Chapter 1 – Real Number System

Subject: *Real Analysis (Mathematics)* ***Level:*** *M.Sc.*

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The rational number system is inadequate for many purposes, both as a field and as an order set for many purpose. This leads to introduction of so called irrational numbers. We can prove in many ways that the rational number system has certain gaps and hence we fail to use it as an ordered set and as a field. ❑

✎ Theorem

There is no rational p such that $p^2 = 2$.

Proof

Let us suppose that there exists a rational p such that $p^2 = 2$.

This implies we can write

$$p = \frac{m}{n} \qquad \text{where } m, n \in \mathbb{Z} \text{ \& } m, n \text{ have no common factor.}$$

Then $p^2 = 2 \quad \Rightarrow \quad \dfrac{m^2}{n^2} = 2 \quad \Rightarrow \quad m^2 = 2n^2$

$\Rightarrow$ m^2 is even

$\Rightarrow$ m is even

$\Rightarrow$ m is divisible by 2 and so m^2 is divisible by 4.

$\Rightarrow$ $2n^2$ is divisible by 4 and so n^2 is divisible by 2. $\qquad \because m^2 = 2n^2$

i.e. n^2 is even $\Rightarrow$ n is even

$\Rightarrow$ m and n both have common factor 2.

Which is contradiction. (because m and n have no common factor.)

Hence $p^2 = 2$ is impossible for rational p. ❑

✎ Theorem

Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$ then A contain no largest member and B contains no smallest member.

Proof

We are to show that for every p in A there exists a rational $q \in A$ such that $p < q$ and for all $p \in B$ we can find rational $q \in B$ such that $q < p$.

Associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \quad \ldots\ldots\ldots\ldots\ldots (i)$$

$$\text{Then } \; q^2 - 2 = \left(\frac{2p + 2}{p + 2}\right)^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2} \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Now if $p \in A$ then $p^2 < 2 \quad \Rightarrow \quad p^2 - 2 < 0$

Since from (i) $\quad q = p - \dfrac{p^2 - 2}{p + 2} \quad \Rightarrow \quad q > p$

And $\quad \dfrac{2(p^2 - 2)}{(p + 2)^2} < 0 \quad \Rightarrow q^2 - 2 < 0 \quad \Rightarrow q^2 < 2 \quad \Rightarrow q \in A$

Now if $p \in B$ then $p^2 > 2 \quad \Rightarrow \quad p^2 - 2 > 0$

Since form (i) $q = p - \dfrac{p^2-2}{p+2} \quad \Rightarrow \; q < p$

And $\dfrac{2(p^2-2)}{(p+2)^2} > 0 \quad \Rightarrow q^2 - 2 > 0 \Rightarrow q^2 > 2 \quad \Rightarrow \; q \in B$

The purpose of above discussion is simply to show that the rational number system has certain gaps, in spite of the fact that the set of rationals is dense i.e. we can always find a rational between any two given rational numbers. These gaps are filled by the irrational number. (e.g. if $r < s$ then $r < \dfrac{r+s}{2} < s$.) ❑

Order on a set

Let S be a non-empty set. An *order* on a set S is a relation denoted by "<" with the following two properties

(i) If $x \in S$ and $y \in S$,

then one and only one of the statement $x < y$, $x = y$, $y < x$ is true.

(ii) If $x, y, z \in S$ and if $x < y$, $y < z$ then $x < z$.

Ordered Set

A set S is said to be *ordered set* if an order is defined on S.

Bound

Let S be an ordered set and $E \subset S$. If there exists a $b \in S$ such that $x \le b \;\; \forall \; x \in E$, then we say that E is bounded above, and b is known as upper bound of E.

Lower bound can be define in the same manner with ≥ in place of ≤.

Least Upper Bound (Supremum)

Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exists an $a \in S$ such that

(i) a is an upper bound of E.

(ii) If $g < a$ then g is not an upper bound of E.

Then a is called the *least upper bound* of E or *supremum* of E and is written as $\sup E = a$.

In other words a is the least member of the set of upper bound of E.

We can define the greatest lower bound or infimum of a set E, which is bounded below, in the same manner. ❑

Example

Consider the sets

$$A = \{p : p \in \mathbb{Q} \wedge p^2 < 2\}$$

$$B = \{p : p \in \mathbb{Q} \wedge p^2 > 2\}$$

where $\mathbb{Q}$ is set of rational numbers.

Then the set A is bounded above. The upper bound of A are the exactly the members of B. Since B contain no smallest member therefore A has no supremum in $\mathbb{Q}$.

Similarly B is bounded below. The set of all lower bounds of B consists of A and $r \in \mathbb{Q}$ with $r \le 0$. Since A has no largest member, therefore, B has no infimum in $\mathbb{Q}$.

Example

If a is supremum of E then a may or may not belong to E.

Let $E_1 = \{r : r \in \mathbb{Q} \wedge r < 0\}$

$E_2 = \{r : r \in \mathbb{Q} \wedge r \ge 0\}$

then $\sup E_1 = \inf E_2 = 0$ and $0 \notin E_1$ and $0 \in E_2$. ❑

✎ Example

Let E be the set of all numbers of the form $\frac{1}{n}$, where n is the natural numbers.

$$\text{i.e. } E=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},.........\right\}$$

Then $\sup E = 1$ which is in E, but $\inf E = 0$ which is not in E. ❑

✎ Least Upper Bound Property

A set S is said to have the *least upper bound property* if the followings is true

(i) S is non-empty and ordered.

(ii) If $E \subset S$ and E is non-empty and bounded above then sup$E$ exists in S.

Greatest lower bound property can be defined in a similar manner. ❑

✎ Example

Let S be set of rational numbers and

$$E=\left\{p : p\in\mathbb{Q} \wedge p^2<2\right\}$$

then $E \subset \mathbb{Q}$, E is non-empty and also bounded above but supremum of E is not in S, this implies that $\mathbb{Q}$ the set of rational numbers does not posses the least upper bound property. ❑

✎ Theorem

Suppose S is an ordered set with least upper bound property. $B \subset S$, B is non-empty and is bounded below. Let L be set of all lower bounds of B then $a = \sup L$ exists in S and also $a = \inf B$.

In particular infimum of B exists in S.

OR

An ordered set which has the least upper bound property has also the greatest lower bound property.

Proof

Since B is bounded below; therefore, L is non-empty.

Since L consists of exactly those $y \in S$ which satisfy the inequality.

$$y \le x \qquad \forall\, x \in B$$

We see that every $x \in B$ is an upper bound of L.

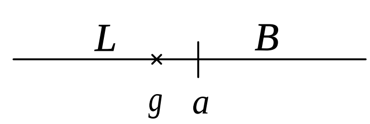
$\Rightarrow$ L is bounded above.

Since S is ordered and non-empty therefore L has a supremum in S. Let us call it a.

If $g < a$, then g is not upper bound of L.

$\Rightarrow \; g \notin B$

$\Rightarrow \; a \le x \quad \forall\, x \in B \quad \Rightarrow a \in L$



Now if $a < b$ then $b \notin L$ because $a = \sup L$.

We have shown that $a \in L$ but $b \notin L$ if $b > a$. In other words, a is a lower bound of B, but b is not if $b > a$. This means that $a = \inf B$. ❑

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✎ Field

A set F with two operations called addition and multiplication satisfying the following axioms is known to be field.

Axioms for Addition:

(i) If $x,\ y\in F$ then $x+y\in F$. *Closure Law*

(ii) $x+y=y+x \quad \forall\ x,y\in F$. *Commutative Law*

(iii) $x+(y+z)=(x+y)+z \quad \forall\ x,y,z\in F$. *Associative Law*

(iv) For any $x\in F$, $\exists\ 0\in F$ such that $x+0=0+x=x$ *Additive Identity*

(v) For any $x\in F$, $\exists\ -x\in F$ such that $x+(-x)=(-x)+x=0$ *+tive Inverse*

Axioms for Multiplication:

(i) If $x,\ y\in F$ then $xy\in F$. *Closure Law*

(ii) $xy=yx \quad \forall\ x,y\in F$ *Commutative Law*

(iii) $x(yz)=(xy)z \quad \forall\ x,y,z\in F$

(iv) For any $x\in F$, $\exists\ 1\in F$ such that $x\cdot 1=1\cdot x=x$ *Multiplicative Identity*

(v) For any $x\in F$, $x\neq 0$, $\exists\ \dfrac{1}{x}\in F$, such that $x\left(\dfrac{1}{x}\right)=\left(\dfrac{1}{x}\right)x=1$ $\times$*tive Inverse.*

Distributive Law

For any $x,y,z\in F$, (i) $x(y+z)=xy+xz$

(ii) $(x+y)z=xz+yz$ ❑

✎ Theorem

The axioms for addition imply the following:

(***a***) If $x+y=x+z$ then $y=z$

(***b***) If $x+y=x$ then $y=0$

(***c***) If $x+y=0$ then $y=-x$.

(***d***) $-(-x)=x$

Proof

(a) Suppose $x+y=x+z$.

Since

$$\begin{aligned} y &= 0+y \\ &= (-x+x)+y && \because\ -x+x=0 \\ &= -x+(x+y) && \text{by Associative law} \\ &= -x+(x+z) && \text{by supposition} \\ &= (-x+x)+z && \text{by Associative law} \\ &= (0)+z && \because\ -x+x=0 \\ &= z \end{aligned}$$

(b) Take $z=0$ in (a)

$$x+y=x+0$$
$$\Rightarrow\ y=0$$

(c) Take $z=-x$ in (a)

$$x+y=x+(-x)$$
$$\Rightarrow\ y=-x$$

(d) Since $(-x)+x=0$

then (c) gives $x=-(-x)$ ❑

Theorem

Axioms of multiplication imply the following.

(***a***) If $x \neq 0$ and $xy = xz$ then $y = z$.

(***b***) If $x \neq 0$ and $xy = x$ then $y = 1$.

(***c***) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$.

(***d***) If $x \neq 0$, then $\frac{1}{1/x} = x$.

Proof

(a) Suppose $xy = xz$

$$\text{Since } y = 1 \cdot y = \left(\frac{1}{x} \cdot x\right) y \qquad \because \frac{1}{x} \cdot x = 1$$

$$= \frac{1}{x}(xy) \qquad \text{by associative law}$$

$$= \frac{1}{x}(xz) \qquad \because xy = xz$$

$$= \left(\frac{1}{x} \cdot x\right) z \qquad \text{by associative law}$$

$$= 1 \cdot z = z$$

(b) Take $z = 1$ in (a)

$$xy = x \cdot 1 \qquad \Rightarrow \; y = 1$$

(c) Take $z = \frac{1}{x}$ in (a)

$$xy = x \cdot \frac{1}{x} \qquad \text{i.e. } xy = 1$$

$$\Rightarrow \; y = \frac{1}{x}$$

(d) Since $\frac{1}{x} \cdot x = 1$

then (c) give

$$x = \frac{1}{1/x}$$

❑

Theorem

The field axioms imply the following.

(***i***) $0 \cdot x = 0$

(***ii***) if $x \neq 0$, $y \neq 0$ then $xy \neq 0$.

(***iii***) $(-x)y = -(xy) = x(-y)$

(***iv***) $(-x)(-y) = xy$

Proof

(***i***) Since $0x + 0x = (0+0)x$

$$\Rightarrow \; 0x + 0x = 0x$$

$$\Rightarrow \; 0x = 0 \qquad \because x + y = x \Rightarrow y = 0$$

(***ii***) Suppose $x \neq 0$, $y \neq 0$ but $xy = 0$

$$\text{Since } 1 = \frac{1}{(x)(y)} \cdot xy$$

$$\Rightarrow 1 = \frac{1}{(x)(y)}(0) \qquad \because\ xy = 0,\ x \neq 0,\ y \neq 0$$

$$\Rightarrow 1 = 0 \qquad \text{from } (i) \quad \because\ x0 = 0$$

a contradiction, thus (*ii*) is true.

(***iii***) Since $(-x)y + xy = (-x + x)y = 0y = 0$ …….. (1)

Also $x(-y) + xy = x(-y + y) = x0 = 0$ ……… (2)

Also $-(xy) + xy = 0$ …………. (3)

Combining (1) and (2)

$$(-x)y + xy = x(-y) + xy$$

$$\Rightarrow (-x)y = x(-y) \quad \text{………… (4)}$$

Combining (2) and (3)

$$x(-y) + xy = -(xy) + xy$$

$$\Rightarrow x(-y) = -xy \quad \text{…………. (5)}$$

From (4) and (5)

$$(-x)y = x(-y) = -xy$$

(***iv***) $(-x)(-y) = -[x(-y)] = -[-xy] = xy$ using (*iii*) ❑

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Ordered Field

An ordered field is a field F which is also an ordered set such that

i) $x+y<x+z$ if $x,y,z\in F$ and $y<z$.

ii) $xy>0$ if $x,y\in F$, $x>0$ and $y>0$.

e.g. the set $\mathbb{Q}$ of rational number is an ordered field. ❑

Theorem

The following statements are true in every ordered field.

i) If $x>0$ then $-x<0$ and vice versa.

ii) If $x>0$ and $y<z$ then $xy<xz$.

iii) If $x<0$ and $y<z$ then $xy>xz$.

iv) If $x\neq 0$ then $x^2>0$ in particular $1>0$.

v) If $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$.

Proof

i) If $x>0$ then $0=-x+x>-x+0$ so that $-x<0$.

If $x<0$ then $0=-x+x<-x+0$ so that $-x>0$.

ii) Since $z>y$ we have $z-y>y-y=0$

which means that $z-y>0$, Also $x>0$

$$\therefore \quad x(z-y)>0$$
$$\Rightarrow xz-xy>0$$
$$\Rightarrow xz-xy+xy>0+xy$$
$$\Rightarrow xz+0>0+xy$$
$$\Rightarrow xz>xy$$

iii) Since $y<z \Rightarrow -y+y<-y+z$

$\Rightarrow z-y>0$

Also $x<0 \Rightarrow -x>0$

Therefore $-x(z-y)>0$

$\Rightarrow -xz+xy>0 \qquad \Rightarrow -xz+xy+xz>0+xz$

$\Rightarrow xy>xz$

iv) If $x>0$ then $x\cdot x>0 \Rightarrow x^2>0$

If $x<0$ then $-x>0 \Rightarrow (-x)(-x)>0 \Rightarrow (-x)^2>0 \Rightarrow x^2>0$

i.e. if $x>0$ then $x^2>0$, since $1^2=1$ then $1>0$.

v) If $y>0$ and $v\leq 0$ then $yv\leq 0$, But $y\left(\frac{1}{y}\right)=1>0 \Rightarrow \frac{1}{y}>0$

Likewise $\frac{1}{x}>0$ as $x>0$

If we multiply both sides of the inequality $x<y$ by the positive quantity $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$

we obtain $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x<\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$

i.e. $\frac{1}{y}<\frac{1}{x}$

finally $0<\frac{1}{y}<\frac{1}{x}$ ❑

Existence of Real Field

There exists an ordered field $\mathbb{R}$ (set of reals) which has the least upper bound property and it contain $\mathbb{Q}$ (set of rationals) as a subfield. ❑

Theorem

a) If $x\in\mathbb{R}$, $y\in\mathbb{R}$ and $x>0$ then there exists a positive integer *n* such that $nx>y$. (*Archimedean Property*)

b) If $x\in\mathbb{R}$, $y\in\mathbb{R}$ and $x<y$ then there exists $p\in\mathbb{Q}$ such that $x<p<y$.

i.e. between any two real numbers there is a rational number *or* $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

a) Let $A=\{nx:\ n\in\mathbb{Z}^{+}\wedge x>0, x\in\mathbb{R}\}$

Suppose the given statement is false i.e. $nx\le y$.

$\Rightarrow$ y is an upper bound of *A*.

Since we are dealing with a set of reals, therefore, it has the least upper bound property.

Let $a=\sup A$

$\Rightarrow$ $a-x$ is not an upper bound of A.

$\Rightarrow$ $a-x<mx$ where $mx\in A$ for some positive integer *m*.

$\Rightarrow$ $a<(m+1)x$ where $m+1$ is integer, therefore $(m+1)x\in A$

Which is impossible because a is least upper bound of A i.e. $a=\sup A$.

Hence we conclude that the given statement is true i.e. $nx>y$.

b) Since $x<y$, therefore $y-x>0$

$\Rightarrow$ $\exists$ a +ive integer *n* such that

$$n(y-x)>1 \qquad (\textit{by Archimedean Property})$$

$$\Rightarrow\ ny>1+nx\ \ldots\ldots\ldots\ldots\ldots\ (i)$$

We apply (*a*) part of the theorem again to obtain two +ive integers m_1 and m_2 such that $m_1\cdot 1>nx$ and $m_2\cdot 1>-nx$

$$\Rightarrow\ -m_2<nx<m_1$$

then there exists an integers $m(-m_2\le m\le m_1)$ such that

$$m-1\le nx<m$$

$$\Rightarrow\ nx<m \quad\text{and}\quad m\le 1+nx$$

$$\Rightarrow\ nx<m<1+nx$$

$$\Rightarrow\ nx<m<ny \qquad \text{from } (i)$$

$$\Rightarrow\ x<\frac{m}{n}<y$$

$$\Rightarrow\ x<p<y \quad\text{where}\quad p={}^{m}\!/_{n}\ \text{is a rational.}$$ ❑

Theorem

Given two real numbers *x* and *y*, $x<y$ there is an irrational number *u* such that

$$x<u<y$$

Proof

Take $x>0$, $y>0$

Then $\exists$ a rational number *q* such that

$$0<\frac{x}{a}<q<\frac{y}{a} \qquad \text{where } a \text{ is an irrational.}$$

$$\Rightarrow\ x<aq<y$$

$$\Rightarrow\ x<u<y$$

Where $u=aq$ is an irrational as product of rational and irrational is irrational. ❑

Theorem
For every real number x there is a set E of rational number such that $x = \sup E$.

Proof

Take $E = \{q \in \mathbb{Q} : q < x\}$ where x is a real.

Then E is bounded above. Since $E \subset \mathbb{R}$ therefore supremum of E exists in $\mathbb{R}$.
Suppose $\sup E = l$.

It is clear that $l \leq x$.
If $l = x$ then there is nothing to prove.
If $l < x$ then $\exists$ $q \in \mathbb{Q}$ such that $l < q < x$

Which can not happen. Hence we conclude that real x is sup$E$. ❑

Theorem
For every real $x > 0$ and every integer $n > 0$, there is one and only one real y such that $y^n = x$.

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof

Take $y_1, y_2 \in \mathbb{R}$ such that $0 < y_1 < y_2$. Then $y_1^n < y_2^n$ i.e. there is at most one $y \in \mathbb{R}$ such that $y^n = x$. This shows the uniqueness of y.

Let us suppose E be the set of all positive real numbers t such that $t^n < x$.

$$\text{i.e. } E = \{t \ : \ t \in \mathbb{R} \wedge t^n < x\}$$

Take $t = \dfrac{x}{1+x}$ then $0 < t < 1$.

Hence $t^n < t$ and we have $t^n < x$

$\Rightarrow$ $t^n < t < x$

$\Rightarrow$ $t \in E$ and E is non-empty.

If $t > 1 + x$ then $t^n > t > x$ so that $t \notin E$.
Thus $1 + x$ is an upper bound of E.
Since E is non-empty and bounded above therefore $\sup E$ exists.

Take $y = \sup E$

To show that $y^n = x$ we will show that each of the inequality $y^n < x$ and $y^n > x$ leads to contradiction.
Consider

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots\cdots\cdots + a^{n-1}) \qquad \text{where } n \in \mathbb{Z}^+.$$

Which yields the inequality (each a is replaced by b on R.H.S of above)

$$b^n - a^n < (b-a)(nb^{n-1}) \ \ldots\ldots\ldots\ldots\ldots (i) \qquad \text{where } 0 < a < b.$$

Now assume $y^n < x$

Choose h so that $0 < h < 1$ and $h < \dfrac{x - y^n}{n(y+1)^{n-1}}$

Put $a = y$ and $b = y + h$ in (i)

$$\begin{aligned} \text{Then } (y+h)^n - y^n &< nh(y+h)^{n-1} \\ &< nh(y+1)^{n-1} \qquad \because h < 1 \\ &< x - y^n \end{aligned}$$

$\Rightarrow$ $(y+h)^n < x$

$\Rightarrow$ $y + h \in E$

Since $y + h > y$ therefore it contradict the fact that y is $\sup E$.

Hence $y^n < x$ is impossible.

Now suppose $y^n > x$

Put $k = \dfrac{y^n - x}{ny^{n-1}}$, then $0 < k < y$

Now if $t \geq y - k$ we get

$$y^n - t^n < y^n - (y-k)^n < y^n - (y^n - nky^{n-1}) \qquad \textit{by binomial expansion}$$
$$< kny^{n-1} = y^n - x$$
$$\Rightarrow\ -t^n < -x \quad \Rightarrow \quad t^n > x \ \text{ and } \ t \notin E$$

It follows that $y - k$ is an upper bound of E but $y - k < y$, which contradict the fact that y is $\sup E$.

Hence we conclude that $y^n = x$. ❑

✍ The Extended Real Numbers

The extended real number system consists of real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$, We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty \quad \forall\ x \in \mathbb{R}.$$

The extended real number system does not form a field. Mostly we write $+\infty = \infty$.

We make following conventions

i) If x is real then $x + \infty = \infty$, $x - \infty = -\infty$, $\dfrac{x}{\infty} = \dfrac{x}{-\infty} = 0$

ii) If $x > 0$ then $x(\infty) = \infty$, $x(-\infty) = -\infty$.

iii) If $x < 0$ then $x(\infty) = -\infty$, $x(-\infty) = \infty$.

✍ Euclidean Space

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\underline{x} = (x_1, x_2, \ldots\ldots\ldots\ldots, x_k)$$

where $x_1, x_2, \ldots\ldots\ldots\ldots, x_k$ are real numbers, called the *coordinates* of $\underline{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$.

If $\underline{y} = (y_1, y_2, \ldots\ldots\ldots\ldots, y_n)$ and a is a real number, put

$$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, \ldots\ldots\ldots\ldots, x_k + y_k)$$

and $$a\,\underline{x} = (a\,x_1, a\,x_2, \ldots\ldots\ldots\ldots, a\,x_k)$$

So that $\underline{x} + \underline{y} \in \mathbb{R}^k$ and $a\,\underline{x} \in \mathbb{R}^k$. These operations make $\mathbb{R}^k$ into a vector space over the real field.

The inner product or scalar product of $\underline{x}$ and $\underline{y}$ is defined as

$$\underline{x}.\underline{y} = \sum_{i=1}^{k} x_i y_i = (x_1 y_1 + x_2 y_2 + \ldots\ldots\ldots + x_k y_k)$$

And the norm of $\underline{x}$ is defined by

$$\|\,\underline{x}\,\| = (x \cdot x)^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}$$

The vector space $\mathbb{R}^k$ with the above inner product and norm is called *Euclidean k-space*. ❑

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Theorem

Let $\underline{x}$, $\underline{y} \in \mathbb{R}^n$ then

i) $\|\underline{x}\|^2 = x \cdot x$

ii) $\|\underline{x} \cdot \underline{y}\| \leq \|\underline{x}\| \|\underline{y}\|$ (*Cauchy-Schwarz's inequality*)

Proof

i) Since $\|\underline{x}\| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}}$ therefore $\|\underline{x}\|^2 = \underline{x} \cdot \underline{x}$

ii) For $l \in \mathbb{R}$ we have

$$
\begin{aligned}
0 \leq \left\|\underline{x} - l\,\underline{y}\right\|^2 &= \left(\underline{x} - l\,\underline{y}\right) \cdot \left(\underline{x} - l\,\underline{y}\right) \\
&= \underline{x} \cdot \left(\underline{x} - l\,\underline{y}\right) + \left(-l\,\underline{y}\right) \cdot \left(\underline{x} - l\,\underline{y}\right) \\
&= \underline{x} \cdot \underline{x} + \underline{x} \cdot (-l\,\underline{y}) + (-l\,\underline{y}) \cdot \underline{x} + (-l\,\underline{y}) \cdot (-l\,\underline{y}) \\
&= \|\underline{x}\|^2 - 2l\,(\underline{x} \cdot \underline{y}) + l^2 \|\underline{y}\|^2
\end{aligned}
$$

Now put $l = \dfrac{\underline{x} \cdot \underline{y}}{\|\underline{y}\|^2}$ (*certain real number*)

$$\Rightarrow\ 0 \leq \|\underline{x}\|^2 - 2\frac{(\underline{x} \cdot \underline{y})(\underline{x} \cdot \underline{y})}{\|\underline{y}\|^2} + \frac{(\underline{x} \cdot \underline{y})^2}{\|\underline{y}\|^4}\|\underline{y}\|^2 \quad \Rightarrow\ 0 \leq \|\underline{x}\|^2 - \frac{(\underline{x} \cdot \underline{y})^2}{\|\underline{y}\|^2}$$

$$\Rightarrow\ 0 \leq \|\underline{x}\|^2 \|\underline{y}\|^2 - \|\underline{x} \cdot \underline{y}\|^2$$

$$\Rightarrow\ 0 \leq \left(\|\underline{x}\| \|\underline{y}\| + \|\underline{x} \cdot \underline{y}\|\right)\left(\|\underline{x}\| \|\underline{y}\| - \|\underline{x} \cdot \underline{y}\|\right)$$

Which hold if and only if

$$0 \leq \|\underline{x}\| \|\underline{y}\| - \|\underline{x} \cdot \underline{y}\|$$

i.e. $\|\underline{x} \cdot \underline{y}\| \leq \|\underline{x}\| \|\underline{y}\|$ ❑

Question

Suppose $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$ the prove that

a) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

b) $\|\underline{x} - \underline{z}\| \leq \|\underline{x} - \underline{y}\| + \|\underline{y} - \underline{z}\|$

Proof

a) Consider

$$
\begin{aligned}
\|\underline{x} + \underline{y}\|^2 &= \left(\underline{x} + \underline{y}\right) \cdot \left(\underline{x} + \underline{y}\right) \\
&= \underline{x} \cdot \underline{x} + \underline{x} \cdot \underline{y} + \underline{y} \cdot \underline{x} + \underline{y} \cdot \underline{y} \\
&= \|\underline{x}\|^2 + 2\left(\underline{x} \cdot \underline{y}\right) + \|\underline{y}\|^2 \\
&\leq \|\underline{x}\|^2 + 2\|\underline{x}\| \|\underline{y}\| + \|\underline{y}\|^2 \qquad \because\ \|\underline{x}\| \|\underline{y}\| \geq \underline{x}.\underline{y} \\
&= \left(\|\underline{x}\| + \|\underline{y}\|\right)^2
\end{aligned}
$$

$$\Rightarrow\ \|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\| \ \ldots\ldots\ldots\ldots\ (i)$$

b) We have

$$
\begin{aligned}
\|\underline{x} - \underline{z}\| &= \|\underline{x} - \underline{y} + \underline{y} - \underline{z}\| \\
&\leq \|\underline{x} - \underline{y}\| + \|\underline{y} - \underline{z}\| \qquad \text{from } (i)
\end{aligned}
$$

❑

Question
If r is rational and x is irrational then prove that $r+x$ and rx are irrational.

Proof
Let $r+x$ be rational.

$\Rightarrow r+x=\dfrac{a}{b}$ where $a,b\in\mathbb{Z}$, $b\neq 0$ such that $(a,b)=1$

$\Rightarrow x=\dfrac{a}{b}-r$

Since r is rational therefore $r=\dfrac{c}{d}$ where $c,d\in\mathbb{Z}$, $d\neq 0$ such that $(c,d)=1$

$\Rightarrow x=\dfrac{a}{b}-\dfrac{c}{d} \quad \Rightarrow x=\dfrac{ad-bc}{bd}$

Which is rational, which can not happened because x is given to be irrational.
Similarly let us suppose that rx is rational then

$rx=\dfrac{a}{b}$ for some $a,b\in\mathbb{Z}$, $b\neq 0$ such that $(a,b)=1$

$\Rightarrow x=\dfrac{a}{b}\cdot\dfrac{1}{r}$

Since r is rational therefore $r=\dfrac{c}{d}$ where $c,d\in\mathbb{Z}$, $d\neq 0$ such that $(c,d)=1$

$\Rightarrow x=\dfrac{a}{b}\cdot\dfrac{1}{c/d}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc}$

Which shows that x is rational, which is again contradiction; hence we conclude that $r+x$ and rx are irrational. ❑

Question
If n is a positive integer which is not perfect square then prove that $\sqrt{n}$ is irrational number.

Solution
There will be two cases

Case I. When n contain no square factor greater then 1.

Let us suppose that $\sqrt{n}$ is a rational number.

$\Rightarrow \sqrt{n}=\dfrac{p}{q}$ where $p,q\in\mathbb{Z}$, $q\neq 0$ and $(p,q)=1$

$\Rightarrow n=\dfrac{p^2}{q^2} \quad \Rightarrow p^2=nq^2$ ....(i)

$\Rightarrow q^2=\dfrac{p^2}{n}$

$\Rightarrow n\,|\,p^2 \quad \Rightarrow n\,|\,p$ ....(ii) ($n\,|\,p$ means " n divides p")

Now suppose $\dfrac{p}{n}=c$ where $c\in\mathbb{Z}$

$\Rightarrow p=nc \quad \Rightarrow p^2=n^2c^2$

Putting this value of p^2 in equation (i)

$n^2c^2=nq^2$

$\Rightarrow nc^2=q^2 \quad \Rightarrow c^2=\dfrac{q^2}{n}$

$\Rightarrow n\,|\,q^2 \quad \Rightarrow n\,|\,q$ ....(iii)

From (ii) and (iii) we get p and q both have common factor n i.e. $(p,q)=n$
Which is a contradiction.
Hence our supposition is wrong.

Case II When n contain a square factor greater then 1.
Let us suppose $n=k^2m>1$

$$\Rightarrow \sqrt{n}=k\sqrt{m}$$

Where k is rational and $\sqrt{m}$ is irrational because m has no square factor greater than one, this implies $\sqrt{n}$, the product of rational and irrational, is irrational. ❑

✎ Question
Prove that $\sqrt{12}$ is irrational.

Proof
Suppose $\sqrt{12}$ is rational.

$\Rightarrow \sqrt{12}=\dfrac{p}{q}$ where $p,q\in\mathbb{Z}$, $q\neq 0$ and $(p,q)=1$

$\Rightarrow 12=\dfrac{p^2}{q^2}$ $\Rightarrow p^2=12q^2$ ………….. (i)

$\Rightarrow q^2=\dfrac{p^2}{12}$ $\Rightarrow q^2=\dfrac{p^2}{2^2\cdot 3}$

$\Rightarrow 2^2\,|\,p^2$ and $3\,|\,p^2$

$\Rightarrow 2\,|\,p$ and $3\,|\,p$

$\Rightarrow$ 2 and 3 are prime divisor of p.

$\Rightarrow 2\cdot 3\,|\,p$ i.e. $6\,|\,p$

$\Rightarrow \dfrac{p}{6}=c$, where c is an integer.

$\Rightarrow p=6c$

Put this value of p in equation (i) to get

$$36c^2=12q^2$$

$\Rightarrow 3c^2=q^2$ $\Rightarrow c^2=\dfrac{q^2}{3}$

$\Rightarrow 3\,|\,q^2$ $\Rightarrow 3\,|\,q$

$\Rightarrow (p,q)=3$, which is a contradiction.

Hence $\sqrt{12}$ is an irrational number. ❑

✎ Question
Let E be a non-empty subset of an ordered set, suppose a is a lower bound of E and b is an upper bound then prove that $a\leq b$.

Proof
Since E is a subset of an ordered set S i.e. $E\subseteq S$.
Also a is a lower bound of E therefore by definition of lower bound

$a\leq x \quad \forall \ x\in E$ …………… (i)

Since b is an upper bound of E therefore by the definition of upper bound

$x\leq b \quad \forall \ x\in E$ …………… (ii)

Combining (i) and (ii)

$a\leq x\leq b$

$\Rightarrow a\leq b$ as required. ❑

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