AA7-1

UNIVERSITY OF THE PUNJAB, LAHORE

Mathematics
Part-II A/2007

Roll No. 027975

Examination:- M.A/M.Sc.

Subject:- Advanced Analysis
Paper - I

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

Attempt any FIVE questions in all selecting at least ONE Question from each Section.

SECTION ONE

Q.1 (a) Let α, β be two cardinal numbers such that $\alpha \le \beta$. Show that for any cardinal γ

(i) $\alpha^{\gamma} \le \beta^{\gamma}$ and (ii) $\gamma^{\alpha} \le \gamma^{\beta}$.

(b) The union of a countable family of countable sets is countable. **(10+10)**

Q.2 (a) Prove that cancellation laws under multiplication and addition do not hold for ordinal numbers.

(b) If $\alpha = \#(A)$ and $\beta = \#(B)$ then what is the relation between A and B so that $\alpha < \beta$.

SECTION-II

Q.3 (a) Define a finite and a $\sigma-finite$ measure on a $\sigma-algebra$. Show that counting measure on N, the set of natural numbers is not finite but a $\sigma-finite$ measure.

(b) Show that if A and B are two sets in a $\sigma-algebra$ with $A \subseteq B$ then $m(A) \le m(B)$. If $m(A) < \infty$ then $m(B-A) = m(B) - m(A)$. **(10+10)**

Q.4 (a) Define a Lebesgue measurable set. Show that the interval (a,∞) is measurable for every $a \in R$.

(b) Find the Lebesgue measure of the following subsets of R, that is, $B = [3,5] \cup [-4,-2], Q, Q'$ and R. **(10+10)**

Q.5 (a) Let f, g be two measurable functions defined on the same measurable set D. Show that the following sets are measurable.

(i) $\{x \in D : f(x) > g(x)\}$

(ii) $\{x \in D : f(x) < g(x)\}$

(iii) $\{x \in D : f(x) \le g(x)\}$

(iv) $\{x \in D : f(x) = g(x)\}$.

(b) Let f be an extended real-valued measurable non-negative function defined over a measurable set D. Show that there is a sequence of simple measurable functions S_n such that $S_n(x) \to f(x)$ for all $x \in D$. **(10+10)**

Q.6 (a) Let $\{f_n\}$ be a sequence of non-negative integable functions defined on a measurable set E such that $\lim\limits_{n\to\infty} f_n = f$ on E and $\liminf\limits_{n\to\infty} \int_E f_n < \infty$. Show that $\int_E f \le \liminf\limits_{n\to\infty} \int_E f_n$. Show by an example that a strict inequality may hold.

(b) Let f and g be two bounded measurable functions defined on a set of finite measure D. If $a \in R$ then prove that

(i) $\int_D af = a\int_D f$ and $\int_D (f+g) = \int_D f + \int_D g$. **(10+10)**

Q.7 (a) Show that $\dfrac{\sin x}{x}$ is R-integrable but is not Lebesgue integrable over $(-\infty, \infty)$.

(b) Let A be any subset of R and $\{E_n\}$ a sequence of pairwise disjoint Lebesgue measurable sets. Show that $m^*[A \cap (\bigcup\limits_{n=1}^{\infty} E_n)] = \sum\limits_{n=1}^{\infty} m^*(A \cap E_n)$. **(10+10)**

SECTION-III

Q.8 (a) Define $J_n(x)$ the Bessel function of first kind and then show that

$$J_{\frac{-3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left\{\frac{\cos x}{x} + \sin x\right\}.$$

(b) Show that $P_n(x) = \sum\limits_{k=0}^{n} ({}^nC_k)^2 \left(\dfrac{x-1}{2}\right)^{n-k} \left(\dfrac{x+1}{2}\right)^k$. **(10+10)**

Q.9 (a) Show that $e^{\frac{x}{2}(t-\frac{1}{t})} = \sum\limits_{n=-\infty}^{\infty} J_n(x) t^n$.

(b) Show that $P_n(x) = \dfrac{1}{2^n . n!} \dfrac{d^n}{dx^n}(x^2-1)^n$. **(10+10)**