Chapter 5 – Function of Several Variables

Subject: *Real Analysis* ***Level:*** *M.Sc.*
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❖ Introduction

There is the basic difference between the calculus of functions of one variable and the calculus of functions of two variables. But there is a slight difference between the calculus of two variable and the calculus of functions of three, four or of many variables. Therefore we shall emphasise mainly on the study of functions of two variables.

❖ Function of two variables

If to each point (x, y) of a certain part of $xy-$plane, there is assigned a real number z, then z is known to be a function of two variable x and y.

e.g. $z = x^2 - y^2$, $z = x^2 + y^2$, $z = xy$ etc.

❖ Neighbourhood (nhood)

A neighbourhood of radius δ of a point (x_0, y_0) of the $xy-$plane is the set of points which lies inside a circle with centre at (x_0, y_0) and has radius δ.

$$N_\delta(x_0, y_0) = (x - x_0)^2 + (y - y_0)^2 < \delta^2$$

Similarly, a nhood of a radius δ of a point (x_0, y_0, z_0) of a space is a sphere with centre at (x_0, y_0, z_0) and radius δ.

$$N_\delta(x_0, y_0, z_0) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2$$

This definition can be extended to the definition of a nhood of a point of a space of any dimension.

❖ Open Set

A set is known to be open set if each point (x_0, y_0) of the set has a nhood which totally lies inside the set.

❖ Domain

A set D which is not empty and open is known to be a domain, if any two points of the set can be joined by a broken line which lies completely with in D.

❖ Region

A domain D is known to be a region if some or all of the boundary points are contained in D.

❖ Closed Region

A region is known to be closed if it contains all the boundary points.

e.g. i) $x^2 + y^2 < 1$ (Domain) ii) $xy < 1$ (Domain)

$x^2 + y^2 = 1$ (Boundary) $xy = 2$ (Boundary)

$x^2 + y^2 \le 1$ (Closed region) $xy \le 1$ (Closed Region)

❖ Limit & Continuity

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Suppose there is a point $(x_0, y_0) \in D$ or is a boundary point then

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = c$$

It means that given $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that

$$|f(x, y) - c| < \varepsilon \quad \text{whenever} \quad |(x, y) - (x_0, y_0)| < \delta \quad \forall \ (x, y) \in N_\delta(x_0, y_0)$$

If limit of a function is equal to actual value of function then f is said to be continuous at the point $\left(x_0, y_0\right)$

$$\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = f(x_0, y_0)$$

If f is continuous at every point of D, then f is said to be continuous on D.

❖ *Theorem*

Let $f(x,y)$ & $g(x,y)$ be defined in a domain D and suppose that

$$\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = u_1 \quad \& \quad \lim_{\substack{x\to x_0 \\ y\to y_0}} g(x,y) = v_1$$

$a)$ then (i) $\displaystyle\lim_{\substack{x\to x_0 \\ y\to y_0}} \left[f(x,y) + g(x,y)\right] = u_1 + v_1$

(ii) $\displaystyle\lim_{\substack{x\to x_0 \\ y\to y_0}} \left[f(x,y)\cdot g(x,y)\right] = u_1 v_1$

(iii) $\displaystyle\lim_{\substack{x\to x_0 \\ y\to y_0}} \frac{f(x,y)}{g(x,y)} = \frac{u_1}{v_1}$

$b)$ If $f(x,y)$ & $g(x,y)$ are defined in D, then

$$\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = f(x_0, y_0) \quad \& \quad \lim_{\substack{x\to x_0 \\ y\to y_0}} g(x,y) = g(x_0, y_0)$$

i.e. $f(x,y)$, $g(x,y)$ are continuous at $\left(x_0, y_0\right)$ then so are the functions

$$f(x,y) + g(x,y) \ , \quad f(x,y)g(x,y) \ \text{ and } \ \frac{f(x,y)}{g(x,y)}, \quad \text{provided } \ g(x,y) \neq 0.$$

Proof

$a)$ (i) $\because$ $\displaystyle\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = u_1$, $\displaystyle\lim_{\substack{x\to x_0 \\ y\to y_0}} g(x,y) = v_1$

$\therefore$ given $\dfrac{\varepsilon}{2} > 0$ $\exists$ a $\delta_1, \delta_2 > 0$ such that

$$\left| f(x,y) - u_1 \right| < \frac{\varepsilon}{2} \qquad \forall \ (x,y) \in N_{\delta_1}(x_0, y_0)$$

$$\& \quad \left| g(x,y) - v_1 \right| < \frac{\varepsilon}{2} \qquad \forall \ (x,y) \in N_{\delta_2}\left(x_0, y_0\right)$$

$$\begin{aligned}
\text{then } \left|\left[f(x,y) + g(x,y)\right] - \left[u_1 + v_1\right]\right| &= \left|\left[f(x,y) - u_1\right] + \left[g(x,y) - v_1\right]\right| \\
&\leq \left| f(x,y) - u_1 \right| + \left| g(x,y) - v_1 \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \forall \ (x,y) \in N_\delta(x_0, y_0) \\
&\qquad \text{where } \delta = \min\left(\delta_1, \delta_2\right)
\end{aligned}$$

Which show that

$$\lim_{\substack{x\to x_0 \\ y\to y_0}} \left[f(x,y) + g(x,y)\right] = u_1 + v_1$$

$$\begin{aligned}
(ii) \quad \left| f(x,y)\cdot g(x,y) - u_1 v_1 \right| &= \left| f(x,y)\cdot g(x,y) - u_1 g(x,y) + u_1 g(x,y) - u_1 v_1 \right| \\
&= \left| g(x,y)\left[f(x,y) - u_1\right] + u_1\left[g(x,y) - v_1\right] \right| \\
&\leq \left| g(x,y)\left[f(x,y) - u_1\right] \right| + \left| u_1\left[g(x,y) - v_1\right] \right| \\
&< \left| g(x,y) \right| \frac{\varepsilon}{2} + u_1 \frac{\varepsilon}{2} \ = \varepsilon_1 \qquad \forall \ (x,y) \in N_\delta(x_0, y_0)
\end{aligned}$$

$$\Rightarrow \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)\cdot g(x,y) = u_1 v_1$$

iii) We prove that $\displaystyle \lim_{\substack{x\to x_0\\ y\to y_0}} \frac{1}{g(x,y)} = \frac{1}{v_1}$

$$\left|\frac{1}{g(x,y)} - \frac{1}{v_1}\right| = \left|\frac{v_1 - g(x,y)}{v_1\, g(x,y)}\right|$$

$$= \frac{|g(x,y) - v_1|}{|v_1||g(x,y)|} < \frac{\varepsilon/2}{|v_1||g(x,y)|}$$

$$< \frac{\varepsilon/2}{|v_1||g(x,y) - v_1 + v_1|} < \frac{\varepsilon/2}{|v_1|\left(|g(x,y) - v_1| + |v_1|\right)}$$

$$< \frac{\varepsilon/2}{|v_1|\left(\varepsilon/2 + |v_1|\right)} = \varepsilon_1 \quad \forall\ (x,y)\in N_{\delta_2}(x_0,y_0)$$

$$\Rightarrow \lim_{\substack{x\to x_0\\ y\to y_0}} \frac{1}{g(x,y)} = \frac{1}{v_1}$$

$$\because \Rightarrow \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = u_1 \quad \& \quad \lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y) = v_1$$

By (*ii*) of theorem

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)\cdot\frac{1}{g(x,y)} = \lim_{\substack{x\to x_0\\ y\to y_0}} \frac{f(x,y)}{g(x,y)} = \frac{u_1}{v_1}$$

b) Since it is given that the limiting values are the same as the actual values of the functions $f(x,y)+g(x,y)$, $f(x,y)\cdot g(x,y)$ and $\dfrac{f(x,y)}{g(x,y)}$ at the point (x_0,y_0) therefore these function are continuous on (x_0,y_0).

Note

It is to be noted that there is a difference between $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ and $\displaystyle\lim_{\substack{x\to a\\ y\to b}} f(x,y)$

i.e. $\displaystyle\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right)$ or $\displaystyle\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$

Obviously in the two cases limits are taken first w.r.t one variable and then w.r.t other variable. These limits are called the repeated limits. Since these are taken along the special path, therefore repeated limits are the special cases of limits.

$\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ exists if and only if limiting vales are not depend upon any path along which $(x,y)\to(a,b)$.

❖ Example

Consider $f:\mathbb{R}^2\to\mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4} & ,\ (x,y)\neq(0,0) \\ 0 & ,\ (x,y)=(0,0) \end{cases}$$

Now $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{y\to 0\\ x\to 0}} f(x,y) = 0$

However along the straight line $y = mx$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{m^4}{1+m^4}$$

which is different for different values of m. Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

❖ *Example*

Consider $f:\mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2\cos x - y^2\cos y}{x^2+y^2} & , \ (x,y)\neq 0 \\ 0 & , \ (x,y)=0 \end{cases}$$

then $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] = \lim\limits_{x\to 0}\cos x = 1$

and $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right] = \lim\limits_{y\to 0}(-\cos y) = -1$

$\Rightarrow \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

❖ *Example*

Consider $f:\mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} (x+y)\dfrac{\sin(x^2+y^2)}{x^2+y^2} & , \ (x,y)\neq (0,0) \\ 0 & , \ (x,y)=(0,0) \end{cases}$$

Use $\dfrac{\sin x}{x} < 1$ to get

$$\| f(x,y) - 0 \| \le |x+y| < |x| + |y|$$

Thus $\quad \| f(x,y) - 0 \| < \varepsilon \quad$ whenever $\quad |x| < \dfrac{\varepsilon}{2} \ , \ |y| < \dfrac{\varepsilon}{2}$

Take $\delta = \dfrac{\varepsilon}{2}$,

It follows that for given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\| f(x,y) - f(0,0) \| < \varepsilon \quad \text{whenever} \ \sqrt{(x-0)^2 + (y-0)^2} < \delta$$

i.e. $\forall \ (x,y) \in N_\delta(0,0)$

Limit of the function at $(0,0)$ is equal to actual value of function at $(0,0)$.

Hence f is continuous at $(0,0)$.

❖ *Partial Derivative*

Let $z = f(x,y)$ be defined in a domain D of xy-plane and take $(x_0,y_0)\in D$, then $f(x,y_0)$ is a function of x alone and its derivative may exist. If it exists then its value at (x_0,y_0) is known to be the partial derivative of $f(x,y)$ at (x_0,y_0) and is denoted as $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ or $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$

The other notations are z_x , f_x , f_1.

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y_0) - f(x,y_0)}{\Delta x}$$

We can define $\dfrac{\partial f}{\partial y}$ in the same manner.

❖ *Geometrical Interpretation*

$z = f(x,y)$ represents a surface in space. $y = y_0$ is a plane. $z = (x, y_0)$ is the curve which arises when $y = y_0$ cuts the surface $z = f(x,y)$. Thus $\frac{\partial f}{\partial x}_{(x_0,y_0)}$ denotes the slope of tangent to the curve $z = f(x, y_0)$ at $x = x_0$. Similarly $\frac{\partial f}{\partial y}_{(x_0,y_0)}$ denotes the slope of the tangent to the curve $z = f(x_0, y)$ at $y = y_0$.

If the point (x_0, y_0) varies, then f_x & f_y are themselves functions of x & y.
In the case of functions of more than three variables it is necessary to indicate the variable held constant during the process of differentiation as a suffix to avoid the confusion.

For example, $z = f(x, y, u, v)$, then partial derivatives are written as $\left(\frac{\partial z}{\partial u}\right)_x$, $\left(\frac{\partial z}{\partial y}\right)_v$ and so on. We take an example: $x = u + v$, $y = u - v$

$$\left(\frac{\partial x}{\partial u}\right)_v = 1,\ \left(\frac{\partial x}{\partial v}\right)_u = 1,\ \left(\frac{\partial y}{\partial u}\right)_v = 1,\ \left(\frac{\partial y}{\partial v}\right)_u = -1$$

Also $x + y = 2u$ and $x = 2u - y$, then

$$\left(\frac{\partial x}{\partial u}\right)_y = 2 \ \&\ \left(\frac{\partial x}{\partial y}\right)_u = -1 \text{ and so on.}$$

❖ *Total Differential*

In the case of partial derivative we have considered increments Δx & Δy separately.

Now take (x, y) & $(x + \Delta x, y + \Delta y)$ two points in the domain of definition of z then if $(z + \Delta z)$ correspond to the point $(x + \Delta x, y + \Delta y)$ we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

If the increment Δz can be expressed as

$$\Delta z = a\Delta x + b\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$, then $a\Delta x + b\Delta y$ is known to be the total differential of z denoted by dz, and we write

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In case when z is differentiable function dz gives very close approximation of Δz.

❖ *Theorem*

If $z = f(x, y)$ has a total differential at a point $(x, y) \in D$, then

$$a = \frac{\partial z}{\partial x} \quad \& \quad b = \frac{\partial z}{\partial y}.$$

Proof

We have

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{where } \varepsilon_1, \varepsilon_2 \to 0 \text{ as } \Delta x, \Delta y \to 0$$

Let us suppose that $\Delta y = 0$

then $\quad \Delta z = a\Delta x + \varepsilon_1 \Delta x$

Taking the limit as $\Delta x \to 0$

$$\frac{\partial z}{\partial x} = a$$

Similarly we can get $\frac{\partial z}{\partial y} = b$.

❖ Theorem (Fundamental Lemma)

If $z = f(x, y)$ has a continuous first order partial derivative in D then z has total differential $dz = \dfrac{\partial z}{\partial x}\Delta x + \dfrac{\partial z}{\partial y}\Delta y$ at every point $(x, y) \in D$.

Proof

Take a point (x, y) as a fixed point in the domain D. Suppose x changes alone. Then we have

$$\Delta z = f(x+\Delta x, y) - f(x, y)$$
$$= f_x(x_1, y)\Delta x \qquad \left(x < x_1 < x+\Delta x\right) \text{ (It is by M. V. Theorem)}$$

$\because$ f_x is continuous

$\therefore$ $\varepsilon_1 = f_x(x_1, y) - f_x(x, y) \to 0$ as $\Delta x \to 0$

$\Rightarrow$ $f(x+\Delta x, y) - f(x, y) = f_x(x, y)\Delta x + \varepsilon_1 \Delta x$ ………. (*i*)

Now if both x, y changes, we obtain a change Δz in z as

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$
$$= \left[f(x+\Delta x, y) - f(x, y)\right] + \left[f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)\right]$$

that is we have expressed Δz as the sum of terms representing the effect of a change in x alone and subsequent change in y alone.

Now $f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) = f_y(x+\Delta x, y_1)\Delta y \quad \left(y < y_1 < y+\Delta y\right)$

(It is by use of M.V. theorem)

$\because$ f_y is given to be continuous

$\therefore$ $\varepsilon_2 = f_y(x+\Delta x, y_1) - f_y(x, y) \to 0$ as $\Delta x, \Delta y \to 0$

$\Rightarrow$ $f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) = f_y(x, y)\Delta y + \varepsilon_2 \Delta y$ ……….. (*ii*)

Using (*i*) & (*ii*), we have

$$\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \quad \text{where } \varepsilon_1, \varepsilon_2 \to 0 \text{ as } \Delta x, \Delta y \to 0$$

which shows that the total differential dz of z exist & is given by

$$dz = f_x(x, y)\Delta x + f_y(x, y)\Delta y \quad \ldots\ldots\ldots\ldots.. (iii)$$

Note

(a) For reasons to be explained later; Δx & Δy can be replaced by dx & dy in (*iii*).

Thus we have $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

Which is the customary way of writing the differential. The preceding analysis extends at once to functions of three or more variables. For example, if

$$w = f(x, y, u, v), \quad \text{then} \quad dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv.$$

(b) In the following discussion, the function and their Ist order partial derivatives will be considered to be continuous in their respective domain of definition.

Example

If $z = x^2 - y^2$, then $dz = 2x dx - 2y dy$.

Example:

If $w = \dfrac{xy}{z}$, then $dw = \dfrac{y}{x}dx + \dfrac{x}{z}dy - \dfrac{xy}{z^2}dz$

PROBLEMS

1) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

a) $z = \frac{x}{x^2 + y^2}$ Ans: $\frac{\partial z}{\partial x} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$, $\frac{\partial z}{\partial x} = \frac{-2xy}{\left(x^2 + y^2\right)^2}$

b) $z = x \sin xy$ Ans: $\frac{\partial z}{\partial x} = \sin xy + xy \cos xy$, $\frac{\partial z}{\partial y} = x^2 \cos xy$

c) $x^3 + xy^2 - x^2 z + z^3 - 2 = 0$ Ans: $\frac{\partial z}{\partial x} = \frac{3x^2 + y^2 - 2xz}{x^2 - 3z^2}$, $\frac{\partial z}{\partial y} = \frac{e^{x+2y} - y}{\sqrt{e^{x+2y} - y^2}}$

2) Evaluate the indicated partial derivatives:

a) $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial v}{\partial y}\right)_x$ if $u = x^2 - y^2$, $v = x + 2y$

b) $\left(\frac{\partial x}{\partial u}\right)_y$ and $\left(\frac{\partial y}{\partial v}\right)_u$ if $u = x - 2y$, $v = u + 2y$ Ans: $\left(\frac{\partial x}{\partial u}\right)_y = 1$, $\left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{2}$

3) Find the differentials of the following functions

a) $z = \frac{x}{y}$ Ans: $\frac{ydx - xdy}{y^2}$

b) $z = \log\sqrt{x^2 + y^2}$ Ans: $\frac{xdx + ydy}{x^2 + y^2}$

c) $z = \tan^{-1}\left(\frac{y}{x}\right)$ Ans: $\frac{-ydx + ydy}{x^2 + y^2}$

d) $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ Ans: $\frac{-(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)^{3/2}}$

4) If $z = x^2 + 2xy$, find Δz in terms of Δx, Δy for $x = 1$, $y = 1$.

Ans: $\Delta z = 4\Delta x + 2\Delta y + \overline{\Delta x^2} + 2\Delta x \Delta y$, $dz = 4\Delta x + 2\Delta y$, $dz = 4\Delta x + 2\Delta y$.

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❖ *Derivative and Differential of functions of functions*

In the following discussion, the function and their first order partial derivatives will be considered to be continuous in their respective domain of definitions.

❖ *Theorem (Chain Rule I)*

Let $z = f(x,y)$, $x = g(t)$ & $y = h(t)$ be defined in a domain D, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Proof

$\because$ $z = f(x,y)$, $x = g(t)$, $y = h(t)$ are defined in D, are continuous and have Ist order partial derivatives.

$\therefore$ By using the fundamental lemma we have

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \;\ldots\ldots\ldots\ldots\; (i)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

Also $\Delta x = g(t + \Delta t) - g(t)$

$\Delta y = h(t + \Delta t) - h(t)$

Dividing (i) by Δt, we get

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

Take the limit as $\Delta t \to 0$, we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad \text{as desired.}$$

❖ *Theorem (Chain Rule II)*

Let $z = f(x,y)$, $x = g(u,v)$, $y = h(u,v)$ be defined in a domain D and have continuous first order partial derivative in D, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Proof

$\because$ the functions are continuous having first order partial derivatives in D, therefore by the fundamental lemma, we have

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \;\ldots\ldots\ldots\ldots\; (i)$$

where $\Delta x = g(u + \Delta u, v) - g(u,v)$, $\Delta y = h(u + \Delta u, v) - h(u,v)$

and $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ i.e. $\Delta u \to 0$

Dividing (i) by Δu throughout to have

$$\frac{\Delta z}{\Delta u} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta u} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta u} + \varepsilon_1\frac{\Delta x}{\Delta u} + \varepsilon_2\frac{\Delta y}{\Delta u}$$

Taking the limit as $\Delta u \to 0$ i.e. $\Delta x, \Delta y \to 0$, we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

Similarly if $\Delta x = g(u, v + \Delta v) - g(u,v)$

$\Delta y = h(u, v + \Delta v) - h(u,v)$

Then dividing (i) by Δv throughout, we obtain

$$\frac{\Delta z}{\Delta v} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta v} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta v} + \varepsilon_1\frac{\Delta x}{\Delta v} + \varepsilon_2\frac{\Delta y}{\Delta v}$$

Taking the limit as $\Delta v \to 0$, we have

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

❖ *Note*

We have proved in chain rule I, that if $z = f(x,y)$, $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad \text{.......... } (i)$$

The three functions of t considered here: $x = g(t)$, $y = h(t)$, $z = f\left(g(t), h(t)\right)$ have differentials $dx = \frac{dx}{dt}\Delta t$, $dy = \frac{dy}{dt}\Delta t$, $dz = \frac{dz}{dt}\Delta t$.

From (i) we conclude that

$$\frac{dz}{dt}\Delta t = \frac{\partial z}{\partial x}\left(\frac{dx}{dt}\Delta t\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\Delta t\right)$$

$$\Rightarrow \; dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{........... } (ii)$$

Similarly,

$$dx = \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v$$

$$dy = \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v$$

$$dz = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v$$

are the corresponding differentials when $z = f(x,y)$, $x = g(u,v)$, $y = h(u,v)$

$$\Rightarrow \; dz = \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)\Delta u + \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)\Delta v$$

$$= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v\right)$$

$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

which is again (ii)

The generalization of this permits to conclude that:

The differential formula

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial t}dt + ...$$

which holds when $z = f(x,y,t,....)$ and $dx = \Delta x$, $dy = \Delta y$, $dt = \Delta t$,……, remain the true when $x, y, t, ...$, and hence z, are all functions of other independent variables and $dx, dy, dt, ..., dz$ are the corresponding differentials.

As a consequence we can conclude:

Any equation in differentials which is correct for one choice of independent variables remains true for any other choice. Another way of saying this is that any equation in differentials treats all variables on an equal basis.

Thus, if $dz = 2dx - 3dy$ at a given point, then $dx = \frac{1}{2}dz + \frac{3}{2}dy$ is the corresponding differentials of x in terms of y and z.

❖ ***Example***

If $z = \dfrac{x^2 - 1}{y}$, then $dz = \dfrac{2xy\,dx - (x^2 - 1)dy}{y^2}$

Hence $\dfrac{\partial z}{\partial x} = \dfrac{2x}{y}$, $\dfrac{\partial z}{\partial y} = \dfrac{1 - x^2}{y^2}$

❖ ***Example***

If $r^2 = x^2 + y^2$, then $rdr = xdx + ydy$

and $\left(\dfrac{\partial r}{\partial x}\right)_y = \dfrac{x}{r}$, $\left(\dfrac{\partial r}{\partial y}\right)_x = \dfrac{y}{r}$, $\left(\dfrac{\partial x}{\partial r}\right)_y = \dfrac{r}{x}$, etc.

❖ ***Example***

If $z = \tan^{-1}\left(\dfrac{y}{x}\right)$ $(x \neq 0)$, then

$$dz = \frac{1}{1 + \left(\dfrac{y}{x}\right)^2} .d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

and hence

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2 + y^2} \quad , \quad \frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2}$$

……………………………………..

❖ *Implicit Function*

If $F(x, y, z)$ is a given function of x, y & z, then the equation $F(x, y, z) = 0$ is a relation which may describe one or several functions z of x & y.

Thus if $x^2 + y^2 + z^2 - 1 = 0$, then

$$z = \sqrt{1 - x^2 - y^2} \quad \text{or} \quad z = -\sqrt{1 - x^2 - y^2}$$

Where both functions being defined for $x^2 + y^2 \le 1$. Either function is said to be implicitly defined by the equation $x^2 + y^2 + z^2 - 1 = 0$.

Similarly, an equation $F(x, y, z, w) = 0$ may define one or more implicit functions w of x, y, z. If two such equations are given;

$$F(x, y, z, w) = 0 \quad , \quad G(x, y, z, w) = 0 \quad ,$$

It is in general possible (at least in theory) to reduce the equations by elimination to the form

$$w = f(x, y) \quad , \quad z = g(x, y)$$

i.e. to obtain two functions of two variables. In general, if m equations in n unknown are given $(m < n)$, it is possible to solve for m of the variables in terms of the remaining $n - m$ variables; the number of dependent variables equals the number of equations

❖ *Example*

If $\quad 3x + 2y + z + 2w = 0$

$\quad\quad 2x + 3y - z - w = 0$

then $w = f(x, y) = -5x - 5y \quad \& \quad z = g(x, y) = 7x + 8y$

❖ *Example*

Suppose that the functions $w = f(x, y)$ & $z = g(x, y)$ are implicitly defined by

$$2x^2 + y^2 + z^2 - zw = 0$$

$$x^2 + y^2 + 2z^2 - 8 + zw = 0$$

Then taking the differentials, we obtain

$$4xdx + 2ydy + 2zdz - wdz - zdw = 0 \quad \ldots\ldots\ldots (i)$$

$$wdz + zdw + 2xdx + 2ydy + 4zdz = 0 \quad \ldots\ldots\ldots (ii)$$

Eliminate dw between (i) and (ii) to have

$$6xdx + 4ydy + 6zdz = 0$$

$$\Rightarrow \; dz = -\frac{x}{z}dx - \frac{2y}{3x}dy$$

$$\Rightarrow \; \frac{\partial z}{\partial x} = -\frac{x}{z} \quad , \quad \frac{\partial z}{\partial y} = -\frac{2y}{3z}$$

Eliminating of dz from (i) and (ii) gives

$$6x(2z + w)dx + 4y(z + w)dy - 6z^2 dw = 0$$

$$\Rightarrow \; dw = \frac{x(2z + w)}{z^2}dx + \frac{2y(z + w)}{3z^2}dy$$

$$\frac{\partial w}{\partial x} = \frac{x(2x + w)}{x^2} \quad , \quad \frac{\partial w}{\partial y} = \frac{2y(z + w)}{z^2}$$

❖ *Examples*

Suppose that the functions $w=f(x,y)$ & $z=g(x,y)$ are implicitly define by

$$F(x,y,z,w)=0 \quad \text{and} \quad G(x,y,z)=0\text{, then}$$

$$F_x dx+F_y dy+F_z dz+F_w dw=0$$

$$\text{and} \quad G_x dx+G_y dy+G_z dz+G_w dw=0$$

$$\Rightarrow F_z dz+F_w dw=-\left[F_x dx+F_y dy\right]$$

$$\text{and} \quad G_z dz+G_w dw=-\left[G_x dx+G_y dy\right]$$

Then by crammer rule, we have

$$dz=\frac{-\begin{vmatrix} F_x dx+F_y dy & F_w \\ G_x dx+G_y dy & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} =-\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}dx-\frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}dy$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} \quad , \quad \frac{\partial z}{\partial y}=-\frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(x,w)}}{\dfrac{\partial(F,G)}{\partial(z,w)}} \quad , \quad \frac{\partial z}{\partial y}=-\frac{\dfrac{\partial(F,G)}{\partial(y,w)}}{\dfrac{\partial(F,G)}{\partial(z,w)}} \qquad \text{provided } \frac{\partial(F,G)}{\partial(z,w)}\neq 0$$

Similarly, we have

$$dw=-\frac{\begin{vmatrix} F_z & F_x dx+F_y dy \\ G_z & G_x dx+G_y dy \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

and we can find $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ in the same manner.

❖ *Particular Cases*

i) One equation in 2 unknowns i.e. $F(x,y)=0$

$$\Rightarrow F_x dx+F_y dy=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{F_x}{F_y} \quad \left(F_y\neq 0\right)$$

ii) One equation in 3 unknowns i.e. $F(x,y,z)=0$

$$F_x dx+F_y dy+F_z dz=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \quad , \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z} \quad (F_z\neq 0)$$

iii) 2 equations in 3 unknown

$$F(x,y,z)=0 \quad , \quad G(x,y,z)=0$$

$$\frac{\partial z}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(y,x)}}{\dfrac{\partial(F,G)}{\partial(y,z)}} \quad , \quad \frac{\partial z}{\partial y}=-\frac{\dfrac{\partial(F,G)}{\partial(w,y)}}{\dfrac{\partial(F,G)}{\partial(z,w)}}$$

………………………………

❖ *Example*

Find the partial derivatives w.r.t x & y, when

$$u + 2v - x^2 + y^2 = 0$$
$$2u - v - 2xy = 0$$

Solution

Take the differentials

$$du + 2\,dv - 2x\,dx + 2y\,dy = 0 \;\ldots\ldots\ldots\ldots\; (i)$$
$$2\,du - dv - 2x\,dy - 2y\,dx = 0 \;\ldots\ldots\ldots\ldots\; (ii)$$

Eliminating dv between (i) and (ii), we have

$$5du - (2x + 4y)dx + (2y - 4x)dy = 0$$
$$\Rightarrow\; du = \frac{1}{5}(2x + 4y)dx - \frac{1}{5}(2y - 4x)dy$$
$$\Rightarrow\; \frac{\partial u}{\partial x} = \frac{1}{5}(2x + 4y) \quad \& \quad \frac{\partial u}{\partial y} = -\frac{1}{5}(2y - 4x)$$

Eliminating du between (i) and (ii), we get

$$5dv - (4x - 2y)dx + (4y + 2x)dy = 0$$
$$\Rightarrow\; dv = \frac{1}{5}(4x - 2y)dx - \frac{1}{5}(4y + 2x)dy$$
$$\Rightarrow\; \frac{\partial v}{\partial x} = \frac{1}{5}(4x - 2y) \quad \& \quad \frac{\partial v}{\partial y} = -\frac{1}{5}(4y + 2x)$$

❖ *Question*

Give that

$$2x + y - 3z - 2u = 0$$
$$\& \quad x + 2y + z + u = 0$$

Find $\left(\frac{\partial x}{\partial y}\right)_z$, $\left(\frac{\partial y}{\partial x}\right)_u$, $\left(\frac{\partial z}{\partial u}\right)_x$, $\left(\frac{\partial y}{\partial z}\right)_x$

Solution

Take the differentials

$$2dx + dy - 3dz - 2du = 0 \;\ldots\ldots\ldots\ldots\; (i)$$
$$dx + 2dy + dz + du = 0 \;\ldots\ldots\ldots\ldots\; (ii)$$

Eliminating du between (i) and (ii), we have

$$4dx + 5dy - dz = 0 \;\ldots\ldots\ldots\; (iii)$$
$$\Rightarrow\; dx = -\frac{5}{4}dy + \frac{1}{4}dz$$
$$\Rightarrow\; \left(\frac{\partial x}{\partial y}\right)_z = -\frac{5}{4}$$

From (iii), we have

$$5dy = dz - 4dx$$
$$\Rightarrow\; dy = \frac{1}{5}dz - \frac{4}{5}dx$$
$$\Rightarrow\; \left(\frac{\partial y}{\partial z}\right)_x = \frac{1}{5}$$

Eliminating dz between (i) & (ii), we get

$$5dx + 7dy + du = 0$$

$$\Rightarrow \ dy = -\frac{5}{7}dx - \frac{1}{7}du$$
$$\Rightarrow \ \left(\frac{\partial y}{\partial x}\right)_u = -\frac{5}{7}$$

Now eliminating dy between (i) & (ii), we get

$$-3dx - 5dz - 3du = 0$$
$$\Rightarrow \ dz = -\frac{3}{5}dx - \frac{3}{5}du$$
$$\Rightarrow \ \left(\frac{\partial z}{\partial u}\right)_x = -\frac{3}{5}$$

❖ *Question*

Given that

$$x^2 + y^2 + z^2 - u^2 + v^2 = 1 \ \ldots\ldots\ldots\ldots\ldots \ (i)$$
$$x^2 - y^2 + z^2 + u^2 + 2v^2 = 2 \ \ldots\ldots\ldots\ldots \ (ii)$$

a) Find du & dv in terms of dx, dy & dz at the point

$$x = 1\ ,\ y = 1\ ,\ z = 2\ ,\ u = 3\ \ \&\ \ v = 2\,.$$

b) Find $\left(\frac{\partial u}{\partial x}\right)_{(y,z)}$, $\left(\frac{\partial v}{\partial y}\right)_{(x,z)}$ at the point given above.

c) Find approximately the values of u & v for $x = 1\cdot 1$, $y = 1.2$, $z = 1.8$

Solutions

Differential gives

$$2xdx + 2ydy + 2zdz - 2udu + 2vdv = 0 \ \ldots\ldots\ldots \ (iii)$$
$$2xdx - 2ydy + 2zdz + 2udu + 2vdv = 0 \ \ldots\ldots\ldots \ (iv)$$

a) Putting $x = 1,\ y = 1,\ z = 2,\ u = 3$ & $v = 2$ in (iii) & (iv), we obtain

$$2dx + 2dy + 4dz - 6du + 4dv = 0 \ \ldots\ldots\ldots\ldots \ (v)$$
$$\& \quad 2dx - 2dy + 4dz + 6du + 8dv = 0 \ \ldots\ldots\ldots\ldots \ (vi)$$

Adding gives

$$12dv = -(4dx + 8dz)$$
$$\Rightarrow \ dv = -\frac{1}{3}(dx + 0\cdot dy + 2dz)$$

Similarly eliminating dv between (v) and (vi), we get

$$du = \frac{1}{9}(dx + 3dy + 2dz)$$

b) $\qquad \because \ du = \frac{1}{9}(dx + 3dy + 2dz)$

$$\therefore \ \left(\frac{\partial u}{\partial x}\right)_{y,z} = \frac{1}{9}$$

$$\& \quad \because \ dv = -\frac{1}{3}(dx + 0\cdot dy + 2dz)$$

$$\therefore \ \left(\frac{\partial v}{\partial y}\right)_{x,z} = 0$$

……………………………………..

❖ *Question*

Find the transformation of $x = r\cos\theta$, $y = r\sin\theta$ from rectangular to polar coordinates. Verify the relations

$a)$ $dx = \cos\theta dr - r\sin\theta d\theta$
$dy = \sin\theta dr + r\cos\theta d\theta$

$b)$ $dr = \cos\theta dx + \sin\theta dy$

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

$c)$ $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$, $\left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$, $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$

Solutions

Given that $x = r\cos\theta$ & $y = r\sin\theta$

$a)$ Differential gives

$$dx = \cos\theta dr - r\sin\theta d\theta \quad \dots\dots\dots (i)$$
$$dy = \sin\theta dr + r\cos\theta d\theta \quad \dots\dots\dots (ii)$$

$b)$ Multiplying (i) by $\cos\theta$ & (ii) by $\sin\theta$ and adding, we get

$$dr = \cos\theta dx + \sin\theta dy$$

Now multiply (i) by $\sin\theta$ & (ii) by $\cos\theta$ and subtract to obtain

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

$c)$ Given $x = r\cos\theta$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

We have already shown that $dr = \cos\theta dx + \sin\theta dy$

Which can be written as $dx = \frac{dr}{\cos\theta} - \tan\theta dy$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta \ = r$$

and $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}\cos^2\theta + \frac{1}{r}\sin^2\theta \ = \frac{1}{r}$$

❖ *Question*

Given that $x^2 - y^2\cos uv + z^2 = 0$
$x^2 + y^2 - \sin uv + 2z^2 = 2$
and $xy - \sin u\cos v + z = 0$

Find $\left(\frac{\partial x}{\partial u}\right)_v$, $\left(\frac{\partial x}{\partial v}\right)_u$ at $x = 1,\ y = 1,\ u = \frac{\pi}{2},\ v = 0,\ z = 0$

Solution

Differential gives

$$2x\,dx - 2y\cos uv\,dy + y^2\sin uv\cdot udv + y^2\sin uv\cdot vdu + 2zdz = 0 \quad \dots\dots (i)$$

$$2x\,dx + 2y\,dy - \cos uv \cdot u dv - \cos uv \cdot v du + 4z dz = 0 \ \ldots\ldots\ldots \ (ii)$$

& $$x\,dy + y\,dx - \cos u \cdot \cos v\,du + \sin u \cdot \sin v\,dv + dz = 0 \ \ldots\ldots\ldots \ (iii)$$

At the given point, these equations reduce to

$$2dx - 2dy = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots \ (iv)$$

$$2dx + 2dy - \frac{\pi}{2}dv = 0 \ \ldots\ldots\ldots \ (v)$$

& $$dx + dy + dz = 0 \ \ldots\ldots\ldots\ldots\ldots \ (vi)$$

Adding (iv) & (v), we have

$$4dx - \frac{\pi}{2}dv = 0$$

$$\Rightarrow \ dx = \frac{\pi}{8}dv + 0 \cdot du \quad \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = 0 \ , \ \left(\frac{\partial x}{\partial v}\right)_u = \frac{\pi}{8}$$

❖ *Question*

Find $\left(\dfrac{\partial u}{\partial x}\right)_y$ if $x^2 - y^2 + u^2 + 2v^2 = 1$

$$x^2 + y^2 - u^2 - v^2 = 2$$

Solution

Taking the differentials, we have

$$2x\,dx - 2y\,dy + 2u\,du + 4v\,dv = 0$$
$$2x\,dx + 2y\,dy - 2u\,du - 2v\,dv = 0$$

Eliminating dv, we get

$$6x\,dx + 2y\,dy - 2u\,du = 0$$

$$\Rightarrow \ du = \frac{3x}{u}dx + \frac{y}{u}dy$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = \frac{3x}{u}$$

❖ *Question*

Given the transformation

$$x = u - 2v$$
$$y = 2u + v$$

a) Write the equations of the inverse transformation

b) Evaluate the Jacobian of the transformation and that of the inverse transformation.

Solution

a) From the equations, we have

$$u = \frac{1}{5}x + \frac{2}{5}y$$

$$v = -\frac{2}{5}x + \frac{1}{5}y$$

which are the equations of the inverse transformation.

b) Jacobian of the given transformation $= \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$$

Jacobian of the inverse transformation $=\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{1}{5} & \dfrac{2}{5} \\ -\dfrac{2}{5} & \dfrac{1}{5} \end{vmatrix} = \frac{1}{5}$$

❖ *Question*

Given the transformation $x = f(u,v)$, $y = g(u,v)$ with Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)}$, show that for the inverse transformation one has

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v} \quad , \quad \frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v} \quad , \quad \frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial u} \quad , \quad \frac{\partial u}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u}$$

Solution

The given equations are

$$f(u,v) - x = 0 \ \ldots\ldots\ldots\ (i)$$
$$g(u,v) - y = 0 \ \ldots\ldots\ldots\ (ii)$$

Differentiating w.r.t. x, we get

$$f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} - 1 = 0$$
$$g_u \frac{\partial u}{\partial x} + g_v \frac{\partial v}{\partial x} - 0 = 0$$

Solving these equations by Crammer's rule, we have

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{g_v}{J} = \frac{1}{J}\frac{\partial y}{\partial v} \qquad \left(\because \ \frac{\partial y}{\partial v} = g_v \right)$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{J} = -\frac{g_u}{J} = -\frac{1}{J}\frac{\partial y}{\partial u}$$

Differentiating (i) & (ii) w.r.t. y, we have

$$f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} - 0 = 0$$
$$g_u \frac{\partial u}{\partial y} + g_v \frac{\partial v}{\partial y} - 1 = 0$$

Solving these equations by Crammer's rule, we get

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{J} = -\frac{f_v}{J} = -\frac{1}{J}\frac{\partial x}{\partial v}$$

$$\frac{\partial v}{\partial y} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{J} = \frac{f_u}{J} = \frac{1}{J}\frac{\partial x}{\partial u}$$

❖ Question

Given the transformation

$$x = u^2 - v^2$$
$$y = 2uv$$

a) Compute its Jacobian.

b) Evaluate $\left(\frac{\partial u}{\partial x}\right)_y$ & $\left(\frac{\partial v}{\partial x}\right)_y$

Solution

The given equations can be written as

$$u^2 - v^2 - x = 0 \; \ldots\ldots\ldots\ldots \; (i)$$
$$2uv - y = 0 \; \ldots\ldots\ldots\ldots \; (ii)$$

Differentiating (*i*) & (*ii*) partially w.r.t. x, we have

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} - 1 = 0 \; \ldots\ldots\ldots\ldots \; (iii)$$

$$2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} - 0 = 0 \; \ldots\ldots\ldots\ldots \; (iv)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2)$$

Solving (*iii*) & (*iv*) by Crammer's rule, we have

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\begin{vmatrix} -1 & -2v \\ 0 & 2u \end{vmatrix}}{J} = \frac{2u}{4(u^2+v^2)} = \frac{u}{2(u^2+v^2)}$$

$$\left(\frac{\partial v}{\partial x}\right)_y = -\frac{\begin{vmatrix} 2u & -1 \\ 2v & 0 \end{vmatrix}}{J} = \frac{-2v}{4(u^2+v^2)} = \frac{-v}{2(u^2+v^2)}$$

Note

$\left(\frac{\partial u}{\partial y}\right)_x$ & $\left(\frac{\partial v}{\partial y}\right)_x$ can be determined in the same manner.

❖ Question

Prove that if $F(x,y,z) = 0$, then

$$\left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x = -1$$

Solution

$$F(x,y,z) = 0$$
$$\Rightarrow \; F_x\,dx + F_y\,dy + F_z\,dz = 0$$
$$\Rightarrow \; dx = -\frac{F_y}{F_x}dy - \frac{F_z}{F_x}dz \qquad \Rightarrow \; \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$$
$$\& \quad dy = -\frac{F_x}{F_y}dx - \frac{F_z}{F_y}dz \qquad \Rightarrow \; \left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$$
$$dz = -\frac{F_x}{F_z}dx - \frac{F_y}{F_z}dy \qquad \Rightarrow \; \left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$$

Hence

$$\left(\frac{\partial z}{\partial x}\right)_y\cdot\left(\frac{\partial x}{\partial y}\right)_z\cdot\left(\frac{\partial y}{\partial z}\right)_x=\left(-\frac{F_x}{F_z}\right)\cdot\left(-\frac{F_y}{F_x}\right)\cdot\left(-\frac{F_z}{F_y}\right)=-1$$

❖ *Question*

Prove that, if $x=f(u,v)$, $y=g(u,v)$, then

$$\left(\frac{\partial x}{\partial u}\right)_v\left(\frac{\partial u}{\partial x}\right)_y=\left(\frac{\partial y}{\partial v}\right)_u\left(\frac{\partial v}{\partial y}\right)_x$$

and $$\left(\frac{\partial x}{\partial v}\right)_u\left(\frac{\partial v}{\partial x}\right)_y=\left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v$$

also that $$\left(\frac{\partial x}{\partial y}\right)_u\left(\frac{\partial y}{\partial x}\right)_u=1$$

Solution

$$\because\ f(u,v)-x=0$$
$$g(u,v)-y=0$$

$$\therefore\ \left(\frac{\partial u}{\partial x}\right)_y=\frac{g_v}{J}\quad,\quad\left(\frac{\partial v}{\partial x}\right)_y=-\frac{g_u}{J}$$

$$\left(\frac{\partial u}{\partial y}\right)_x=-\frac{f_v}{J}\quad,\quad\left(\frac{\partial v}{\partial y}\right)_x=\frac{f_u}{J}$$ as already shown

Taking differentials of the given equations, we have

$$f_u\,du+f_v\,dv-dx=0$$
$$g_u\,du+g_v\,dv-dy=0$$
$$\Rightarrow\ dx=f_u\,du+f_v\,dv\ \ldots\ldots\ldots\ldots\ (i)$$
$$dy=g_u\,du+g_v\,dv\ \ldots\ldots\ldots\ldots\ (ii)$$

$$\Rightarrow\ \left(\frac{\partial x}{\partial u}\right)_v=f_u\quad,\quad\left(\frac{\partial x}{\partial v}\right)_u=f_v$$

$$\left(\frac{\partial y}{\partial u}\right)_v=g_u\quad,\quad\left(\frac{\partial y}{\partial v}\right)_u=g_v$$

Now $$\left(\frac{\partial x}{\partial u}\right)_v\cdot\left(\frac{\partial u}{\partial x}\right)_y=\left(\frac{\partial y}{\partial v}\right)_u\cdot\left(\frac{\partial v}{\partial y}\right)_x$$

$$\Rightarrow\ f_u\cdot\frac{g_v}{J}=g_v\cdot\frac{f_u}{J},$$ which is true

Similarly, we have the second relation.

Eliminating dv between (i) & (ii), we get

$$\left(f_u\cdot g_v-f_v\cdot g_u\right)du-g_v\,dx+f_v\,dy=0$$

$$\Rightarrow\ dx=\frac{f_u\,g_v-f_v\,g_u}{g_v}\cdot du+\frac{f_v}{g_v}\,dy$$

and $$dy=\frac{g_v}{f_v}dx-\frac{f_u\,g_v-f_v\,g_u}{f_v}\,du$$

$$\Rightarrow\ \left(\frac{\partial x}{\partial y}\right)_u=\frac{f_v}{g_v}\quad\&\quad\left(\frac{\partial y}{\partial x}\right)_u=\frac{g_v}{f_v}$$

$$\Rightarrow\ \left(\frac{\partial x}{\partial y}\right)_u\cdot\left(\frac{\partial y}{\partial x}\right)_u=\frac{f_v}{g_v}\cdot\frac{g_v}{f_v}=1$$

❖ *Question*

Given that $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$ with the Jacobian $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$, show that for the inverse transformation one has

i) $\dfrac{\partial u}{\partial x} = \dfrac{1}{J}\dfrac{\partial(y,z)}{\partial(v,w)}$, $\dfrac{\partial u}{\partial y} = \dfrac{1}{J}\dfrac{\partial(z,x)}{\partial(v,w)}$, $\dfrac{\partial u}{\partial z} = \dfrac{1}{J}\dfrac{\partial(x,y)}{\partial(v,w)}$

ii) $\dfrac{\partial v}{\partial x} = \dfrac{1}{J}\dfrac{\partial(y,z)}{\partial(w,u)}$, $\dfrac{\partial v}{\partial y} = \dfrac{1}{J}\dfrac{\partial(z,x)}{\partial(w,u)}$, $\dfrac{\partial v}{\partial z} = \dfrac{1}{J}\dfrac{\partial(x,y)}{\partial(w,u)}$

iii) $\dfrac{\partial w}{\partial x} = \dfrac{1}{J}\dfrac{\partial(y,z)}{\partial(u,v)}$, $\dfrac{\partial w}{\partial y} = \dfrac{1}{J}\dfrac{\partial(z,x)}{\partial(u,v)}$, $\dfrac{\partial w}{\partial z} = \dfrac{1}{J}\dfrac{\partial(x,y)}{\partial(u,v)}$

Solution

We have

$$f(u,v,w) - x = 0$$
$$g(u,v,w) - y = 0$$
$$h(u,v,w) - z = 0$$

Differentiating w.r.t. to x, we get

$$f_u\frac{\partial u}{\partial x} + f_v\frac{\partial v}{\partial x} + f_w\frac{\partial w}{\partial x} - 1 = 0$$
$$g_u\frac{\partial u}{\partial x} + g_v\frac{\partial v}{\partial x} + g_w\frac{\partial w}{\partial x} - 0 = 0$$
$$h_u\frac{\partial u}{\partial x} + h_v\frac{\partial v}{\partial x} + h_w\frac{\partial w}{\partial x} - 0 = 0$$

By Crammer's rule, we have

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -1 & f_v & f_w \\ 0 & g_v & g_w \\ 0 & h_v & h_w \end{vmatrix}}{J} = \frac{\begin{vmatrix} g_v & g_w \\ h_v & h_w \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial(g,h)}{\partial(v,w)} = \frac{1}{J}\frac{\partial(y,z)}{\partial(v,w)}$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} f_u & -1 & f_w \\ g_u & 0 & g_w \\ h_u & 0 & h_w \end{vmatrix}}{J} = -\frac{\begin{vmatrix} g_u & g_w \\ h_u & h_w \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial(g,h)}{\partial(w,u)} = \frac{1}{J}\frac{\partial(y,z)}{\partial(w,u)}$$

$$\frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} f_u & f_v & -1 \\ g_u & g_v & 0 \\ h_u & h_v & 0 \end{vmatrix}}{J} = \frac{\begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}}{J} = \frac{1}{J}\frac{\partial(g,h)}{\partial(u,v)} = \frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)}$$

We can find the other relations in the same way by differentiating given relation w.r.t. y and w.r.t. z respectively.

…………………………………….

❖ *Partial Derivative of Higher Order*

Let a function $z = f(x,y)$ be given. Then its two partial derivatives $\dfrac{\partial z}{\partial x}$ & $\dfrac{\partial z}{\partial y}$ are themselves functions of x & y.

$$\text{i.e.} \quad \frac{\partial z}{\partial x} = f_x(x,y) \quad , \quad \frac{\partial z}{\partial y} = f_y(x,y)$$

Hence each can be differentiable w.r.t. x & y.

Thus, we obtain four partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y) \quad , \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

$\dfrac{\partial^2 z}{\partial x^2}$ is the result of differentiating $\dfrac{\partial z}{\partial x}$ w.r.t. x, where $\dfrac{\partial^2 z}{\partial y \partial x}$ is the result of differentiating $\dfrac{\partial z}{\partial x}$ w.r.t. y. If all the derivatives concerned are continuous in the domain considered, then $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ i.e. order of differentiation is immaterial.

Third and higher order partial derivatives are defined in the same manner and under appropriate assumptions of continuity the order of differentiation does not matter.

❖ *Laplacian of z*

If $z = f(x,y)$, then the Laplacian of z is denoted by $\nabla^2 z$ is the expression

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

if $w = f(x,y,z)$, the Laplacian of w is the expression

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

The symbol "∇" is a vector differential operator define as

$$\nabla = \frac{\partial}{\partial x}\underline{\hat{i}} + \frac{\partial}{\partial y}\underline{\hat{j}} + \frac{\partial}{\partial x}\underline{\hat{k}}$$

We then have symbolically

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

❖ *Harmonic Function*

If $z = f(x,y)$ has continuous second order derivatives in a domain D and $\nabla^2 z = 0$ in D, then z is said to be Harmonic in D. The same term is used for the function of three variables which has continuous 2nd derivatives in a domain D in space and whose Laplacian is zero in D. The two equations for harmonic functions

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

are known as the Laplace equations in two and three dimensions respectively.

❖ Bi-Harmonic Equations

Another important combination of derivatives occurs in the equation

$$\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2\partial y^2}+\frac{\partial^4 z}{\partial y^4}=0$$

which is known to be the Bi-harmonic equation. This combination can be expressed in terms of Laplacian as

$$\nabla^2\left(\nabla^2 z\right) = \nabla^4 z = 0$$

The solutions of $\nabla^4 z=0$ are termed as Pri-harmonic functions.

❖ Higher Derivatives of Functions of Functions

(1) Let $z=f(x,y)$ and $x=g(t)$, $y=h(t)$ so that z can be expressed in terms of t alone. Then

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt} \quad \ldots\ldots\ldots.. (i)$$

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)=\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{dx}{dt}\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}+\frac{dy}{dt}\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) \quad \ldots\ldots\ldots (ii)$$

Using (i), we have

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y\,\partial x}\frac{dy}{dt}$$

& $$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial z}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}$$

Putting these values in (ii), we have

$$\frac{d^2z}{dt^2}=\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}\cdot\frac{dy}{dt}+\frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

(2) If $z=f(x,y)$ and $x=g(u,v)$, $y=h(u,v)$, then

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \ldots\ldots\ldots.. (iii)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \ldots\ldots\ldots.. (iv)$$

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\left(\frac{\partial^2 y}{\partial u^2}\right)+\frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) \quad \ldots\ldots\ldots\ldots. (iv)$$

Using (iii), we have

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial u}+\frac{\partial^2 z}{\partial y\,\partial x}\cdot\frac{\partial y}{\partial u}$$

and $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}$$

Putting these values in (iv), we get

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)+2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2+\frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

We can find the values of $\dfrac{\partial^2 z}{\partial u\,\partial v}$ & $\dfrac{\partial^2 z}{\partial v^2}$ in the same manner.

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❖ *The Laplacian in Polar, Cylindrical and Spherical Co-ordinate*

We consider first the two-dimensional Laplacian

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

and its expression in terms of polar co-ordinates r & θ.

Thus we are given $w = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$ and we wish to express $\nabla^2 w$ in terms of r, θ and derivatives of w with respect to r and θ. The solution is as follows. One has

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \quad \text{by chain rule}$$

To evaluate $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}$, we use the equations

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

These can be solved for dr and $d\theta$ by determinants or by elimination to give

$$dr = \cos\theta\, dx + \sin\theta\, dy$$

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

Hence $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$ and $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

Putting these values above in expressions of $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y}$, we have

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= \cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial w}{\partial y} &= \sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta} \end{aligned}\right\} \text{.............. } (i)$$

These equations provide general rules for expressing derivatives w.r.t. x or y in terms of derivatives w.r.t. r and θ. By applying the first equation to the function $\frac{\partial w}{\partial x}$, one finds that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial x}\right)$$

By (i) this can be written as follows:

$$\frac{\partial^2 w}{\partial x^2} = \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right)$$

The rule for differentiation of a product gives finally

$$\frac{\partial^2 w}{\partial x^2} = \cos^2\theta\cdot\frac{\partial^2 w}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial \theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{\sin^2\theta}{r}\cdot\frac{\partial w}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial w}{\partial \theta} \text{ .... } (ii)$$

In the same manner one finds

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right)$$

$$= \sin^2\theta \cdot \frac{\partial^2 w}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\cdot\frac{\partial w}{\partial r} - \frac{2\sin\theta\,\cos\theta}{r^2}\cdot\frac{\partial w}{\partial\theta} \quad \ldots\ldots\ (iii)$$

Adding (*ii*) & (*iii*), we conclude

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} \quad \ldots\ldots\ldots\ (iv)$$

This is the desired result.

Equation (*iv*) at once permits one to write the expression for the 3-demensional Laplacian in cylindrical co-ordinates for the transformation of coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad z = z$$

involves only x & y. In the same way as above, we have

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$= \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}$$

❖ *Laplacian in Spherical Polar Coordinates*

The transformation form rectangular to spherical polar coordinates is

$$x = \rho\sin\varphi\cos\theta \ , \ y = \rho\sin\varphi\sin\theta \ , \ z = \rho\cos\varphi$$

Writing $r = \rho\sin\varphi$, we have

$$x = r\cos\theta \ , \ y = r\sin\theta \ , \ z = z$$

Which can be considered as a transformation from rectangular to cylindrical coordinates (r,θ,z)

We have

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \quad \ldots\ldots\ldots.\ (i)$$

$$\text{where} \quad \left.\begin{aligned} z &= \rho\cos\varphi \\ r &= \rho\sin\varphi \end{aligned}\right\} \ \ldots\ldots\ldots\ (ii)$$

We have transformation from (x,y) to (r,θ) as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial\theta}$$

Now if we take transformation from (z,r) to (ρ,φ), then

$$\Rightarrow \ \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{\rho}\frac{\partial w}{\partial\varphi}$$

$$\text{Also} \quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial\rho}\cdot\frac{\partial\rho}{\partial r} + \frac{\partial w}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial r}$$

$$\text{Where} \quad \rho^2 = z^2 + r^2 \quad , \quad \tan\varphi = \frac{r}{z}$$

$$\Rightarrow \ 2\rho\frac{\partial\rho}{\partial r} = 2r \ \Rightarrow \ \frac{\partial\rho}{\partial r} = \frac{r}{\rho} = \frac{\rho\sin\varphi}{\rho} = \sin\varphi$$

$$\& \quad \sec^2\varphi\cdot\frac{\partial\varphi}{\partial r} = \frac{1}{z} \quad \Rightarrow \ \frac{\partial\varphi}{\partial r} = \frac{\cos^2\varphi}{z} = \frac{\cos^2\varphi}{\rho\cos\varphi} = \frac{\cos\varphi}{\rho}$$

$$\Rightarrow \ \frac{\partial w}{\partial r} = \frac{\partial w}{\partial\rho}\cdot\sin\varphi + \frac{\partial w}{\partial\varphi}\cdot\frac{\cos\varphi}{\rho} \quad \ldots\ldots\ldots\ldots\ (iv)$$

Substituting (*iii*) & (*iv*) in (*i*), we have

$$\nabla^2 w = \left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}$$

$$= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \sin^2\varphi}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho\sin\varphi}\left(\frac{\partial w}{\partial \rho}\sin\varphi + \frac{\partial w}{\partial \varphi}\frac{\cos\varphi}{\rho}\right)$$

$$= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \sin^2\varphi}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{\cot\varphi}{\rho^2}\cdot\frac{\partial w}{\partial \varphi}$$

$$= \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial \varphi^2} + \frac{2}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \sin^2\varphi}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{\cot\varphi}{\rho^2}\cdot\frac{\partial w}{\partial \varphi}$$

❖ ***Question***

If u & v are functions of x & y defined by the equations

$$xy + uv = 1 \ , \ \ xu + yv = 1$$

then find $\dfrac{\partial^2 u}{\partial x^2}$.

Solution

$$y\,dx + x\,dy + v\,du + u\,dv = 0 \ \ \ldots\ldots\ldots\ (i)$$

$$u\,dx + v\,dy + x\,du + y\,dv = 0 \ \ \ldots\ldots..\ (ii)$$

Eliminating dv between (i) & (ii)

$$\left(y^2 - u^2\right)dx + \left(xy - uv\right)dy + \left(vy - ux\right)du = 0$$

$$\Rightarrow\ du = \frac{u^2 - y^2}{vy - ux}dx + \frac{uv - xy}{vy - ux}dy$$

$$\Rightarrow\ \frac{\partial u}{\partial x} = \frac{u^2 - y^2}{vy - ux} = \frac{u^2 - y^2}{1 - 2ux} \qquad \text{(using given eq.)}$$

$$\Rightarrow\ \frac{\partial^2 u}{\partial x^2} = \frac{(1-2ux)\cdot 2u\cdot\frac{\partial u}{\partial x} - (u^2 - y^2)\left[(-2u) - 2x\frac{\partial u}{\partial x}\right]}{\left(1-2ux\right)^2}$$

❖ ***Question***

Find $\dfrac{\partial^2 w}{\partial x^2}, \ \dfrac{\partial^2 w}{\partial y^2}$ when

$i)$ $w = \dfrac{1}{\sqrt{x^2 + y^2}}$

$ii)$ $w = \tan^{-1}\dfrac{y}{x}$

$iii)$ $w = e^{x^2 - y^2}$

❖ ***Question***

Show that the following functions are harmonic in x & y

$i)$ $e^x \cos y$

$ii)$ $x^3 - 3xy^2$

$iii)$ $\log\sqrt{x^2 + y^2}$

…………………………………………

❖ *Sufficient Condition for the Validity of Reversal in the Order of Derivation*

We now prove two theorems which lay sufficient conditions for the equality of f_{xy} and f_{yx}.

❖ *Schawarz's Theorem*

If (a,b) be a point of the domain of a function $f(x,y)$ such that

i) $f_x(x,y)$ exists in a certain nhood of (a,b).

ii) $f_{xy}(x,y)$ is continuous at (a,b).

then $f_{yx}(a,b)$ exists and is equal to $f_{xy}(a,b)$.

Proof

The given conditions imply that there exists a certain nhood of (a,b) at every point (x,y) of which $f_x(x,y)$, $f_y(x,y)$ and $f_{xy}(x,y)$ exist. Let $(a+h,b+k)$ be any point of this nhood. We write

$$\phi(h,k)=f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)$$
$$g(y)=f(a+h,y)-f(a,y)$$

so that $\phi(h,k)=g(b+k)-g(b)$ ……….. (i)

$\because$ f_y exists in a nhood of (a,b), the function $g(y)$ in derivable in $[b,b+k]$, and, therefore, by applying the M.V. theorem to the expression on R.H.S of (i), we have

$$\phi(h,k)=kg'(b+\theta k) \qquad (0<\theta<1)$$
$$=k\left(f_y(a+h,b+\theta k)-f_y(a,b+\theta k)\right) \text{ ……….. } (ii)$$

Again since f_{xy} exists in a nhood of (a,b), the function $f_y(x,b+\theta k)$ of x is derivable w.r.t. x in interval $(a,a+h)$ and, therefore, by applying the M.V. theorem to the right of (ii), we have

$$\phi(h,k)=hk\, f_{xy}(a+\theta' h,b+\theta k) \qquad (0<\theta'<1)$$

or $$\frac{1}{k}\left(\frac{f(a+h,b+k)-f(a,b+k)}{h}-\frac{f(a+h,b)-f(a,b)}{h}\right)=f_{xy}(a+\theta' h,b+\theta k)$$

Since $f_x(x,y)$ exists in a nhood of (a,b), this gives when $h\to 0$,

$$\frac{f_x(a,b+k)-f_x(a,b)}{k}=\lim_{h\to 0} f_{xy}(a+\theta' h,b+\theta k)$$

Let, now, $k\to 0$. Since $f_{xy}(x,y)$ is continuous at (a,b), we obtain

$$f_{yx}(a,b)=\lim_{k\to 0}\lim_{h\to 0} f_{xy}(a+\theta' h,b+\theta k)=f_{xy}(a,b)$$

❖ *Young's Theorem*

If (a,b) be a point of the domain of definition of a function $f(x,y)$ such that $f_x(x,y)$ and $f_y(x,y)$ are both differentiable at (a,b), then

$$f_{xy}(a,b)=f_{yx}(a,b)$$

Proof

The differentiability of f_x and f_y at (a,b) implies that they exist in a certain nhood of (a,b) and that f_{xx}, f_{yx}, f_{xy}, f_{yy} exist at (a,b).

Let $(a+h,b+h)$ be a point of this nhood. We write

$$\phi(h,h)=f(a+h,b+h)-f(a+h,b)-f(a,b+h)+f(a,b)$$
$$g(y)=f(a+h,y)-f(a,y)$$

so that $\phi(h,h)=g(b+h)-g(b)$ ………. (i)

Since f_y exists in a nhood of (a,b), the function $g(y)$ is derivable in $(b, b+h)$, and, therefore, by applying the M.V. theorem to the expression on the right of (i), we have

$$\phi(h,h) = h\,g'(b+\theta h) \qquad (0<\theta<1)$$

$$= h\left(f_y(a+h, b+\theta h) - f_y(a, b+\theta h)\right) \;\ldots\ldots\ldots\ldots\; (ii)$$

Since $f_y(x,y)$ is differentiable at (a,b), we have, by definition,

$$f_y(a+h, b+\theta h) - f_y(a,b) = h\, f_{xy}(a,b) + \theta h\, f_{yy}(a,b)$$

$$+ h\varphi_1(h,h) + \theta h \psi_1(h,h) \;\ldots\ldots\; (iii)$$

and $\quad f_y(a, b+\theta h) - f_y(a,b) = \theta h\, f_{yy}(a,b) + \theta h \psi_2(h,h) \;\ldots\ldots\ldots\; (iv)$

where φ_1, ψ_1, ψ_2 all $\to 0$ as $h \to 0$

From (ii), (iii) and (iv), we obtain

$$\frac{\phi(h,h)}{h^2} = f_{xy}(a,b) + \phi_1(h,h) + \theta\psi_1(h,h) - \theta\psi_2(h,h) \;\ldots\ldots\ldots\; (v)$$

By a similar argument and on considering

$$g(x) = f(x, b+k) - f(x,b)$$

We can show that

$$\frac{\phi(h,h)}{h^2} = f_{yx}(a,b) + \psi_3(h,h) + \theta'\varphi_2(h,h) - \theta'\varphi_3(h,h) \;\ldots\ldots\ldots\ldots\; (vi)$$

where φ_2, φ_3, ψ_3 all $\to 0$ as $h \to 0$

Equating the right hand side of (v) and (vi) and making $h \to 0$, we obtain

$$f_{xy}(a,b) = f_{yx}(a,b)$$

……………………………………

❖ *Maxima and Minima for Functions of Two Variables*

Let $(x_0.y_0)$ be the point of the domain of a function $f(x,y)$, then $f(x_0,y_0)$ said to an extreme value of the function $f(x,y)$, if the expression

$$\Delta f = f(x_0+h, y_0+h) - f(x_0,y_0)$$

preserves its sign for all h and k.

The extreme value of $f(x_0,y_0)$ being called a maximum or a minimum value according as this difference is positive or negative respectively.

Necessary Condition

The Necessary Condition for $f(x_0,y_0)$ to be an extreme value of function $f(x,y)$ is that $f_x(x_0,y_0)=0=f_y(x_0,y_0)$, provided that these partial derivatives exist.

It is to be noted that it is impossible to determine the nature of a critical point by studying the function $f(x,y_0)$ and $f(x_0,y)$.

e.g. Let $f(x,y)=1+x^2-y^2$

then $f(0,y)=1-y^2 \quad \Rightarrow \quad f'(0,y)=-2y=0 \quad \Rightarrow \quad (0,0)$ is a turning point.

Now $f''(0,y)=-2 \quad \Rightarrow \quad (0,0)$ is a point of maximum value.

But $f(x,0)=1+x^2$

$\Rightarrow \quad f'(x,0)=2x=0 \quad \Rightarrow \quad x=0 \quad \Rightarrow \quad (0,0)$ is the critical point

$\Rightarrow \quad f''(x,0)=2>0 \quad \Rightarrow \quad (0,0)$ is the maximum value

Hence we fail to decide the nature of the critical point in this way.

Sufficient Condition

Let $z=f(x,y)$ be defined and have continuous 1st and 2nd order partial derivatives in a domain D. Suppose (x_0,y_0) is a point of D for which f_x and f_y are both zero.

Let $A=f_{xx}(x_0,y_0)$, $B=f_{xy}(x_0,y_0)$, $C=f_{yy}(x_0,y_0)$,

then we have the following cases

i) $B^2-AC<0$ and $A+C<0 \quad \Rightarrow$ relative maximum at (x_0,y_0).

ii) $B^2-AC<0$ and $A+C>0 \quad \Rightarrow$ relative minimum at (x_0,y_0)

iii) $B^2-AC>0 \quad \Rightarrow$ saddle point at (x_0,y_0)

iv) $B^2-AC=0 \quad \Rightarrow$ nature of the critical point is undetermined

Proof

By the application of M.V. theorem for function of two variables we have

$$\Delta f = hf_x(x_0+\theta h, y_0+\theta k)+kf_y(x_0+\theta h, y_0+\theta k) \qquad (0<\theta<1)$$

$$= h\left[f_x(x_0+\theta h, y_0+\theta k)-f_x(x_0,y_0)\right]+k\left[f_y(x_0+\theta h, y_0+\theta k)-f_y(x_0,y_0)\right]$$

(it is because $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, a turning point)

$$= h\left[\theta h f_{xx}(x_0,y_0)+\theta k f_{yx}(x_0,y_0)+\varepsilon_1\theta h+\varepsilon_2\theta k\right]$$

$$+k\left[\theta h f_{xy}(x_0,y_0)+\theta k f_{yy}(x_0,y_0)+\varepsilon_3\theta h+\varepsilon_4\theta k\right]$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ & $\varepsilon_4 \to 0$ as $h,k\to 0$

$$\Delta f = h^2 f_{xx}(x_0,y_0)+2hk\, f_{xy}(x_0,y_0)+k^2 f_{yy}(x_0,y_0)+\varepsilon_1 h^2+(\varepsilon_2+\varepsilon_3)hk+\varepsilon_4 k^2$$

$$\Rightarrow \quad \Delta f = h^2A+2hkB+k^2C+\varepsilon_1 h^2+(\varepsilon_2+\varepsilon_3)hk+\varepsilon_4 k^2$$

The sign of Δf depends upon the quadratic $d^2 f = h^2A+2hk\,B+k^2C$

i & ii) Let $B^2-AC<0$, $(A\neq 0)$

$$\Rightarrow \quad d^2 f = \frac{1}{A}\left(h^2A+2hk\,AB+k^2AC\right)$$

$$= \frac{1}{A}\left(h^2A^2 + 2hk\,AB + k^2B^2 + (k^2AC - k^2B^2)\right)$$
$$= \frac{1}{A}\left((hA + kB)^2 + k^2(AC - B^2)\right)$$

Since $(hA+kB)^2$ is positive and $AC - B^2$ (supposed) is +ive, therefore the sign of $d^2 f$ depends upon the sign of A.

$\Rightarrow\ \Delta f > 0$ if $A > 0$ & $\Delta f < 0$ if $A < 0$

Again, since $B^2 - AC < 0 \Rightarrow B^2 < AC \Rightarrow AC > 0$

$\Rightarrow$ A and C are either both +ive or both –ive.

If $A > 0$, $C > 0$ then $A + C > 0$ and if $A < 0$, $C < 0$ then $A + C < 0$.

Hence we have the following result

a) $\Delta f > 0$ when $A + C > 0 \Rightarrow (x_0, y_0)$ is a point of minimum value.

b) $\Delta f < 0$ when $A + C < 0 \Rightarrow (x_0, y_0)$ is a point of maximum value.

iii) Let $B^2 - AC > 0$, then

$$d^2 f = \frac{1}{A}\left((hA + kB)^2 + k^2(AC - B^2)\right)$$
$$= \frac{1}{A}\left((hA + kB)^2 - k^2(B^2 - AC)\right)$$

which may be +ive or –ive for certain value of h & k, therefore (x_0, y_0) is a saddle point.

iv) Let $B^2 - AC = 0$, $A \neq 0$

$$\Rightarrow\ d^2 f = \frac{1}{A}\left(hA + kB\right)^2$$

which may vanish for certain values of h and k, implies that nature of the point remain undetermined.

❖ *Question*

Test for maxima and minima

$$z = 1 - x^2 - y^2$$

Solution

$$\frac{\partial z}{\partial x} = -2x = 0 \quad\Rightarrow\quad x = 0$$
$$\frac{\partial z}{\partial y} = -2y = 0 \quad\Rightarrow\quad y = 0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = -2 \quad , \quad B = \frac{\partial^2 z}{\partial x\,\partial y} = 0 \quad , \quad C = \frac{\partial^2 z}{\partial y^2} = -2$$

$B^2 - AC = 0 - 4 = -4 < 0$ and $A + C = -2 - 2 = -4 < 0$

$\Rightarrow$ the function has maximum value at $(0,0)$.

❖ *Question*

Test for maxima and minima

$$z = x^3 - 3xy^2$$

Solution

$$\frac{\partial z}{\partial y} = 3x^2 - 3y^2 = 0 \quad\Rightarrow\quad x = -y \ \ \& \ \ x = y$$
$$\frac{\partial z}{\partial y} = -6xy = 0 \quad\Rightarrow\quad xy = 0$$

$\Rightarrow$ $(0,0)$ is the critical point.

$A = \dfrac{\partial^2 z}{\partial x^2} = 6x = 0$ at $(0,0)$

$B = \dfrac{\partial^2 z}{\partial x \partial y} = -6y = 0$ at $(0,0)$

$C = \dfrac{\partial^2 z}{\partial y^2} = -6x = 0$ at $(0,0)$

$B^2 - 4AC = 0$ also $A + C = 0$

Therefore we need further consideration for the nature of point

$$\begin{aligned}\Delta z &= z(0+h, 0+k) - z(0,0)\\ &= z(h,k) - z(0,0)\\ &= h^3 - 2hk^2\end{aligned}$$

For $h = k$

$$\Delta z = h^3 - 3h^3 = -2h^3$$

$$\Rightarrow \Delta z > 0 \text{ if } h < 0 \quad \& \quad \Delta z < 0 \text{ if } h > 0$$

Hence $(0,0)$ is a saddle point.

❖ *Question*

Examine the function

$$z = f(x,y) = x^2 y^2$$

Solution

$f_x = 0 \Rightarrow 2xy^2 = 0$

$f_y = 0 \Rightarrow 2yx^2 = 0$

implies that $(0,0)$ is the critical point

$A = f_{xx} = 2y^2 = 0$ at $(0,0)$

$B = f_{xy} = -4xy = 0$ at $(0,0)$

$C = f_{yy} = 2x^2 = 0$ at $(0,0)$

Since $B^2 - 4AC = 0$ and also $A + C = 0$

Therefore we need further consideration for the nature of point.

$$\begin{aligned}\Delta f &= f(h,k) - f(0,0)\\ &= h^2 k^2\end{aligned}$$

$\Delta f > 0$ for all h & k

Hence $(0,0)$ is the point where function has minimum value.

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❖ *Lagrange's Multiplier*
(Maxima & Minima for Function with Side Condition)

A problem of considerable importance for application is that of maximizing and minimizing of function (optimization) of several variables where the variables are related by one or more equations, which are turned as side condition. e.g. the problem of finding the radius of largest sphere inscribable in the ellipsoid $x^2+2y^2+3z^2=6$ is equivalent to minimizing the function $w=x^2+y^2+z^2$ with the side condition $x^2+2y^2+z^2=6$.

To handle such problem, we can, if possible, eliminate some of the variables by using the side conditions and reduce the problem to an ordinary maximum and minimum problem such as that consider previously.
This procedure is not always feasible and following procedure often is more convenient which treat the variable in more symmetrical manner, so that various simplifications may be possible.

Consider the problem of finding the extreme values of the function $f(x_1,x_2,....,x_n)$ when the variable are restricted by a certain number of side conditions say

$$g_1(x_1,x_2,....,x_n) = 0$$
$$g_2(x_1,x_2,....,x_n) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$g_m(x_1,x_2,....,x_n) = 0$$

We then form the linear combination

$$\varphi(x_1,...,x_n)= f(x_1,...,x_n)+\lambda_1 g_1(x_1,...,x_n)+\lambda_2 g_2(x_2,...,x_n)+ ....+\lambda_m g_m(x_1,...,x_n)$$

where $\lambda_1,\lambda_2,....,\lambda_m$ are m constants.

We then differentiate φ w.r.t. each coordinate and consider the following system of $n+m$ equations.

$$D_r\varphi(x_1,x_2,....,x_n)=0 \quad , \quad r=1,2,....,n$$
$$g_k(x_1,x_2,....,x_n)=0 \quad , \quad k=1,2,....,m$$

Lagrange discovered that if the point $(x_1,x_2,....,x_n)$ is a solution of the extreme problem then it will also satisfy the system of $n+m$ equation.

In practise, we attempt to solve this system for $n+m$ unknowns, which are $\lambda_1,\lambda_2,....,\lambda_m$ & $x_1,x_2,....,x_n$

The point so obtain must then be tested to determine whether they yield a maximum, a minimum or neither.

The numbers $\lambda_1,\lambda_2,....,\lambda_m$, which are introduced only to help to solve the system for $x_1,x_2,....,x_n$ are known as Lagrange's multiplier. One multiplier is introduced for each side condition.

❖ *Question*

Find the critical points of $w=xyz$, subject to condition $x^2+y^2+z^2=1$.

Solution

We form the function

$$\varphi = xyz+\lambda(x^2+y^2+z^2-1)$$

then

$$\frac{\partial\varphi}{\partial x}=yz+2\lambda x=0$$

$$\frac{\partial\varphi}{\partial y}=xz+2\lambda y=0$$

$$\frac{\partial \varphi}{\partial z} = xy + 2\lambda z = 0$$

$$\& \quad x^2 + y^2 + z^2 - 1 = 0$$

Multiplying the first three equations by x, y & z respectively, adding and using the fourth equation, we find

$$\lambda = -\frac{3xyz}{2}$$

using this relation we find that $(0,0,\pm1)$, $(0,\pm1,0)$, $(\pm1,0,0)$ and $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ are the critical points.

❖ ***Question***

Find the critical points of $w = xyz$, where $x^2 + y^2 = 1$ & $x - z = 0$. Also test for maxima and minima.

Solution

Consider $F = xyz + \lambda_1\left(x^2 + y^2 + 1\right) + \lambda_2\left(x - z\right)$

For the critical points, we have

$$F_x = yz + 2\lambda_1 x + \lambda_2 = 0 \quad \ldots\ldots\ldots\ldots (i)$$

$$F_y = xz + 2\lambda_1 y = 0 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

$$F_z = xy - \lambda_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

and

$$x^2 + y^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

$$x - z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

From (iii), $\lambda_2 = xy$ & from (ii) $\lambda_1 = -\dfrac{xz}{2y}$

Use these values in equation (i) to have

$$yz - \frac{x^2 z}{y} + xy = 0$$

$$\Rightarrow \; y^2 z - x^2 z + xy^2 = 0$$

$\because \quad x = z$ from (v)

$\therefore \; y^2 x - x^3 + xy^2 = 0 \quad \Rightarrow \; 2xy^2 - x^3 = 0$

But $y^2 = 1 - x^2$, from (iv)

$\therefore \; 2x\left(1 - x^2\right) - x^3 = 0 \quad \Rightarrow \; 2x - 3x^3 = 0 \quad \Rightarrow \; x = 0 \;,\; \pm\sqrt{\frac{2}{3}}$

This implies the critical points are $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$, $\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$, $(0,1,0)$, $(0,-1,0)$

$$A = F_{xx} = 2\lambda_1$$

$$B = F_{xy} = z$$

$$C = F_{yy} = 2\lambda_1$$

$$B^2 - AC = z^2 - 4\lambda_1^2$$

$$= z^2 - 4\frac{x^2 z^2}{4y^2} = \frac{z^2\left(y^2 - x^2\right)}{y^2}$$

(i) At $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$, we have

$$B^2 - AC = \frac{\frac{2}{3}\left(\frac{1}{3}-\frac{2}{3}\right)}{1/3} < 0$$

$$\& \quad A = F_{xx} = 2\lambda_1 = -\frac{xz}{y} = -\left(\frac{2/3}{1/\sqrt{3}}\right) < 0$$

$\Rightarrow$ function has maximum value at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

Similarly, we can show that F is also maximum at $(0,-1,0)$ and is minimum at remaining points. (Check yourself)

❖ *Question*

Find the point of the curve

$$x^2 - xy + y^2 - z^2 = 1\ ,\ \ x^2 + y^2 = 1$$

which is nearest to the origin.

Solution

Let a point on a given curve be (x,y,z)

Implies that we are to minimize the function

$$f = d^2 = x^2 + y^2 + z^2$$

subject to the conditions

$$x^2 - xy + y^2 - z^2 = 1$$
$$x^2 + y^2 = 1$$

Consider

$$F = x^2 + y^2 + z^2 + \lambda_1\left(x^2 - xy + y^2 - z^2 - 1\right) + \lambda_2\left(x^2 + y^2 - 1\right)$$

For the critical points

$$F_x = 2x(1+\lambda_1+\lambda_2) - \lambda_1 y = 0 \ \ldots\ldots\ldots..\ (i)$$
$$F_y = 2y(1+\lambda_1+\lambda_2) - \lambda_1 x = 0 \ \ldots\ldots\ldots..\ (ii)$$
$$F_z = 2z(1-\lambda_1) = 0 \ \ldots\ldots\ldots\ldots\ (iii)$$
$$x^2 - xy + y^2 - z^2 = 1 \ \ldots\ldots\ldots\ldots\ (iv)$$
$$x^2 + y^2 = 1 \ \ldots\ldots\ldots\ldots..\ (v)$$

From equation (iii), we have

$$z = 0 \ \text{ and } \ \lambda_1 = 1$$

Put $z = 0$ in equation (iv), gives

$$x^2 - xy + y^2 - 1 = 0$$
$$\Rightarrow\ xy = x^2 + y^2 - 1$$
$$\Rightarrow\ xy = 0 \quad \text{by } (v)$$
$$\Rightarrow\ x = 0 \ \text{ or } \ y = 0 \ \text{ or both are zero.}$$

$z = 0$, $x = 0$ in (v) gives, $y^2 = 1 \quad \Rightarrow\ y = \pm 1$

$\Rightarrow\ (0,\pm1,0)$ are the critical points.

$z = 0$, $y = 0 \quad \Rightarrow\ x = \pm1 \quad \Rightarrow\ (\pm1,0,0)$ are the critical points.

We can not take $x = 0$, $y = 0$ at the same time, because it gives $(0,0,0)$ which is origin itself as a critical point.

$\because\ d^2 = 1$ at all these four points.

$\therefore$ these are the required point at which function is nearest to origin.

❖ ***Question***

Find the point on the curve

$$x^2+y^2+z^2=1$$

which is farthest from the point $(1,2,3)$

Solution

We are the maximize the function

$$f=(x-1)^2+(y-2)^2+(z-3)^2$$

subject to the condition

$$x^2+y^2+z^2=1$$

Let

$$F=(x-1)^2+(y-2)^2+(z-3)^2+\lambda\left(x^2+y^2+z^2-1\right)$$

For the critical points, we have

$$x-1+\lambda x=0 \;\ldots\ldots\ldots\ldots\; (i)$$

$$y-2+\lambda y=0 \;\ldots\ldots\ldots\ldots\; (ii)$$

$$z-3+\lambda z=0 \;\ldots\ldots\ldots\ldots\; (iii)$$

$$\& \quad x^2+y^2+z^2=1 \;\ldots\ldots\ldots\ldots\; (iv)$$

$$\Rightarrow\; x=\frac{1}{1+\lambda}\;,\;\; y=\frac{2}{1+\lambda}\;,\;\; z=\frac{1}{3+\lambda}$$

Putting in (iv)

$$\left(\frac{1}{1+\lambda}\right)^2(1+4+9)=1 \quad\Rightarrow\quad (1+\lambda)^2=14 \quad\Rightarrow\quad \lambda=-1\pm\sqrt{14}$$

$$\Rightarrow\; x=\frac{1}{\pm\sqrt{14}}\;,\;\; y=\frac{2}{\pm\sqrt{14}}\;,\;\; z=\frac{3}{\pm\sqrt{14}}$$

$\Rightarrow$ critical points are

$$\left(\pm\frac{1}{\sqrt{14}},\pm\frac{2}{\sqrt{14}},\pm\frac{3}{\sqrt{14}}\right)$$

Its clear that the required point which is farthest from the point $(1,2,3)$ is $\left(-\frac{1}{\sqrt{14}},-\frac{2}{\sqrt{14}},-\frac{3}{\sqrt{14}}\right)$

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❖ *Directional Derivative*

i) Let $f: V \to \mathbb{R}$, where $V \subset \mathbb{R}^n$, is nhood of $\underline{a} \in \mathbb{R}^n$. Then the directional derivative $D_\beta f$ at $\underline{a}$ in the direction of $\underline{\beta} \in \mathbb{R}^n$, is defined by the limit, if it exists,

$$D_\beta f(\underline{a}) = \lim_{h \to 0} \frac{f\left(\underline{a} + h\underline{\beta}\right) - f(\underline{a})}{h}$$

ii) The directional derivative of $f\left(x_1, x_2, ..., x_i, ..., x_n\right)$ at $\underline{a} = \left(a_1, a_2, ..., a_i, ..., a_n\right)$ in the direction of the unit vector $\left(0,0,...,1,0,0,...,0\right)$ is called partial derivative of f at $\underline{a}$ w.r.t. the *i*th component x_i and is denoted by

$$D_i f(\underline{a}) \quad \text{or} \quad \frac{\partial f(\underline{a})}{\partial x} \quad \text{or} \quad f_{x_i}(a)$$

where $D_i f(\underline{a}) = \lim\limits_{h \to 0} \dfrac{f\left(a_1, a_2, ..., a_i + h, ..., a_n\right) - f\left(a_1, a_2, ..., a_i, ..., a_n\right)}{h}$

❖ *Example*

Let $f(x,y) = x^2 + y^2 + x + y$, then f has a directional derivative in every direction and at every point in $\mathbb{R}^2$.

Since, if $\beta = \left(a,b\right) \in \mathbb{R}^2$, we have

$$\begin{aligned} D_\beta f\left(x,y\right) &= \lim_{h \to 0} \frac{\left(x+ha\right)^2 + \left(y+hb\right)^2 + \left(x+ha\right) + \left(y+hb\right) - x^2 - y^2 - x - y}{h} \\ &= \lim_{h \to 0} \left(2ax + 2by + ha^2 + hb^2 + a + b\right) \\ &= 2\left(ax + by\right) + a + b \end{aligned}$$

❖ *Exercise*

$$\text{Let} \quad f(x,y) = \begin{cases} \dfrac{xy\left(x^2 - y^2\right)}{x^4 + y^4} & ; \quad x^4 + y^4 \neq 0 \\ 0 & ; \quad \left(x,y\right) \neq (0,0) \end{cases}$$

Note that if $\beta = \left(a,b\right) \in \mathbb{R}^2$,

$$\begin{aligned} D_\beta f(0,0) &= \lim_{h \to 0} \frac{\left(0+ah\right)\left(0+bh\right)\left[\left(0+ah\right)^2 - \left(0+bh\right)^2\right]}{h\left[\left(0+ah\right)^4 + \left(0+bh\right)^4\right]} \\ &= \lim_{h \to 0} \frac{ab\left(a^2 - b^2\right)}{h\left(a^4 + b^4\right)} \end{aligned}$$

This limit obviously exists only if $\beta = \left(1,0\right)$ or $\left(0,1\right)$. Hence the directional derivatives of f at $\left(0,0\right)$ that exists are the partial derivatives f_x and f_y given by $f_x = 0$, $f_y = 0$.

❖ *Example*

Let

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^4 + y^4} & ; \quad \left(x,y\right) = \left(0,0\right) \\ 0 & ; \quad \left(x,y\right) \neq \left(0,0\right) \end{cases}$$

It is discontinuous at $\left(0,0\right)$. To see it, note that

$$\lim_{(x,y) \to (0,0)} f\left(x,y\right) \text{ is zero along } y = 0 \text{ and is } \frac{1}{2} \text{ along } y^2 = x.$$

However, if $\beta=(a,b)$, then

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(0+ah)(0+bh)^2}{h\left[(0+ah)^2+(0+bh)^4\right]}$$

$$=\lim_{h\to 0}\frac{ah\cdot b^2h^2}{h\left[a^2h^2+b^4h^4\right]}\quad=\lim_{h\to 0}\frac{ab^2}{a^2+h^2b^4}$$

$$=\begin{cases} b^2/a & ,\ a\neq 0\\ 0 & ,\ a=0\end{cases}$$

Hence the directional derivative of f at $(0,0)$ exists in every direction.

❖ *Question*

Let $z=f(x,y)$, $x=u^2-v^2$, $y=2uv$. Then show that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{4(u^2+v^2)}\left\{\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right\}$$

Solution

We have

$$\frac{\partial x}{\partial u}=2u\quad,\quad\frac{\partial x}{\partial v}=-2v\quad,\quad\frac{\partial y}{\partial u}=2v\quad,\quad\frac{\partial y}{\partial v}=2v$$

Also

$$1=2u\frac{\partial u}{\partial x}-2v\frac{\partial v}{\partial x}\quad,\quad 0=2u\frac{\partial u}{\partial y}-2v\frac{\partial v}{\partial y}$$

and

$$0=2v\frac{\partial u}{\partial x}+2u\frac{\partial v}{\partial x}\quad,\quad 1=2v\frac{\partial u}{\partial y}+2u\frac{\partial v}{\partial y}$$

Solving these four equations for $\frac{\partial u}{\partial x},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial x}=\frac{u}{2(u^2+v^2)}\quad,\quad\frac{\partial v}{\partial x}=\frac{-v}{2(u^2+v^2)}$$

$$\frac{\partial u}{\partial y}=\frac{v}{2(u^2+v^2)}\quad,\quad\frac{\partial v}{\partial y}=\frac{u}{2(u^2+v^2)}$$

And

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{1}{2(u^2+v^2)}\left[u\cdot\frac{\partial z}{\partial u}-v\cdot\frac{\partial z}{\partial v}\right]$$

&

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{1}{2(u^2+v^2)}\left[v\cdot\frac{\partial z}{\partial u}+u\cdot\frac{\partial z}{\partial v}\right]$$

Hence

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{4(u^2+v^2)}\left\{\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right\}$$

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❖ *Question*

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & ; \ (x,y)\neq(0,0) \\ 0 & ; \ (x,y)=(0,0) \end{cases}$$

Show that f_x, f_y exist at $(0,0)$ but f is discontinuous at $(0,0)$.

Solution

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(ah)(bh)}{h\left[(ah)^2+(bh)^2\right]} \qquad \text{where } \beta=(a,b)$$

$$=\lim_{h\to 0}\frac{ab}{h\left(a^2+b^2\right)}$$

Which exists only when $\beta=(1,0)$ or $(0,1)$.

$\Rightarrow$ f_x & f_y exist at $(0,0)$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$$

Let $y=mx$, then

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{mx^2}{x^2+m^2x^2}$$

$$=\lim_{x\to 0}\frac{m}{1+m^2}$$

Which is different for different m.

$\Rightarrow$ $f(x,y)$ is discontinuous at $(0,0)$.

❖ *Question*

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\begin{cases} \dfrac{x^2y}{x^4+y^2} & ; \ (x,y)\neq(0,0) \\ 0 & ; \ (x,y)=(0,0) \end{cases}$$

Show that f_x, f_y exist at $(0,0)$ but f is discontinuous at $(0,0)$.

Solution

$$f_\beta(0,0)=\lim_{h\to 0}\frac{\left(a^2h^2\right)(bh)}{h\left[a^4h^4+b^2h^2\right]} \quad , \quad \beta=(a,b)$$

$$=\lim_{h\to 0}\frac{a^2b}{a^4h^2+b^2}$$

$$=\begin{cases} \dfrac{a^2}{b} & , \ b\neq 0 \\ 0 & , \ b=0 \end{cases}$$

Now $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ is zero along $x=0$ and is $\dfrac{1}{2}$ along $y=x^2$

$\Rightarrow$ it is discontinuous at $(0,0)$.

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❖ *Question*

Find the greatest volume of the box contained in the ellipsoid $3x^2 + 2y^2 + z^2 = 18$, when each of its edges is parallel to one of the coordinate axes.

Solution

$$V = \text{volume of the box} = (12x)(2y)(2z) = 8xyz$$

We need to find maximum of V subject to $3x^2 + 2y^2 + z^2 - 18 = 0$

Consider $\varphi(x, y, z) = 8xyz + \lambda\left(3x^2 + 2y^2 + z^2 - 18\right) = 0$

Then

$$\varphi_x = 8yz + 6\lambda x = 0$$
$$\varphi_y = 8xz + 4\lambda y = 0$$
$$\varphi_z = 8xy + 2\lambda z = 0$$
$$\Rightarrow \ 4xyz + 3\lambda x^2 = 0$$
$$2xyz + \lambda y^2 = 0$$
$$4xyz + \lambda z^2 = 0$$
$$\Rightarrow \ \lambda\left(3x^2 - 2y^2\right) = 0$$
$$\lambda\left(3x^2 - z^2\right) = 0$$
$$\Rightarrow \ x^2 = \frac{2y^2}{3} = \frac{z^2}{3}$$

Substituting these values in

$$3x^2 + 2y^2 + z^2 - 18 = 0$$

We get

$$3x^2 + 3x^2 + 3x^2 = 18 \qquad \Rightarrow \ 9x^2 = 18$$
$$\Rightarrow \ x = \sqrt{2}\ , \ \ y = \sqrt{3} \ \text{ and } \ z = \sqrt{6}$$

Which gives

$$f(x, y, z) = 8xyz = 48$$

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❖ *Definition*

If $f:\mathbb{R}^n \to \mathbb{R}$, $\underline{a} \in \mathbb{R}^n$ then

$$\nabla f(\underline{a}) = \sum_{k=1}^{n} \frac{\partial f(\underline{a})}{\partial x_k} = \frac{\partial f(\underline{a})}{\partial x_1} + \frac{\partial f(\underline{a})}{\partial x_2} + ... + \frac{\partial f(\underline{a})}{\partial x_n}$$

❖ *Definition*

Let $f:G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$.

i) f is said to have a local maximum at $\underline{a} \in G$, if there is a nhood $V_\varepsilon(\underline{a})$ such that $f(\underline{x}) \le f(\underline{a})$ $\forall$ $\underline{x} \in V_\varepsilon$.

ii) f is said to have a local minimum at $\underline{a} \in G$, if there is a nhood $V_\varepsilon(\underline{a})$ such that $f(\underline{x}) \ge f(\underline{a})$ $\forall$ $\underline{x} \in V_\varepsilon$.

❖ *Theorem*

Let $f:G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$. If f has a local extremum at $\underline{a} \in G$, then $\nabla f(\underline{a}) = 0$.

Proof

It is clear that $\nabla f(\underline{a}) = 0$ iff $\dfrac{\partial(\underline{a})}{\partial x_i} = 0$, $i = 1,2,3,....,n$

Write $f(x_i + t) = f(x_1, x_2, ..., x_i + t, ..., x_n) = f(\underline{x})$

If f has a local maximum at $\underline{a}$, then

$$\frac{f(a_i + t) - f(a_i)}{t} \le 0 \quad \text{if} \quad t > 0$$

$$\Rightarrow \lim_{t \to 0} \frac{f(a_i + t) - f(a_i)}{t} \le 0 \quad \text{if} \quad t > 0$$

So that $\dfrac{\partial f(\underline{a})}{\partial x_i} \le 0$

Similarly,

$$\lim_{t \to 0} \frac{f(a_i + t) - f(a_i)}{t} \ge 0 \quad \text{if} \quad t < 0$$

So that $\dfrac{\partial f(\underline{a})}{\partial x_i} \ge 0$

Hence $\dfrac{\partial f(\underline{a})}{\partial x_i} = 0$, $i = 1,2,3,....,n$

$$\Rightarrow \nabla f(\underline{a}) = 0$$

Note

There are situations when $\nabla f(\underline{a}) = 0$ but f has no local maximum or minimum at $\underline{a}$. If so and if the sign of $f(\underline{x}) - f(\underline{a})$ depends upon the direction of $\underline{x}$ and $\underline{a}$, f is said to have a saddle point at a.

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