University of Sargodha

M.A/M.Sc Part- II/Composite, 1st-A/2014

Mathematics: VI **Advanced Analysis**

Maximum Marks: 100 **Time Allowed: 3 Hours**

Objective Part Compulsory

Note: Write short answers of following questions each in 2-3 lines only on the answer book having 2 marks each(To be attempted on answer book).

Q. 1 $10\times 2 = 20$

(i) State Russel's Paradox

(ii) Let $(X, \mathcal{A})$ be a measuable space. Show that if f and g are $\mathcal{A}$-measurable, then so is fg.

(iii) Prove that empty set is subset of every set.

(iv) Show that the set of rational numbers $\mathbb{Q}$ is null in the Lebesgue measure space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ and $\mathbb{Q}^c \in \mathcal{M}_L$ with $\mu_L(Q^c) = \infty$.

(v) Give a bijection to prove that $\mathbb{N} \approx 2\mathbb{N}$.

(vi) Show by giving an example that a step function need not be a simple function.

(vii) Write Fourier Legender series for $f(x) = \sin x$ (leave answer in integral form).

(viii) Let $\mathcal{A}$ be σ-algebra of subsets of a set X. For every sequence $\{A_k\}_1^\infty$ in $\mathcal{A}$, the two sets $\liminf\limits_{k\to\infty} A_k, \limsup\limits_{k\to\infty} A_k \in \mathcal{A}$.

(ix) Prove that $J_\nu(0) = 0$.

(x) Let $\{E_n\}_1^\infty$ be a sequence in a σ-algebra $\mathcal{A}$ of subsets of a set X. Let $\{F_n\}_1^\infty$ be the sequence defined by $F_1 = E_1, F_n = E_n \setminus \bigcup\limits_1^{n-1} E_i$ for $n \geq 2$. Show that $\bigcup\limits_1^\infty E_n = \bigcup\limits_1^\infty F_n$.

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(Subjective Part)

Note: Attempt four out of six questions. All questions carry equal marks.

Q. 2 (a)Given a measure space $(X, \mathcal{A}, \mu)$. Let $\{f_n\}_1^\infty$ be a bounded sequence of real valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. If for every $\eta > 0$, there exists an $\mathcal{A}$-measurable subset E of D with $\mu(E) < \frac{1}{\eta}$ such that $\lim\limits_{n\to\infty} f_n(x)$ exists for every $x \in D\setminus E$, then prove that $\lim\limits_{n\to\infty} f_n(x)$ exists a.e. on D. 10

	(b) Show that the Legender polynomials are orthogonal over $[-1,1]$.	10
Q.3	(a)Prove that if f and g are $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ in a measure space $(X, \mathcal{A}, \mu)$, $f+g$ is $\mathcal{A}$-measurable.	10
	(b) Solve the Legender equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ of degree n about the point $x = 0$ using Frobenius method.	10
Q.4	(a) Prove that every interval is Lebesgue measurable.	10
	(b) Prove that $J_1(x) = \frac{2}{\pi}\int\limits_0^{\frac{\pi}{2}} \sin(x\sin\theta)\sin\theta d\theta$	10
Q. 5	(a)Let ψ be simple function defined on a set D in a measure space $(X, \mathcal{A}, \mu)$. If D_1 and D_2 are disjoint measurable subsets of D with $D = D_1 \cup D_2$, then prove that $\int\limits_D \psi d\mu = \int\limits_{D_1} \psi d\mu + \int\limits_{D_2} \psi d\mu$.	10
	(b) Let X be a set of ordinals, then prove that • $\bigcup X$ is an ordinal • $\bigcup X$ is least upper bound for X.	10
Q. 6	(a) Given a measure space $(X, \mathcal{A}, \mu)$. Let f_1 and f_2 be bounded real valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. If $f_1 = f_2$ a.e. on D, then $\int_D f_1 d\mu = \int_D f_2 d\mu$	10
	(b)Let $S(A)$ be the collection of all initial segments in a well-ordered set A and let $S(A)$ be ordered by set inclusion, then prove that $A \approx S(A)$	10
Q. 7	(a) Given a measure space $(X, \mathcal{A}, \mu)$. Let $\{f_n\}_1^\infty$ be a bounded sequence of real valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. Let f be a bounded real valued $\mathcal{A}$-measurable function on D. If $f_n \to f$ a.e. on D, then prove that $\lim\limits_{n\to\infty} \int\limits_D \lvert f_n - f \rvert d\mu = 0.$	10
	(b)Prove that $2^{\aleph_0} = c$	10