Chapter 3 – Limit and Continuity

Subject: *Real Analysis (Mathematics)* ***Level:*** *M.Sc.*
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❖ *Limit of the function*

Suppose

(i) (X, d_X) and (Y, d_Y) be two metric spaces

(ii) $E \subset X$

(iii) $f : E \to Y$ i.e. f maps E into Y.

(iv) p is the limit point of E.

We write $f(x) \to q$ as $x \to p$ or $\lim\limits_{x\to p} f(x) = q$, if there is a point q with the following property;

For every $e > 0$, there exists a $d > 0$ such that $d_Y\left(f(x), q\right) < e$ for all points $x \in E$ for which $d_X(x, p) < d$.

If X and Y are replaced by a real line, complex plane or by Euclidean space $\mathbb{R}^k$, then the distances d_X and d_Y are replaced by absolute values or by appropriate norms. ❑

Note: *i*) It is to be noted that $p \in X$ but that p need not a point of E in the above definition (p is a limit point of E which may or may not belong to E.)

ii) Even if $p \in E$, we may have $f(p) \neq \lim\limits_{x\to p} f(x)$. ❑

❖ *Example*

$$\lim_{x\to\infty} \frac{2x}{1+x} = 2$$

We have $\left|\frac{2x}{x-1} - 2\right| = \left|\frac{2x-2-2x}{1+x}\right| = \left|\frac{-2}{1+x}\right| < \frac{2}{x}$

Now if $e > 0$ is given we can find $d = \frac{2}{e}$ so that

$$\left|\frac{2x}{1+x} - 2\right| < e \quad \text{whenever} \quad x > d .$$ ❑

❖ *Example*

Consider the function $f(x) = \frac{x^2 - 1}{x - 1}$.

It is to be noted that f is not defined at $x = 1$ but if $x \neq 1$ and is very close to 1 or less then $f(x)$ equals to 2. ❑

❖ *Definitions*

i) Let X and Y be subsets of $\mathbb{R}$, a function $f : X \to Y$ is said to tend to limit l as $x \to \infty$, if for a real number $e > 0$ however small, $\exists$ a positive number d which depends upon e such that distance

$$\left| f(x) - l \right| < e \quad \text{when } x > d \quad \text{and we write } \lim_{x\to\infty} f(x) = l .$$

ii) f is said to tend to a right limit l as $x \to c$ if for $e > 0$, $\exists$ $d > 0$ such that $\left| f(x) - l \right| < e$ whenever $x \in G$ and $0 < x < c + d$.

And we write $f(c+) = \lim\limits_{x\to c+} f(x) = l$

iii) f is said to tend to a left limit l as $x \to c$ if for $e > 0$, $\exists$ a $d > 0$ such that $\left| f(x) - l \right| < e$ whenever $x \in G$ and $0 < c - d < x < c$.

And we write $f(c-) = \lim\limits_{x\to c-} f(x) = l$. ❑

❖ Theorem

Suppose

(i) (X, d_x) and (Y, d_y) be two metric spaces

(ii) $E \subset X$

(iii) $f : E \to Y$ i.e. f maps E into Y.

(iv) p is the limit point of E.

Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

Proof

Suppose $\lim_{x \to p} f(x) = q$ holds.

Choose $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$, we are to show that $\lim_{n \to \infty} f(p_n) = q$

Then there exists a $d > 0$ such that

$$d_y\left(f(x), q\right) < e \quad \text{if } x \in E \text{ and } 0 < d_x(x, p) < d \quad \ldots\ldots\ldots (i)$$

Also $\exists$ a positive integer n_0 such that $n > n_0$

$$\Rightarrow d_x\left(p_n, p\right) < d \quad \ldots\ldots\ldots\ldots (ii)$$

from (*i*) and (*ii*), we have for $n > n_0$

$$d_y\left(f(p_n), q\right) < e$$

Which shows that limit of the sequence

$$\lim_{n \to \infty} f\left(p_n\right) = q$$

Conversely, suppose that $\lim_{n \to \infty} f\left(p_n\right) = q$ is false.

Then $\exists$ some $e > 0$ such that for every $d > 0$, there is a point $x \in E$ for which $d_y\left(f(x), q\right) \geq e$ but $0 < d_x(x, p) < d$.

In particular, taking $d_n = \dfrac{1}{n}$, $n = 1, 2, 3, \ldots\ldots$

We find a sequence in E satisfied $p_n \neq p$, $\lim_{n \to \infty} p_n = p$ for which $\lim_{n \to \infty} f(p_n) = q$ is false. ❑

❖ Example

$$\lim_{x \to \infty} \sin \frac{1}{x} \quad \text{does not exist.}$$

Suppose that $\lim_{x \to \infty} \sin \frac{1}{x}$ exists and take it to be l, then there exist a positive real number d such that

$$\left| \sin \frac{1}{x} - l \right| < 1 \quad \text{when} \quad 0 < \left| x - 0 \right| < d \quad (\text{we take } e = 1 > 0 \text{ here})$$

We can find a positive integer n such that

$$\frac{2}{np} < d \quad \text{then} \quad \frac{2}{(4n+1)p} < d \quad \text{and} \quad \frac{2}{(4n+3)p} < d$$

It thus follows

$$\left| \sin \frac{(4n+1)p}{2} - l \right| < 1 \quad \Rightarrow \left| 1 - l \right| < 1$$

$$\text{and} \quad \left| \sin \frac{(4n+3)p}{2} - l \right| < 1 \quad \Rightarrow \left| -1 - l \right| < 1 \quad \text{or} \quad \left| 1 + l \right| < 1$$

So that

$$2 = \left|1+l+1-l\right| \le \left|1+l\right| + \left|1-l\right| < 1+1 \quad \Rightarrow \ 2 < 2$$

This is impossible; hence limit of the function does not exist. ❑

Alternative:

Consider $x_n = \dfrac{2}{(2n-1)p}$ then $\lim\limits_{x\to\infty} x_n = 0$

But $\{f(x_n)\}$ i.e. $\left\{\sin\dfrac{1}{x_n}\right\}$ is an oscillatory sequence

i.e. $\{1,-1,1,-1,..........\}$ therefore $\left\{\sin\dfrac{1}{x_n}\right\}$ diverges.

Hence we conclude that $\lim\limits_{x\to\infty} \sin\dfrac{1}{x}$ does not exit. ❑

❖ ***Example***

Consider the function

$$f(x) = \begin{cases} x & ; \quad x<1 \\ 2+(x-1)^2 & ; \quad x \ge 1 \end{cases}$$

We show that $\lim\limits_{x\to 1} f(x)$ does not exist.

To prove this take $x_n = 1-\dfrac{1}{n}$, then $\lim\limits_{x\to\infty} x_n = 1$ and $\lim\limits_{n\to\infty} f(x_n) = 1$

But if we take $x_n = 1+\dfrac{1}{n}$ then $x_n \to 1$ as $n\to\infty$

and $\lim\limits_{x\to\infty} f(x_n) = \lim\limits_{x\to\infty} 2+\left(1+\dfrac{1}{n}-1\right)^2 = 2$

This show that $\{f(x_n)\}$ does not tend to a same limit as for all sequences $\{S_n\}$ such that $x_n \to 1$.

Hence this limit does not exist. ❑

❖ ***Example***

Consider the function $f:[0,1]\to\mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & if\ \ x\ is\ rational \\ 1 & if\ \ x\ is\ irratioanl \end{cases}$$

Show that $\lim\limits_{x\to p} f(x)$ where $p\in[0,1]$ does not exist.

Solution

Let $\lim\limits_{x\to p} f(x) = q$, if given $e>0$ we can find $d>0$ such that

$$\left|f(x)-q\right| < e \quad \text{whenever} \quad \left|x-p\right| < d \ .$$

Consider the irrational $(r-s, r+s) \subset [0,1]$ such that r is rational and s is irrational.

Then $f(r)=0$ & $f(s)=1$

Suppose $\lim\limits_{x\to p} f(x) = q$ then

$$\begin{aligned} \left|f(s)\right| &= 1 \\ \Rightarrow\ 1 &= \left|f(s)-q+q\right| \\ &= \left|(f(s)-q+q-0\right| \\ &= \left|f(s)-q+q-f(r)\right| \qquad \because\ 0=f(r) \end{aligned}$$

$$\leq \left| f(s) - q \right| + \left| f(r) - q \right| < e + e$$

i.e. $1 < e + e$

$$\Rightarrow 1 < \frac{1}{4} + \frac{1}{4} \qquad \text{if } e = \frac{1}{4}$$

Which is absurd.

Hence the limit of the function does not exist. ❑

❖ *Exercise*

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

We have

$$\left| x \sin \frac{1}{x} - 0 \right| < e \quad \text{where } e > 0 \text{ is a pre-assigned positive number.}$$

$$\Rightarrow \left| x \sin \frac{1}{x} \right| < e$$

$$\Rightarrow |x| \left| \sin \frac{1}{x} \right| < e$$

$$\Rightarrow |x| < e \qquad \because \left| \sin \frac{1}{x} \right| \leq 1$$

$$\Rightarrow |x - 0| < e = d$$

It shows that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Same the case for function for $f(x) = x \cos \frac{1}{x}$

Also we can derived the result that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$. ❑

❖ *Theorem*

If $\lim_{x \to c} f(x)$ exists then it is unique.

Proof

Suppose $\lim_{x \to c} f(x)$ is not unique.

Take $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} f(x) = l_2$ where $l_1 \neq l_2$.

$\Rightarrow \exists$ real numbers d_1 and d_2 such that

$$\left| f(x) - l_1 \right| < e \quad \text{whenever} \quad |x - c| < d_1$$

$$\& \quad \left| f(x) - l_2 \right| < e \quad \text{whenever} \quad |x - c| < d_2$$

Now

$$|l_1 - l_2| = \left| \left(f(x) - l_1 \right) - \left(f(x) - l_2 \right) \right|$$

$$\leq \left| f(x) - l_1 \right| + \left| f(x) - l_2 \right|$$

$$< e + e \quad \text{whenever} \quad |x - c| < \min(d_1, d_2)$$

$$\Rightarrow l_1 = l_2$$ ❑

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❖ *Theorem*

Suppose that a real valued function f is defined on an open interval G except possibly at $c \in G$. Then $\lim_{x \to c} f(x) = l$ if and only if for every positive real number e, there is $d > 0$ such that $\left| f(t) - f(s) \right| < e$ whenever s & t are in $\{x : \left| x - c \right| < d\}$.

Proof

Suppose $\lim_{x \to c} f(x) = l$

$\therefore$ for every $e > 0$, $\exists$ $d > 0$ such that

$$\left| f(s) - l \right| < \frac{1}{2}e \quad \text{whenever} \quad 0 < \left| s - c \right| < d$$

$$\& \quad \left| f(t) - l \right| < \frac{1}{2}e \quad \text{whenever} \quad 0 < \left| t - c \right| < d$$

$$\Rightarrow \left| f(s) - f(t) \right| \le \left| f(s) - l \right| + \left| f(t) - l \right|$$

$$< \frac{e}{2} + \frac{e}{2} \qquad \text{whenever} \quad \left| s - c \right| < d \quad \& \quad \left| t - c \right| < d$$

$$\left| f(t) - f(s) \right| < e \quad \text{whenever} \quad s \ \& \ t \ \text{are in} \ \{x : \left| x - c \right| < d\}.$$

Conversely, suppose that the given condition holds.

Let $\{x_n\}$ be a sequence of distinct elements of G such that $x_n \to c$ as $n \to \infty$.

Then for $d > 0$ $\exists$ a natural number n_0 such that

$$\left| x_n - l \right| < d \quad \text{and} \quad \left| x_m - l \right| < d \qquad \forall \ m, n > n_0.$$

And for $e > 0$

$$\left| f(x_n) - f(x_m) \right| < e \qquad \text{whenever} \quad m, n > n_0$$

$\Rightarrow$ $\{f(x_n)\}$ is a Cauchy sequence and therefore it is convergent. ❑

❖ *Theorem (Sandwiching Theorem)*

Suppose that f, g and h are functions defined on an open interval G except possibly at $c \in G$. Let $f \le h \le g$ on G.

If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = l$, then $\lim_{x \to c} h(x) = l$.

Proof

For $e > 0$ $\exists$ $d_1, d_2 > 0$ such that

$$\left| f(x) - l \right| < e \quad \text{whenever} \quad 0 < \left| x - c \right| < d_1$$

$$\& \ \left| g(x) - l \right| < e \quad \text{whenever} \quad 0 < \left| x - c \right| < d_2$$

$$\Rightarrow \ l - e < f(x) < l + e \quad \text{for} \quad 0 < \left| x - c \right| < d_1$$

$$\& \quad l - e < g(x) < l + e \quad \text{for} \quad 0 < \left| x - c \right| < d_2$$

$$\Rightarrow \ l - e < f(x) \le h(x) \le g(x) < l + e$$

$$\Rightarrow \ l - e < h(x) < l + e \quad \text{for} \quad 0 < \left| x - c \right| < \min(d_1, d_2)$$

$$\Rightarrow \ \lim_{x \to c} h(x) = l$$ ❑

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❖ Theorem

Let (*i*) (X, d) , (Y, d_y) be two metric spaces.

(*ii*) $E \subset X$

(*iii*) p is a limit point of E.

(*iv*) $f : E \to Y$.

(*v*) $g : E \to Y$

and $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$ then

i- $\lim_{x\to p}\left(f(x) \pm g(x)\right) = A \pm B$

ii- $\lim_{x\to p}(fg)(x) = AB$

iii- $\lim_{x\to p}\left(\frac{f(x)}{g(x)}\right) = \frac{A}{B}$ provided $B \neq 0$.

Proof

Do yourself ❑

❖ Continuity

Suppose

i) (X, d_X), (Y, d_Y) are two metric spaces

ii) $E \subset X$

iii) $p \in E$

iv) $f : E \to Y$

Then f is said to be continuous at p if for every $e > 0$ $\exists$ a $d > 0$ such that $d_Y\left(f(x), f(p)\right) < e$ for all points $x \in E$ for which $d_X\left(x, p\right) < d$.

Note:

(*i*) If f is continuous at every point of E. Then f is said to be continuous on E.

(*ii*) It is to be noted that f has to be defined at p iff $\lim_{x\to p} f(x) = f(p)$. ❑

❖ Examples

$f(x) = x^2$ is continuous $\forall$ $x \in \mathbb{R}$.

Here $f(x) = x^2$, Take $p \in \mathbb{R}$

Then $\left| f(x) - f(p) \right| < e$

$\Rightarrow \left| x^2 - p^2 \right| < e$

$\Rightarrow \left| (x-p)(x+p) \right| < e$

$\Rightarrow \left| x - p \right| < e = d$

$\because$ p is arbitrary real number

$\therefore$ the function $f(x)$ is continuous $\forall$ real numbers. ❑

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❖ *Theorem*

Let

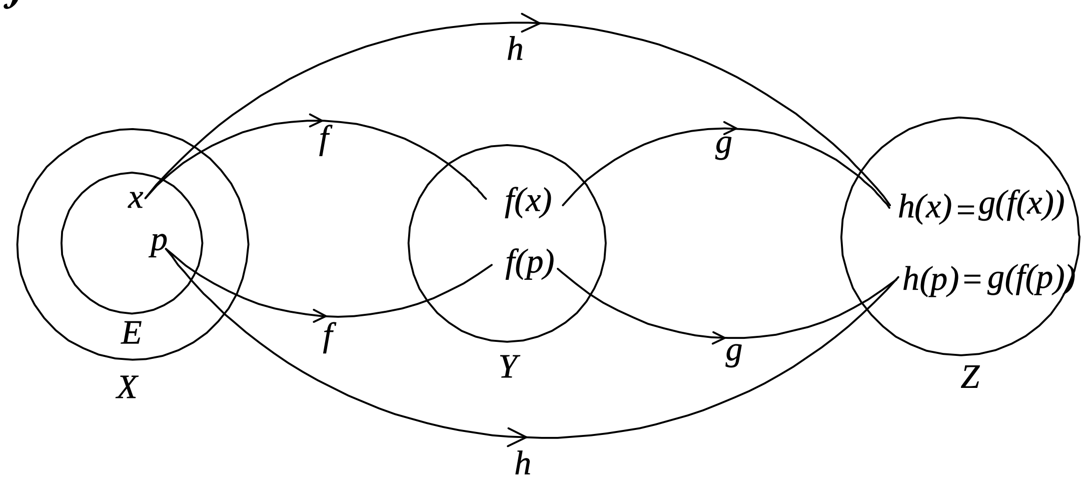
i) X, Y, Z be metric spaces

ii) $E \subset X$

iii) $f : E \to Y$, $g : f(E) \to Z$ and $h : E \to Z$ defined by $h(x) = g\big(f(x)\big)$

If f is continuous at $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at *p.*

Proof



$\because$ g is continuous at $f(p)$

$\therefore$ for every $e > 0$, $\exists$ a $d > 0$ such that

$$d_Z\big(g(y), g(f(p))\big) < e \quad \text{whenever} \quad d_Y\big(y, f(p)\big) < d_1 \quad \ldots\ldots\ldots.. (i)$$

$\because$ f is continuous at $p \in E$

$\therefore$ $\exists$ a $d > 0$ such that

$$d_Y\big(f(x), f(p)\big) < d_1 \quad \text{whenever} \quad d_X(x, p) < d \quad \ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii), we have

$$d_Z\big(g(y), g(f(p))\big) < e \quad \text{whenever} \quad d_X(x, p) < d$$

$$\Rightarrow d_Z\big(h(x), h(p)\big) < e \quad \text{whenever} \quad d_X(x, p) < d$$

which shows that the function h is continuous at p. ❑

❖ *Example*

(i) $f(x) = (1 - x^2)$ is continuous $\forall$ $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ is continuous $\forall$ $x \in [0, \infty]$, then $g\big(f(x)\big) = \sqrt{1 - x^2}$ is continuous $x \in (-1, 1)$.

(ii) Let $g(x) = \sin x$ and $f(x) = \begin{cases} x - p & , \ x \le 0 \\ x + p & , \ x > 0 \end{cases}$

Then $g\big(f(x)\big) = -\sin x$ $\forall$ x

Then the function $g\big(f(x)\big)$ is continuous at $x = 0$, although f is discontinuous at $x = 0$. ❑

❖ *Theorem*

Let f be defined on X. If f is continuous at $c \in X$ then $\exists$ a number $d > 0$ such that f is bounded on the open interval $(c - d, c + d)$.

Proof

Since f is continuous at $c \in X$.

Therefore for a real number $e > 0$, $\exists$ a real number $d > 0$ such that

$$\left| f(x) - f(c) \right| < e \quad \text{whenever} \quad x \in X \quad \text{and} \quad \left| x - c \right| < d .$$

$$\Rightarrow \left| f(x) \right| = \left| f(x) - f(c) + f(c) \right|$$

$$\le \left| f(x)-f(c) \right| - \left| f(c) \right|$$
$$< e + \left| f(c) \right| \quad \text{whenever} \quad \left| x-c \right| < d \,.$$

It shows that f is bounded on the open interval $\left]c-d, c+d\right[$. ❑

❖ Theorem

Suppose f is continuous on $[a,b]$. If $f(c)>0$ for some $c\in[a,b]$ then there exist an open interval $G\subset[a,b]$ such that $f(x)>0 \;\; \forall \;\; x\in G$.

Proof

Take $e=\dfrac{1}{2}f(c)$

$\because$ f is continuous on $[a,b]$

$\therefore$ $\left| f(x)-f(c) \right| < e$ whenever $\left| x-c \right| < d$, $x\in[a,b]$

Take $G=\left\{ x\in[a,b] : \left| x-c \right| < d \right\}$

$$\Rightarrow \left| f(x) \right| = \left| f(x)-f(c)+f(c) \right|$$
$$\le \left| f(x)-f(c) \right| + \left| f(c) \right|$$
$$< e + \left| f(c) \right| \quad \text{whenever} \quad \left| x-c \right| < d$$

For $x\in G$, we have

$$f(x) = f(c)-\big(f(c)-f(x)\big) \ge f(c)-\left| f(c)-f(x) \right|$$
$$\ge f(c)-\left| f(x)-f(c) \right| > f(c)-\frac{1}{2}f(c)$$
$$\Rightarrow f(x) > \frac{1}{2}f(c) > 0$$ ❑

❖ Example

Define a function f by

$$f(x)=\begin{cases} x\cos x & ; \;\; x\neq o \\ 0 & ; \;\; x=0 \end{cases}$$

This function is continuous at $x=0$ because

$$\left| f(x)-f(0) \right| = \left| x\cos x \right| \le \left| x \right| \qquad \left(\because \left|\cos x\right| \le 1 \right)$$

Which shows that for $e>0$, we can find $d>0$ such that

$$\left| f(x)-f(0) \right| < e \quad \text{whenever} \quad 0 < \left| x-c \right| < d = e$$ ❑

❖ Example

$f(x)=\sqrt{x}$ is continuous on $\left[0,\infty\right[$.

Let c be an arbitrary point such that $0 < c < \infty$

For $e>0$, we have

$$\left| f(x)-f(c) \right| = \left| \sqrt{x}-\sqrt{c} \right| = \frac{\left| x-c \right|}{\sqrt{x}+\sqrt{c}} < \frac{\left| x-c \right|}{\sqrt{c}}$$

$$\Rightarrow \left| f(x)-f(c) \right| < e \quad \text{whenever} \quad \frac{\left| x-c \right|}{\sqrt{c}} < e$$

i.e. $\left| x-c \right| < \sqrt{c}\, e = d$

$\Rightarrow$ f is continuous for $x=c$.

$\because$ c is an arbitrary point lying in $\left[0,\infty\right[$

$\therefore$ $f(x)=\sqrt{x}$ is continuous on $\left[0,\infty\right[$ ❑

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❖ *Example*

Consider the function f defined on $\mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & , \ x \text{ is rational} \\ -1 & , \ x \text{ is irrational} \end{cases}$$

This function is discontinuous every where but $|f(x)|$ is continuous on $\mathbb{R}$. ❑

❖ *Theorem*

A mapping of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Proof

Suppose f is continuous on X and V is open in Y.

We are to show that $f^{-1}(V)$ is open in X i.e. every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

Let $p \in X$ and $f(p) \in V$

$\because$ V is open

$\therefore$ $\exists$ $e > 0$ such that $y \in V$ if $d_Y(y, f(p)) < e$ …….. (i)

$\because$ f is continuous at p

$\therefore$ $\exists$ $d > 0$ such that $d_Y(f(x), f(p)) < e$ when $d_X(x, p) < d$ ……… (ii)

From (i) and (ii), we conclude that

$$x \in f^{-1}(V) \text{ as soon as } d_X(x, p) < d$$

Which shows that $f^{-1}(V)$ is open in X.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y.

We are to prove that f is continuous for this.

Fix $p \in X$ and $e > 0$.

Let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < e$

V is open, $f^{-1}(V)$ is open

$\Rightarrow$ $\exists$ $d > 0$ such that $x \in f^{-1}(V)$ as soon as $d_X(x, p) < d$.

But if $x \in f^{-1}(V)$ then $f(x) \in V$ so that $d_Y(f(x), f(y)) < e$

Which proves that f is continuous. ❑

Note

The above theorem can also be stated as a mapping $f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y. ❑

❖ *Theorem*

Let $f_1, f_2, f_3, \ldots, f_k$ be real valued functions on a metric space X and $\underline{f}$ be a mapping from X on to $\mathbb{R}^k$ defined by

$$\underline{f}(x) = (f_1(x), f_2(x), f_3(x), \ldots, f_k(x)) \ , \quad x \in X$$

then $\underline{f}$ is continuous on X if and only if $f_1, f_2, f_3, \ldots, f_k$ are continuous on X.

Proof

Let us suppose that the function $\underline{f}$ is continuous on X, we are to show that $f_1, f_2, f_3, \ldots, f_k$ are continuous on X.

If $p \in X$, then $d_{\mathbb{R}^k}(\underline{f}(x), \underline{f}(p)) < e$ whenever $d_X(x, p) < d$

$\Rightarrow$ $\|\underline{f}(x) - \underline{f}(p)\| < e$ whenever $\|x - p\| < d$

$\Rightarrow \| f_1(x)-f_1(p), f_1(x)-f_1(p),......f_k(x)-f_k(p)\| < e$ whenever $\|x-p\| < d$

$\Rightarrow \left[\left(f_1(x)-f_1(p)\right)^2, \left(f_2(x)-f_2(p)\right)^2, ..., \left(f_k(x)-f_k(p)\right)^2\right]^{1/2} < e$

whenever $\|x-p\| < d$

i.e. $\Rightarrow \left[\sum_{i=1}^{k}\left(f_i(x)-f_i(p)\right)^2\right]^{1/2} < e$ whenever $\|x-p\| < d$

$\Rightarrow \| f_1(x)-f_1(p)\| < e$ whenever $\|x-p\| < d$

$\| f_2(x)-f_2(p)\| < e$ whenever $\|x-p\| < d$

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$\| f_k(x)-f_k(x)\| < e$ whenever $\|x-p\| < d$

$\Rightarrow$ all the functions $f_1, f_2, f_3,....., f_k$ are continuous at p.

$\because$ p is arbitrary point of x, therefore $f_1, f_2, f_3,....., f_k$ are continuous on X .

Conversely, suppose that the function $f_1, f_2, f_3,....., f_k$ are continuous on X , we are to show that $\underline{f}$ is continuous on X .

For $p \in X$ and given $e_i > 0$, $i = 1,2,.....k$ $\exists$ $d_i > 0$, $i = 1,2,....,k$

Such that

$\| f_1(x)-f_1(p)\| < e_1$ whenever $\|x-p\| < d_1$

$\| f_2(x)-f_2(p)\| < e_2$ whenever $\|x-p\| < d_2$

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$\| f_k(x)-f_k(x)\| < e_k$ whenever $\|x-p\| < d_k$

Take $d = \min(d_1, d_2, d_3,....,d_k)$ then

$\| f_i(x)-f_i(p)\| < e_i$ whenever $\|x-p\| < d$

$\Rightarrow \left[\left(f_1(x)-f_1(p)\right)^2 + \left(f_2(x)-f_2(p)\right)^2 + ... + \left(f_k(x)-f_k(p)\right)^2\right]^{1/2} < \left(e_1^2 + e_2^2 + ... + e_k^2\right)^{1/2}$

i.e. $\Rightarrow \left[\left(f_1(x)-f_1(p)\right)^2 + \left(f_2(x)-f_2(p)\right)^2 + ... + \left(f_k(x)-f_k(p)\right)^2\right]^{1/2} < e$

whenever $\|x-p\| < d$

where $\left(e_1^2 + e_2^2 + ... + e_k^2\right)^{1/2} = e$

Then $d_{\mathbb{R}^k}\left(\underline{f}(x), \underline{f}(p)\right) < e$ whenever $d_X(x,p) < d$

$\Rightarrow$ $\underline{f}(x)$ is continuous at p.

$\because$ p is an arbitrary point therefore we conclude that $\underline{f}$ is continuous on X . ❑

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❖ *Theorem*

Suppose f is continuous on $[a,b]$

i) If $f(a)<0$ and $f(b)>0$ then there is a point c, $a<c<b$ such that $f(c)=0$.

ii) If $f(a)>0$ and $f(b)<0$, then there is a point c, $a<c<b$ such that $f(c)=0$.

Proof

i) Bisect $[a,b]$ then f must satisfy the given condition on at least one of the sub-interval so obtained. Denote this interval by $[a_2,b_2]$

If f satisfies the condition on both sub-interval then choose the right hand one $[a_2,b_2]$.

It is obvious that $a\le a_2\le b_2\le b$. By repeated bisection we can find nested intervals $\{I_n\}$, $I_{n+1}\subseteq I_n$, $I_n=[a_n,b_n]$ so that f satisfies the given condition on $[a_n,b_n]$, $n=1,2,.......$

And $a=a_1\le a_2\le a_3\le.....\le a_n\le b_n\le.....\le b_2\le b_1=b$

Where $b_n-a_n=\left(\frac{1}{2}\right)^n(b-a)$

Then $\bigcap_{i=1}^{n} I_n$ contain one and only one point. Let that point be c such that $f(c)=0$

If $f(c)\ne 0$, let $f(c)>0$ then there is a subinterval $[a_m,b_m]$ such that $a_m<b_m<c$ Which can not happen. Hence $f(c)=0$

ii) *Do yourself as above* ❑

❖ *Example*

Show that $x^3-2x^2-3x+1=0$ has a solution $c\in[-1,1]$

Solution

Let $f(x)=x^3-2x^2-3x+1$

$\because$ $f(x)$ is polynomial

$\therefore$ it is continuous everywhere. (for being a polynomial continuous everywhere)

Now $f(-1)=(-1)^3-2(-1)^2-3(-1)+1$

$=-1-2+3+1=1>0$

$f(1)=(1)^3-2(1)^2-3(1)+1$

$=1-2-3+1=-3<0$

Therefore there is a point $c\in[-1,1]$ such that $f(c)=0$

i.e. c is the root of the equation. ❑

❖ *Theorem (The intermediate value theorem)*

Suppose f is continuous on $[a,b]$ and $f(a)\ne f(b)$, then given a number l that lies between $f(a)$ and $f(b)$, $\exists$ a point c , $a<c<b$ with $f(c)=l$.

Proof

Let $f(a)<f(b)$ and $f(a)<l<f(b)$.

Suppose $g(x)=f(x)-l$

Then $g(a)=f(a)-l<0$ and $g(b)=f(b)-l>0$

$\Rightarrow$ $\exists$ a point c between a and b such that $g(c)=0$

$\Rightarrow$ $f(c)-l=0$ $\Rightarrow$ $f(c)=l$

If $f(a)>f(b)$ then take $g(x)=l-f(x)$ to obtain the required result. ❑

❖ *Theorem*

Suppose f is continuous on $[a,b]$, then f is bounded on $[a,b]$

(Continuity implies boundedness)

Proof

Suppose that f is not bounded on $[a,b]$,

We can, therefore, find a sequence $\{x_n\}$ in the interval $[a,b]$ such that

$$f(x_n) > n \quad \text{for all} \quad n \geq 1.$$

$\Rightarrow \{f(x_n)\}$ diverges.

But $a \leq x_n \leq b$; $n \geq 1$

$\Rightarrow \exists$ a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges to l .

$\Rightarrow \{f(x_{n_k})\}$ also converges to l .

$\Rightarrow \{f(x_n)\}$ converges to l .

Which is contradiction

Hence our supposition is wrong. ❑

❖ *Uniform continuity*

Let f be a mapping of a metric space X into a metric space Y. We say that f is uniformly continuous on X if for every $e > 0$ there exists $d > 0$ such that

$$d_Y\left(f(p), f(q)\right) < e \quad \forall \quad p,q \in X \quad \text{for which} \quad d_x\left(p,q\right) < d$$

The uniform continuity is a property of a function on a set i.e. it is a global property but continuity can be defined at a single point i.e. it is a local property. Uniform continuity of a function at a point has no meaning.

If f is continuous on X then it is possible to find for each $e > 0$ and for each point p of X, a number $d > 0$ such that $d_Y\left(f(x), f(p)\right) < e$ whenever $d_X(x,p) < d$. Then number d depends upon e and on p in this case but if f is uniformly continuous on X then it is possible for each $e > 0$ to find one number $d > 0$ which will do for all point p of X.

It is evident that every uniformly continuous function is continuous.

To emphasize a difference between continuity and uniform continuity on set S, we consider the following examples. ❑

❖ *Example*

Let S be a half open interval $0 < x \leq 1$ and let f be defined for each x in S by the formula $f(x) = x^2$. It is uniformly continuous on S. To prove this observe that we have

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x-y||x+y| \\ &< 2|x-y| \end{aligned}$$

If $|x-y| < d$ then $|f(x) - f(y)| < 2d = e$

Hence if e is given we need only to take $d = \dfrac{e}{2}$ to guarantee that

$$|f(x) - f(y)| < e \quad \text{for every pair} \quad x, y \quad \text{with} \quad |x-y| < d$$

Thus f is uniformly continuous on the set S. ❑

❖ ***Example***

$f(x) = x^n$, $n \geq 0$ is uniformly continuous of $[0,1]$

Solution

For any two values x_1, x_2 in $[0,1]$ we have

$$\left|x_1^n - x_2^n\right| = \left|(x_1 - x_2)\left(x_1^{n-1} + x_1^{n-2}x_2 + x_1^{n-3}x_2^2 + \dots + x_2^{n-1}\right)\right|$$
$$\leq n\left|x_1 - x_2\right|$$

Given $e > 0$, we can find $d = \frac{e}{n}$ independent of x_1 and x_2 such that

$$\left|x_1^2 - x_2^2\right| < n\left|x_1 - x_2\right| < e \quad \text{whenever} \quad x_1, x_2 \in [0,1] \quad \text{and} \quad \left|x_1 - x_2\right| < d = \frac{e}{n}$$

Hence the function f is uniformly continuous on $[0,1]$. ❑

❖ ***Example***

Let S be the half open interval $0 < x \leq 1$ and let a function f be defined for each x in S by the formula $f(x) = \frac{1}{x}$. This function is continuous on the set S, however we shall prove that this function is not uniformly continuous on S.

Solution

Let suppose $e = 10$ and suppose we can find a d , $0 < d < 1$, to satisfy the condition of the definition.

Taking $x = d$, $y = \frac{d}{11}$, we obtain

$$\left|x - y\right| = \frac{10d}{11} < d$$

and

$$\left|f(x) - f(y)\right| = \left|\frac{1}{d} - \frac{11}{d}\right| = \frac{10}{d} > 10$$

Hence for these two points we have $\left|f(x) - f(y)\right| > 10$ (always)

Which contradict the definition of uniform continuity.

Hence the given function being continuous on a set S is not uniformly continuous on S. ❑

❖ ***Example***

$f(x) = \sin\frac{1}{x}$; $x \neq 0$. is not uniformly continuous on $0 < x \leq 1$ i.e $(0,1]$.

Proof

Suppose that f is uniformly continuous on the given interval then for $e = 1$, there is $d > 0$ such that

$$\left|f(x_1) - f(x_2)\right| < 1 \quad \text{whenever} \quad \left|x_1 - x_2\right| < d$$

Take $x_1 = \frac{1}{(n - \frac{1}{2})p}$ and $x_2 = \frac{1}{3(n - \frac{1}{2})p}$, $n \geq 1$.

So that $\left|x_1 - x_2\right| < d = \frac{2}{3(n - \frac{1}{2})p}$

But $\left|f(x_1) - f(x_2)\right| = \left|\sin(n - \frac{1}{2})p - \sin 3(n - \frac{1}{2})p\right| = 2 > 1$

Which contradict the assumption.

Hence f is not uniformly continuous on the interval. ❑

❖ *Example*

Prove that $f(x)=\sqrt{x}$ is uniformly continuous on $[0,1]$.

Solution

Suppose $e=1$ and suppose we can find d, $0<d<1$ to satisfy the condition of the definition.

Taking $x=d^2$, $y=\frac{d^2}{4}$

Then $|x-y| = d^2-\frac{d^2}{4} = \frac{3d^2}{4} < d$

And $|f(x)-f(y)| = \left|\sqrt{d^2}-\sqrt{\frac{d^2}{4}}\right|$

$$= \left|d-\frac{d}{2}\right| = \left|\frac{d}{2}\right| < 1=e$$

Hence f is uniformly continuous on $[0,1]$. ❑

❖ *Theorem*

If f is continuous on a closed and bounded interval $[a,b]$, then f is uniformly continuous on $[a,b]$.

Proof

Suppose that f is not uniformly continuous on $[a,b]$ then $\exists$ a real number $e>0$ such that for every real number $d>0$.

We can find a pair u, v satisfying

$$|u-v|<d \quad \text{but} \quad |f(u)-f(v)| \geq e > 0$$

If $d=\frac{1}{n}$, $n=1,2,3,....$

We can determine two sequence $\{u_n\}$ and $\{v_n\}$ such that

$$|u_n-v_n| < \frac{1}{n} \quad \text{but} \quad |f(u_n)-f(v_n)| \geq e$$

$\because$ $a \leq u_n \leq b$ $\forall$ $n=1,2,3.......$

$\therefore$ there is a subsequence $\{u_{n_k}\}$ which converges to some number u_0 in $[a,b]$

$\Rightarrow$ for some $l>0$, we can find an integer n_0 such that

$$|u_{n_k}-u_0|< l \quad \forall \quad n \geq n_0$$

$$\Rightarrow |v_{n_k}-u_0| \leq |v_{n_k}-u_{n_k}|+|u_{n_k}-u_0| < \frac{1}{n}+l$$

$\Rightarrow$ $\{v_{n_k}\}$ also converges to u_0.

$\Rightarrow$ $\{f(u_{n_k})\}$ and $\{f(v_{n_k})\}$ converge to $f(u_0)$.

Consequently, $|f(u_{n_k})-f(v_{n_k})| < e$ whenever $|u_{n_k}-v_{n_k}| < e$

Which contradict our supposition.

Hence we conclude that f is uniformly continuous on $[a,b]$. ❑

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❖ Theorem

Let $\underline{f}$ and $\underline{g}$ be two continuous mappings from a metric space X into $\mathbb{R}^k$, then the mappings $\underline{f}+\underline{g}$ and $\underline{f}\cdot\underline{g}$ are also continuous on X.

i.e. the sum and product of two continuous vector valued function are also continuous.

Proof

$i)$ $\because$ $\underline{f}$ & $\underline{g}$ are continuous on X.

$\therefore$ by the definition of continuity, we have for a point $p\in X$.

$$\left\|\underline{f}(x)-\underline{f}(p)\right\| < \frac{e}{2} \quad \text{whenever} \quad \left\|x-p\right\| < d_1$$

and $$\left\|\underline{g}(x)-\underline{g}(p)\right\| < \frac{e}{2} \quad \text{whenever} \quad \left\|x-p\right\| < d_2$$

Now consider

$$\begin{aligned}
&\left\|\underline{f}(x)+\underline{g}(x)-\underline{f}(x)-\underline{g}(p)\right\| \\
&=\left\|\underline{f}(x)-\underline{f}(p)+\underline{g}(x)-\underline{g}(p)\right\| \\
&\leq\left\|\underline{f}(x)-\underline{f}(p)\right\|+\left\|\underline{g}(x)-\underline{g}(p)\right\| \\
&<\frac{e}{2}+\frac{e}{2}=e \quad \text{whenever} \quad \left\|x-p\right\|<d \quad \text{where } d=\min\left(d_1,d_2\right)
\end{aligned}$$

which shows that the vector valued function $\underline{f}+\underline{g}$ is continuous at $x=p$ and hence on X.

$ii)$ $$\underline{f}\cdot\underline{g} = \sum_{i=1}^{k} f_i\cdot g_i$$

$$= f_1g_1+f_2g_2+f_3g_3+.....+f_kg_k$$

$\because$ the function $\underline{f}$ and $\underline{g}$ are continuous on X

$\therefore$ their components f_i and g_i are continuous on X. ❑

❖ Question

Suppose f is a real valued function define on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}\left[f(x+h)-f(x-h)\right] = 0 \quad \forall \ x\in\mathbb{R}$$

Does this imply that the function f is continuous on $\mathbb{R}$.

Solution

$$\because \lim_{h\to 0}\left[f(x+h)-f(x-h)\right]=0 \quad \forall \ x\in\mathbb{R}$$

$$\Rightarrow \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h)$$

$$\Rightarrow f(x+0) = f(x-0) \ \forall \ x\in\mathbb{R}$$

Also it is given that $f(x) = f(x+0) = f(x-0)$

It means f is continuous on $x\in\mathbb{R}$. ❑

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❖ Discontinuities

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is discontinuous at x or that f has a discontinuity at x.

If the function f is defined on an interval, the discontinuity is divided into two types

1. Let f be defined on (a,b). If f is discontinuous at a point x and if $f(x+)$ and $f(x-)$ exist then f is said to have a discontinuity of first kind or a simple discontinuity at x.
2. Otherwise the discontinuity is said to be second kind.

For simple discontinuity

i. either $f(x+) \neq f(x-)$ [$f(x)$ is immaterial]

ii. or $f(x+) = f(x-) \neq f(x)$ ❑

❖ Example

i) Define $f(x) = \begin{cases} 1 & , x \text{ is rational} \\ 0 & , x \text{ is irrational} \end{cases}$

The function f has discontinuity of second kind on every point x because neither $f(x+)$ nor $f(x-)$ exists. ❑

ii) Define $f(x) = \begin{cases} x & , x \text{ is rational} \\ 0 & , x \text{ is irrational} \end{cases}$

Then f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point. ❑

iii) Define $f(x) = \begin{cases} x+2 & (-3 < x < -2) \\ -x-2 & (-2 < x < 0) \\ x+2 & (0 < x < 1) \end{cases}$

The function has simple discontinuity at $x = 0$ and it is continuous at every other point of the interval $(-3,1)$ ❑

iv) Define $f(x) = \begin{cases} \sin\dfrac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

$\because$ neither $f(0+)$ nor $f(0-)$ exists, therefore the function f has discontinuity of second kind.

f is continuous at every point except $x = 0$. ❑

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