University of Sargodha

M.A/M.Sc Part-1 / Composite, 1st-A/2015

Mathematics: V Topology & Functional Analysis

Maximum Marks: 100 **Time Allowed: 3 Hours**

Note: Objective part is compulsory. Attempt any four questions from subjective part.

Objective Part

Q1. Write short answers of the following: **(10x2=20)**

(i) Define homeomorphism and Topological Property.

(ii) Show that any two equivalent norms on N define the same topology on N.

(iii) Prove that interior of a subset of topological space is an open set.

(iv) Give example of a reflexive space.

(v) Prove that e very metric space is a Hausdorff space.

(vi) What do you know about singletons in a T_1-space.

(vii) State the condition for a normed space to be an inner product space and give an example.

(viii) Write all closed subsets of X= {a, b,c } with respect to its indiscrete topology

(ix) For a subset A of a Hilbert space H show that $A^{\perp}$ is closed subspace of H.

(x) Prove Bessel's inequality.

Subjective Part

Q2. (a) Prove that dual space X' of a normed space X is a Banach space (whether or not X is). 10

(b) Show that a bounded linear operator T: $D(T) \to Y$, where X is a normed space and Y a Banach space, has an extension T': $(T) \to Y$ where T' is a bounded linear operator of norm $\|T'\|=\|T\|$. 10

Q3. (a) Prove that a family □ of subsets of a topological space (X,τ) is a base for τ if and only if

(i) X = U ⁻, that is, every point of X is in some B ϵ □

(ii) For B_1, B_2 ϵ □ and $x \in B_1 \cap B_2$, there is a B ϵ □ where $x \in B \subseteq B_1 \cap B_2$ 10

(b) Let $X= \Pi X\alpha$ be the product space of topological spaces $X\alpha$, $\alpha \in \Omega$. Prove that the αth projection $\pi_\alpha: X \to X\alpha$ is both open and continuous. 10

Q4. (a) Prove that in a countably compact space X, every infinite subset of X has a limit point in X. 10

(b) State and prove Cantor Intersection Theorem 10

Q5. (a) Prove that any two norms defined on a finite dimensional linear space are equivalent. 10

(b) Show that if M is a Banach space, so is B(N,M) under the norm defined by

$$\|T\| = \operatorname{Sup}_{\|x\|=1} \|Tx\|, \quad x \in N, \quad T \in B(N,M)$$ 10

Q6. (a) Prove that dual of □^{1} is □$^\infty$. 12

(b) Show that relation of ' being equivalent to ' among norms is an equivalence relation. 8

Q7. (a) Using Gram-Schmidt process show that if $\{x_1, x_2, \ldots, x_n\}$ be any (countable) linearly independent set of vectors in an inner product space V, then V contains an is orthonormal set $\{e_1, e_2, \ldots, e_n\}$ such that the spaces generated by $\{x_1, x_2, \ldots, x_n\}$ and $\{e_1, e_2, \ldots, e_n\}$ are the same. 10

(b) State and prove Riesz Representation theorem. 10