15658

University of Sargodha

M.A/M.Sc Part-1 / Composite, 1st -A/2013

Mathematics-V Topology & Functional Analysis

Maximum Marks: 100 **Time Allowed: 3 Hours**

Note: **Objective part is compulsory. Attempt any four questions from subjective part.**

Objective Part

Q.1. **Write short answer of the following.** (20)

i. Prove that every point p in a discrete space X has a finite local base. **ii.** If X be a co finite topological space, then find the closure of any subset A of X in both cases when A is finite or infinite. **iii.** Show that any subspace of first countable space is first countable. **iv.** Define Hausdorff space. **v.** Show that every finite T_1 space is discrete. **vi.** If (X, d) be a metric space, then for any $x, y \in X$ show that $d(x,y) \geq |d(x,A) - d(y,A)|$. **vii.** Find the largest possible value of c for linearly independent set of vectors $\{x_1 = (1, 0, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1)\}$ satisfying $a_1x_1 + a_2x_2 + a_3x_3 \geq c(|a_1| + |a_2| + |a_3|)$. **viii.** Prove by giving an example, that an infinite dimensional subspace need not to be closed. **ix.** Define Dual space. **x.** If Y is closed subspace of a Hilbert space H, then show that $Y = Y^{\perp\perp}$.

Subjective Part

Q.2. a. Show that $\bar{A} = int(A) \cup b(A)$. (10)

b. If β is a class of subsets of a non-empty set X. Then show that β is a base for some topology on X if following two properties exist in X. (10)
i. $X = \cup\{B : B \in \beta\}$ ii. For any $B, B^* \in \beta, B \cap B^*$ is the union of members of β.

Q.3. a. Prove that the function $f : X \to Y$ is continuous if and only if for every subset A of X, $f[\bar{A}] \subset \overline{f[A]}$. (10)

b. If d be a metric on a non-empty set X, then show that the function $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is also a metric on X. (10)

Q.4. a. State and prove Bair's Category theorem. (10)

b. In an inner product and the corresponding normed space prove the inequality $|< x, y >| \leq \|x\|\|y\|$. (10)

Q.5. a. Show that the space $C[a, b]$ is not an inner product space and hence not a Hilbert space. (10)

b. If $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces, then show that T is continuous iff T is bounded. (10)

Q.6. a. State and prove F. Riesz's lemma. (10)

b. Show that every closed ball is a closed set. (10)

Q.7. a. If a normed space X is finite dimensional, then prove that every linear operator on X is bounded. (10)

b. For any subset $M \neq \theta$ of a Hilbert space H, then show that the span of M is dense in H if and only if $M^{\perp} = 0$ (10)
