University of Sargodha

M.A/M.Sc Part-1 / Composite, 1st-A/2014

Mathematics: V Topology & Functional Analysis

Maximum Marks: 100 **Time Allowed: 3 Hours**

Note: **Objective part is compulsory. Attempt any four questions from subjective part.**

Objective Part

Q.1. Write short answer of the following. $(10\times 2=20)$

i. Define Co-finite topology.

ii. Show that any subspace of first countable space is first countable.

iii. Show that an orthonormal set is linearly independent.

iv. Show that a convergent sequence in a metric space (X, d) is Cauchy.

v. Give an example of a topological space which is T_1 but not T_2.

vi. Let X be a topological space and $A, B \subset X$, show that $A \subset B \Rightarrow A^{\circ} \subset B^{\circ}$.

vii. Define T_4-space.

viii. State F. Riesz's Lemma.

ix. Show that the operator $T: C[a,b] \to C[a,b]$ by $Tx(t) = tx(t)$ is linear.

x. Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ be a subset of R (set of real numbers). Show that A is nowhere dense in R.

Subjective Part

Q.2. **(a)** If d is a metric on a non-empty set X, then show that the function $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ is also a metric on X. **(10)**

(b) Let X be a topological space, A be a subset of X, A' denote the derived set of A, then show that $\overline{A} = A \cup A'$. **(10)**

Q.3. **(a)** State and prove Cantor's Intersection Theorem. **(10)**

(b) Show that every finite dimensional subspace Y of a normed space X is complete. **(10)**

Q.4. **(a)** Show that for any subset $M \neq \phi$ of a Hilbert space H, the span of M is dense in H iff $M^{\perp} = \{0\}$. M = H **(10)**

(b) Show that if a normed space X is finite dimensional, then every linear operator on X is bounded. **(10)**

Q.5. **(a)** Show that a function define on a first countable space X is continuous at $p \in X$ iff it is sequentially continuous at p. **(10)**

(b) Show that the continuous image of a compact set is compact. **(10)**

Q.6. **(a)** Suppose that Y be any closed subspace of a Hilbert space H. Then $H = Y \oplus Z$, where $Z = Y^{\perp}$. **(10)**

(b) Show that the equivalent norms on a vector space X induced the same topology for X. **(10)**

Q.7. **(a)** Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ be a topology on X, let $A = \{a, c, d\}$. Find interior, exterior, closure, boundary and limit points of A. **(10)**

(b) Show that a topological space X is a T_1-space iff every singleton subset of X is closed. **(10)**
