# Chapter 2 – Sequences and Series

**Subject:** *Real Analysis* **Level:** *M.Sc.*
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### Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Or it can also be defined as an ordered set.

### Notation:

An infinite sequence is denoted as

$\{S_n\}_{n=1}^{\infty}$ or $\{S_n : n\in\mathbb{N}\}$ or $\{S_1, S_2, S_3, ..........\}$ or simply as $\{S_n\}$

e.g. i) $\{n\} = \{1, 2, 3, .........\}$

ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, ..............\right\}$

iii) $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, ............\}$

### Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{S_n\}_{n=1}^{\infty}$ is usually written as $\{S_{n_k}\}^{\infty}$.

### Increasing Sequence

A sequence $\{S_n\}$ is said to be an increasing sequence if $S_{n+1} \geq S_n \quad \forall\ n\geq 1$.

### Decreasing Sequence

A sequence $\{S_n\}$ is said to be an decreasing sequence if $S_{n+1} \leq S_n \quad \forall\ n\geq 1$.

### Monotonic Sequence

A sequence $\{S_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.

$\{S_n\}$ is monotonically increasing if $S_{n+1} - S_n \geq 0$ or $\frac{S_{n+1}}{S_n} \geq 1, \ \forall\ n\geq 1$

$\{S_n\}$ is monotonically decreasing if $S_n - S_{n+1} \geq 0$ or $\frac{S_n}{S_{n+1}} \geq 1, \ \forall\ n\geq 1$

### Strictly Increasing or Decreasing

$\{S_n\}$ is called strictly increasing or decreasing according as

$$S_{n+1} > S_n \quad \text{or} \quad S_{n+1} < S_n \qquad \forall\ n\geq 1.$$

### Bernoulli's Inequality

Let $p\in\mathbb{R}$, $p\geq -1$ and $p\neq 0$ then for $n\geq 2$ we have

$$(1+p)^n > 1+np$$

### Proof:

We shall use mathematical induction to prove this inequality.

If $n = 2$

$L.H.S = (1+p)^2 = 1+2p+p^2$

$R.H.S = 1+2p$

$\Rightarrow\ L.H.S > R.H.S$

i.e. condition *I* of mathematical induction is satisfied.

Suppose $(1+p)^k > 1+kp$ .................. $(i)$ where $k \geq 2$

Now $(1+p)^{k+1} = (1+p)(1+p)^k$

$> (1+p)(1+kp)$ using $(i)$

$= 1+kp+p+kp^2$

$= 1+(k+1)p+kp^2$

$\geq 1+(k+1)p$ ignoring $kp^2 \geq 0$

$\Rightarrow (1+p)^{k+1} > 1+(k+1)p$

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore condition *II* of mathematical induction is satisfied. Hence we conclude that $(1+p)^n > 1+np$.

## *Example*

Let $S_n = \left(1+\frac{1}{n}\right)^n$ where $n \geq 1$

To prove that this sequence is an increasing sequence, we use $p = \frac{-1}{n^2}$, $n \geq 2$ in Bernoulli's inequality to have

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{n}{n^2}$$

$$\Rightarrow \left(\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right)^n > 1-\frac{1}{n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1} = \left(1+\frac{1}{n-1}\right)^{n-1}$$

$$\Rightarrow S_n > S_{n-1} \qquad \forall \; n \geq 1$$

which shows that $\{S_n\}$ is increasing sequence.

## *Example*

Let $t_n = \left(1+\frac{1}{n}\right)^{n+1}$ ; $n \geq 1$

then the sequence is decreasing sequence.

We use $p = \frac{1}{n^2-1}$ in Bernoulli's inequality.

$$\left(1+\frac{1}{n^2-1}\right)^n > 1+\frac{n}{n^2-1} \;\ldots\ldots\ldots\; (i)$$

where

$$1+\frac{1}{n^2-1} = \frac{n^2}{n^2-1} = \left(\frac{n}{n-1}\right)\left(\frac{n}{n+1}\right)$$

$$\Rightarrow \left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right) = \left(\frac{n}{n-1}\right) \;\ldots\ldots\ldots\ldots\; (ii)$$

Now $t_{n-1} = \left(1+\frac{1}{n-1}\right)^n = \left(\frac{n}{n-1}\right)^n$

$$= \left(\left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right)\right)^n \qquad \text{from } (ii)$$

$$=\left(1+\frac{1}{n^2-1}\right)^n\left(\frac{n+1}{n}\right)^n$$

$$>\left(1+\frac{n}{n^2-1}\right)\left(\frac{n+1}{n}\right)^n \qquad \text{from } (i)$$

$$>\left(1+\frac{1}{n}\right)\left(\frac{n+1}{n}\right)^n \qquad \because \frac{n}{n^2-1}>\frac{n}{n^2}=\frac{1}{n}$$

$$=\left(\frac{n+1}{n}\right)^{n+1}=t_n$$

i.e. $t_{n-1}>t_n$

Hence the given sequence is decreasing sequence.

## Bounded Sequence

A sequence $\{S_n\}$ is said to be bounded if there exists a positive real number $l$ such that $|S_n|<l \quad \forall\ n\in\mathbb{N}$

If $S$ and $s$ are the supremum and infimum of elements forming the bounded sequence $\{S_n\}$ we write $S=\sup S_n$ and $s=\inf S_n$

All the elements of the sequence $S_n$ such that $|S_n|<l \quad \forall\ n\in\mathbb{N}$ lie with in the strip $\{y: -l<y<l\}$. But the elements of the unbounded sequence can not be contained in any strip of a finite width.

## Examples

***(i)*** $\{U_n\}=\left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence

***(ii)*** $\{V_n\}=\{\sin nx\}$ is also bounded sequence. Its supremum is 1 and infimum is $-1$.

***(iii)*** The geometric sequence $\{ar^{n-1}\}$, $r>1$ is an unbounded above sequence. It is bounded below by $a$.

***(iv)*** $\left\{\tan\frac{np}{2}\right\}$ is an unbounded sequence.

## Convergence of the Sequence

A sequence $\{S_n\}$ of real numbers is said to convergent to limit '$s$' as $n\to\infty$, if for every positive real number $e>0$, however small, there exists a positive integer $n_0$, depending upon $e$, such that $|S_n-s|<e \quad \forall\ n>n_0$.

## Theorem

A convergent sequence of real number has one and only one limit (i.e. Limit of the sequence is unique.)

### Proof:

Suppose $\{S_n\}$ converges to two limits $s$ and $t$, where $s\neq t$.

Put $e=\frac{|s-t|}{2}$ then there exits two positive integers $n_1$ and $n_2$ such that

$$|S_n-s|<e \qquad \forall\ n>n_1$$

and $$|S_n-t|<e \qquad \forall\ n>n_2$$

$\Rightarrow$ $|S_n-s|<e$ and $|S_n-t|<e$ hold simultaneously $\forall\ n>\max(n_1,n_2)$.

Thus for all $n>\max(n_1,n_2)$ we have

$$|s-t|=|s-S_n+S_n-t|$$

$$\leq |S_n - s| + |S_n - t|$$
$$< e + e = 2e$$
$$\Rightarrow |s-t| < 2\left(\frac{|s-t|}{2}\right)$$
$$\Rightarrow |s-t| < |s-t|$$

Which is impossible, therefore the limit of the sequence is unique.

***Note:*** If $\{S_n\}$ converges to $s$ then all of its infinite subsequence converge to $s$.

## Cauchy Sequence

A sequence $\{x_n\}$ of real number is said to be a *Cauchy sequence* if for given positive real number $e$, $\exists$ a positive integer $n_0(e)$ such that

$$|x_n - x_m| < e \qquad \forall\ m, n > n_0$$

## Theorem

A Cauchy sequence of real numbers is bounded.

### Proof

Let $\{S_n\}$ be a Cauchy sequence.

Take $e = 1$, then there exits a positive integers $n_0$ such that

$$|S_n - S_m| < 1 \qquad \forall\ m, n > n_0.$$

Fix $m = n_0 + 1$ then

$$|S_n| = |S_n - S_{n_0+1} + S_{n_0+1}|$$
$$\leq |S_n - S_{n_0+1}| + |S_{n_0+1}|$$
$$< 1 + |S_{n_0+1}| \qquad \forall\ n > n_0$$
$$< l \qquad \forall\ n > 1 \text{ , and } l = 1 + |S_{n_0+1}| \quad (n_0 \text{ changes as } e \text{ changes})$$

Hence we conclude that $\{S_n\}$ is a Cauchy sequence, which is bounded one.

***Note:***

(i) Convergent sequence is bounded.

(ii) The converse of the above theorem does not hold.
i.e. every bounded sequence is not Cauchy.

Consider the sequence $\{S_n\}$ where $S_n = (-1)^n$, $n \geq 1$. It is bounded sequence because

$$|(-1)^n| = 1 < 2 \qquad \forall\ n \geq 1$$

But it is not a Cauchy sequence if it is then for $e = 1$ we should be able to find a positive integer $n_0$ such that $|S_n - S_m| < 1$ for all $m, n > n_0$

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$|S_n - S_m| = |(-1)^{2n+2} - (-1)^{2k+1}|$$
$$= |1+1| = 2 < 1 \quad \text{is absurd.}$$

Hence $\{S_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence. (it is an oscillatory sequence)

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## Divergent Sequence

A $\{S_n\}$ is said to be divergent if it is not convergent or it is unbounded.

e.g. $\{n^2\}$ is divergent, it is unbounded.

***(ii)*** $\{(-1)^n\}$ tends to 1 or -1 according as $n$ is even or odd. It oscillates finitely.

***(iii)*** $\{(-1)^n n\}$ is a divergent sequence. It oscillates infinitely.

***Note:*** If two subsequence of a sequence converges to two different limits then the sequence itself is a divergent.

## Theorem

If $S_n < U_n < t_n \quad \forall \; n \geq n_0$ and if both the $\{S_n\}$ and $\{t_n\}$ converge to same limits as $s$, then the sequence $\{U_n\}$ also converges to $s$.

### Proof

Since the sequence $\{S_n\}$ and $\{t_n\}$ converge to the same limit $s$, therefore, for given $e > 0$ there exists two positive integers $n_1,\, n_2 > n_0$ such that

$$|S_n - s| < e \qquad \forall \; n > n_1$$

$$|t_n - s| < e \qquad \forall \; n > n_2$$

i.e. $$s - e < S_n < s + e \qquad \forall \; n > n_1$$

$$s - e < t_n < s + e \qquad \forall \; n > n_2$$

Since we have given

$$S_n < U_n < t_n \qquad \forall \; n > n_0$$

$$\therefore \; s - e < S_n < U_n < t_n < s + e \qquad \forall \; n > \max(n_0, n_1, n_2)$$

$$\Rightarrow \; s - e < U_n < s + e \qquad \forall \; n > \max(n_0, n_1, n_2)$$

i.e. $$|U_n - s| < e \qquad \forall \; n > \max(n_0, n_1, n_2)$$

i.e. $$\lim_{n\to\infty} U_n = s$$

## Example

Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

## Solution

Using Bernoulli's Inequality

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \frac{n}{\sqrt{n}} \geq \sqrt{n} \geq 1 \qquad \forall \; n.$$

Also

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 = \left[\left(1 + \frac{1}{\sqrt{n}}\right)^n\right]^{\frac{2}{n}} > \left(\sqrt{n}\right)^{\frac{2}{n}} > n^{\frac{1}{n}} \geq 1$$

$$\Rightarrow \; 1 \leq n^{\frac{1}{n}} < \left(1 + \frac{1}{\sqrt{n}}\right)^2$$

$$\Rightarrow \; \lim_{n\to\infty} 1 \leq \lim_{n\to\infty} n^{\frac{1}{n}} < \lim_{n\to\infty}\left(1 + \frac{1}{\sqrt{n}}\right)^2$$

$$\Rightarrow \; 1 \leq \lim_{n\to\infty} n^{\frac{1}{n}} < 1$$

i.e. $$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

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### *Example*

Show that $\lim_{n\to\infty}\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots\ldots\ldots\ldots+\frac{1}{(2n)^2}\right)=0$

### *Solution*

We have

$$S_n=\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots\ldots\ldots\ldots+\frac{1}{(2n)^2}\right)$$

and

$$\frac{n}{(2n)^2} < S_n < \frac{n}{n^2}$$

$$\Rightarrow \frac{1}{4n} < S_n < \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{4n} < \lim_{n\to\infty}S_n < \lim_{n\to\infty}\frac{1}{n}$$

$$\Rightarrow 0 < \lim_{n\to\infty}S_n < 0$$

$$\Rightarrow \lim_{n\to\infty}S_n=0$$

### *Theorem*

If the sequence $\{S_n\}$ converges to $s$ then $\exists$ a positive integer $n$ such that $|S_n|>\frac{1}{2}s$.

### *Proof*

We fix $e=\frac{1}{2}|s|>0$

$\Rightarrow \exists$ a positive integer $n_1$ such that

$$|S_n-s|<e \qquad \text{for } n>n_1$$

$$\Rightarrow |S_n-s|<\frac{1}{2}|s|$$

Now

$$\frac{1}{2}|s|=|s|-\frac{1}{2}|s|$$

$$<|s|-|S_n-s| \le |s+(S_n-s)|$$

$$\Rightarrow \frac{1}{2}|s|<|S_n|$$

### *Theorem*

Let $a$ and $b$ be fixed real numbers if $\{S_n\}$ and $\{t_n\}$ converge to $s$ and $t$ respectively, then

(*i*) $\{aS_n+bt_n\}$ converges to $as + bt$.

(*ii*) $\{S_n t_n\}$ converges to $st$.

(*iii*) $\left\{\frac{S_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n\neq 0\ \forall\ n$ and $t\neq 0$.

### *Proof*

Since $\{S_n\}$ and $\{t_n\}$ converge to $s$ and $t$ respectively,

$$\therefore\ |S_n-s|<e \qquad \forall\ n>n_1\in\mathbb{N}$$

$$\left|t_n - t\right| < e \qquad \forall\ n > n_2 \in \mathbb{N}$$

Also $\exists\ l > 0$ such that $\left|S_n\right| < l \quad \forall\ n > 1$ $\qquad$ ( $\because\ \{S_n\}$ is bounded )

($i$) We have

$$\begin{aligned}\left|\left(aS_n + bt_n\right) - \left(as + bt\right)\right| &= \left|a\left(S_n - s\right) + b\left(t_n - t\right)\right| \\ &\leq \left|a\left(S_n - s\right)\right| + \left|b\left(t_n - t\right)\right| \\ &< \left|a\right|e + \left|b\right|e \qquad \forall\ n > \max(n_1, n_2) \\ &= e_1 \qquad \text{Where } e_1 = \left|a\right|e + \left|b\right|e \text{ a certain number.}\end{aligned}$$

This implies $\{aS_n + bt_n\}$ converges to *as* + *bt*.

($ii$) $$\begin{aligned}\left|S_n t_n - st\right| &= \left|S_n t_n - S_n t + S_n t - st\right| \\ &= \left|S_n\left(t_n - t\right) + t\left(S_n - s\right)\right| \leq \left|S_n\right| \cdot \left|\left(t_n - t\right)\right| + \left|t\right| \cdot \left|\left(S_n - s\right)\right| \\ &< l\,e + \left|t\right|e \qquad \forall\ n > \max(n_1, n_2) \\ &= e_2 \qquad \text{where } e_2 = l\,e + \left|t\right|e \text{ a certain number.}\end{aligned}$$

This implies $\{S_n t_n\}$ converges to *st.*

($iii$) $$\begin{aligned}\left|\frac{1}{t_n} - \frac{1}{t}\right| &= \left|\frac{t - t_n}{t_n t}\right| \\ &= \frac{\left|t_n - t\right|}{\left|t_n\right|\left|t\right|} < \frac{e}{\frac{1}{2}\left|t\right|\left|t\right|} \qquad \forall\ n > \max(n_1, n_2) \qquad \because\ \left|t_n\right| > \frac{1}{2}t \\ &= \frac{e}{\frac{1}{2}\left|t\right|^2} = e_3 \qquad \text{where } e_3 = \frac{e}{\frac{1}{2}\left|t\right|^2} \text{ a certain number.}\end{aligned}$$

This implies $\left\{\frac{1}{t_n}\right\}$ converges to $\frac{1}{t}$.

Hence $\left\{\frac{S_n}{t_n}\right\} = \left\{S_n \cdot \frac{1}{t_n}\right\}$ converges to $s \cdot \frac{1}{t} = \frac{s}{t}$. $\quad$ ( from (ii) )

## *Theorem*

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim\limits_{n\to\infty} r_n = x$.

### *Proof*

Since *x* and *x* + 1 are two different real numbers

$\because\ \exists$ a rational number $r_1$ such that

$$x < r_1 < x + 1$$

Similarly $\exists$ a rational number $r_2 \neq r_1$ such that

$$x < r_2 < \min\left(r_1, x + \frac{1}{2}\right) < x + 1$$

Continuing in this manner we have

$$x < r_3 < \min\left(r_2, x + \frac{1}{3}\right) < x + 1$$

$$x < r_4 < \min\left(r_3, x + \frac{1}{4}\right) < x + 1$$

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$$x < r_n < \min\left(r_{n-1}, x + \frac{1}{n}\right) < x + 1$$

This implies that $\exists$ a sequence $\{r_n\}$ of the distinct rational number such that

$$x-\frac{1}{n}<x<r_n<x+\frac{1}{n}$$

Since

$$\lim_{n\to\infty}\left(x-\frac{1}{n}\right)=\lim_{n\to\infty}\left(x+\frac{1}{n}\right)=x$$

Therefore

$$\lim_{n\to\infty} r_n = x$$

### Theorem

Let a sequence $\{S_n\}$ be a bounded sequence.

($i$) If $\{S_n\}$ is monotonically increasing then it converges to its supremum.

($ii$) If $\{S_n\}$ is monotonically decreasing then it converges to its infimum.

### Proof

Let $S=\sup S_n$ and $s=\inf S_n$

Take $e>0$

($i$) Since $S=\sup S_n$

$\therefore\ \exists\ S_{n_0}$ such that $S-e<S_{n_0}$

Since $\{S_n\}$ is $\uparrow$ ( $\uparrow$ stands for monotonically increasing )

$\therefore\ S-e<S_{n_0}<S_n<S<S+e$ for $n>n_0$

$\Rightarrow\ S-e<S_n<S+e$ for $n>n_0$

$\Rightarrow\ |S_n-S|<e$ for $n>n_0$

$\Rightarrow\ \lim_{n\to\infty} S_n = S$

($ii$) Since $s=\inf S_n$

$\therefore\ \exists\ S_{n_1}$ such that $S_{n_1}<s+e$

Since $\{S_n\}$ is $\downarrow$. ( $\downarrow$ stands for monotonically decreasing )

$\therefore\ s-e<s<S_n<S_{n_1}<s+e$ for $n>n_1$

$\Rightarrow\ s-e<S_n<s+e$ for $n>n_1$

$\Rightarrow\ |S_n-s|<e$ for $n>n_1$

Thus $\lim_{n\to\infty} S_n = s$

### Note

A monotonic sequence can not oscillate infinitely.

### Example:

Consider $\{S_n\}=\left\{\left(1+\frac{1}{n}\right)^n\right\}$

As shown earlier it is an increasing sequence

Take $S_{2n}=\left(1+\frac{1}{2n}\right)^{2n}$

Then $\sqrt{S_{2n}}=\left(1+\frac{1}{2n}\right)^{n}$

$$\Rightarrow \frac{1}{\sqrt{S_{2n}}}=\left(\frac{2n}{2n+1}\right)^n \qquad \Rightarrow \frac{1}{\sqrt{S_{2n}}}=\left(1-\frac{1}{2n+1}\right)^n$$

Using Bernoulli's Inequality we have

$$\Rightarrow \frac{1}{\sqrt{S_{2n}}} \geq 1-\frac{n}{2n+1} > 1-\frac{n}{2n} = \frac{1}{2} \qquad \because \left(1-\frac{1}{2n+1}\right)^n \geq 1-\frac{n}{2n+1}$$

$$\Rightarrow \sqrt{S_{2n}} < 2 \qquad \forall\ n=1,2,3,\ldots\ldots\ldots$$

$$\Rightarrow S_{2n} < 4 \qquad \forall\ n=1,2,3,\ldots\ldots\ldots$$

$$\Rightarrow S_n < S_{2n} < 4 \qquad \forall\ n=1,2,3,\ldots\ldots\ldots$$

Which show that the sequence $\{S_n\}$ is bounded one.

Hence $\{S_n\}$ is a convergent sequence the number to which it converges is its supremum, which is denoted by '$e$' and $2<e<3$.

## Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

## Example

Let $t_1 > 0$ and let $\{t_n\}$ be defined by $t_{n+1} > 2-\dfrac{1}{t_n}$ ; $n \geq 1$

$$\Rightarrow t_n > 0 \qquad \forall\ n \geq 1$$

Also $\quad t_n - t_{n+1} = t_n - 2 + \dfrac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n-1)^2}{t_n} > 0$$

$$\Rightarrow t_n > t_{n+1} \qquad \forall\ n \geq 1$$

This implies that $t_n$ is monotonically decreasing.

Since $t_n > 1 \qquad \forall\ n \geq 1$

$\Rightarrow t_n$ is bounded below $\quad \Rightarrow t_n$ is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$

Then $\quad \lim\limits_{n\to\infty} t_{n+1} = \lim\limits_{n\to\infty} t_n$

$$\Rightarrow \lim_{n\to\infty}\left(2-\frac{1}{t_n}\right) = \lim_{n\to\infty} t_n$$

$$\Rightarrow 2-\frac{1}{t} = t \quad \Rightarrow \frac{2t-1}{t} = t \quad \Rightarrow 2t-1 = t^2 \quad \Rightarrow t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1$$

## Example

Let $\{S_n\}$ be defined by $S_{n+1} = \sqrt{S_n + b}$ ; $n \geq 1$ and $S_1 = a > b$.

It is clear that $S_n > 0\ \forall\ n \geq 1$ and $S_2 > S_1$ and

$$S_{n+1}^2 - S_n^2 = (S_n + b) - (S_{n-1} + b)$$

$$= S_n - S_{n-1}$$

$$\Rightarrow (S_{n+1} + S_n)(S_{n+1} - S_n) = S_n - S_{n-1}$$

$$\Rightarrow S_{n+1} - S_n = \frac{S_n - S_{n-1}}{S_{n+1} + S_n}$$

Since $S_{n+1} + S_n > 0 \quad \forall\ n \geq 1$

Therefore $S_{n+1} - S_n$ and $S_n - S_{n-1}$ have the same sign.

i.e. $S_{n+1} > S_n$ if and only if $S_n > S_{n-1}$ and

$S_{n+1} < S_n$ if and only if $S_n < S_{n-1}$.

But we know that $S_2 > S_1$ therefore $S_3 > S_2$, $S_4 > S_3$, and so on.
This implies the sequence is an increasing sequence.

Also $S_{n+1}^2 - S_n^2 = \left(\sqrt{S_n + b}\right)^2 - S_n^2 = S_n + b - S_n^2$

$$= -\left(S_n^2 - S_n - b\right)$$

Since $S_n > 0 \quad \forall\ n \ge 1$, therefore $S_n$ is the root (+ive) of the

$$S_n^2 - S_n - b = 0$$

Take this value of $S_n$ as $a$ where $a = \dfrac{1+\sqrt{1+4b}}{2}$

> *For equation* $ax^2 + bx + c = 0$
> *The product of roots is* $ab = {}^{c}\!/\!_{a}$
> *i.e. the other root* $b = \dfrac{c}{aa}$

the other root of equation is therefore $\dfrac{-b}{a}$

Since $S_{n+1} > S_n \ \forall\ n \ge 1$

Also $-\left(S_n - a\right)\left(S_n + \dfrac{b}{a}\right) = S_{n+1}^2 - S_n^2 > 0$

$$\therefore\ S_n + \frac{b}{a} > 0 \quad \text{or} \quad -\left(S_n - a\right) \ge 0$$

$$\Rightarrow\ S_n < a \qquad \forall\ n \ge 1$$

which shows that $S_n$ is bounded and hence it is convergent.

Suppose $\lim\limits_{n\to\infty} S_n = s$

Then $\lim\limits_{n\to\infty}(S_{n+1})^2 = \lim\limits_{n\to\infty}(S_n + b)$

$$\Rightarrow\ s^2 = s + b \quad \Rightarrow\ s^2 - s - b = 0$$

Which shows that $a = \dfrac{1+\sqrt{1+4b}}{2}$ is the limit of the sequence.

## *Theorem*

Every Cauchy sequence of real numbers has a convergent subsequence.

### *Proof*

Suppose $\{S_n\}$ is a Cauchy sequence.

Let $e > 0$ then $\exists$ a positive integer $n_0 \ge 1$ such that

$$\left|S_{n_k} - S_{n_{k-1}}\right| < \frac{e}{2^k} \qquad \forall\ n_k, n_{k-1},\ k = 1,2,3,.........$$

Put $b_k = \left(S_{n_1} - S_{n_0}\right) + \left(S_{n_2} - S_{n_1}\right) + ........... + \left(S_{n_k} - S_{n_{k-1}}\right)$

$$\Rightarrow\ \left|b_k\right| = \left|\left(S_{n_1} - S_{n_0}\right) + \left(S_{n_2} - S_{n_1}\right) + ........... + \left(S_{n_k} - S_{n_{k-1}}\right)\right|$$

$$\le \left|\left(S_{n_1} - S_{n_0}\right)\right| + \left|\left(S_{n_2} - S_{n_1}\right)\right| + ........... + \left|\left(S_{n_k} - S_{n_{k-1}}\right)\right|$$

$$< \frac{e}{2} + \frac{e}{2^2} + ........... + \frac{e}{2^k}$$

$$= e\left(\frac{1}{2} + \frac{1}{2^2} + ........... + \frac{1}{2^k}\right) = e\left(\frac{\frac{1}{2}\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\right) = e\left(1 - \frac{1}{2^k}\right)$$

$$\Rightarrow\ \left|b_k\right| < e \qquad \forall\ k \ge 1$$

$$\Rightarrow\ \{b_k\} \text{ is convergent}$$

$$\because\ b_k = S_{n_k} - S_{n_0} \quad \therefore\ S_{n_k} = b_k + S_{n_0}$$

Where $S_{n_0}$ is a certain fix number therefore $\{S_{n_k}\}$ which is a subsequence of $\{S_n\}$ is convergent.

**Theorem (Cauchy's General Principle for Convergence)**

A sequence of real number is convergent if and only if it is a Cauchy sequence.

**Proof**

**Necessary Condition**

Let $\{S_n\}$ be a convergent sequence, which converges to $s$.

Then for given $e>0$ $\exists$ a positive integer $n_0$, such that

$$\left|S_n - s\right| < \frac{e}{2} \quad \forall \ n > n_0$$

Now for $n > m > n_0$

$$\begin{aligned}\left|S_n - S_m\right| &= \left|S_n - s + S_m - s\right| \\ &\le \left|S_n - s\right| + \left|S_m - s\right| \\ &< \frac{e}{2} + \frac{e}{2} = e\end{aligned}$$

Which shows that $\{S_n\}$ is a Cauchy sequence.

**Sufficient Condition**

Let us suppose that $\{S_n\}$ is a Cauchy sequence then for $e>0$, $\exists$ a positive integer $m_1$ such that

$$\left|S_n - S_m\right| < \frac{e}{2} \qquad \forall \ n,m > m_1 \ \ldots\ldots\ldots.. (i)$$

Since $\{S_n\}$ is a Cauchy sequence

therefore it has a subsequence $\{S_{n_k}\}$ converging to $s$ (say).

$\Rightarrow$ $\exists$ a positive integer $m_2$ such that

$$\left|S_{n_k} - s\right| < \frac{e}{2} \qquad \forall \ n > m_2 \ \ldots\ldots\ldots.. (ii)$$

Now

$$\begin{aligned}\left|S_n - s\right| &= \left|S_n - S_{n_k} + S_{n_k} - s\right| \\ &\le \left|S_n - S_{n_k}\right| + \left|S_{n_k} - s\right| \\ &< \frac{e}{2} + \frac{e}{2} = e \qquad \forall \ n > \max(m_1, m_2)\end{aligned}$$

which shows that $\{S_n\}$ is a convergent sequence.

**Example**

Let $\{S_n\}$ be define by $0 < a < S_1 < S_2 < b$ and also

$$S_{n+1} = \sqrt{S_n \cdot S_{n-1}}\,, \quad n > 2 \ \ldots\ldots\ldots \text{(i)}$$

Here $S_n > 0$, $\forall \ n \ge 1$ and $a < S_1 < b$

Let for some $k > 2$

$$a < S_k < b$$

then $a^2 < a S_k < S_k S_{k-1} = (S_{k+1})^2 < b^2$ $\qquad \because \ S_{n+1} = \sqrt{S_n S_{n-1}}$

i.e. $a^2 < S_{k+1}^2 < b^2$

$\Rightarrow \ a < S_{k+1} < b$

$\Rightarrow \ a < S_n < b \quad \forall \ n \in \mathbb{N}$

$$\because \ \frac{S_n}{S_{n+1}} > \frac{a}{b}$$

$$\therefore \ \frac{S_n}{S_{n+1}} + 1 > \frac{a}{b} + 1$$

$$\Rightarrow \frac{S_n + S_{n+1}}{S_{n+1}} > \frac{a+b}{b}$$

$$\Rightarrow \frac{S_n + S_{n+1}}{S_n} > \frac{a+b}{b} \qquad S_{n+1} \text{ is replace by } S_n \quad \therefore\ S_n < S_{n+1}$$

And $\quad S_{n+1}^2 - S_n^2 = S_n \cdot S_{n-1} - S_n^2 \qquad \because\ S_{n+1} = \sqrt{S_n S_{n-1}}$

$$= S_n\left(S_{n-1} - S_n\right)$$

$$\Rightarrow \left|S_{n+1} - S_n\right| = \frac{S_n}{S_n + S_{n+1}}\left|S_{n-1} - S_n\right|$$

$$< \frac{b}{a+b}\left|S_{n-1} - S_n\right|$$

$$\Rightarrow \left|S_{n+1} - S_n\right| < \frac{b}{a+b}\left|S_n - S_{n-1}\right| \qquad \because\ \left|S_{n-1} - S_n\right| = \left|S_n - S_{n-1}\right|$$

$$< \left(\frac{b}{a+b}\right)^2 \left|S_{n-1} - S_{n-2}\right|$$

$$< \left(\frac{b}{a+b}\right)^3 \left|S_{n-2} - S_{n-3}\right|$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$< \left(\frac{b}{a+b}\right)^{n-1} (b-a)$$

Take $\quad r = \dfrac{b}{a+b} < 1$

Then for $n > m$ we have

$$\left|S_n - S_m\right| = \left|S_n - S_{n-1} + S_{n-1} - S_{n-2} + \ldots\ldots\ldots\ldots + S_{m+1} - S_m\right|$$

$$\leq \left|S_n - S_{n-1}\right| + \left|S_{n-1} - S_{n-2}\right| + \ldots\ldots\ldots\ldots + \left|S_{m+1} - S_m\right|$$

$$< \left(r^{n-2} + r^{n-3} + \ldots\ldots\ldots\ldots + r^{m-1}\right)(b-a)$$

$$= e$$

This implies that $\{S_n\}$ is a Cauchy sequence, therefore it is convergent.

## *Example*

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots + \frac{1}{n}$$

For $\ m, n \in \mathbb{N},\ n > m$ we have

$$\left|t_n - t_m\right| = \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots\ldots\ldots + \frac{1}{n}$$

$$> (n-m)\frac{1}{n} = 1 - \frac{m}{n}$$

In particular if $n = 2m$ then

$$\left|t_n - t_m\right| > \frac{1}{2}$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent.

***Theorem (nested intervals)***

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n \ \forall \ n \geq 1$, and $(b_n - a_n) \to 0$ as $n \to \infty$ then $\bigcap I_n$ contains one and only one point.

***Proof***

Since $I_{n+1} \subset I_n$

$$\therefore \ a_1 < a_2 < a_3 < \ldots\ldots\ldots\ldots < a_{n-1} < a_n < b_n < b_{n-1} < \ldots\ldots\ldots < b_3 < b_2 < b_1$$

$\{a_n\}$ is increasing sequence, bounded above by $b_1$ and bounded below by $a_1$.

And $\{b_n\}$ is decreasing sequence bounded below by $a_1$ and bounded above by $b_1$.

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ both are convergent.

Suppose $\{a_n\}$ converges to $a$ and $\{b_n\}$ converges to $b$.

But $|a - b| = |a - a_n + a_n - b_n + b_n - b|$

$$\leq |a_n - a| + |a_n - b_n| + |b_n - b| \to 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow$ $a = b$

and $a_n < a < b_n \quad \forall \ n \geq 1$.

***Theorem (Bolzano-Weierstrass theorem)***

Every bounded sequence has a convergent subsequence.

***Proof***

Let $\{S_n\}$ be a bounded sequence.

Take $a_1 = \inf S_n$ and $b_1 = \sup S_n$

Then $a_1 < S_n < b_1 \quad \forall \ n \geq 1$.

Now bisect interval $[a_1, b_1]$ such that at least one of the two sub-intervals contains infinite numbers of terms of the sequence.

Denote this sub-interval by $[a_2, b_2]$.

If both the sub-intervals contain infinite number of terms of the sequence then choose the one on the right hand.

Then clearly $a_1 \leq a_2 < b_2 \leq b_1$.

Suppose there exist a subinterval $[a_k, b_k]$ such that

$$a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq a_k < b_k \leq \ldots\ldots\ldots\ldots \leq b_2 \leq b_1$$

$$\Rightarrow \ (b_k - a_k) = \frac{1}{2^k}(b_1 - a_1)$$

Bisect the interval $[a_k, b_k]$ in the same manner and choose $[a_{k+1}, b_{k+1}]$ to have

$$a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq a_k \leq a_{k+1} < b_{k+1} \leq b_k \leq \ldots\ldots\ldots\ldots \leq b_2 \leq b_1$$

and
$$b_{k+1} - a_{k+1} = \frac{1}{2^{k+1}}(b_1 - a_1)$$

This implies that we obtain a sequence of interval $[a_n, b_n]$ such that

$$b_n - a_n = \frac{1}{2^n}(b_1 - a_1) \to 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow$ we have a unique point $s$ such that

$$s = \bigcap [a_n, b_n]$$

there are infinitely many terms of the sequence whose length is $e > 0$ that contain $s$.
For $e = 1$ there are infinitely many values of $n$ such that

$$|S_n - s| < 1$$

Let $n_1$ be one of such value then

$$|S_{n_1} - s| < 1$$

Again choose $n_2 > n_1$ such that

$$\left|S_{n_2} - s\right| < \frac{1}{2}$$

Continuing in this manner we find a sequence $\left\{S_{n_k}\right\}$ for each positive integer $k$ such that $n_k < n_{k+1}$ and

$$\left|S_{n_k} - s\right| < \frac{1}{k} \qquad \forall\ k = 1, 2, 3, \ldots\ldots\ldots\ldots$$

Hence there is a subsequence $\left\{S_{n_k}\right\}$ which converges to $s$.

## *Limit Inferior of the sequence*

Suppose $\left\{S_n\right\}$ is bounded then we define limit inferior of $\left\{S_n\right\}$ as follow

$$\lim_{n\to\infty}\left(\inf S_n\right) = \lim_{n\to\infty} U_k \quad \text{where} \quad U_k = \inf\left\{S_n : n \geq k\right\}$$

If $S_n$ is bounded below then

$$\lim_{n\to\infty}\left(\inf S_n\right) = -\infty$$

## *Limit Superior of the sequence*

Suppose $\left\{S_n\right\}$ is bounded above then we define limit superior of $\left\{S_n\right\}$ as follow

$$\lim_{n\to\infty}\left(\sup S_n\right) = \lim_{n\to\infty} V_k \quad \text{where} \quad V_k = \inf\left\{S_n : n \geq k\right\}$$

If $S_n$ is not bounded above then we have

$$\lim_{n\to\infty}\left(\sup S_n\right) = +\infty$$

***Note:***

($i$) A bounded sequence has unique limit inferior and superior

($ii$) Let $\left\{S_n\right\}$ contains all the rational numbers, then every real number is a subsequencial limit then limit superior of $S_n$ is $+\infty$ and limit inferior of $S_n$ is $-\infty$

($iii$) Let $\left\{S_n\right\} = (-1)^n\left(1 + \frac{1}{n}\right)$

then limit superior of $S_n$ is 1 and limit inferior of $S_n$ is $-1$.

($iv$) Let $U_k = \inf\left\{S_n : n \geq k\right\}$

$$= \inf\left\{\left(1 + \frac{1}{k}\right)\cos kp\,, \left(1 + \frac{1}{k+1}\right)\cos(k+1)p\,, \left(1 + \frac{1}{k+2}\right)\cos(k+2)p\,, \ldots\ldots\ldots\ldots\ldots\right\}$$

$$= \begin{cases} \left(1 + \frac{1}{k}\right)\cos kp & \textit{if } k \textit{ is odd} \\ \left(1 + \frac{1}{k+1}\right)\cos(k+1)p & \textit{if } k \textit{ is even} \end{cases}$$

$\Rightarrow \lim_{n\to\infty}\left(\inf S_n\right) = \lim_{n\to\infty} U_k = -1$

Also $V_k = \sup\left\{S_n : n \geq k\right\}$

$$= \begin{cases} \left(1 + \frac{1}{k+1}\right)\cos(k+1)p & \textit{if } k \textit{ is odd} \\ \left(1 + \frac{1}{k}\right)\cos kp & \textit{if } k \textit{ is even} \end{cases}$$

$\Rightarrow \lim_{n\to\infty}\left(\inf S_n\right) = \lim_{n\to\infty} V_k = 1$

### Theorem

If $\{S_n\}$ is a convergent sequence then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\inf S_n) = \lim_{n\to\infty}(\sup S_n)$$

### Proof

Let $\lim_{n\to\infty} S_n = s$ then for a real number $e > 0$, $\exists$ a positive integer $n_0$ such that

$$|S_n - s| < e \qquad \forall\ n \ge n_0 \ \ldots\ldots\ldots\ (i)$$

i.e. $s - e < S_n < s + e \qquad \forall\ n \ge n_0$

If $V_k = \sup\{S_n : n \ge k\}$

Then $s - e < V_n < s + e \qquad \forall\ k \ge n_0$

$$\Rightarrow\ s - e < \lim_{k\to\infty} V_n < s + e \qquad \forall\ k \ge n_0 \ \ldots\ldots\ldots\ (ii)$$

from (*i*) and (*ii*) we have

$$s = \lim_{k\to\infty} \sup\{S_n\}$$

We can have the same result for limit inferior of $\{S_n\}$ by taking

$$U_k = \inf\{S_n : n \ge k\}$$

## Infinite Series

Given a sequence $\{a_n\}$, we use the notation $\sum_{i=1}^{\infty} a_n$ or simply $\sum a_n$ to denotes the sum $a_1 + a_2 + a_3 + \dots\dots\dots\dots$ and called a infinite series or just series.

The numbers $S_n = \sum_{k=1}^{n} a_k$ are called the partial sum of the series.

If the sequence $\{S_n\}$ converges to *s*, we say that the series converges and write $\sum_{n=1}^{\infty} a_n = s$, the number *s* is called the sum of the series but it should be clearly understood that the '*s*' is the limit of the sequence of sums and is not obtained simply by addition.

If the sequence $\{S_n\}$ diverges then the series is said to be diverge.

***Note:***

The behaviors of the series remain unchanged by addition or deletion of the certain terms

## Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

### Proof

Let $S_n = a_1 + a_2 + a_3 + \dots\dots\dots + a_n$

Take $\lim_{n\to\infty} S_n = s = \sum a_n$

Since $a_n = S_n - S_{n-1}$

Therefore
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0$$

***Note:***

The converse of the above theorem is false

## Example

Consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

We know that the sequence $\{S_n\}$ where $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots\dots + \frac{1}{n}$ is divergent therefore $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent series, although $\lim_{n\to\infty} a_n = 0$.

This implies that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

It is know as basic divergent test.

## Theorem (General Principle of Convergence)

A series $\sum a_n$ is convergent if and only if for any real number $e > 0$, there exists a positive integer $n_0$ such that

$$\left|\sum_{i=m+1}^{\infty} a_i\right| < e \qquad \forall \;\; n > m > n_0$$

### Proof

Let $S_n = a_1 + a_2 + a_3 + \dots\dots\dots\dots + a_n$

then $\{S_n\}$ is convergent if and only if for $e > 0$ $\exists$ a positive integer $n_0$ such that

$$| S_n - S_m | < e \qquad \forall \; n > m > n_0$$

$$\Rightarrow \left| \sum_{i=m+1}^{\infty} a_i \right| = | S_n - S_m | < e$$

## Example

If $|x| < 1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

And if $|x| \geq 1$ then $\sum_{n=0}^{\infty} x^n$ is divergent.

## Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{S_n\}$ be a sequence of its partial sums then $\sum a_n$ is convergent if $\{S_n\}$ is bounded and it diverges if $\{S_n\}$ is unbounded.

## Proof

Since $a_n \geq 0 \quad \forall \; n \geq 0$

$$S_n = S_{n-1} + a_n > S_{n-1} \qquad \forall \; n \geq 0$$

therefore the sequence $\{S_n\}$ is monotonic increasing and hence it is converges if $\{S_n\}$ is bounded and it will diverge if it is unbounded.

Hence we conclude that $\sum a_n$ is convergent if $\{S_n\}$ is bounded and it divergent if $\{S_n\}$ is unbounded.

## Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0 \quad \forall \; n$. Also suppose that for a fixed positive number $l$ and positive integer $k$, $a_n < l\, b_n \quad \forall \; n \geq k$

Then $\sum a_n$ converges if $\sum b_n$ is converges and $\sum b_n$ is diverges if $\sum a_n$ is diverges.

## Proof

Suppose $\sum b_n$ is convergent and

$$a_n < l\, b_n \qquad \forall \; n \geq k \; \dots\dots\dots\dots \; (i)$$

then for any positive number $e > 0$ there exists $n_0$ such that

$$\sum_{i=m+1}^{n} b_i < \frac{e}{l} \qquad n > m > n_0$$

from $(i)$

$$\Rightarrow \sum_{i=m+1}^{n} a_i < l \sum_{i=m+1}^{n} b_i < e \qquad , \qquad n > m > n_0$$

$$\Rightarrow \sum a_n \text{ is convergent.}$$

Now suppose $\sum a_n$ is divergent then $\{S_n\}$ is unbounded.

$\Rightarrow \; \exists$ a real number $b > 0$ such that

$$\sum_{i=m+1}^{n} b_i > l\; b \qquad , \qquad n > m$$

from $(i)$

$$\Rightarrow \sum_{i=m+1}^{n} b_i > \frac{1}{l} \sum_{i=m+1}^{n} a_i > b \qquad , \qquad n > m$$

$$\Rightarrow \sum b_n \text{ is convergent.}$$

**Example**

We know that $\sum \frac{1}{n}$ is divergent and

$$n \geq \sqrt{n} \quad \forall \; n \geq 1$$

$$\Rightarrow \; \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

$\Rightarrow \; \sum \frac{1}{\sqrt{n}}$ is divergent as $\sum \frac{1}{n}$ is divergent.

**Example**

The series $\sum \frac{1}{n^a}$ is convergent if $a > 1$ and diverges if $a \leq 1$.

Let $S_n = 1 + \frac{1}{2^a} + \frac{1}{3^a} + .................. + \frac{1}{n^a}$

If $a > 1$ then

$$S_n < S_{2n} \quad \text{and} \quad \frac{1}{n^a} < \frac{1}{(n-1)^a}$$

Now $S_{2n} = \left[1 + \frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} ........... + \frac{1}{(2n)^a}\right]$

$$= \left[1 + \frac{1}{3^a} + \frac{1}{5^a} + ........... + \frac{1}{(2n-1)^a}\right] + \left[\frac{1}{2^a} + \frac{1}{4^a} + \frac{1}{6^a} + ........... + \frac{1}{(2n)^a}\right]$$

$$= \left[1 + \frac{1}{3^a} + \frac{1}{5^a} + ........... + \frac{1}{(2n-1)^a}\right] + \frac{1}{2^a}\left[1 + \frac{1}{2^a} + \frac{1}{3^a} + ........... + \frac{1}{(n)^a}\right]$$

$$< \left[1 + \frac{1}{2^a} + \frac{1}{4^a} + ........... + \frac{1}{(2n-2)^a}\right] + \frac{1}{2^a} S_n$$

replacing 3 by 2, 5 by 4 and so on.

$$= 1 + \frac{1}{2^a}\left[1 + \frac{1}{2^a} + ........... + \frac{1}{(n-1)^a}\right] + \frac{1}{2^a} S_n$$

$$= 1 + \frac{1}{2^a} S_{n-1} + \frac{1}{2^a} S_n \quad = 1 + \frac{1}{2^a} S_{2n} + \frac{1}{2^a} S_{2n} \qquad \because S_{n-1} < S_n < S_{2n}$$

$$= 1 + \frac{2}{2^a} S_{2n}$$

$$\Rightarrow \; S_{2n} < 1 + \frac{1}{2^{a-1}} S_{2n}$$

$$\Rightarrow \; \left(1 - \frac{1}{2^{a-1}}\right) S_{2n} < 1 \quad \Rightarrow \; \left(\frac{2^{a-1} - 1}{2^{a-1}}\right) S_{2n} < 1 \quad \Rightarrow \; S_{2n} < \frac{2^{a-1}}{2^{a-1} - 1}$$

i.e. $S_n < S_{2n} < \frac{2^{a-1}}{2^{a-1} - 1}$

$\Rightarrow \; \{S_n\}$ is bounded and also monotonic. Hence we conclude that $\sum \frac{1}{n^a}$ is convergent when $a > 1$.

If $a \leq 1$ then

$$n^a \leq n \quad \forall \; n \geq 1$$

$$\Rightarrow \; \frac{1}{n^a} \geq \frac{1}{n} \quad \forall \; n \geq 1$$

$\because \; \sum \frac{1}{n}$ is divergent therefore $\sum \frac{1}{n^a}$ is divergent when $a \leq 1$.

## Theorem

Let $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = l \neq 0$ then the series $\sum a_n$ and $\sum b_n$ behave alike.

## Proof

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = l$

$$\Rightarrow \left| \frac{a_n}{b_n} - l \right| < e \quad \forall \ n \geq n_0 .$$

Use $e = \frac{l}{2}$

$$\Rightarrow \left| \frac{a_n}{b_n} - l \right| < \frac{l}{2} \quad \forall \ n \geq n_0 .$$

$$\Rightarrow \ l - \frac{l}{2} < \frac{a_n}{b_n} < l + \frac{l}{2}$$

$$\Rightarrow \ \frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2}$$

then we got

$$a_n < \frac{3l}{2} b_n \quad \text{and} \quad b_n < \frac{2}{l} a_n$$

Hence by comparison test we conclude that $\sum a_n$ and $\sum b_n$ converge or diverge together.

## Example

To check $\sum \frac{1}{n} \sin^2 \frac{x}{n}$ diverges or converges consider

$$a_n = \frac{1}{n} \sin^2 \frac{x}{n} \quad \text{and take} \quad b_n = \frac{1}{n^3}$$

then $\frac{a_n}{b_n} = n^2 \sin^2 \frac{x}{n}$

$$= \frac{\sin^2 \frac{x}{n}}{\frac{1}{n^2}} \quad = x^2 \left( \frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^2$$

Applying limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} x^2 \left( \frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^2 = x^2 \left( \lim_{n\to\infty} \frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^2 = x^2(1) = x^2$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have the similar behavior $\forall$ finite values of $x$ except $x = 0$.

Since $\sum \frac{1}{n^3}$ is convergent series therefore the given series is also convergent for finite values of $x$ except $x = 0$.

**Theorem ( Cauchy Condensation Test )**

Let $a_n \ge 0$ , $a_n > a_{n+1}$ $\forall$ $n \ge 1$, then the series $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

**Proof**

Let us suppose

$$S_n = a_1 + a_2 + a_3 + ............... + a_n$$

and $t_n = a_1 + 2a_2 + 2^2 a_{2^2} + ................ + 2^{n-1} a_{2^{n-1}}$ .

$\because$ $a_n \ge 0$ and $n < 2^{n-1} < 2^n - 1$

$\therefore$ $S_n < S_{2^{n-1}} < S_{2^n - 1}$ for $n > 2$

then

$$S_{2^n-1} = a_1 + a_2 + a_3 + ...... + a_{2^n-1}$$

$$= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + ....... + \left(a_{2^{n-1}} + a_{2^{n-1}+1} + a_{2^{n-1}+2} + ..... + a_{2^n-1}\right)$$

$$< a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + ....... + \left(a_{2^{n-1}} + a_{2^{n-1}} + a_{2^{n-1}} + ..... + a_{2^{n-1}}\right)$$

$$< a_1 + 2a_2 + 2^2 a_4 + ....... + 2^{n-1} a_{2^{n-1}} = t_n$$

$\Rightarrow$ $S_n < t_n$

$\Rightarrow$ $S_n < t_n < 2S_{2^n}$ …………. $(i)$

Now consider

$$S_{2^n} = a_1 + a_2 + a_3 + ............... + a_{2^n}$$

$$= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + ....... + \left(a_{2^{n-1}+1} + a_{2^{n-1}+2} + a_{2^{n-1}+3} + .... + a_{2^n}\right)$$

$$> \frac{1}{2} a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + ...... + \left(a_{2^n} + a_{2^n} + a_{2^n} + ..... + a_{2^n}\right)$$

$$= \frac{1}{2} a_1 + a_2 + 2a_4 + 2^2 a_8 + ................ + 2^{n-1} a_{2^n}$$

$$= \frac{1}{2}\left(a_1 + 2a_2 + 2^3 a_4 + 2^3 a_8 + ................ + 2^n a_{2^n}\right)$$

$\Rightarrow$ $S_{2n} > \frac{1}{2} t_n$ ………… $(ii)$

$\Rightarrow$ $2S_{2n} > t_n$

From $(i)$ and $(ii)$ we see that the sequence $S_n$ and $t_n$ are either both bounded or both unbounded, implies that $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

**Example**

Consider the series $\sum \frac{1}{n^p}$

If $p \le 0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p} \ne 0$

therefore the series diverges when $p \le 0$.

If $p > 0$ then the condensation test is applicable and we are lead to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{(p-1)}}\right)^k$$

$$= \sum_{k=0}^{\infty} 2^{(1-p)k}$$

Now $2^{1-p} < 1$ iff $1 - p < 0$ i.e. when $p > 1$

And the result follows by comparing this series with the geometric series having common ratio less than one.

The series diverges when $2^{1-p}=1$ ( i.e. when $p=1$)

The series is also divergent if $0<p<1$.

## ***Example***

If $p>1$, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converges and

If $p\le 1$ the series is divergent.

$\because$ $\{\ln n\}$ is increasing $\therefore$ $\left\{\frac{1}{n\ln n}\right\}$ decreases

and we can use the condensation test to the above series.

We have $a_n=\frac{1}{n(\ln n)^p}$

$$\Rightarrow \quad a_{2^n}=\frac{1}{2^n\left(\ln 2^n\right)^p} \qquad \Rightarrow \quad 2^n a_{2^n}=\frac{1}{\left(n\ln 2\right)^p}$$

$\Rightarrow$ we have the series

$$\sum 2^n a_{2^n}=\sum\frac{1}{(n\ln 2)^p}=\frac{1}{\left(\ln 2\right)^p}\sum\frac{1}{n^p}$$

which converges when $p>1$ and diverges when $p\le 1$.

## ***Example***

Consider $\sum\frac{1}{\ln n}$

Since $\{\ln n\}$ is increasing there $\left\{\frac{1}{\ln n}\right\}$ decreases.

And we can apply the condensation test to check the behavior of the series

$$\because \; a_n=\frac{1}{\ln n} \qquad \therefore \; a_{2^n}=\frac{1}{\ln 2^n}$$

so $2^n a_{2^n}=\frac{2^n}{\ln 2^n} \quad \Rightarrow \quad 2^n a_{2^n}=\frac{2^n}{n\ln 2}$

since $\frac{2^n}{n}>\frac{1}{n} \qquad \forall \; n\ge 1$

and $\sum\frac{1}{n}$ is diverges therefore the given series is also diverges.

## Alternating Series

A series in which successive terms have opposite signs is called an alternating series.

e.g. $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...............$ is an alternating series.

## Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + ................$ converges.

### Proof

Looking at the odd numbered partial sums of this series we find that

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + ........... + (a_{2n-1} - a_{2n}) + a_{2n+1}$$

Since $\{a_n\}$ is decreasing therefore all the terms in the parenthesis are non-negative

$$\Rightarrow S_{2n+1} > 0 \quad \forall \ n$$

Moreover

$$S_{2n+3} = S_{2n+1} - a_{2n+2} + a_{2n+3}$$
$$= S_{2n+1} - \left(a_{2n+2} - a_{2n+3}\right)$$

Since $a_{2n+2} - a_{2n+3} \geq 0$ therefore $S_{2n+3} \leq S_{2n+1}$

Hence the sequence of odd numbered partial sum is decreasing and is bounded below by zero. (as it has +ive terms)

It is therefore convergent.

Thus $S_{2n+1}$ converges to some limit $l$ (say).

Now consider the even numbered partial sum. We find that

$$S_{2n+2} = S_{2n+1} - a_{2n+2}$$

and

$$\lim_{n\to\infty} S_{2n+2} = \lim_{n\to\infty}\left(S_{2n+1} - a_{2n+2}\right)$$
$$= \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} a_{2n+2}$$
$$= l - 0 = l \qquad \because \lim_{n\to\infty} a_n = 0$$

so that the even partial sum is also convergent to $l$.

$\Rightarrow$ both sequences of odd and even partial sums converge to the same limit.

Hence we conclude that the corresponding series is convergent.

## Absolute Convergence

$\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

## Theorem

An absolutely convergent series is convergent.

### Proof:

If $\sum |a_n|$ is convergent then for a real number $e > 0$, $\exists$ a positive integer $n_0$ such that

$$\left|\sum_{i=m+1}^{n} a_i\right| < \sum_{i=m+1}^{n} |a_i| < e \qquad \forall \ n, m > n_0$$

$\Rightarrow$ the series $\sum a_n$ is convergent. (Cauchy Criterion has been used)

**Note**

The converse of the above theorem does not hold.

e.g. $\sum \frac{(-1)^{n+1}}{n}$ is convergent but $\sum \frac{1}{n}$ is divergent.

### *Theorem (The Root Test)*

Let $\lim\limits_{n\to\infty} Sup|a_n|^{1/n} = p$

Then $\sum a_n$ converges absolutely if $p<1$ and it diverges if $p>1$.

### *Proof*

Let $p<1$ then we can find the positive number $e>0$ such that $p+e<1$

$\Rightarrow |a_n|^{1/n} < p+e < 1 \quad \forall\ n>n_0$

$\Rightarrow |a_n| < (p+e)^n < 1$

$\because \sum (p+e)^n$ is convergent because it is a geometric series with $|r|<1$.

$\therefore \sum |a_n|$ is convergent

$\Rightarrow \sum a_n$ converges absolutely.

Now let $p>1$ then we can find a number $e_1>0$ such that $p-e_1>1$.

$\Rightarrow |a_n|^{1/n} > p+e > 1$

$\Rightarrow |a_n| > 1$ for infinitely many values of *n.*

$\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ is divergent.

***Note:***

The above test give no information when $p=1$.

e.g. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

For each of these series $p=1$, but $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent.

### *Theorem (Ratio Test)*

The series $\sum a_n$

(i) Converges if $\lim\limits_{n\to\infty} Sup\left|\frac{a_{n+1}}{a_n}\right|<1$

(ii) Diverges if $\left|\frac{a_{n+1}}{a_n}\right|>1$ for $n \geq n_0$, where $n_0$ is some fixed integer.

### *Proof*

If (i) holds we can find $b<1$ and integer $N$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < b \quad \text{for } n \geq N$$

In particular

$$\left|\frac{a_{N+1}}{a_N}\right| < b$$

$\Rightarrow |a_{N+1}| < b|a_N|$

$\Rightarrow |a_{N+2}| < b|a_{N+1}| < b^2|a_N|$

$\Rightarrow |a_{N+3}| < b^3|a_N|$

……………………

……………………

……………………

$\Rightarrow |a_{N+p}| < b^p|a_N|$

$$\Rightarrow \ |a_n| < b^{n-N}|a_N| \qquad \text{we put } N+p=n.$$

i.e. $|a_n| < |a_N|\, b^{-N} b^n$ for $n \geq N$.

$\because \sum b^n$ is convergent because it is geometric series with common ration $<1$.

Therefore $\sum a_n$ is convergent (by comparison test)

Now if

$$|a_{n+1}| \geq |a_n| \qquad \text{for } n \geq n_0$$

then $\lim\limits_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ is divergent.

***Note***

The knowledge $\left|\dfrac{a_{n+1}}{a_n}\right| = 1$ implies nothing about the convergent or divergent of series.

## *Example*

Consider the series $\sum a_n$ with $a_n = \left[\dfrac{n}{n+1} - \left(\dfrac{n}{n+1}\right)^{n+1}\right]^{-n}$

$$\because \ \frac{n}{n+1} < 1 \qquad \therefore \ a_n > 0 \quad \forall \ n.$$

Also

$$(a_n)^{\frac{1}{n}} = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-1}$$

$$= \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n}{n+1}\right)^{n}\right]^{-1} = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n+1}{n}\right)^{-n}\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty}\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= 1\cdot\left[1-e^{-1}\right]^{-1} = \left[1-\frac{1}{e}\right]^{-1} = \left[\frac{e-1}{e}\right]^{-1} = \left[\frac{e}{e-1}\right] \ > 1$$

$\Rightarrow$ the series is divergent.

## *Theorem (Dirichlet)*

Suppose that $\{S_n\}$, $S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$ is bounded. Let $\{b_n\}$ be positive term decreasing sequence such that $\lim\limits_{n\to\infty} b_n = 0$, then $\sum a_n b_n$ is convergent.

## *Proof*

$\because \{S_n\}$ is bounded

$\therefore \exists$ a positive number $l$ such that

$$|S_n| < l \qquad \forall \ n \geq 1.$$

Then

$$a_i b_i = (S_i - S_{i-1}) b_i \qquad \text{for } i \geq 2$$

$$= S_i b_i - S_{i-1} b_i$$

$$= S_i b_i - S_{i-1} b_i + S_i b_{i+1} - S_i b_{i+1}$$

$$= S_i\left(b_i - b_{i+1}\right) - S_{i-1}\, b_i + S_i\, b_{i+1}$$

$$\Rightarrow \sum_{i=m+1}^{n} a_i\, b_i = \sum_{i=m+1}^{n} S_i\left(b_i - b_{i+1}\right) - \left(S_m\, b_{m+1} - S_n\, b_{n+1}\right)$$

$\because$ $\{b_n\}$ is decreasing

$$\therefore \left|\sum_{i=m+1}^{n} a_i\, b_i\right| = \left|\sum_{i=m+1}^{n} S_i\left(b_i - b_{i+1}\right) - S_m\, b_{m+1} + S_n\, b_{n+1}\right|$$

$$< \sum_{i=m+1}^{n} \left\{\left|S_i\right|\left(b_i - b_{i+1}\right)\right\} + \left|S_m\right| b_{m+1} + \left|S_n\right| b_{n+1}$$

$$< \sum_{i=m+1}^{n} \left\{l\left(b_i - b_{i+1}\right)\right\} + l\, b_{m+1} + l\, b_{n+1} \qquad \because \left|S_i\right| < l$$

$$= l\left(\sum_{i=m+1}^{n} \left(b_i - b_{i+1}\right) + b_{m+1} + b_{n+1}\right)$$

$$= l\left(\left(b_{m+1} - b_{n+1}\right) + b_{m+1} + b_{n+1}\right) = 2\, l\left(b_{m+1}\right)$$

$\Rightarrow \left|\sum_{i=m+1}^{n} a_i\, b_i\right| < e$ where $e = 2\, l\left(b_{m+1}\right)$ a certain number

$\Rightarrow$ The $\sum a_n\, b_n$ is convergent. ( We have use Cauchy Criterion here. )

### *Theorem*

Suppose that $\sum a_n$ is convergent and that $\{b_n\}$ is monotonic convergent sequence then $\sum a_n\, b_n$ is also convergent.

### *Proof*

Suppose $\{b_n\}$ is decreasing and it converges to $b$.

Put $c_n = b_n - b$

$\Rightarrow$ $c_n \geq 0$ and $\lim\limits_{n\to\infty} c_n = 0$

$\because$ $\sum a_n$ is convergent

$\therefore$ $\{S_n\}$, $S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots\ldots + a_n$ is convergent

$\Rightarrow$ It is bounded

$\Rightarrow$ $\sum a_n\, c_n$ is bounded.

$\because$ $a_n\, b_n = a_n\, c_n + a_n\, b$ and $\sum a_n\, c_n$ and $\sum a_n\, b$ are convergent.

$\therefore$ $\sum a_n\, b_n$ is convergent.

Now if $\{b_n\}$ is increasing and converges to $b$ then we shall put $c_n = b - b_n$.

### *Example*

$\sum \frac{1}{(n\ln n)^a}$ is convergent if $a > 1$ and divergent if $a \leq 1$.

To see this we proceed as follows

$$a_n = \frac{1}{(n\ln n)^a}$$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{\left(2^n \ln 2^n\right)^a} = \frac{2^n}{\left(2^n\; n\,\ln 2\right)^a}$

$$= \frac{2^n}{2^{na}\; n^a \left(\ln 2\right)^a} = \frac{1}{2^{na-n}\; n^a \left(\ln 2\right)^a}$$

$$= \frac{1}{(\ln 2)^a}\cdot\frac{\left(\frac{1}{2}\right)^{(a-1)n}}{n^a}$$

Since $\sum \frac{1}{n^a}$ is convergent when $a > 1$ and $\left(\frac{1}{2}\right)^{(a-1)n}$ is decreasing for $a > 1$ and it converges to 0. Therefore $\sum b_n$ is convergent

$\Rightarrow \sum a_n$ is also convergent.

Now $\sum b_n$ is divergent for $a \le 1$ therefore $\sum a_n$ diverges for $a \le 1$.

***Example***

To check $\sum \frac{1}{n^a \ln n}$ is convergent or divergent.

We have $a_n = \frac{1}{n^a \ln n}$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{(2^n)^a (\ln 2^n)} = \frac{2^n}{2^{na}(n \ln 2)}$

$$= \frac{1}{\ln 2}\cdot\frac{2^{(1-a)n}}{n} = \frac{1}{\ln 2}\cdot\frac{\left(\frac{1}{2}\right)^{(a-1)n}}{n}$$

$\because \sum \frac{1}{n}$ is divergent although $\left\{\left(\frac{1}{2}\right)^{n(a-1)}\right\}$ is decreasing, tending to zero for $a > 1$ therefore $\sum b_n$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

The series also divergent if $a \le 1$.
i.e. it is always divergent.

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