I_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (i)
\]

\( I_2 \), the second integral, converges for every real \( p \) as proved earlier.

To test \( I_1 \), put \( t = \frac{1}{x} \) \( \Rightarrow \) \( dx = -\frac{1}{t^2} \, dt \)
\[ I_1 = \lim_{e \to 0} \int_{e}^{1} e^{-x} x^{p-1} \, dx = \lim_{e \to 0} \int_{e}^{1} e^{-\frac{1}{t}} t^{1-p} \left( -\frac{1}{t^2} \, dt \right) = \lim_{e \to 0} \int_{1}^{e} e^{-\frac{1}{t}} t^{-p-1} \, dt \]

Take \( f(t) = e^{-\frac{1}{t}} t^{-p-1} \) and \( g(t) = t^{-p} \)

Then \( \lim_{t \to \infty} \frac{f(t)}{g(t)} = \lim_{t \to \infty} \frac{e^{-\frac{1}{t}} t^{-p-1}}{t^{-p}} = 1 \) and since \( \int_{1}^{\infty} t^{-p} \, dt \) converges when \( p > 0 \)

\[ \therefore \int_{0+}^{\infty} e^{-x} x^{p-1} \, dx \text{ converges when } p > 0 \]

Thus \( \int_{0+}^{\infty} e^{-x} x^{p-1} \, dx \) converges when \( p > 0 \).

When \( p > 0 \), the value of the sum in (i) is denoted by \( \Gamma(p) \). The function so defined is called the Gamma function.

**Note**

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

**A Useful Comparison Integral**

\[ \int_{a}^{b} \frac{dx}{(x-a)^n} \]

We have, if \( n \neq 1 \),

\[ \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \left[ \frac{1}{(1-n)(x-a)^{n-1}} \right]_{a+\varepsilon}^{b} = \frac{1}{(1-n)} \left( \frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}} \right) \]

Which tends to \( \frac{1}{(1-n)(b-a)^{n-1}} \) or \( +\infty \) according as \( n < 1 \) or \( n > 1 \), as \( \varepsilon \to 0 \).

Again, if \( n = 1 \),

\[ \int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log \varepsilon \to +\infty \text{ as } \varepsilon \to 0. \]

Hence the improper integral \( \int_{a}^{b} \frac{dx}{(x-a)^n} \) converges iff \( n < 1 \).
**Question**

Examine the convergence of

(i) \[ \int_0^1 \frac{1}{x^{3/2}(1 + x^2)} \, dx \]  

(ii) \[ \int_0^1 \frac{1}{x^2(1 + x)^2} \, dx \]  

(iii) \[ \int_0^1 \frac{1}{x^{1/2}(1 - x)^{3/2}} \, dx \]

**Solution**

(i) \[ \int_0^1 \frac{1}{x^{3/2}(1 + x^2)} \, dx \]

Here ‘0’ is the only point of infinite discontinuity of the integrand. We have

\[ f(x) = \frac{1}{x^{3/2}(1 + x^2)} \]

Take \[ g(x) = \frac{1}{x^{3/2}} \]

Then \[ \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1 + x^2} = 1 \]

\[ \Rightarrow \int_0^1 f(x) \, dx \quad \text{and} \quad \int_0^1 g(x) \, dx \quad \text{have identical behaviours.} \]

\[ \therefore \int_0^1 \frac{dx}{0 x^{3/2}(1 + x^2)} \quad \text{converges} \quad \therefore \int_0^1 \frac{dx}{0 x^{1/2}(1 + x^2)} \quad \text{also converges.} \]

(ii) \[ \int_0^1 \frac{1}{x^2(1 + x)^2} \, dx \]

Here ‘0’ is the only point of infinite discontinuity of the given integrand. We have

\[ f(x) = \frac{1}{x^2(1 + x)^2} \]

Take \[ g(x) = \frac{1}{x^2} \]

Then \[ \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1 + x} = 1 \]

\[ \Rightarrow \int_0^1 f(x) \, dx \quad \text{and} \quad \int_0^1 g(x) \, dx \quad \text{behave alike.} \]

But \[ n = 2 \] being greater than 1, the integral \[ \int_0^1 g(x) \, dx \] does not converge. Hence the given integral also does not converge.

(iii) \[ \int_0^1 \frac{1}{x^{1/2}(1 - x)^{3/2}} \, dx \]

Here ‘0’ and ‘1’ are the two points of infinite discontinuity of the integrand. We have

\[ f(x) = \frac{1}{x^{1/2}(1 - x)^{3/2}} \]

We take any number between 0 and 1, say \[ 1/2, \] and examine the convergence of
the improper integrals $\int_{0}^{1/2} f(x) \, dx$ and $\int_{1/2}^{1} f(x) \, dx$.

To examine the convergence of $\int_{0}^{1/2} \frac{1}{x^{1/2}(1-x)^{3/5}} \, dx$, we take $g(x) = \frac{1}{x^{1/2}}$.

Then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{x^{1/2}(1-x)^{3/5}} = 1$$

$\therefore \int_{0}^{1/2} \frac{1}{x^{1/2}} \, dx$ converges $\therefore \int_{0}^{1/2} \frac{1}{x^{1/2}(1-x)^{3/5}} \, dx$ is convergent.

To examine the convergence of $\int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{3/5}} \, dx$, we take $g(x) = \frac{1}{(1-x)^{3/5}}$.

Then

$$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{1}{x^{1/2}(1-x)^{3/5}} = 1$$

$\therefore \int_{1/2}^{1} \frac{1}{(1-x)^{3/5}} \, dx$ converges $\therefore \int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{3/5}} \, dx$ is convergent.

Hence $\int_{0}^{1} f(x) \, dx$ converges.

**Question**

Show that $\int_{0}^{1} x^{m-1}(1-x)^{n-1} \, dx$ exists iff $m, n$ are both positive.

**Solution**

The integral is proper if $m \geq 1$ and $n \geq 1$.

The number ‘0’ is a point of infinite discontinuity if $m < 1$ and the number ‘1’ is a point of infinite discontinuity if $n < 1$.

Let $m < 1$ and $n < 1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of the improper integrals $\int_{0}^{1/2} x^{m-1}(1-x)^{n-1} \, dx$ and $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1} \, dx$ at ‘0’ and ‘1’ respectively.

**Convergence at 0:**

We write

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}}$$

and take $g(x) = \frac{1}{x^{1-m}}$.

Then $\frac{f(x)}{g(x)} \to 1$ as $x \to 0$.

As $\int_{0}^{1/2} x^{m-1} \, dx$ is convergent at 0 iff $1 - m < 1$ i.e. $m > 0$.

We deduce that the integral $\int_{0}^{1/2} x^{m-1}(1-x)^{n-1} \, dx$ is convergent at 0, iff $m$ is +ive.
Convergence at 1:
We write \( f(x) = x^{m-1}(1-x)^{-1-n} = \frac{x^{m-1}}{(1-x)^{1-n}} \) and take \( g(x) = \frac{1}{(1-x)^{1-n}} \)

Then \( \frac{f(x)}{g(x)} \to 1 \) as \( x \to 1 \)

As \( \int_{\frac{1}{2}}^{1} \frac{1}{(1-x)^{1-n}} \, dx \) is convergent, iff \( 1 - n < 1 \) i.e. \( n > 0 \).

We deduce that the integral \( \int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1} \, dx \) converges iff \( n > 0 \).

Thus \( \int_{0}^{1} x^{m-1}(1-x)^{n-1} \, dx \) exists for positive values of \( m, n \) only.

It is a function which depends upon \( m \) & \( n \) and is defined for all positive values of \( m \) & \( n \). It is called Beta function.

> **Question**
> Show that the following improper integrals are convergent.

\((i)\) \( \int_{1}^{\infty} \sin^{2} \frac{1}{x} \, dx \) \((ii)\) \( \int_{1}^{\infty} \frac{\sin^{2} x}{x^{2}} \, dx \) \((iii)\) \( \int_{0}^{\infty} \frac{x \log x}{(1+x)^{2}} \, dx \) \((iv)\) \( \int_{0}^{\infty} \log x \cdot \log(1+x) \, dx \)

> **Solution**

\((i)\) Let \( f(x) = \sin^{2} \frac{1}{x} \) and \( g(x) = \frac{1}{x^{2}} \)

then \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\sin^{2} \frac{1}{x}}{x^{2}} = \lim_{y \to 0} \left( \frac{\sin y}{y} \right)^{2} = 1 \)

\( \Rightarrow \int_{1}^{\infty} f(x) \, dx \) and \( \int_{1}^{\infty} \frac{1}{x^{2}} \, dx \) behave alike.

\( \therefore \int_{1}^{\infty} \frac{1}{x^{2}} \, dx \) is convergent \( \because \int_{1}^{\infty} \sin^{2} \frac{1}{x} \, dx \) is also convergent.

\((ii)\) \( \int_{1}^{\infty} \frac{\sin^{2} x}{x^{2}} \, dx \)

Take \( f(x) = \frac{\sin^{2} x}{x^{2}} \) and \( g(x) = \frac{1}{x^{2}} \)

\( \sin^{2} x \leq 1 \) \( \Rightarrow \frac{\sin^{2} x}{x^{2}} \leq \frac{1}{x^{2}} \) \( \forall \ x \in (1, \infty) \)

and \( \int_{1}^{\infty} \frac{1}{x^{2}} \, dx \) converges \( \therefore \int_{1}^{\infty} \frac{\sin^{2} x}{x^{2}} \, dx \) converges.

> **Note**

\( \int_{0}^{\infty} \frac{\sin^{2} x}{x^{2}} \, dx \) is a proper integral because \( \lim_{x \to 0} \frac{\sin^{2} x}{x^{2}} = 1 \) so that ‘0’ is not a point of infinite discontinuity. Therefore \( \int_{0}^{\infty} \frac{\sin^{2} x}{x^{2}} \, dx \) is convergent.
(iii) $\int_{0}^{1} \frac{x \log x}{(1 + x)^{2}} \, dx$

$\because \log x < x, \quad x \in (0, 1)$

$\therefore x \log x < x^2$

$\Rightarrow \frac{x \log x}{(1 + x)^2} < \frac{x^2}{(1 + x)^2}$

Now $\int_{0}^{1} \frac{x^2}{(1 + x)^2} \, dx$ is a proper integral.

$\therefore \int_{0}^{1} \frac{x \log x}{(1 + x)^2} \, dx$ is convergent.

(iv) $\int_{0}^{1} \log x \cdot \log(1 + x) \, dx$

$\therefore \log x < x \therefore \log(x + 1) < x + 1$

$\Rightarrow \log x \cdot \log(1 + x) < x(x + 1)$

$\therefore \int_{0}^{1} x(x + 1) \, dx$ is a proper integral $\therefore \int_{0}^{1} \log x \cdot \log(1 + x) \, dx$ is convergent.

**Note**

(i) $\int_{0}^{a} \frac{1}{x^p} \, dx$ diverges when $p \geq 1$ and converges when $p < 1$.

(ii) $\int_{a}^{\infty} \frac{1}{x^p} \, dx$ converges iff $p > 1$.

**UNIFORM CONVERGENCE OF IMPROPER INTEGRALS**

**Definition**

Let $f$ be a real valued function of two variables $x \& y$, $x \in [a, +\infty)$, $y \in S$ where $S \subset \mathbb{R}$. Suppose further that, for each $y$ in $S$, the integral $\int_{a}^{\infty} f(x, y) \, d\alpha(x)$ is convergent. If $F$ denotes the function defined by the equation

$$F(y) = \int_{a}^{\infty} f(x, y) \, d\alpha(x) \quad \text{if} \quad y \in S$$

the integral is said to converge **pointwise** to $F$ on $S$.

**Definition**

Assume that the integral $\int_{a}^{\infty} f(x, y) \, d\alpha(x)$ converges pointwise to $F$ on $S$. The integral is said to converge **Uniformly** on $S$ if, for every $\varepsilon > 0$ there exists a $B > 0$ (depending only on $\varepsilon$) such that $b > B$ implies

$$\left| F(y) - \int_{a}^{b} f(x, y) \, d\alpha(x) \right| < \varepsilon \quad \forall \quad y \in S.$$  

(Pointwise convergence means convergence when $y$ is fixed but uniform convergence is for every $y \in S$).
Theorem (Cauchy condition for uniform convergence.)

The integral \( \int_a^\infty f(x,y)\,d\alpha(x) \) converges uniformly on \( S \), iff, for every \( \varepsilon > 0 \) there exists a \( B > 0 \) (depending on \( \varepsilon \)) such that \( c > b > B \) implies
\[
\left| \int_b^c f(x,y)\,d\alpha(x) \right| < \varepsilon \quad \forall \ y \in S.
\]

Proof

Proceed as in the proof for Cauchy condition for infinite integral \( \int_a^\infty f\,d\alpha \).

Theorem (Weierstrass M-test)

Assume that \( \alpha \uparrow \) on \( [a, +\infty) \) and suppose that the integral \( \int_a^b f(x,y)\,d\alpha(x) \) exists for every \( b \geq a \) and for every \( y \) in \( S \). If there is a positive function \( M \) defined on \( [a, +\infty) \) such that the integral \( \int_a^\infty M(x)\,d\alpha(x) \) converges and \( |f(x,y)| \leq M(x) \) for each \( x \geq a \) and every \( y \) in \( S \), then the integral \( \int_a^\infty f(x,y)\,d\alpha(x) \) converges uniformly on \( S \).

Proof

Given \( \varepsilon > 0 \), \( \exists \ B > 0 \) such that \( b > B \) implies
\[
\left| \int_b^c M\,d\alpha - L \right| < \frac{\varepsilon}{2} \quad \text{.......... (ii)}
\]
Also if \( c > b > B \), then
\[
\left| \int_a^c M\,d\alpha - L \right| < \frac{\varepsilon}{2} \quad \text{.......... (iii)}
\]
Then
\[
\int_b^c M\,d\alpha = \left\| \int_b^c M\,d\alpha - \int_a^b M\,d\alpha \right\| \\
= \left\| \int_a^c M\,d\alpha - L + \int_a^b M\,d\alpha \right\| \\
\leq \left\| \int_a^c M\,d\alpha - L \right\| + \left\| \int_a^b M\,d\alpha - L \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{(By ii & iii)}
\]
\[
\Rightarrow \left| \int_b^c f(x,y)\,d\alpha(x) \right| < \varepsilon, \quad c > b > B \ & \text{for each} \ y \in S
\]
Cauchy condition for convergence (uniform) being satisfied.

Therefore the integral \( \int_a^\infty f(x,y)\,d\alpha(x) \) converges uniformly on \( S \).
Example

Consider \( \int_0^\infty e^{-xy} \sin x \, dx \)

\[ |e^{-xy} \sin x| \leq |e^{-xy}| = e^{-xy} \quad (\because |\sin x| \leq 1) \]

and \( e^{-xy} \leq e^{-x} \) if \( c \leq y \)

Now take \( M(x) = e^{-cx} \)

The integral \( \int_0^\infty M(x) \, dx = \int_0^\infty e^{-c} \, dx \) is convergent & converging to \( \frac{1}{c} \).

\[ \therefore \text{The conditions of M-test are satisfied and } \int_0^\infty e^{-xy} \sin x \, dx \text{ converges uniformly on } [c, +\infty) \text{ for every } c > 0. \]

Theorem (Dirichlet’s test for uniform convergence)

Assume that \( f \) is bounded on \([a, +\infty)\) and suppose the integral \( \int_a^b f(x, y) \, d\alpha(x) \) exists for every \( b \geq a \) and for every \( y \) in \( S \). For each fixed \( y \) in \( S \), assume that \( f(x, y) \leq f(x', y) \) if \( a \leq x' < x < +\infty \). Furthermore, suppose there exists a positive function \( g \), defined on \([a, +\infty)\), such that \( g(x) \to 0 \) as \( x \to +\infty \) and such that \( x \geq a \) implies

\[ |f(x, y)| \leq g(x) \quad \text{for every } y \text{ in } S. \]

Then the integral \( \int_a^\infty f(x, y) \, d\alpha(x) \) converges uniformly on \( S \).

Proof

Let \( M > 0 \) be an upper bound for \( |\alpha| \) on \([a, +\infty)\).

Given \( \varepsilon > 0 \), choose \( B > a \) such that \( x \geq B \) implies

\[ g(x) < \frac{\varepsilon}{4M} \]

( \( \because g(x) \) is +ive and \( \to 0 \) as \( x \to +\infty \) \( \therefore |g(x) - 0| < \frac{\varepsilon}{4M} \) for \( x \geq B \) )

If \( c > B \), integration by parts yields

\[ \int_b^c f \, d\alpha = \left| f(x, y) \right| \cdot \alpha(x) \bigg|_b^c - \int_b^c \alpha \cdot df \]

\[ = f(c, y)\alpha(c) - f(b, y)\alpha(b) + \int_b^c \alpha \cdot df \] \hspace{1cm} (i)

But, since \(-f\) is increasing (for each fixed \( y \)), we have

\[ \int_b^c \alpha \cdot df \leq M \int_b^c df \] \hspace{1cm} (\( \because \text{upper bound of } |\alpha| \text{ is } M \))

\[ = M \left( f(b, y) - M f(c, y) \right) \] \hspace{1cm} (ii)

Now if \( c > b > B \), we have from (i) and (ii)

\[ \left| \int_b^c f \, d\alpha \right| \leq |f(c, y)\alpha(c) - f(b, y)\alpha(b)| + \int_b^c |\alpha \cdot df| \]

\[ \leq |\alpha(c)| |f(c, y)| + |f(b, y)| |\alpha(b)| + M \left| f(b, y) - f(c, y) \right| \]

\[ \leq |\alpha(c)| |f(c, y)| + |\alpha(b)| |f(b, y)| + M \left| f(b, y) \right| + M \left| f(c, y) \right| \]
\[ \leq M \, g(c) + M \, g(b) + M \, g(b) + M \, g(c) = 2M \left[ g(b) + g(c) \right] \]
\[ < 2M \left[ \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \right] = \varepsilon \]
\[ \Rightarrow \left| \int_a^b f(x) \, dx \right| < \varepsilon \quad \text{for every} \ y \in S. \]

Therefore the Cauchy condition is satisfied and \( \int_a^b f(x,y) \, d\alpha(x) \) converges uniformly on \( S \).

**Example**

Consider \( \int_0^\infty \frac{e^{-xy}}{x} \, \sin x \, dx \)

Take \( \alpha(x) = \cos x \) and \( f(x,y) = \frac{e^{-xy}}{x} \) if \( x > 0, \ y \geq 0 \).

If \( S = [0, +\infty) \) and \( g(x) = \frac{1}{x} \) on \( [\varepsilon, +\infty) \) for every \( \varepsilon > 0 \) then

i) \( f(x,y) \leq f(x', y) \) if \( x' \leq x \) and \( \alpha(x) \) is bounded on \( [\varepsilon, +\infty) \).

ii) \( g(x) \to 0 \) as \( x \to +\infty \)

iii) \( \left| f(x,y) \right| = \left| \frac{e^{-xy}}{x} \right| \leq \frac{1}{x} = g(x) \quad \forall \ y \in S. \)

So that the conditions of Dirichlet’s theorem are satisfied. Hence

\[ \int_0^\infty \frac{e^{-xy}}{x} \, \sin x \, dx = + \int_0^\infty \frac{e^{-xy}}{x} \, d(-\cos x) \] converges uniformly on \( [\varepsilon, +\infty) \) if \( \varepsilon > 0 \).

\[ \lim_{x \to \infty} \frac{\sin x}{x} = 1 \quad \therefore \int_0^\infty \frac{e^{-xy}}{x} \, \sin x \, dx \] converges being a proper integral.

\[ \Rightarrow \int_0^\infty \frac{e^{-xy}}{x} \, \sin x \, dx \] also converges uniformly on \( [0, +\infty) \).

**Remarks**

Dirichlet’s test can be applied to test the convergence of the integral of a product. For this purpose the test can be modified and restated as follows:

Let \( \phi(x) \) be bounded and monotonic in \( [a, +\infty) \) and let \( \phi(x) \to 0 \), when \( x \to \infty \). Also let \( \int_a^X f(x) \, dx \) be bounded when \( X \geq a \).

Then \( \int_a^\infty f(x) \phi(x) \, dx \) is convergent.

**Example**

Consider \( \int_0^\infty \frac{\sin x}{x} \, dx \)

\[ \therefore \frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0. \]
\[ 0 \text{ is not a point of infinite discontinuity.} \]

Now consider the improper integral \( \int_{1}^{\infty} \frac{\sin x}{x} \, dx \).

The factor \( \frac{1}{x} \) of the integrand is monotonic and \( \to 0 \) as \( x \to \infty \).

Also \( \int_{1}^{X} \sin x \, dx = \cos X + \cos(1) - \cos X - \cos(1) < 2 \)

So that \( \int_{1}^{X} \sin x \, dx \) is bounded above for every \( X \geq 1 \).

\[ \Rightarrow \int_{1}^{\infty} \frac{\sin x}{x} \, dx \text{ is convergent. Now since } \int_{0}^{\infty} \frac{\sin x}{x} \, dx \text{ is a proper integral, we see that } \int_{0}^{\infty} \frac{\sin x}{x} \, dx \text{ is convergent.} \]

\[ \textbf{Example} \]

Consider \( \int_{0}^{\infty} \sin x^2 \, dx \).

We write \( \sin x^2 = \frac{1}{2x} \cdot 2x \cdot \sin x^2 \)

Now \( \int_{1}^{X} \sin x^2 \, dx = \int_{1}^{X} \frac{1}{2x} \cdot 2x \cdot \sin x^2 \, dx \)

\( \frac{1}{2x} \) is monotonic and \( \to 0 \) as \( x \to \infty \).

Also \( \left| \int_{1}^{X} 2x \sin x^2 \, dx \right| = \left| -\cos x^2 + \cos(1) \right| < 2 \)

So that \( \int_{1}^{X} 2x \sin x^2 \, dx \) is bounded for \( X \geq 1 \).

Hence \( \int_{1}^{\infty} \frac{1}{2x} \cdot 2x \cdot \sin x^2 \, dx \) i.e. \( \int_{1}^{\infty} \sin x^2 \, dx \) is convergent.

Since \( \int_{0}^{\infty} \sin x^2 \, dx \) is only a proper integral, we see that the given integral is convergent.

\[ \textbf{Example} \]

Consider \( \int_{0}^{\infty} e^{-ax} \frac{\sin x}{x} \, dx \), \( a > 0 \)

Here \( e^{-ax} \) is monotonic and bounded and \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \) is convergent.

Hence \( \int_{0}^{\infty} e^{-ax} \frac{\sin x}{x} \, dx \) is convergent.
Example

Show that \( \int_0^\infty \frac{\sin x}{x} \, dx \) is not absolutely convergent.

Solution

Consider the proper integral
\[
\int_0^\pi \left| \frac{\sin x}{x} \right| \, dx
\]
where \( n \) is a positive integer. We have
\[
\int_0^\pi \left| \frac{\sin x}{x} \right| \, dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \left| \frac{\sin x}{x} \right| \, dx
\]
Put \( x = (r-1)\pi + y \) so that \( y \) varies in \([0,\pi]\).
We have \( |\sin((r-1)\pi + y)| = |(-1)^{r-1} \sin y| = \sin y \)
\[
\therefore \int_{(r-1)\pi}^{r\pi} \frac{\sin x}{x} \, dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} \, dy
\]
\[
\therefore r\pi \text{ is the max. value of } [(r-1)\pi + y] \text{ in } [0,\pi]
\]
\[
\Rightarrow \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} \, dy \geq \frac{1}{r\pi} \int_0^{\pi} \sin y \, dy = \frac{\pi}{r\pi} \sum_{r=1}^{n} \frac{1}{r} \Rightarrow \sum_{r=1}^{n} \frac{1}{r} \to \infty \quad \text{as } n \to \infty
\]
Let, now, \( X \) be any real number.
There exists a positive integer \( n \) such that \( n\pi \leq X < (n+1)\pi \).
We have
\[
\int_0^X \left| \frac{\sin x}{x} \right| \, dx \geq \int_0^\pi \left| \frac{\sin x}{x} \right| \, dx
\]
Let \( X \to \infty \) so that \( n \) also \( \to \infty \). Then we see that
\[
\int_0^\infty \left| \frac{\sin x}{x} \right| \, dx \to \infty
\]
So that \( \int_0^\infty \left| \frac{\sin x}{x} \right| \, dx \) does not converge.

Questions

Examine the convergence of
\[
(i) \int_1^\infty \frac{x}{(1+x)^3} \, dx \quad (ii) \int_1^\infty \frac{1}{(1+x)\sqrt{x}} \, dx \quad (iii) \int_1^\infty \frac{dx}{x^{3/2} (1+x)^{1/2}}
\]

Solution

(i) Let \( f(x) = \frac{x}{(1+x)^3} \) and take \( g(x) = \frac{x^3}{x^2} = \frac{1}{x} \)
As \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{(1+x)^3} = 1 \)
Therefore the two integrals \( \int_{1}^{\infty} \frac{x}{(1 + x)^3} \, dx \) and \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) have identical behaviour for convergence at \( \infty \).

\[ \therefore \int_{1}^{\infty} \frac{1}{x^2} \, dx \text{ is convergent} \quad \therefore \int_{1}^{\infty} \frac{x}{(1 + x)^3} \, dx \text{ is convergent.} \]

(ii) Let \( f(x) = \frac{1}{(1 + x)^{\frac{3}{2}}} \) and take \( g(x) = \frac{1}{x^{\frac{5}{2}}} \).

We have \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 1 \)

and \( \int_{1}^{\infty} \frac{1}{x^{\frac{5}{2}}} \, dx \) is convergent. Thus \( \int_{1}^{\infty} \frac{1}{(1 + x)^{\frac{3}{2}}} \, dx \) is convergent.

(iii) Let \( f(x) = \frac{1}{x^{\frac{3}{2}} (1 + x)^{\frac{1}{2}}} \)

we take \( g(x) = \frac{1}{x^2} \).

We have \( \lim_{x \to \infty} f(x) = 1 \) and \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) is convergent \( \therefore \int_{1}^{\infty} f(x) \, dx \) is convergent.

\[ \text{Question} \]

Show that \( \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \) is convergent.

\[ \text{Solution} \]

We have

\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{a \to \infty} \left[ \int_{-a}^{0} \frac{1}{1 + x^2} \, dx + \int_{0}^{a} \frac{1}{1 + x^2} \, dx \right] \]

\[ = \lim_{a \to \infty} \left[ \int_{0}^{a} \frac{1}{1 + x^2} \, dx + \int_{0}^{a} \frac{1}{1 + x^2} \, dx \right] = 2 \lim_{a \to \infty} \left[ \frac{1}{1 + x^2} \right] \]

\[ = 2 \lim_{a \to \infty} \tan^{-1} x \bigg|_{0}^{a} = 2 \left( \frac{\pi}{2} \right) = \pi \]

therefore the integral is convergent.

\[ \text{Question} \]

Show that \( \int_{0}^{\infty} \frac{\tan^{-1} x}{1 + x^2} \, dx \) is convergent.

\[ \text{Solution} \]

\[ \therefore (1 + x^2) \cdot \frac{\tan^{-1} x}{(1 + x^2)} = \tan^{-1} x \to \frac{\pi}{2} \text{ as } x \to \infty \]

\( \int_{0}^{\infty} \frac{\tan^{-1} x}{1 + x^2} \, dx \) & \( \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx \) behave alike.

\[ \therefore \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx \text{ is convergent} \quad \therefore \text{A given integral is convergent.} \]

\[ \text{Here } f(x) = \frac{\tan^{-1} x}{1 + x^2} \text{ and } g(x) = 1 + x^2 \]
Question
Show that \( \int_{0}^{\infty} \frac{\sin x}{(1 + x)^\alpha} \, dx \) converges for \( \alpha > 0 \).

Solution
\[
\int_{0}^{\infty} \sin x \, dx
\]
is bounded because \( \int_{0}^{x} \sin x \, dx \leq 2 \quad \forall \ x > 0 \).

Furthermore the function \( \frac{1}{(1 + x)^\alpha} \), \( \alpha > 0 \) is monotonic on \([0, +\infty)\).

\[\Rightarrow\] the integral \( \int_{0}^{\infty} \frac{\sin x}{(1 + x)^\alpha} \, dx \) is convergent.

Question
Show that \( \int_{0}^{\infty} e^{-x} \cos x \, dx \) is absolutely convergent.

Solution
\[\because \left| e^{-x} \cos x \right| < e^{-x} \quad \text{and} \quad \int_{0}^{\infty} e^{-x} \, dx = 1\]

\[\therefore\] the given integral is absolutely convergent. (comparison test)

Question
Show that \( \int_{0}^{1} \frac{e^{-x}}{\sqrt{1 - x^4}} \, dx \) is convergent.

Solution
\[\because e^{-x} < 1 \quad \text{and} \quad 1 + x^2 > 1\]

\[\therefore \frac{e^{-x}}{\sqrt{1 - x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1 - x^2}}\]

Also \( \int_{0}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx = \lim_{\varepsilon \to 0} \int_{0}^{1-\varepsilon} \frac{1}{\sqrt{1 - x^2}} \, dx \)

\[= \lim_{\varepsilon \to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}\]

\[\Rightarrow \int_{0}^{1} \frac{e^{-x}}{\sqrt{1 - x^4}} \, dx \] is convergent. (by comparison test)

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