Metric Spaces

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❖ Metric Spaces

Let X be a non-empty set and $\mathbb{R}$ denotes the set of real numbers. A function $d: X \times X \to \mathbb{R}$ is said to be metric if it satisfies the following axioms $\forall\ \ x, y, z \in X$.

[M_1] $d(x,y) \geq 0$ i.e. d is finite and non-negative real valued function.

[M_2] $d(x,y) = 0$ if and only if $x = y$.

[M_3] $d(x,y) = d(y,x)$ (Symmetric property)

[M_4] $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular inequality)

The pair (X, d) is then called *metric space*.

d is also called *distance function* and $d(x, y)$ is the distance from x to y.

Note: If (X, d) be a metric space then X is called *underlying set*.

❖ Examples:

i) Let X be a non-empty set. Then $d: X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1 & if \ \ x \neq y \\ 0 & if \ \ x = y \end{cases}$$

is a metric on X and is called *trivial metric* or *discrete metric*.

ii) Let $\mathbb{R}$ be the set of real number. Then $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x,y) = |x - y| \text{ is a metric on } \mathbb{R}.$$

The space $(\mathbb{R}, d)$ is called *real line* and d is called *usual metric on* $\mathbb{R}$.

iii) Let X be a non-empty set and $d: X \times X \to \mathbb{R}$ be a metric on X. Then $d': X \times X \to \mathbb{R}$ defined by $d'(x,y) = \min(1, d(x,y))$ is also a metric on X.

Proof:

[M_1] Since d is a metric so $d(x,y) \geq 0$
as $d'(x,y)$ is either 1 or $d(x,y)$ so $d'(x,y) \geq 0$.

[M_2] If $x = y$ then $d(x,y) = 0$ and then $d'(x,y)$ which is $\min(1, d(x,y))$ will be zero.

Conversely, suppose that $d'(x,y) = 0 \quad \Rightarrow \min(1, d(x,y)) = 0$

$\Rightarrow d(x,y) = 0 \qquad \Rightarrow x = y$ as d is metric.

[M_3] $d'(x,y) = \min(1, d(x,y)) = \min(1, d(y,x)) = d'(y,x) \qquad \because d(x,y) = d(y,x)$

[M_4] We have $d'(x,z) = \min(1, d(x,z))$

$\Rightarrow d'(x,z) \leq 1$ or $d'(x,z) \leq d(x,z)$

We wish to prove $d'(x,z) \leq d'(x,y) + d'(y,z)$

now if $d(x,z) \geq 1$, $d(x,y) \geq 1$ and $d(y,z) \geq 1$

then $d'(x,z) = 1$, $d'(x,y) = 1$ and $d'(y,z) = 1$

and $d'(x,y) + d'(y,z) = 1 + 1 = 2$

therefore $\Rightarrow\ d'(x,z) \le d'(x,y)+d'(y,z)$

Now if $d(x,z)<1$, $d(x,y)<1$ and $d(y,z)<1$

Then $d'(x,z)=d(x,z)$, $d'(x,y)=d(x,y)$ and $d'(y,z)=d(y,z)$

As d is metric therefore $d(x,z) \le d(x,y)+d(y,z)$

$$\Rightarrow\ d'(x,z) \le d'(x,y)+d'(y,z)$$

Q.E.D

iv) Let $d : X\times X \to \mathbb{R}$ be a metric space. Then $d' : X\times X \to \mathbb{R}$ defined by

$$d'(x,y)=\frac{d(x,y)}{1+d(x,y)} \text{ is also a metric.}$$

Proof.

[M$_1$] Since $d(x,y)\ge 0$ therefore $\dfrac{d(x,y)}{1+d(x,y)}=d'(x,y)\ge 0$

[M$_2$] Let $d'(x,y)=0\ \Rightarrow\ \dfrac{d(x,y)}{1+d(x,y)}=0\ \Rightarrow\ d(x,y)=0\ \Rightarrow\ x=y$

Now conversely suppose $x=y$ then $d(x,y)=0$.

Then $d'(x,y)=\dfrac{d(x,y)}{1+d(x,y)}=\dfrac{0}{1+0}=0$

[M$_3$] $d'(x,y)=\dfrac{d(x,y)}{1+d(x,y)}=\dfrac{d(y,x)}{1+d(y,x)}=d'\left(y,x\right)$

[M$_4$] Since d is metric therefore $d(x,z) \le d(x,y)+d(y,z)$

Now by using inequality $a<b\ \Rightarrow\ \dfrac{a}{1+a}<\dfrac{b}{1+b}$.

We get $\dfrac{d(x,z)}{1+d(x,z)} \le \dfrac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$

$$\Rightarrow\ d'(x,z) \le \frac{d(x,y)}{1+d(x,y)+d(y,z)}+\frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\Rightarrow\ d'(x,z) \le \frac{d(x,y)}{1+d(x,y)}+\frac{d(y,z)}{1+d(y,z)}$$

$$\Rightarrow\ d'(x,z) \le d'(x,y)+d'(y,z)$$

Q.E.D

v) The space **C**[a, b] is a metric space and the metric d is defined by

$$d(x,y)=\max_{t\in J}\left|x(t)-y(t)\right|$$

where J = [a, b] and x, y are continuous real valued function defined on [a, b].

Proof.

[M$_1$] Since $\left|x(t)-y(t)\right|\ge 0$ therefore $d(x,y)\ge 0$.

[M$_2$] Let $d(x,y)=0\ \Rightarrow\ \left|x(t)-y(t)\right|=0\ \Rightarrow\ x(t)=y(t)$

Conversely suppose $x=y$

Then $d(x,y)=\max\limits_{t\in J}\left|x(t)-y(t)\right|=\max\limits_{t\in J}\left|x(t)-x(t)\right|=0$

[M3] $d(x,y)=\max_{t\in J}\left|x(t)-y(t)\right|=\max_{t\in J}\left|y(t)-x(t)\right|=d(y,x)$

[M4] $d(x,z)=\max_{t\in J}\left|x(t)-z(t)\right|=\max_{t\in J}\left|x(t)-y(t)+y(t)-z(t)\right|$

$$\leq \max_{t\in J}\left|x(t)-y(t)\right|+\max_{t\in J}\left|y(t)-z(t)\right|$$

$$= d(x,y)+d(y,z)$$

Q.E.D

vi) $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a metric, where $\mathbb{R}$ is the set of real number and *d* defined by

$$d(x,y)=\sqrt{\left|x-y\right|}$$

vii) Let $x=(x_1,y_1)$, $y=(x_2,y_2)$ we define

$d(x,y)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ is a metric on $\mathbb{R}$

and called *Euclidean metric on* $\mathbb{R}^2$ or *usual metric on* $\mathbb{R}^2$.

viii) $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is not a metric, where $\mathbb{R}$ is the set of real number and *d* defined by

$$d(x,y)=(x-y)^2$$

Proof.

[M1] Square is always positive therefore $(x-y)^2=d(x,y)\geq 0$

[M2] Let $d(x,y)=0 \;\Rightarrow\; (x-y)^2=0 \;\Rightarrow\; x-y=0 \;\Rightarrow\; x=y$

Conversely suppose that $x=y$

then $d(x,y)=(x-y)^2=(x-x)^2=0$

[M3] $d(x,y)=(x-y)^2=(y-x)^2=d(y,x)$

[M4] Suppose that triangular inequality holds in *d*. then for any $x,y,z\in\mathbb{R}$

$$d(x,z)\leq d(x-y)+d(y,z)$$

$$\Rightarrow\; (x-z)^2\leq(x-y)^2+(y-z)^2$$

Since $x,y,z\in\mathbb{R}$ therefore consider $x=0,\; y=1$ and $z=2$.

$$\Rightarrow\; (0-2)^2\leq(0-1)^2+(1-2)^2$$

$$\Rightarrow\; 4\leq 1+1 \qquad \Rightarrow\; 4\leq 2$$

which is not true so triangular inequality does not hold and *d* is not metric.

ix) Let $x=(x_1,x_2)$, $y=(y_1,y_2)\in\mathbb{R}^2$. We define

$$d(x,y)=\left|x_1-y_1\right|+\left|x_2-y_2\right|$$

is a metric on $\mathbb{R}^2$, called *Taxi-Cab metric* on $\mathbb{R}^2$.

x) Let $\mathbb{R}^n$ be the set of all real *n*-tuples. For

$x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ in $\mathbb{R}^n$

we define $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2}$

then *d* is metric on $\mathbb{R}^n$, called *Euclidean metric on* $\mathbb{R}^n$ or *usual metric on* $\mathbb{R}^n$.

xi) The space l^∞. As points we take bounded sequence

$x=(x_1,x_2,\ldots)$, also written as $x=(x_i)$, of complex numbers such that

$$\left|x_i\right| \leq C_x \quad \forall\ i=1,2,3,\ldots$$

where C_x is fixed real number. The metric is defined as

$$d(x,y)=\sup_{i\in\mathbb{N}}\left|x_i-y_i\right| \qquad \text{where } y=(y_i)$$

xii) The space l^p, $p\geq 1$ is a real number, we take as member of l^p, all sequence $x=\left(\xi_j\right)$ of complex number such that $\sum_{j=1}^{\infty}\left|\xi_j\right|^p<\infty$.

The metric is defined by $d\left(x,y\right)=\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}$

Where $y=\left(\eta_j\right)$ such that $\sum_{j=1}^{\infty}\left|\eta_j\right|^p<\infty$

Proof.

[M1] Since $\left|\xi_j-\eta_j\right|\geq 0$ therefore $\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}=d\left(x,y\right)\geq 0$.

[M2] If $x=y$ then

$$d\left(x,y\right)=\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}=\left(\sum_{j=1}^{\infty}\left|\xi_j-\xi_j\right|^p\right)^{\frac{1}{p}}==\left(\sum_{j=1}^{\infty}\left|0\right|^p\right)^{\frac{1}{p}}=0$$

Conversely, if $d\left(x,y\right)=0$

$$\Rightarrow \left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}=0 \;\Rightarrow\; \left|\xi_j-\eta_j\right|=0 \;\Rightarrow\; \left(\xi_j\right)=\left(\eta_j\right) \;\Rightarrow\; x=y$$

[M3] $d\left(x,y\right)=\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}=\left(\sum_{j=1}^{\infty}\left|\eta_j-\xi_j\right|^p\right)^{\frac{1}{p}}=d\left(y,x\right)$

[M4] Let $z=\left(\zeta_j\right)$, such that $\sum_{j=1}^{\infty}\left|\zeta_j\right|^p<\infty$

then $d\left(x,z\right)=\left(\sum_{j=1}^{\infty}\left|\xi_j-\zeta_j\right|^p\right)^{\frac{1}{p}}$

$$=\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j+\eta_j-\zeta_j\right|^p\right)^{\frac{1}{p}}$$

Using *Minkowski's Inequality

$$\leq\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{\infty}\left|\eta_j-\zeta_j\right|^p\right)^{\frac{1}{p}}$$

$$=d\left(x,y\right)+d\left(y,z\right)$$

Q.E.D

❖ *Pseudometric*

Let X be a non-empty set. A function $d: X\times X \to \mathbb{R}$ is called pseudometric if and only if

i) $d(x,x)=0$ for all $x\in X$.
ii) $d(x,y)=d(y,x)$ for all $x,y\in X$.
iii) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

OR

A pseudometric satisfies all axioms of a metric except $d(x,y)=0$ may not imply $x=y$ but $x=y$ implies $d(x,y)=0$.

Example

Let $x,y\in\mathbb{R}^2$ and $x=(x_1,x_2)$, $y=(y_1,y_2)$

Then $d(x,y)=\left|x_1-y_1\right|$ is a pseudometric on $\mathbb{R}^2$.

Let $x=(2,3)$ and $y=(2,5)$

Then $d(x,y)=\left|2-2\right|=0$ but $x\neq y$

NOTE: Every metric is a pseudometric, but pseudometric is not metric.

$*$ ***Minkowski's Inequality***

If $\xi_i=\left(\xi_1,\xi_2,...,\xi_n\right)$ and $\eta_i=\left(\eta_1,\eta_2,...,\eta_n\right)$ are in $\mathbb{R}^n$ and $p>1$, then

$$\left(\sum_{i=1}^{\infty}\left|\xi_i+\eta_i\right|^p\right)^{\frac{1}{p}}\le\left(\sum_{i=1}^{\infty}\left|\xi_i\right|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}\left|\eta_i\right|^p\right)^{\frac{1}{p}}$$

❖ *Distance between sets*

Let (X,d) be a metric space and $A,\ B\subset X$. The distance between A and B denoted by $d(A,B)$ is defined as $d(A,B)=\inf\left\{d(a,b)\,|\,a\in A,\ b\in B\right\}$

If $A=\left\{x\right\}$ is a singleton subset of X, then $d(A,B)$ is written as $d(x,B)$ and is called distance of point x from the set B.

❖ *Theorem*

Let (X,d) be a metric space. Then for any $x,\ y\in X$

$$\left|d(x,A)-d(y,A)\right|\le d(x,y)$$

Proof.

Let $z\in A$ then $d(x,z)\le d(x,y)+d(y,z)$

then $d(x,A)=\inf\limits_{z\in A} d(x,z)\le d(x,y)+\inf\limits_{z\in A} d(y,z)$

$$=d(x,y)+d(y,A)$$

$$\Rightarrow\ d(x,A)-d(y,A)\le d(x,y)\ \cdots\cdots\cdots\cdots(i)$$

Next

$$d(y,A)=\inf_{z\in A} d(y,z)\le d(y,x)+\inf_{z\in A} d(x,z)$$
$$=d(y,x)+d(x,A)$$
$$\Rightarrow\ -d(x,A)+d(y,A)\le d(y,x)$$
$$\Rightarrow\ -\big(d(x,A)-d(y,A)\big)\le d(x,y)\cdots\cdots\cdots\cdots(ii) \qquad \because\ d(x,y)=d(y,x)$$

Combining equation (i) and (ii)

$$\big|d(x,A)-d(y,A)\big|\le d(x,y) \qquad \text{Q.E.D}$$

❖ *Diameter of a set*

Let (X,d) be a metric space and $A\subset X$, we define diameter of A denoted by

$$d(A)=\sup_{a,b\in A} d(a,b)$$

NOTE: For an empty set φ, following convention are adopted

(i) $d(\varphi)=-\infty$, some authors take $d(\varphi)$ also as 0.

(ii) $d(p,\varphi)=\infty$ i.e distance of a point p from empty set is ∞.

(iii) $d(A,\varphi)=\infty$, where A is any non-empty set.

❖ *Bounded Set*

Let (X,d) be a metric space and $A\subset X$, we say A is bounded if diameter of A is finite i.e. $d(A)<\infty$.

❖ *Theorem*

The union of two bounded set is bounded.

Proof.

Let (X,d) be a metric space and $A, B\subset X$ be bounded. We wish to prove $A\cup B$ is bounded.

Let $x,y\in A\cup B$

If $x, y\in A$ then since A is bounded therefore $d(x,y)<\infty$

and hence $d(A\cup B)=\sup\limits_{x,y\in A\cup B} d(x,y)<\infty$ then $A\cup B$ is bounded.

Similarity if $x, y\in B$ then $A\cup B$ is bounded.

Now if $x\in A$ and $y\in B$ then

$$d(x,y)\le d(x,a)+d(a,b)+d(b,y) \qquad \text{where } a\in A,\ b\in B.$$

Since $d(x,a)$, $d(a,b)$ and $d(b,y)$ are finite

Therefore $d(x,y)<\infty$ i.e $A\cup B$ is bounded. Q.E.D

❖ *Open Ball*

Let (X,d) be a metric space. An open ball in (X,d) is denoted by

$$B(x_0;r)=\{x\in X \mid d(x_0,x)<r\}$$

x_0 is called centre of the ball and r is called radius of ball and $r\ge 0$.

❖ *Closed Ball*

The set $\overline{B}(x_0;r) = \{x \in X \mid d(x_0,x) \le r\}$ is called closed ball in (X,d).

❖ *Sphere*

The set $S(x_0;r) = \{x \in X \mid d(x_0,x) = r\}$ is called sphere in (X,d).

❖ *Examples*

Consider the set of real numbers with usual metric $d = |x - y| \ \forall\ x, y \in \mathbb{R}$
then $B(x_\circ;r) = \{x \in \mathbb{R} \mid d(x_\circ,x) < r\}$
i.e. $B(x_\circ;r) = \{x \in \mathbb{R} : |x - x_\circ| < r\}$
i.e. $x_0 - r < x < x + r = (x_0 - r, x_0 + r)$
i.e. open ball is the real line with usual metric is an open interval.
And $\overline{B}(x_\circ;r) = \{x \in \mathbb{R} : |x - x_0| \le r\}$
i.e. $x_0 - r \le x \le x_0 + r = [x_0 - r, x_0 + r]$
i.e. closed ball in a real line is a closed interval.
And $S(x_\circ;r) = \{x \in \mathbb{R} : |x - x_0| = r\} = \{x_0 - r, x_0 + r\}$
i.e. two point $x_0 - r$ and $x_0 + r$ only.

❖ *Open Set*

Let (X,d) be a metric space and set G is called open in X if for every $x \in G$, there exists an open ball $B(x;\, r) \subset G$.

❖ *Theorem*

An open ball in metric space X is open.

Proof.

Let $B(x_0;\, r)$ be an open ball in (X,d).
Let $y \in B(x_0;\, r)$ then $d(x_0,y) = r_1 < r$
Let $r_2 < r - r_1$, then $B(y;r_2) \subset B(x_0;r)$
Hence $B(x_0;\, r)$ is an open set.

Alternative:

Let $B(x_0;\, r)$ be an open ball in (X,d).
Let $x \in B(x_0;r)$ then $d(x_0,x) = r_1 < r$
Take $r_2 = r - r_1$ and consider the open ball $B(x\,;\, r_2)$
we show that $B(x\,;\, r_2) \subset B(x\,;\, r)$.
For this let $y \in B(x;r_2)$ then $d(x,y) < r_2$
and $d(x_0,y) \le d(x_0,x) + d(x,y)$
$< r_1 + r_2 \ = r$
hence $y \in B(x_0;r)$ so that $B(x;r_2) \subset B(x_0;r)$. Thus $B(x_0;r)$ is an open.

Q.E.D

NOTE: Let (X,d) be a metric space then

i) X and φ are open sets.
ii) Union of any number of open sets is open.
iii) Intersection of a finite number of open sets is open.

❖ *Limit point of a set*

Let (X,d) be a metric space and $A\subset X$, then $x\in X$ is called a *limit point* or *accumulation point* of A if for every open ball $B(x;r)$ with centre x,

$$B(x;r)\cap\{A-\{x\}\}\neq\varphi.$$

i.e. every open ball contain a point of A other than x.

❖ *Closed Set*

A subset A of metric space X is *closed* if it contains every limit point of itself. The set of all limit points of A is called the *derived set of A* and denoted by A'.

❖ *Theorem*

A subset A of a metric space is closed if and only if its complement A^c is open.

Proof.

Suppose A is closed, we prove A^c is open.
Let $x\in A^c$ then $x\notin A$.
$\Rightarrow x$ is not a limit point of A.
then by definition of a limit point there exists an open ball $B(x;r)$ such that

$$B(x;r)\cap A=\varphi.$$

This implies $B(x;r)\subset A^c$. Since x is an arbitrary point of A^c. So A^c is open.

Conversely, assume that A^c is an open then we prove A is closed.
i.e. A contain all of its limit points.
Let x be an accumulation point of A. and suppose $x\in A^c$.
then there exists an open ball $B(x;r)\subset A^c \quad\Rightarrow\quad B(x;r)\cap A=\varphi$.
This shows that x is not a limit point of A. this is a contradiction to our assumption.
Hence $x\in A$. Accordingly A is closed.

The proof is complete.

❖ *Theorem*

A closed ball is a closed set.

Proof.

Let $\overline{B}(x;r)$ be a closed ball. We prove $\overline{B}^c(x;r)=C$ (say) is an open ball.
Let $y\in C$ then $d(x,y)>r$.
Let $r_1=d(x,y)$ then $r_1>r$. And take $r_2=r_1-r$
Consider the open ball $B\left(y;\frac{r_2}{2}\right)$ we prove $B\left(y;\frac{r_2}{2}\right)\subset C$.

For this let $z \in B\left(y;\frac{r_2}{2}\right)$ then $d(z,y) < \frac{r_2}{2}$

By the triangular inequality

$$d(x,y) \le d(x,z) + d(z,y)$$

$$\Rightarrow d(x,y) \le d(z,x) + d(z,y) \qquad \because d(y,z) = d(z,y)$$

$$\Rightarrow d(z,x) \ge d(x,y) - d(z,y)$$

$$\Rightarrow d(z,x) > r_1 - \frac{r_2}{2} = \frac{2r_1 - r_2}{2} = \frac{2r_1 - r_1 + r}{2} = \frac{r_1 + r}{2} \qquad \because r_2 = r_1 - r$$

$$\Rightarrow d(z,x) > \frac{r+r}{2} = r \qquad \because r_1 - r = r_2 > 0 \quad \therefore r_1 > r$$

$\Rightarrow z \notin \overline{B}(x;r)$ This shows that $z \in C$

$$\Rightarrow B\left(y;\frac{r_2}{2}\right) \subset C$$

Hence C is an open set and consequently $\overline{B}(x;r)$ is closed. **Q.E.D**

❖ *Theorem*

Let (X,d) be a metric space and $A \subset X$. If $x \in X$ is a limit point of *A*. then every open ball $B(x;r)$ with centre *x* contain an infinite numbers of point of *A*.

Proof.

Suppose $B(x;r)$ contain only a finite number of points of *A*.

Let $a_1, a_2, ..., a_n$ be those points.

and let $d(x,a_i) = r_i$ where $i = 1,2,...,n$.

also consider $r' = \min(r_1, r_2, ..., r_n)$

Then the open ball $B(x;r')$ contain no point of *A* other than *x*. then *x* is not limit point of *A*. This is a contradiction therefore $B(x;r)$ must contain infinite numbers of point of *A*.

❖ *Closure of a Set*

Let (X,d) be a metric space and $M \subset X$. Then *closure of M* is denoted by $\overline{M} = M \cup M'$ where M' is the set of all limit points of *M*. It is the smallest closed superset of *M*.

❖ *Dense Set*

Let (X, d) be a metric space the a set $M \subset X$ is called dense in *X* if $\overline{M} = X$.

❖ *Countable Set*

A set *A* is *countable* if it is finite or there exists a function $f: A \to \mathbb{N}$ which is one-one and onto, where $\mathbb{N}$ is the set of natural numbers.

e.g. $\mathbb{N}, \mathbb{Q}$ and $\mathbb{Z}$ are countable sets . The set of real numbers, the set of irrational numbers and any interval are not countable sets.

❖ *Separable Space*

A space *X* is said to be *separable* if it contains a countable dense subsets.

e.g. the real line $\mathbb{R}$ is separable since it contain the set $\mathbb{Q}$ of rational numbers, which is dense is $\mathbb{R}$.

❖ *Theorem*

Let (*X*, *d*) be a metric space, $A \subset X$ is dense if and only if *A* has non-empty intersection with any open subset of *X*.

Proof.

Assume that A is dense in X. then $\overline{A} = X$.
Suppose there is an open set $G \subset X$ such that $A \cap G = \varphi$.
Then if $x \in G$ then $A \cap \left(G - \{x\}\right) = \varphi$
which show that *x* is not a limit point of A.
This implies $x \notin A$ but $x \in X \quad \Rightarrow \quad \overline{A} \neq X$
This is a contradiction.
Consequently $A \cap G \neq \varphi$ for any open $G \subset X$.

Conversely suppose that $A \cap G \neq \varphi$ for any open $G \subset X$.
We prove $\overline{A} = X$, for this let $x \in X$.
If $x \in A$ then $x \in A \cup A' = \overline{A}$ then $X = \overline{A}$.
If $x \notin A$ then let $\{G_i\}$ be the family of all the open subset of *X* such that $x \in G_i$ for every *i*.
Then by hypothesis $A \cap G_i \neq \varphi$ for any *i*. i.e G_i contain point of A other then *x*.
This implies that *x* is an accumulation point of *A*. i.e. $x \in A'$
Accordingly $x \in A \cup A' = \overline{A}$ and $X = \overline{A}$.
The proof is complete.

❖ *Neighbourhood of a Point*

Let (*X*, *d*) be a metric space and $x_0 \in X$ and a subset $N \subset X$ is called a *neighbourhood of* x_0 if there exists an open ball $B(x_0;\varepsilon)$ with centre x_0 such that $B(x_0;\varepsilon) \subset N$.

Shortly "*neighbourhood*" is written as "*nhood*".

❖ *Interior Point*

Let (*X*, *d*) be a metric space and $A \subset X$, a point $x_0 \in X$ is called an *interior point* of *A* if there is an open ball $B(x_0;r)$ with centre x_0 such that $B(x_0;r) \subset A$.
The set of all interior points of *A* is called *interior of A* and is denoted by *int(A)* or A°.
It is the largest open set contain in A. i.e. $A^\circ \subset A$.

❖ *Continuity*

A function $f:\left(X,d\right) \to \left(Y,d'\right)$ is called continuous at a point $x_0 \in X$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $d'\left(f(x), f(x_0)\right) < \varepsilon$ for all *x* satisfying $d(x,x_0) < \delta$.

ALTERNATIVE:

$f: X \to Y$ is continuous at $x_0 \in X$ if for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x \in B(x_0;\delta) \qquad \Rightarrow f(x) \in B\left(f(x_0);\varepsilon\right).$$

❖ *Theorem*

$f:(X,d)\to(Y,d')$ is continuous at $x_0\in X$ if and only if $f^{-1}(G)$ is open is X. wherever G is open in Y.

NOTE : Before proving this theorem note that if $f:X\to Y$, $f^{-1}:Y\to X$ and $A\subset X$, $B\subset Y$ then $f^{-1}f(A)\supset A$ and $f\,f^{-1}(B)\subset B$

Proof.

Assume that $f:X\to Y$ is continuous and $G\subset Y$ is open. We will prove $f^{-1}(G)$ is open in X.

Let $x\in f^{-1}(G)\ \Rightarrow\ f(x)\in f\,f^{-1}(G)\subset G$

When G is open, there is an open ball $B\big(f(x);\varepsilon\big)\subset G$.

Since $f:X\to Y$ is continuous, therefore for $\varepsilon>0$ there is a $\delta>0$ such that

$$y\in B(x;\delta)\ \Rightarrow\ f(y)\in B\big(f(x);\varepsilon\big)\subset G\ \text{ then }\ y\in f^{-1}f(G)\subset f^{-1}(G)$$

Since y is an arbitrary point of $B(x;\delta)\subset f^{-1}(G)$. Also x was arbitrary, this show that $f^{-1}(G)$ is open in X.

Conversely, for any $G\subset Y$ we prove $f:X\to Y$ is continuous.

For this let $x\in X$ and $\varepsilon>0$ be given. Now $f(x)\in Y$ and let $B\big(f(x);\varepsilon\big)$ be an open ball in Y. then by hypothesis $f^{-1}\Big(B\big(f(x);\varepsilon\big)\Big)$ is open in X and $x\in f^{-1}\Big(B\big(f(x);\varepsilon\big)\Big)$

As $y\in B\big(x;\delta\big)\subset f^{-1}\Big(B\big(f(x);\varepsilon\big)\Big)$

$\Rightarrow\ f(y)\in f\,f^{-1}\Big(B\big(f(x);\varepsilon\big)\Big)\subset B\big(f(x);\varepsilon\big)$ i.e. $f(y)\in B\big(f(x);\varepsilon\big)$

Consequently $f:X\to Y$ is continuous.

The proof is complete.

❖ *Convergence of Sequence:*

Let $(x_n)=(x_1,x_2,...)$ be a sequence in a metric space (X,d), we say (x_n) converges to $x\in X$ if $\lim\limits_{n\to\infty} d(x_n,x)=0$.

We write $\lim\limits_{n\to\infty} x_n = x$ or simply $x_n\to x$ as $n\to\infty$.

Alternatively, we say $x_n\to x$ if for every $\varepsilon>0$ there is an $n_0\in\mathbb{N}$, such that

$$\forall\ n>n_0,\qquad d(x_n,x)<\varepsilon.$$

❖ *Theorem*

If (x_n) is converges then limit of (x_n) is unique.

Proof.

Suppose $x_n\to a$ and $x_n\to b$,

Then $0\le d(a,b)\le d(a,x_n)+d(x_n,b)\to 0+0$ as $n\to\infty$ $\quad\Rightarrow\ d(a,b)=0\ \Rightarrow\ a=b$

Hence the limit is unique. ☉

ALTERNATIVE

Suppose that a sequence (x_n) converges to two distinct limits a and b.

and $d(a,b)=r>0$

Since $x_n\to a$, given any $\varepsilon>0$, there is a natural number n_1 depending on ε

such that

$$d(x_n,a)<\frac{\varepsilon}{2} \qquad \text{whenever} \quad n>n_1$$

Also $x_n \to b$, given any $\varepsilon>0$, there is a natural number n_2 depending on ε such that

$$d(x_n,b)<\frac{\varepsilon}{2} \qquad \text{whenever} \quad n>n_2$$

Take $n_0=\max(n_1,n_2)$ then

$$d(x_n,a)<\frac{\varepsilon}{2} \quad \text{and} \quad d(x_n,b)<\frac{\varepsilon}{2} \qquad \text{whenever } n>n_0$$

Since ε is arbitrary, take $\varepsilon=r$ then

$$r=d(a,b)\le d(a,x_n)+d(x_n,b)$$

$$<\frac{r}{2}+\frac{r}{2}=r \qquad \because\ d(a,x_n)=d(x_n,a)<\frac{\varepsilon}{2}$$

Which is a contradiction, Hence $a=b$ i.e. limit is unique.

❖ *Theorem*

i) A convergent sequence is bounded.
ii) If $x_n\to x$ and $y_n\to y$ then $d(x_n,y_n)\to d(x,y)$.

Proof.

(i) Suppose $x_n\to x$, therefore for any $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\forall\ n>n_0,\quad d(x_n,x)<\varepsilon$$

Let $a=\max\{d(x_1,x),d(x_2,x),............,d(x_n,x)\}$ and $k=\max\{\varepsilon,a\}$

Then by using triangular inequality for arbitrary $x_i,x_j\in(x_n)$

$$0\le d\left(x_i,x_j\right)\le d(x_i,x)+d(x,x_j)$$

$$\le k+k=2k$$

Hence (x_n) is bounded.

(ii) By using triangular inequality

$$d\left(x_n,y_n\right)\le d\left(x_n,x\right)+d\left(x,y\right)+d\left(y,y_n\right)$$

$$\Rightarrow\ d\left(x_n,y_n\right)-d\left(x,y\right)\le d\left(x_n,x\right)+d\left(y,y_n\right)\to 0+0 \quad \text{as} \quad n\to\infty \ ....(i)$$

Next $$d\left(x,y\right)\le d\left(x,x_n\right)+d\left(x_n,y_n\right)+d\left(y_n,y\right)$$

$$\Rightarrow\ d\left(x,y\right)-d\left(x_n,y_n\right)\le d\left(x,x_n\right)+d\left(y_n,y\right)\to 0+0 \quad \text{as} \quad n\to\infty \ ....(ii)$$

From (i) and (ii)

$$\left|d\left(x_n,y_n\right)-d\left(x,y\right)\right|\to 0 \text{ as } n\to\infty$$

Hence

$$\lim_{n\to\infty} d\left(x_n,y_n\right)=d(x,y) \qquad \textbf{Q.E.D}$$

❖ *Cauchy Sequence*

A sequence (x_n) in a metric space (X,d) is called *Cauchy* if any $\varepsilon>0$ there is a $n_0\in\mathbb{N}$ such that $\forall\ m,n>n_0,\quad d(x_m,x_n)<\varepsilon$.

❖ *Theorem*

A convergent sequence in a metric space (X,d) is Cauchy.

Proof.

Let $x_n \to x \in X$, therefore any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\forall\ m,n > n_0,\quad d(x_n,x) < \frac{\varepsilon}{2} \quad \text{and} \quad d(x_m,x) < \frac{\varepsilon}{2}.$$

Then by using triangular inequality

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,x) + d(x,x_n) \\ &\le d(x_m,x) + d(x_n,x) \qquad \because\ d(x,y) = d(y,x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus every convergent sequence in a metric space is Cauchy.

❖ *Example*

Let (x_n) be a sequence in the discrete space (X,d). If (x_n) be a Cauchy sequence, then for $\varepsilon = \frac{1}{2}$, there is a natural number n_0 depending on ε such that

$$d(x_m,x_n) < \tfrac{1}{2} \qquad \forall\ m,n \ge n_0$$

Since in discrete space *d* is either 0 or 1 therefore $d(x_m,x_n) = 0 \ \Rightarrow\ x_m = x_n = x$ (say)

Thus a Cauchy sequence in (X,d) become constant after a finite number of terms,

i.e. $(x_n) = (x_1, x_2, \dots, x_{n_0}, x, x, x, \dots)$

❖ *Subsequence*

Let $(a_1, a_2, a_3, \dots)$ be a sequence (X,d) and let $(i_1, i_2, i_3, \dots)$ be a sequence of positive integers such that $i_1 < i_2 < i_3 < \dots$ then $(a_{i_1}, a_{i_2}, a_{i_3}, \dots)$ is called s*ubsequence* of $(a_n : n \in \mathbb{N})$.

❖ *Theorem*

(i) Let (x_n) be a Cauchy sequence in (X,d), then (x_n) converges to a point $x \in X$ if and only if (x_n) has a convergent subsequence (x_{n_k}) which converges to $x \in X$.

(ii) If (x_n) converges to $x \in X$, then every subsequence (x_{n_k}) also converges to $x \in X$.

Proof.

(i) Suppose $x_n \to x \in X$ then (x_n) itself is a subsequence which converges to $x \in X$.

Conversely, assume that (x_{n_k}) is a subsequence of (x_n) which converges to *x*.

Then for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\forall\ n_k > n_0,\ d(x_{n_k}, x) < \frac{\varepsilon}{2}$.

Further more (x_n) is Cauchy sequence

Then for the $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that $\forall\ m,n > n_1,\ d(x_m,x_n) < \frac{\varepsilon}{2}$.

Suppose $n_2 = \max(n_0, n_1)$ then by using the triangular inequality we have

$$d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall\ n_k, n > n_2$$

This show that $x_n \to x$.

(ii) $x_n \to x$ implies for any $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$

Then in particular $d(x_{n_k}, x) < \varepsilon \quad \forall\ n_k > n_0$

Hence $x_{n_k} \to x \in X$.

❖ *Example*

Let $X = (0,1)$ then $(x_n) = (x_1, x_2, x_3, ...) = (1/2, 1/3, 1/4, ...)$ is a sequence in *X*. Then $x_n \to 0$ but 0 is not a point of *X*.

❖ *Theorem*

Let (X, d) be a metric space and $M \subset X$.

(i) Then $x \in \overline{M}$ if and only if there is a sequence (x_n) in *M* such that $x_n \to x$.

(ii) If for any sequence (x_n) in *M*, $x_n \to x \Rightarrow x \in M$, then *M* is closed.

Proof.

(i) Suppose $x \in \overline{M} = M \cup M'$

If $x \in M$, then there is a sequence $(x, x, x, ...)$ in *M* which converges to *x*.

If $x \notin M$, then $x \in M'$ i.e. *x* is an accumulation point of *M*, therefore each $n \in \mathbb{N}$ the open ball $B\left(x; \frac{1}{n}\right)$ contain infinite number of point of *M*.

We choose $x_n \in M$ from each $B\left(x; \frac{1}{n}\right)$

Then we obtain a sequence (x_n) of points of *M* and since $\frac{1}{n} \to 0$ as $n \to \infty$.

Then $x_n \to x$ as $n \to \infty$.

Conversely, suppose (x_n) such that $x_n \to x$.

We prove $x \in \overline{M}$

If $x \in M$ then $x \in \overline{M}$. $\quad \because\ \overline{M} = M \cup M'$

If $x \notin M$, then every neighbourhood of *x* contain infinite number of terms of (x_n).

Then *x* is a limit point of M i.e. $x \in M'$

Hence $x \in \overline{M} = M \cup M'$.

(ii) If (x_n) is in *M* and $x_n \to x$, then $x \in \overline{M}$ then by hypothesis $M = \overline{M}$, then *M* is closed.

❖ *Complete Space*

A metric space (X, d) is called *complete* if every Cauchy sequence in *X* converges to a point of *X*.

❖ *Subspace*

Let (X,d) be a metric space and $Y \subset X$ then Y is called *subspace* if Y is itself a metric space under the metric d.

❖ *Theorem*

A subspace of a complete metric space (X,d) is complete if and only if Y is closed in X.

Proof.

Assume that Y is complete we prove Y is closed.

Let $x \in \overline{Y}$ then there is a sequence (x_n) in Y such that $x_n \to x$.

Since convergent sequence is a Cauchy and Y is complete then $x_n \to x \in Y$.

Since x was arbitrary point of Y $\Rightarrow$ $\overline{Y} \subset Y$

Therefore $\overline{Y} = Y$ $\because Y \subset \overline{Y}$

Consequently Y is closed.

Conversely, suppose Y is closed and (x_n) is a Cauchy sequence. Then (x_n) is Cauchy in X and since X is complete so $x_n \to x \in X$.

Also $x \in \overline{Y}$ and $\overline{Y} \subset X$.

Since Y is closed i.e. $Y = \overline{Y}$ therefore $x \in Y$.

Hence Y is complete. ⊙

❖ *Nested Sequence:*

A sequence sets $A_1, A_2, A_3, \ldots$ is called *nested* if $A_1 \supset A_2 \supset A_3 \supset \ldots$

❖ *Theorem (Cantor's Intersection Theorem)*

A metric space (X,d) is complete if and only if every nested sequence of non-empty closed subset of X, whose diameter tends to zero, has a non-empty intersection.

Proof.

Suppose (X,d) is complete and let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a nested sequence of closed subsets of X.

Since A_i is non-empty we choose a point a_n from each A_n. And then we will prove $(a_1, a_2, a_3, \ldots)$ is Cauchy in X.

Let $\varepsilon > 0$ be given, since $\lim\limits_{n\to\infty} d(A_n) = 0$ then there is $n_0 \in \mathbb{N}$ such that $d\left(A_{n_0}\right) < 0$

Then for $m,n > n_0$, $d(a_m, a_n) < \varepsilon$.

This shows that (a_n) is Cauchy in X.

Since X is complete so $a_n \to p \in X$ (say)

We prove $p \in \bigcap\limits_n A_n$,

Suppose the contrary that $p \notin \bigcap\limits_n A_n$ then $\exists$ a $k \in \mathbb{N}$ such that $p \notin A_k$.

Since A_k is closed, $d(p, A_k) = \delta > 0$.

Consider the open ball $B\left(p;\frac{\delta}{2}\right)$ then A_k and $B\left(p;\frac{\delta}{2}\right)$ are disjoint

Now $a_k, a_{k+1}, a_{k+2}, ...$ all belong to A_k then all these points do not belong to $B\left(p;\frac{\delta}{2}\right)$

This is a contradiction as p is the limit point of $\left(a_n\right)$.

Hence $p \in \bigcap_n A_n$.

Conversely, assume that every nested sequence of closed subset of X has a non-empty intersection. Let (x_n) be Cauchy in X, where $(x_n) = (x_1, x_2, x_3, ...)$

Consider the sets

$$A_1 = \{x_1, x_2, x_3, ...\}$$
$$A_2 = \{x_2, x_3, x_4, ...\}$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$A_k = \{x_n : n \geq k\}$$

Then we have $A_1 \supset A_2 \supset A_3 \supset ...$

We prove $\lim_{n\to\infty} d(A_n) = 0$

Since (x_n) is Cauchy, therefore $\exists\ n_0 \in \mathbb{N}$ such that

$$\forall\ m,n > n_0,\ d(x_m, x_n) < \varepsilon, \quad \text{i.e.} \quad \lim_{n\to\infty} d(A_n) = 0.$$

Now $d\left(\overline{A_n}\right) = d\left(A_n\right)$ then $\lim_{n\to\infty} d(A_n) = \lim_{n\to\infty} d(\overline{A_n}) = 0$

Also $\overline{A_1} \supset \overline{A_2} \supset \overline{A_3} \supset ...$

Then by hypothesis $\bigcap_n \overline{A_n} \neq \varphi$. Let $p \in \bigcap_n \overline{A_n}$

We prove $x_n \to p \in X$

Since $\lim_{n\to\infty} d(\overline{A_n}) = 0$ therefore $\exists\ k_0 \in \mathbb{N}$ such that $d\left(\overline{A_{k_0}}\right) < \varepsilon$

Then for $n > k_0$, $x_n, p \in \overline{A_n} \quad \Rightarrow d(x_n, p) < \varepsilon \quad \forall\ n > k_0$

This proves that $x_n \to p \in X$.

The proof is complete.

❖ *Complete Space (Examples)*

(i) The discrete space is complete.

Since in discrete space a Cauchy sequence becomes constant after finite terms i.e. (x_n) is Cauchy in discrete space if it is of the form

$$(x_1, x_2, x_3, ..., x_n = b, b, b, ...)$$

(ii) The set $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ of integers with usual metric is complete.

(iii) The set of rational numbers with usual metric is not complete.

$\because$ $(1.1, 1.41, 1.412, ...)$ is a Cauchy sequence of rational numbers but its limit is $\sqrt{2}$, which is not rational.

(iv) The space of irrational number with usual metric is not complete.

We take $(-1,1),(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{3},\frac{1}{3}),...,(-\frac{1}{n},\frac{1}{n})$

We choose one irrational number from each interval and these irrational tends to zero as we goes toward infinity, as zero is a rational so space of irrational is not complete.

❖ Theorem

The real line is complete.

Proof.

Let (x_n) be any Cauchy sequence of real numbers.

We first prove that (x_n) is bounded.

Let $\varepsilon = 1 > 0$ then $\exists$ $n_0 \in \mathbb{N}$ such that $\forall$ $m,n \geq n_0$, $d(x_m, x_n) = |x_m - x_n| < 1$

In particular for $n \geq n_0$ we have

$$|x_{n_0} - x_n| \leq 1 \quad \Rightarrow x_{n_0} - 1 \leq x_n \leq x_{n_0} + 1$$

Let $\alpha = \max\{x_1, x_2, ..., x_{n_0} + 1\}$ and $\beta = \min\{x_1, x_2, ..., x_{n_0} - 1\}$

then $\beta \leq x_n \leq \alpha \quad \forall\ n$.

this shows that (x_n) is bounded with β as lower bound and α as upper bound.

Secondly we prove (x_n) has convergent subsequence (x_{n_i}).

If the range of the sequence is $\{x_n\} = \{x_1, x_2, x_3, ...\}$ is finite, then one of the term is the sequence say b will repeat infinitely i.e. $b, b, b, ...$

Then $(b, b, b, ...)$ is a convergent subsequence which converges to b.

If the range is infinite then by the Bolzano Weirestrass theorem, the bounded infinite set $\{x_n\}$ has a limit point, say b.

Then each of the open interval $S_1 = (b-1, b+1)$, $S_2 = (b - \frac{1}{2}, b + \frac{1}{2})$, $S_2 = (b - \frac{1}{3}, b + \frac{1}{3})$, … has an infinite numbers of points of the set $\{x_n\}$.

i.e. there are infinite numbers of terms of the sequence (x_n) in every open interval S_n.

We choose a point x_{i_1} from S_1, then we choose a point x_{i_2} from S_2 such that $i_1 < i_2$ i.e. the terms x_{i_2} comes after x_{i_1} in the original sequence (x_n). Then we choose a term x_{i_3} such that $i_2 < i_3$, continuing in this manner we obtain a subsequence

$$(x_{i_n}) = (x_{i_1}, x_{i_2}, x_{i_3}, \ldots).$$

It is always possible to choose a term because every interval contain an infinite numbers of terms of the sequence (x_n).

Since $b - \frac{1}{n} \to b$ and $b + \frac{1}{n} \to b$ as $n \to \infty$. Hence we have convergent subsequence (x_{i_n}) whose limit is b.

Lastly we prove that $x_n \to b \in \mathbb{R}$.

Since (x_n) is a Cauchy therefore for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\forall \ m,n > n_0 \quad |x_m - x_n| < \frac{\varepsilon}{2}$$

Also since $x_{i_n} \to b$ there is a natural number i_m such that $i_m > n_0$

Then $\forall \ m,n,i_m > n_0$

$$d(x_n,b) = |x_n - b| = |x_n - x_{i_m} + x_{i_m} - b|$$
$$\leq |x_n - x_{i_m}| + |x_{i_m} - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $x_n \to b \in \mathbb{R}$ and the proof is complete.

❖ *Theorem*

The Euclidean space $\mathbb{R}^n$ is complete.

Proof.

Let (x_m) be any Cauchy sequence in $\mathbb{R}^n$.

Then for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\forall \ m,r > n_0$

$$d(x_m,x_r) = \left(\sum \left(\overset{(m)}{\xi}_j - \overset{(r)}{\xi}_j \right)^2 \right)^{1/2} < \varepsilon \ ....(i)$$

where $x_m = \left(\overset{(m)}{\xi}_j\right) = \left(\overset{(m)}{\xi}_1, \overset{(m)}{\xi}_2, \overset{(m)}{\xi}_3, \ldots, \overset{(m)}{\xi}_n\right)$ and $x_r = \left(\overset{(r)}{\xi}_j\right) = \left(\overset{(r)}{\xi}_1, \overset{(r)}{\xi}_2, \overset{(r)}{\xi}_3, \ldots, \overset{(r)}{\xi}_n\right)$

Squaring both sided of (i) we obtain

$$\sum \left(\overset{(m)}{\xi}_j - \overset{(r)}{\xi}_j \right)^2 < \varepsilon^2$$

$$\Rightarrow \quad \left| \overset{(m)}{\xi}_j - \overset{(r)}{\xi}_j \right| < \varepsilon \quad \forall \quad j = 1,2,3,\ldots,n$$

This implies $\left(\overset{(m)}{\xi}_j\right) = \left(\overset{(1)}{\xi}_j, \overset{(2)}{\xi}_j, \overset{(3)}{\xi}_j, \ldots\right)$ is a Cauchy sequence of real numbers for every $j = 1,2,3,\ldots,n$.

Since $\mathbb{R}$ is complete therefore $\overset{(m)}{\xi}_j \to \xi_j \in \mathbb{R}$ (say)

Using these *n* limits we define

$$x = (\xi_j) = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n) \text{ then clearly } x \in \mathbb{R}^n.$$

We prove $x_m \to x$

In (i) as $r \to \infty$, $d(x_m, x) < \varepsilon \ \forall \ m > n_0$ which show that $x_m \to x \in \mathbb{R}^n$

And the proof is complete.

NOTE: In the above theorem if we take $n = 2$ then we see complex plane $\mathbb{C} = \mathbb{R}^2$ is complete. Moreover the unitary space $\mathbb{C}^n$ is complete.

❖ *Theorem*

The space l^∞ is complete.

Proof.

Let (x_m) be any Cauchy sequence in l^∞.

Then for any $\varepsilon > 0$ there is $n_0 > \mathbb{N}$ such that $\forall\ m,n > n_0$

$$d(x_m, x_n) = \sup_j \left| \overset{(m)}{\xi}_j - \overset{n}{\xi}_j \right| < \varepsilon \text{ .... } (i)$$

Where $x_m = \left(\overset{(m)}{\xi}_j\right) = \left(\overset{(m)}{\xi}_1, \overset{(m)}{\xi}_2, \overset{(m)}{\xi}_3, \ldots\right)$ and $x_n = \left(\overset{(n)}{\xi}_j\right) = \left(\overset{(n)}{\xi}_1, \overset{(n)}{\xi}_2, \overset{(n)}{\xi}_3, \ldots\right)$

Then from (i)

$$\left| \overset{(m)}{\xi}_j - \overset{(n)}{\xi}_j \right| < \varepsilon \text{ ....}(ii) \qquad \forall\ j = 1,2,3,\ldots \text{ and } \forall\ m,n > n_0$$

It means $\left(\overset{(m)}{\xi}_j\right) = \left(\overset{(1)}{\xi}_j, \overset{(2)}{\xi}_j, \overset{(3)}{\xi}_j, \ldots\right)$ is a Cauchy sequence of real or complex numbers for every $j = 1,2,3,\ldots$

And since $\mathbb{R}$ and $\mathbb{C}$ are complete therefore $\overset{(m)}{\xi}_j \to \xi_j \in \mathbb{R}$ or $\mathbb{C}$ (say).

Using these infinitely many limits we define $x = \left(\xi_j\right) = \left(\xi_1, \xi_2, \xi_3, \ldots\right)$.

We prove $x \in l^\infty$ and $x_m \to x$.

In (i) as $n \to \infty$ we obtain $\left| \overset{(m)}{\xi}_j - \xi_j \right| < \varepsilon$ ....(iii) $\forall\ m > n_0$

We prove x is bounded.

By using the triangular inequality

$$\left|\xi_j\right| = \left|\xi_j - \overset{(m)}{\xi}_j + \overset{(m)}{\xi}_j\right| \le \left|\xi_j - \overset{(m)}{\xi}_j\right| + \left|\overset{(m)}{\xi}_j\right| < \varepsilon + k_m$$

Where $\left|\overset{(m)}{\xi}_j\right| < k_m$ as x_m is bounded.

Hence $\left(\xi_j\right) = x$ is bounded.

This shows that $x_n \to x \in l^\infty$.

And the proof is complete.

❖ *Theorem*

The space **C** of all convergent sequence of complex number is complete.

Note: It is subspace of l^∞.

Proof.

First we prove **C** is closed in l^∞.

Let $x = \left(\xi_j\right) \in \overline{\mathbf{C}}$, then there is a sequence (x_n) in **C** such that $x_n \to x$,

where $x_n = \left(\overset{(n)}{\xi}_j\right) = \left(\overset{(n)}{\xi}_1, \overset{(n)}{\xi}_2, \overset{(n)}{\xi}_3, \ldots\right)$.

Then for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\forall\ n \ge n_0$

$$d\left(x_n, x\right) = \sup_j \left|\overset{(n)}{\xi}_j - \xi_j\right| < \frac{\varepsilon}{3}$$

Then in particular for $n = n_0$ and $\forall \; j = 1, 2, 3, ...$

$$\left| \overset{(n_0)}{\xi}_j - \xi_j \right| < \frac{\varepsilon}{3}$$

Now $x_{n_0} \in \mathbf{C}$ then x_{n_0} is a convergent sequence therefore $\exists \; n_1 \in \mathbb{N}$ such that $\forall \; j, k > n_1$

$$\left| \overset{(n_0)}{\xi}_j - \overset{(n_0)}{\xi}_k \right| < \frac{\varepsilon}{3}$$

Then by using triangular inequality we have

$$\begin{aligned} \left| \xi_j - \xi_k \right| &= \left| \xi_j - \overset{(n_0)}{\xi}_j + \overset{(n_0)}{\xi}_j - \overset{(n_0)}{\xi}_k + \overset{(n_0)}{\xi}_k - \xi_k \right| \\ &\leq \left| \xi_j - \overset{(n_0)}{\xi}_j \right| + \left| \overset{(n_0)}{\xi}_j - \overset{(n_0)}{\xi}_k \right| + \left| \overset{(n_0)}{\xi}_k - \xi_k \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \forall \; j, k > n_1 \end{aligned}$$

Hence x is Cauchy in l^∞ and x is convergent

Therefore $x \in \mathbf{C}$ and $\Rightarrow \; \overline{\mathbf{C}} = \mathbf{C}$.

i.e. $\mathbf{C}$ is closed in l^∞ and l^∞ is complete.

Since we know that a subspace of complete space is complete if and only if it is closed in the space.

Consequently **C** is complete.

❖ *Theorem*

The space l^p, $p \geq 1$ is a real number, is complete.

Proof.

Let (x_n) be any Cauchy sequence in l^p.

Then for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\forall \; m, n > n_0$

$$d\left(x_m, x_n\right) = \left(\sum_{j=1}^{\infty} \left| \overset{(m)}{\xi}_j - \overset{(n)}{\xi}_j \right|^p \right)^{\frac{1}{p}} < \varepsilon \; \text{…………..} \; (i)$$

where $x_m = \left(\overset{(m)}{\xi}_j \right) = \left(\overset{(m)}{\xi}_1, \overset{(m)}{\xi}_2, \overset{(m)}{\xi}_3, \ldots \right)$

Then from (i) $\quad \left| \overset{(m)}{\xi}_j - \overset{(n)}{\xi}_j \right| < \varepsilon$ ……… (ii) $\quad \forall \; m, n > n_0$ and for any fixed j.

This shows that $\left(\overset{(m)}{\xi}_j \right)$ is a Cauchy sequence of numbers for the fixed j.

Since $\mathbb{R}$ and $\mathbb{C}$ are complete therefore $\overset{(m)}{\xi}_j \to \xi_j \in \mathbb{R}$ or $\mathbb{C}$ (say) as $m \to \infty$.

Using these infinite many limits we define $x = \left(\xi_j\right) = \left(\xi_1, \xi_2, \xi_3, \ldots\right)$.

We prove $x \in l^p$ and $x_m \to x$ as $m \to \infty$.

From (i) we have

$$\left(\sum_{j=1}^{k}\left|\overset{(m)}{\xi}_j-\overset{(n)}{\xi}_j\right|^p\right)^{\frac{1}{p}}<\varepsilon$$

i.e. $$\sum_{j=1}^{k}\left|\overset{(m)}{\xi}_j-\overset{(n)}{\xi}_j\right|^p<\varepsilon^p \quad \ldots\ldots\ldots\ldots (iii)$$

Taking as $n\to\infty$, we get

$$\sum_{j=1}^{k}\left|\overset{(m)}{\xi}_j-\xi_j\right|^p<\varepsilon^p\ , \qquad k = 1, 2, 3, \ldots\ldots$$

Now taking $k\to\infty$, we obtain

$$\sum\left|\overset{(m)}{\xi}_j-\xi_j\right|^p<\varepsilon^p \quad \ldots\ldots\ldots\ldots (iv) \qquad \forall\ j=1,2,3,\ldots\ldots\ldots$$

This shows that $(x_m - x)\in l^p$

Now l^p is a vector space and $x_m\in l^p$, $x-x_m\in l^p$ then $x_m+(x-x_m)=x\in l^p$.

Also from (iv) we see that

$$\left(d(x_m,x)\right)^p<\varepsilon^p \qquad \forall\ m>n_0$$

i.e. $$d(x_m,x)<\varepsilon \qquad \forall\ m>n_0$$

This shows that $x_m\to x\in l^p$ as $x\to\infty$.

And the proof is complete.

❖ *Theorem*

The space **C**[a, b] is complete.

Proof.

Let (x_n) be a Cauchy sequence in **C**[a, b].

Therefore for every $\varepsilon>0$, there is $n_0\in\mathbb{N}$ such that $\forall\ m,n>n_0$

$$d(x_m,x_n)=\max_{t\in J}\left|x_m(t)-x_n(t)\right|<\varepsilon \quad \ldots\ldots\ldots\ldots (i) \qquad \text{where } J=[a,b].$$

Then for any fix $t=t_0\in J$

$$\left|x_m(t_0)-x_n(t_0)\right|<\varepsilon \qquad \forall\ m,n>n_0$$

It means $\left(x_1(t_0),x_2(t_0),x_3(t_0),\ldots\right)$ is a Cauchy sequence of real numbers. And since $\mathbb{R}$ is complete therefore $x_m(t_0)\to x(t_0)\in\mathbb{R}$ (say) as $m\to\infty$.

In this way for every $t\in J$, we can associate a unique real number $x(t)$ with $x_n(t)$.

This defines a function $x(t)$ on J.

We prove $x(t)\in\mathbf{C}[a, b]$ and $x_m(t)\to x(t)$ as $m\to\infty$.

From (i) we see that

$$\left|x_m(t)-x_n(t)\right|<\varepsilon \quad \text{for every } t\in J \text{ and } \forall\ m,n>n_0.$$

Letting $n\to\infty$, we obtain for all $t\in J$

$$\left|x_m(t)-x(t)\right|<\varepsilon \quad \forall\ m<n_0.$$

Since the convergence is uniform and the x_n's are continuous, the limit function $x(t)$ is continuous, as it is well known from the calculus.

Then $x(t)$ is continuous.

Hence $x(t)\in \mathbf{C}[a,b]$, also $\left|x_m(t)-x(t)\right|<\varepsilon$ as $m\to\infty$

Therefore $x_m(t)\to x(t)\in \mathbf{C}[a,b]$.

The proof is complete.

❖ *Theorem*

If (X,d_1) and (Y,d_2) are complete then $X\times Y$ is complete.

NOTE: The metric d (say) on $X\times Y$ is defined as $d(x,y)=\max\left(d_1(\xi_1,\xi_2),d_2(\eta_1,\eta_2)\right)$ where $x=(\xi_1,\eta_1)$, $y=(\xi_2,\eta_2)$ and $\xi_1,\xi_2\in X$, $\eta_1,\eta_2\in Y$.

Proof.

Let (x_n) be a Cauchy sequence in $X\times Y$.

Then for any $\varepsilon>0$, there is $n_0\in\mathbb{N}$ such that $\forall\ m,n>n_0$

$$d(x_m,x_n)=\max\left(d_1\left(\overset{(m)}{\xi},\overset{(n)}{\xi}\right),d_2\left(\overset{(m)}{\eta},\overset{(n)}{\eta}\right)\right)<\varepsilon$$

$$\Rightarrow\ d_1\left(\overset{(m)}{\xi},\overset{(n)}{\xi}\right)<\varepsilon \text{ and } d_2\left(\overset{(m)}{\eta},\overset{(n)}{\eta}\right)<\varepsilon\ \ \forall\ m,n>n_0$$

This implies $\left(\overset{(m)}{\xi}\right)=\left(\overset{(1)}{\xi},\overset{(2)}{\xi},\overset{(3)}{\xi},\ldots\right)$ is a Cauchy sequence in *X*.

and $\left(\overset{(m)}{\eta}\right)=\left(\overset{(1)}{\eta},\overset{(2)}{\eta},\overset{(3)}{\eta},\ldots\right)$ is a Cauchy sequence in *Y*.

Since *X* and *Y* are complete therefore $\overset{(m)}{\xi}\to\xi\in X$ (say) and $\overset{(m)}{\eta}\to\eta\in Y$ (say)

Let $x=(\xi,\mu)$ then $x\in X\times Y$.

Also $$d(x_m,x)=\max\left(d_1\left(\overset{(m)}{\xi},\xi\right),d_2\left(\overset{(m)}{\eta},\eta\right)\right)\to 0 \quad\text{as}\quad n\to\infty .$$

Hence $x_m\to x\in X\times Y$.

This proves completeness of $X\times Y$.

❖ *Theorem*

$f:(X,d)\to(Y,d')$ is continuous at $x_0\in X$ if and only if $x_n\to x$ implies $f(x_n)\to f(x_0)$.

Proof.

Assume that f is continuous at $x_0\in X$ then for given $\varepsilon>0$ there is a $\delta>0$ such that

$$d(x,x_0)<\delta \quad \Rightarrow\ d'\left(f(x),f(x_0)\right)<\varepsilon .$$

Let $x_n\to x_0$, then for our $\delta>0$ there is $n_0\in\mathbb{N}$ such that

$$d(x_n,x_0)<\delta ,\quad \forall\ n>n_0$$

Then by hypothesis $d'\left(f(x_n),f(x_0)\right)<\varepsilon ,\quad \forall\ n>n_0$

i.e. $f(x_n)\to f(x_0)$

Conversely, assume that $x_n\to x_0 \quad\Rightarrow\ f(x_n)\to f(x_0)$

We prove $f:X\to Y$ is continuous at $x_0\in X$, suppose this is false

Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is an $x \in X$ such that

$$d\left(x, x_0\right) < \delta \quad \text{but} \quad d'\left(f(x), f(x_0)\right) \geq \varepsilon$$

In particular when $\delta = \dfrac{1}{n}$, there is $x_n \in X$ such that

$$d\left(x_n, x_0\right) < \delta \quad \text{but} \quad d\left(f(x_n), f(x_0)\right) \geq \varepsilon .$$

This shows that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$ as $n \to \infty$.
This is a contradiction.
Consequently $f : X \to Y$ is continuous at $x_0 \in X$.
The proof is complete.

❖ *Rare (or nowhere dense in X)*

Let X be a metric, a subset $M \subset X$ is called *rare* (or *nowhere dense in X*) if $\overline{M}$ has no interior point i.e. $\text{int}\left(\overline{M}\right) = \varphi$.

❖ *Meager (or of the first category)*

Let X be a metric, a subset $M \subset X$ is called *meager* (or *of the first category*) if M can be expressed as a union of countably many rare subset of X.

❖ *Non-meager (or of the second category)*

Let X be a metric, a subset $M \subset X$ is called *non-meager* (or *of the second category*) if it is not meager (of the first category) in X.

❖ *Example:*

Consider the set $\mathbb{Q}$ of rationales as a subset of a real line $\mathbb{R}$. Let $q \in \mathbb{Q}$, then $\{q\} = \overline{\{q\}}$ because $\mathbb{R} - \{q\} = (-\infty, q) \cup (q, \infty)$ is open. Clearly $\{q\}$ contain no open ball. Hence $\mathbb{Q}$ is nowhere dense in $\mathbb{R}$ as well as in $\mathbb{Q}$. Also since $\mathbb{Q}$ is countable, it is the countable union of subsets $\{q\}$, $q \in \mathbb{Q}$. Thus $\mathbb{Q}$ is of the first category.

❖ *Bair's Category Theorem*

If $X \neq \varphi$ is complete then it is non-meager in itself.

OR

A complete metric space is of second category.

Proof.

Suppose that X is meager in itself then $X = \bigcup\limits_{k=1}^{\infty} M_k$, where each M_k is rare in X.

Since M_1 is rare then $\text{int}(M) = M^\circ = \varphi$

i.e. $\overline{M_1}$ has non-empty open subset

But X has a non-empty open subset (i.e. X itself) then $\overline{M_1} \neq X$.

This implies $\overline{M_1}^{\,c} = X - \overline{M_1}$ is a non-empty and open.

We choose a point $p_1 \in \overline{M_1}^{\,c}$ and an open ball $B_1 = B\left(p_1; \varepsilon_1\right) \subset \overline{M_1}^{\,c}$, where $\varepsilon_1 < \frac{1}{2}$.

Now $\overline{M_2}^c$ is non-empty and open

Then $\exists$ a point $p_2 \in \overline{M_2}^c$ and open ball $B_2 = B(p_2;\varepsilon_2) \in \overline{M_2}^c \cap B\left(p_1;\frac{1}{2}\varepsilon_1\right)$

($\overline{M_2}$ has no non-empty open subset then $\overline{M_2}^c \cap B\left(p_1;\frac{1}{2}\varepsilon_1\right)$ is non-empty and open.)

So we have chosen a point p_2 from the set $\overline{M_2}^c \cap B\left(p_1;\frac{1}{2}\varepsilon_1\right)$ and an open ball $B(p_2,\varepsilon_2)$ around it, where $\varepsilon_2 < \frac{1}{2}\varepsilon_1 < \frac{1}{2}\cdot\frac{1}{2} < 2^{-1}$.

Proceeding in this way we obtain a sequence of balls B_k such that

$$B_{k+1} \subset B\left(p_k;\frac{1}{2}\varepsilon_k\right) \subset B_k \quad \text{where} \quad B_k = B(p_k;\varepsilon_k) \quad \forall\ k = 1,2,3,\ldots\ldots.$$

Then the sequence of centres p_k is such that for $m > n$

$$d(p_m, p_n) < \frac{1}{2}\varepsilon_m < \frac{1}{2^{m+1}} \to 0 \quad \text{as} \quad m \to \infty.$$

Hence the sequence (p_k) is Cauchy.

Since X is complete therefore $p_k \to p \in X$ (say) as $k \to \infty$.

Also

$$\begin{aligned} d(p_m, p) &\le d(p_m, p_n) + d(p_n, p) \\ &< \frac{1}{2}\varepsilon_m + d(p_n, p) \\ &< \varepsilon_m + d(p_n, p) \to \varepsilon_m + 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

$\Rightarrow\ p \in B_m \quad \forall\ m \quad$ i.e. $p \in \overline{M_m}^c \quad \forall\ m \qquad \because\ B_m = \overline{M_2}^c \cap B\left(p_{m-1};\tfrac{1}{2}\varepsilon_{m-1}\right)$

$\Rightarrow\ B_m \subset \overline{M_m}^c \quad \Rightarrow\ B_m \cap M_m = \varphi$

$\Rightarrow p \notin M_m \quad \forall\ m \quad \Rightarrow p \notin X$

This is a contradiction.

Bair's Theorem is proof.

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