Theory of Relativity & Analytic Dynamics: Handwritten Notes

by

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These notes cover few topics of first section "theory of relativity", so it is advised to be careful while using these notes. These cover most of "analytic dynamics". Please don't use these notes for reference, try to find some book.

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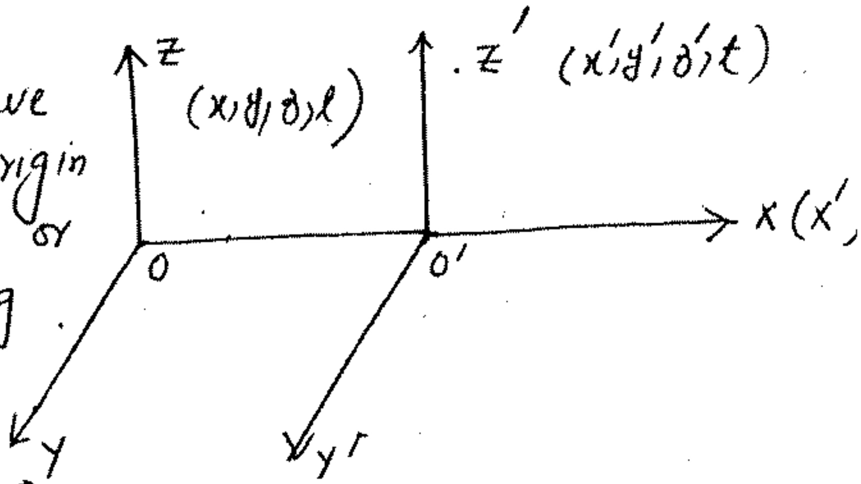
LORENTZ TRANSFORMATION

Consider two frames (x, y, z, t) and (x', y', z', t') Coincides with each other at instant $t = t' = 0$
The Origin of the latter frame moving along the Common x-axis with Uniform Speed v, the remaining pairs of axis are parallel
Let the transformation equation Connecting the two frames are

$$x' = \alpha(x - vt) \to (1) \quad y' = y \to (2) \quad z' = z \to (3)$$
$$t' = (\beta t + \gamma x) \to (4)$$

Let at $t = t' = 0$
a Spherical electrical wave Starts from the Common Origin with Speed c. At t or t' it will describing the Spheres



$$x^2 + y^2 + z^2 = c^2 t^2 \to (5)$$
$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \to (6)$$

Using (1) (2) (3) and (4) in (6)

$$\alpha^2 (x - vt)^2 + y^2 + z^2 = c^2 (\beta t + \gamma x)^2$$
$$\alpha^2 x^2 + \alpha^2 v^2 t^2 - 2xvt\alpha^2 + y^2 + z^2 = c^2\beta^2 t^2 + c^2\gamma^2 x^2 + 2c^2\beta\gamma x t$$

$$(\alpha^2 - c^2\gamma^2) x^2 + y^2 + z^2 - 2(v\alpha^2 + \beta\gamma c^2) x t = (c^2\beta^2 - \alpha^2 v^2) t^2 \to (7)$$

Comparing (5) and (7)

$$\alpha^2 - c^2\gamma^2 = 1 \to (i)$$
$$-2(v\alpha^2 + \beta\gamma c^2) = 0 \Rightarrow v\alpha^2 + \beta\gamma c^2 = 0 \to (ii)$$
$$c^2\beta^2 - \alpha^2 v^2 = c^2 \to (iii)$$

from (ii) $\Rightarrow \alpha^2 = -\dfrac{\beta\gamma c^2}{v} \to (iv)$

$$\therefore (i) \Rightarrow -\frac{\beta\gamma c^2}{v} - c^2\gamma^2 = 1$$
$$\Rightarrow -c^2\gamma\left(\frac{\beta + v\gamma}{v}\right) = 1 \Rightarrow c^2\gamma(\beta + v\gamma) = -v \to (v)$$

$$(iii) \Rightarrow c^2\beta^2 - \left(-\frac{\beta\gamma c^2}{v}\right) v^2 = c^2 \Rightarrow \cancel{c^2}\beta(\beta + \gamma v) = \cancel{c^2}$$
$$\Rightarrow \beta(\beta + \gamma v) = 1 \to (vi)$$

$$(vi) \Rightarrow \beta^2 + \beta\gamma v = 1 \Rightarrow \therefore \beta\gamma v = 1-\beta^2 \Rightarrow \gamma = \frac{1-\beta^2}{\beta v} \rightarrow (a)$$

$$\therefore (iv) \rightarrow \quad \alpha^2 = -\frac{\cancel{\beta} c^2}{v}\left(\frac{1-\beta^2}{\cancel{\beta} v}\right) \Rightarrow \alpha^2 = -\frac{(1-\beta^2)c^2}{v^2} \rightarrow (b)$$

$$\therefore (i) \Rightarrow -\left(\frac{1-\beta^2}{v^2}\right)c^2 - c^2\left(\frac{1-\beta^2}{\beta v}\right)^2 = 1 \qquad \text{using (a) \& (b)}$$

$$\Rightarrow -\frac{c^2(1-\beta^2)}{v^2}\left(1 + \frac{(1-\beta^2)}{\beta^2}\right) = 1$$

$$\Rightarrow -c^2\left(\frac{1-\beta^2}{v^2}\right)\left(\frac{\cancel{\beta^2}+1-\cancel{\beta^2}}{\beta^2}\right) = 1 \Rightarrow -c^2(1-\beta^2) = v^2\beta^2$$

$$\Rightarrow -c^2 + c^2\beta^2 = v^2\beta^2$$

$$\Rightarrow (c^2-v^2)\beta^2 = c^2 \Rightarrow \beta^2 = \frac{c^2}{c^2-v^2} \Rightarrow \beta = \frac{c}{\sqrt{c^2-v^2}}$$

$$\Rightarrow \beta = \frac{\cancel{c}}{\cancel{c}\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}} \rightarrow (c).$$

Now

$$\beta^2 = \frac{1}{(1-v^2/c^2)} \Rightarrow \beta^2 = \frac{1}{\frac{c^2-v^2}{c^2}} = \frac{c^2}{(c^2-v^2)}$$

$$1-\beta^2 = 1 - \frac{c^2}{c^2-v^2} = \frac{\cancel{c^2}-v^2-\cancel{c^2}}{(c^2-v^2)} = \frac{-v^2}{(c^2-v^2)}$$

$$\therefore (a) \Rightarrow \gamma = \frac{-v^{\cancel{2}}}{(c^2-v^2)} \cdot \frac{1}{\cancel{v}\frac{1}{\sqrt{1-v^2/c^2}}} = \frac{-v}{c^2(1-v^2/c^2)} \cdot \sqrt{1-v^2/c^2} = \frac{-\frac{v}{c^2}}{\sqrt{1-v^2/c^2}} \rightarrow (d)$$

$$(b) \Rightarrow \alpha^2 = -\left(\frac{-\cancel{v^2}}{c^2-v^2}\right)\frac{c^2}{\cancel{v^2}} = \frac{1}{\cancel{c^2}(1-v^2/c^2)}\cancel{c^2} = \frac{1}{1-v^2/c^2}$$

$$\Rightarrow \alpha = \frac{1}{\sqrt{1-v^2/c^2}} \rightarrow (e)$$

$\therefore$ the transformation Equations $x' = \frac{1}{\sqrt{1-v^2/c^2}}(x-vt)$, $y = y'$, $z = z'$, $t' = \frac{t}{\sqrt{1-v^2/c^2}} - \frac{\frac{v}{c^2}x}{\sqrt{1-v^2/c^2}}$

$$= \frac{t - \frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}$$

This is Called Lorentz transformation
When v is Small as Compared to c
then transformation Equation $\Rightarrow$ $x' = x - vt \quad y' = y \quad z' = z \quad t' = t$
($\because v/c$ is Small so take it Zero)
Which is also Called Galelian transformation

Effect of Lorentz transformation ON OUR MOTION OF TIME AND SPACE

Consider a clock moving with frame S' placed at a Point (x_0', y_0', z_0') relative to frame S, its time is,

$$t = \frac{t' + \frac{v x_0}{c^2}}{\sqrt{1 - v^2/c^2}}$$

In S interval, $t_2 - t_1$ is related to the Corresponding S' interval $t_2' - t_1'$ is given by

$$t_2 - t_1 = t_2' - t_1' \Big/ \sqrt{1 - v^2/c^2}$$

then according to an observer in a frame S, the clock of the frame S' appears to slow down by the factor. If a clock is fixed in the S-frame, then according to S', its time is given by $t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$

$$\therefore \quad t_2' - t_1' = t_2 - t_1 \Big/ \sqrt{1 - v^2/c^2}$$

According to an observer in S' frame, the clock of S frame appears to be slow down by the $\sqrt{1 - v^2/c^2}$. This is Called <u>CLOCK PARADOX</u>

Let the length of a rigid rod placed along x-axis be l, l' relative to S and S' respectively & Let the Co-ordinate of the ends points $(x_1, 0, 0), (x_2, 0, 0)$, & $(x_1', 0, 0), (x_2', 0, 0)$ w.r.t. S and S'

therefore $x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \qquad x_2' = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}}$

$$\Rightarrow \quad x_2' - x_1' = (x_2 - x_1) \Big/ \sqrt{1 - v^2/c^2} \quad \Rightarrow \quad l' = \frac{l}{\sqrt{1 - v^2/c^2}}$$

Thus according to an observer in the S-frame, the rod appears to be Contracted.

CONCLUSIONS :- 1. Every clock appears to go its fastest rate, when it is at rest relative to an observer. If it moves relatively to the observer with velocity v, its rate appears to be slow down by $\sqrt{1-v^2/c^2}$

2/ Every rod appears to be largest when it is at rest relative to the observer. If it moves relatively to the observer with velocity v, its length to be contracted by $\sqrt{1-v^2/c^2}$

Relativity of Simultaneity

Two Events which are simultaneously relative to S but occur at different places are not simultaneously relative to S'

Let that two Events occur at $(x_1,0,0)$ and $(x_2,0,0)$ and simultaneous w.r.t. S ie

$$t_1 = t_2$$

$$\therefore \quad t_1' = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1-v^2/c^2}} \qquad t_2' = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow t_2' - t_1' = \frac{v/c^2\,(x_1 - x_2)}{\sqrt{1-v^2/c^2}} \qquad \left(\begin{matrix} t_1 \,\&\, t_2 \\ \text{cancel} \\ \because t_1 = t_2 \end{matrix}\right)$$

RELATIVISTIC LAW OF ADDTION OF VELOCITIES
OR
PRODUCT OF TWO LORENTZ TRANSFORMATION IS A LORENTZ TRANSFORMATION

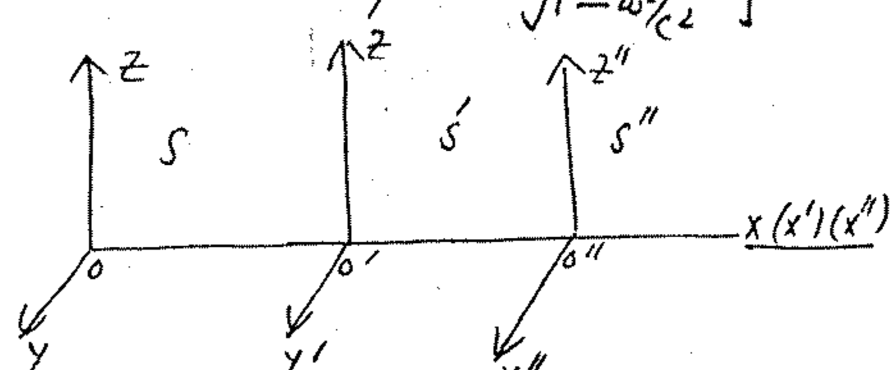
Let S' move relatively to S with speed v and S'' move relative to S' with speed ω

Thus transformation from S to S' is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad y' = y \quad z' = z \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}} \quad \Bigg] \rightarrow (I)$$

and the transformation from S' to S'' is

$$x'' = \frac{x' - \omega t'}{\sqrt{1 - \omega^2/c^2}} \quad y'' = y' \quad z'' = z' \quad t'' = \frac{t' - \frac{\omega x'}{c^2}}{\sqrt{1 - \omega^2/c^2}} \quad \Bigg] \rightarrow (2)$$



using (1) in (2)

$$x'' = \frac{\frac{x - vt}{\sqrt{1 - v^2/c^2}} - \omega \frac{\left(t - \frac{vx}{c^2}\right)}{\sqrt{1 - v^2/c^2}}}{\sqrt{1 - \omega^2/c^2}} = \frac{x - vt - \omega t + \frac{\omega v x}{c^2}}{\sqrt{1 - v^2/c^2}\sqrt{1 - \omega^2/c^2}}$$

$$= \frac{x\left(1 + \frac{\omega v}{c^2}\right) - (v + \omega)t}{\sqrt{\left(1 - v^2/c^2\right)\left(1 - \omega^2/c^2\right)}} = \frac{\left(1 + \frac{\omega v}{c^2}\right)\left[x - \left(\frac{v + \omega}{1 + \frac{\omega v}{c^2}}\right)t\right]}{\sqrt{1 - \omega^2/c^2 - v^2/c^2 + \frac{\omega^2 v^2}{c^4}}}$$

$$= \frac{\left(1 + \frac{\omega v}{c^2}\right)\left(x - \left(\frac{v + \omega}{1 + \frac{v\omega}{c^2}}\right)t\right)}{\sqrt{\left(1 + \frac{\omega^2 v^2}{c^4} + \frac{2\omega v}{c^2}\right) - \left(\frac{\omega^2}{c^2} + v^2/c^2 + \frac{2\omega v}{c^2}\right)}} = \frac{\left(1 + \frac{\omega v}{c^2}\right)\left[x - \frac{(v + \omega)}{\left(1 + \frac{v\omega}{c^2}\right)}t\right]}{\sqrt{\left(1 + \frac{\omega v}{c^2}\right)^2 - \frac{1}{c^2}(\omega + v)^2}}$$

$$x'' = \frac{\cancel{\left(1+\frac{wv}{c^2}\right)}\left(x - \frac{(v+w)}{\left(1+\frac{wv}{c^2}\right)}t\right)}{\cancel{\left(1+\frac{wv}{c^2}\right)}\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{wv}{c^2}}\right)^2}} = \frac{x-Ut}{\sqrt{1-U^2/c^2}}$$

where $U = \dfrac{v+w}{1+\frac{wv}{c^2}}$

$$t'' = \frac{\dfrac{t-\frac{vx}{c^2}}{\sqrt{1-v^2/c^2}} - \dfrac{w}{c^2}\left(\dfrac{x-vt}{\sqrt{1-v^2/c^2}}\right)}{\sqrt{1-w^2/c^2}} = \frac{t-\frac{vx}{c^2}-\frac{wx}{c^2}+\frac{wvt}{c^2}}{\sqrt{1-v^2/c^2}\sqrt{1-w^2/c^2}}$$

$$= \frac{t\left(1+\frac{wv}{c^2}\right) - \frac{x}{c^2}(v+w)}{\sqrt{\left(1+\frac{wv}{c^2}\right)^2 - \frac{1}{c^2}(v+w)^2}} = \frac{\cancel{\left(1+\frac{wv}{c^2}\right)}\left[t - \left(\frac{v+w}{1+\frac{wv}{c^2}}\right)\frac{x}{c^2}\right]}{\cancel{\left(1+\frac{wv}{c^2}\right)}\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{wv}{c^2}}\right)^2}}$$

$$= \frac{t-\frac{Ux}{c^2}}{\sqrt{1-U^2/c^2}}$$

$\Rightarrow$ S'' moves relative to S with velocity $U = \dfrac{v+w}{1+\frac{vw}{c^2}}$

MINKOWSKI SPACE

The Equation

$$x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2$$

$$\Rightarrow \sum_{i=1}^{3} x_i^2 + i^2c^2t^2 = \sum_{i=1}^{3} x_i'^2 + i^2c^2t'^2$$

where $x_1 = x$ $\qquad x_1' = x'$
$x_2 = y$ $\qquad x_2' = y'$
$x_3 = z$ $\qquad x_3' = z'$

Let $x_4 = ict$ $\qquad x_4' = ict'$

$$\sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} x_i'^2$$

This shows that in terms of Co-ordinate x_1, x_2, x_3, x_4 the Lorentz transformation is an orthogonal transformation. The Co-ordinate x_1, x_2, x_3, x_4 are called Minkowski Co-ordinates and the Space determined by them is called Minkowski Space.

Note It is an abstract Space or Complex Space $\because$ $x_4 = ict$

$x_4' = ict'$

Interms of Minkowski Co-ordinate the L.T. is

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \quad \text{—— (i)} \qquad \because x_4 = ict, \quad t = \frac{x_4}{ic}$$

$$= \frac{x_1 - \frac{v}{ic} x_4}{\sqrt{1 - v^2/c^2}} \rightarrow (1) \qquad x_2' = x_2, \quad y_2' = y_2$$

$$t' = \frac{t - \frac{v x_1}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$\therefore x_4' = ict'$

$t' = \frac{x_4'}{ic}$

$$\therefore \frac{x_4'}{ic} = \frac{\frac{x_4}{ic} - \frac{v}{c^2} x_1}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow x_4' = \frac{x_4 - \frac{v}{c^{\cancel{2}}} x_1 (i\cancel{c})}{\sqrt{1 - v^2/c^2}} = \frac{x_4 - \frac{ivx_1}{c}}{\sqrt{1 - v^2/c^2}} \rightarrow (2)$$

x_1, x_4 — plane

$$(1) \Rightarrow x_1' = \frac{x_1 + \frac{i^2 v}{i c} x_4}{\sqrt{1 - v^2/c^2}} = \frac{x_1}{\sqrt{1 - v^2/c^2}} + \frac{iv/c \; x_4}{\sqrt{1 - v^2/c^2}}$$

$$(2) \Rightarrow \text{and } x_4' = \frac{x_4}{\sqrt{1 - v^2/c^2}} - \frac{iv/c}{\sqrt{1 - v^2/c^2}} x_1$$

$$\Rightarrow \quad x_1' = x_1 \cos\phi + \sin\phi \, x_4 = x_1\cos\phi + x_4 \sin\phi$$
$$x_4' = x_4 \cos\phi - \sin\phi \, x_1 = x_4 \cos\phi - x_1 \sin\phi$$

where $\cos\phi = \dfrac{1}{\sqrt{1-v^2/c^2}}$ $\quad \sin\phi = \dfrac{iv/c}{\sqrt{1-v^2/c^2}}$

Therefore Lorentz transformation through an angle ϕ in the $x_1 x_4$ — plane and Lorentz transformation are

$$x_1' = x_1 \cos\phi + x_4 \sin\phi$$
$$x_2' = x_2$$
$$x_3' = x_3$$
$$x_4' = -x_1 \sin\phi + x_4 \cos\phi$$

the matrix of Lorentz transformation is

$$\begin{bmatrix} \cos\phi & 0 & 0 & \sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\phi & 0 & 0 & \cos\phi \end{bmatrix}$$

WORLD TIME OR PROPER TIME

We define an invariant $d\tau^2 = -\frac{1}{c^2} \sum_{i=1}^{4} (dx_i)^2 \rightarrow ①$

where x_i is the Minkowski Co-ordinate of an Event. (Invariance of $d\tau^2$ is evident from the fact that it is propotional to the scalar square of a vector dx_i); Let the particle is at rest relative to s' So that the Components of dx_i' are $(0,0,0, ic\,dt')$ $\quad ① \Rightarrow d\tau^2 = -\frac{1}{c^2}\left[0+0+0-c^2 dt'^2\right]$

$$\Rightarrow d\tau^2 = dt'^2$$

$\Rightarrow d\tau$ is an infinitesimal time

in which the Particle is at rest
When $d\tau$ is real It is Called time like interval
when $d\tau$ is Imaginary, It is said to be Space like

$① \Rightarrow$

$$d\tau^2 = -\frac{1}{c^2}\left(dx^2 + dy^2 + dz^2 + i^2c^2dt^2\right)$$
$$= \frac{-1}{c^2}\left(dx^2 + dy^2 + dz^2 - c^2dt^2\right)$$
$$= dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right)$$

$r = xi + yj + zk$
$\frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$
$|v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

$$\therefore \left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)$$

$$\therefore \left(\frac{d\tau}{dt}\right)^2 = 1 - v^2/c^2 \qquad \because v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\therefore \frac{d\tau}{dt} = \sqrt{1 - v^2/c^2} \longrightarrow ⓐ$$

$$\int d\tau = \int \sqrt{1 - v^2/c^2}\, dt \Rightarrow \tau = \int \sqrt{1 - v^2/c^2}\, dt$$

Now
$$\frac{dx_i}{d\tau} = \frac{dx_i}{dt} \cdot \frac{dt}{d\tau} = \frac{dx_i}{dt} \cdot \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \text{using ⓐ}$$

$$\frac{dx_i}{d\tau} = (1 - v^2/c^2)^{-1/2}\, \dot{x}_i$$

We define a vector $\underline{V}$ as $\bar{V} = \left(\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}, \frac{dx_4}{dt}\right)$

$$\bar{V} = \left((1 - v^2/c^2)^{-1/2}v_x, (1 - v^2/c^2)^{-1/2}v_y, (1 - v^2/c^2)^{-1/2}v_z, (1 - v^2/c^2)^{-1/2}ic\right)$$

$$\bar{V} = (1 - v^2/c^2)^{-1/2}\left(v_x, v_y, v_z, ic\right) \rightarrow ②$$

$\bar{V}$ is Called 4-velocity vector $\therefore$ ② is written as

$$= (1 - v^2/c^2)^{-1/2}(v, ic)$$

To Find the Component of $\bar{V}$ w.r.t. $\bar{S}$ moving with velocity u relative to S

By L.T.

$$\left.\begin{aligned} \bar{x}_1 &= x_1\cos\alpha + x_4\sin\alpha \\ \bar{x}_2 &= x_2 \qquad \bar{x}_3 = x_3 \\ \bar{x}_4 &= -x_1\sin\alpha + x_4\cos\alpha \end{aligned}\right] \to (1)$$

$$(1) \Rightarrow \left.\begin{aligned} \frac{d\bar{x}_1}{dT} &= \frac{dx_1}{dT}\cos\alpha + \frac{dx_4}{dT}\sin\alpha \\ \frac{d\bar{x}_2}{dT} = \frac{dx_2}{dT} \qquad \frac{d\bar{x}_3}{dT} &= \frac{dx_3}{dT} \qquad \frac{d\bar{x}_4}{dT} = -\frac{dx_1}{dT}\sin\alpha + \frac{dx_4}{dT}\cos\alpha \end{aligned}\right] \to (2)$$

$$(2) \Rightarrow \left.\begin{aligned} \bar{V}_1 &= V_1\cos\alpha + V_4\sin\alpha \\ \bar{V}_2 &= V_2 \qquad \bar{V}_3 = V_3 \qquad \bar{V}_4 = -V_1\sin\alpha + V_4\cos\alpha \end{aligned}\right] \to (3)$$

(Note $\frac{d\bar{x}_1}{dT} = \frac{d\bar{x}_1}{dt}\,\frac{dt}{dT} = \bar{v}_x\left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2}$

$$(3) \Rightarrow \left.\begin{aligned} \left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2}\bar{v}_x &= \left(1-v^2/c^2\right)^{-1/2} v_x\cos\alpha + ic\left(1-v^2/c^2\right)^{-1/2}\sin\alpha \\ \left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2}\bar{v}_y &= \left(1-v^2/c^2\right)^{-1/2} v_y \\ \left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2}\bar{v}_z &= \left(1-v^2/c^2\right)^{-1/2} v_z \\ \left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2} ic &= -v_x\left(1-\frac{v^2}{c^2}\right)^{-1/2}\sin\alpha + ic\left(1-v^2/c^2\right)^{-1/2}\cos\alpha \end{aligned}\right] \to (4)$$

But $\cos\alpha = \dfrac{1}{\sqrt{1-u^2/c^2}}$ $\quad \sin\alpha = \dfrac{\frac{iu}{c}}{\sqrt{1-u^2/c^2}}$

$$(4) \Rightarrow \bar{v}_x = \frac{\left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2}}{\left(1-v^2/c^2\right)^{1/2}\left(1-u^2/c^2\right)^{1/2}}\left(v_x + i\cancel{c}\left(\frac{iu}{\cancel{c}}\right)\right) = \frac{\sqrt{1-\frac{\bar{v}^2}{c^2}}}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}\left(v_x-u\right)$$

$$= \lambda\left(v_x-u\right) \to (i)$$

$$\bar{v}_y = \frac{\left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2}}{\left(1-v^2/c^2\right)^{1/2}\left(1-u^2/c^2\right)^{1/2}}\left(1-u^2/c^2\right)^{1/2} v_y = \frac{\sqrt{1-\bar{v}^2/c^2}}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}\left(1-u^2/c^2\right)^{1/2} v_y$$

$$\bar{v}_y = \lambda\left(1-u^2/c^2\right)^{1/2} v_y \to (ii)$$

Similarly

$$\bar{v}_z = \lambda\left(1-\frac{u^2}{c^2}\right)^{1/2} v_z \to (iii)$$

$$ic = \frac{\left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2}}{\left(1-v^2/c^2\right)^{1/2}\left(1-u^2/c^2\right)^{1/2}}\left[\frac{-v_x\, icu}{c} + ic\right]$$

$$1 = \sqrt{\frac{\left(1-\frac{\bar{v}^2}{c^2}\right)}{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}\left(-\frac{v_x u}{c^2}+1\right)$$

$$\therefore \quad 1 = \lambda\left(1-\frac{v_x u}{c^2}\right) \rightarrow \text{(iv)}$$

dividing (iv) by (i), (ii) & (iii)

$$\bar{v}_x = \frac{v_x - u}{\left(1-\frac{v_x u}{c^2}\right)} \qquad \bar{v}_y = \frac{\left(1-u^2/c^2\right)^{1/2} v_y}{\left(1-\frac{v_x u}{c^2}\right)}$$

$$\bar{v}_z = \frac{\left(1-u^2/c^2\right)^{1/2} v_z}{\left(1-\frac{v_x u}{c^2}\right)^{1/2}}$$

Note 1 If u and v are small as compared to c
then $\bar{v}_x = v_x - u$
$\bar{v}_y = v_y \qquad \bar{v}_z = v_z$

or $\bar{v} = \bar{v} - u$ which is classical law for relative velocity

2// Let a light pulse travels along the x-axis with velocity c relative to S, we find its velocity relative to $\bar{S}$

$$\therefore \quad v_x = c \qquad v_y = v_z = 0$$

$$\therefore \quad \bar{v}_x = \frac{c-u}{1-\frac{cu}{c^2}} = \frac{c-u}{1-u/c} = c\frac{(c-u)}{(c-u)} = c$$

$$\bar{v}_y = \bar{v}_z = 0$$

MASS AND MOMENTUM :- Since Space and time are both relative in relativity. Classical Conservation laws can not be expected to valid in relativistic mechanics

Let that there are two invariant quantities associated with two interacting particles such That $M_1U_1 + M_2U_2 = M_1V_1 + M_2V_2 \longrightarrow (1)$

where U's & V's are 4-velocity vectors before and after the Collision

$\Rightarrow$

$$\sum MV = \text{Constant}$$

$$\Rightarrow \quad \sum (1 - v^2/c^2)^{-1/2} (\underline{v}, ic) M = \text{Const}$$

$$\Rightarrow \quad \sum \frac{M}{\sqrt{1 - v^2/c^2}} (\underline{v}, ic) = \text{Const}$$

$$\sum m(\underline{v}, ic) = \text{Const} \quad \text{where } m = \frac{M}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad \sum m\underline{v} = \text{Const}$$

$$\Rightarrow \quad \sum m = \text{Const}$$

"M" will be called the mass of the Particle & we replace it by m_0 $\quad m_0 = M$

$$\therefore \quad m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where v is small as compared to c

$$m = m_0$$

4-MOMENTUM VECTOR. We define 4-momentum vector by

$$\bar{P} = m_0 \bar{V}$$
$$= m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} (\underline{v}, ic)$$
$$= \frac{m_0}{(1-v^2/c^2)^{1/2}} (\underline{v}, ic) = m(v, ic)$$

$$\bar{P} = (m\underline{v}, imc)$$

ω $m\underline{v}$ = relativistic linear momentum $\Rightarrow \bar{P} = (P, imc)$

$$(P_1, P_2, P_3, P_4) = (P_x, P_y, P_z, imc)$$
$$P_1 = P_x, \quad P_2 = P_y, \quad P_3 = P_z, \quad P_4 = imc$$

The transformation Equation from frame S to $\bar{S}$ is

$$\bar{P}_1 = P_1 \cos\alpha + P_4 \sin\alpha \qquad \bar{P}_4 = -P_1 \sin\alpha + P_4 \cos\alpha$$
$$\bar{P}_2 = P_2 \qquad \bar{P}_3 = P_3$$

$$\Rightarrow \left.\begin{aligned} \bar{P}_x &= P_x \cos\alpha + imc \sin\alpha \\ \bar{P}_y &= P_y \quad \bar{P}_z = P_z \quad i\bar{m}c = -P_x \sin\alpha + imc\cos\alpha \end{aligned}\right] \rightarrow ①$$

Now $\cos\alpha = \dfrac{1}{\sqrt{1-u^2/c^2}}$ $\quad \sin\alpha = \dfrac{iu/c}{\sqrt{1-u^2/c^2}}$

$\therefore ① \Rightarrow$

$$\bar{P}_x = \frac{P_x + imc\frac{iu}{c}}{\sqrt{1-u^2/c^2}} = \frac{P_x - mu}{\sqrt{1-u^2/c^2}}$$

$$\bar{P}_y = P_y \qquad \bar{P}_z = P_z$$

$$i\bar{m}c = \frac{-\frac{iu}{c}P_x + imc}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \bar{m} = \frac{-\frac{P_x u}{c^2} + m}{\sqrt{1-u^2/c^2}} = \frac{-\frac{m v_x u}{c^2} + m}{\sqrt{1-u^2/c^2}} = \frac{m\left(1 - \frac{u v_x}{c^2}\right)}{\sqrt{1-u^2/c^2}}$$

the last Equation shows that mass of the particle depends not on its velocity but also the velocity of frame refrence. If u and v_x are small as compared to c the last Equation $\Rightarrow \bar{m} = m$

The 4- Force.

the force f is acting on a particle of mass m moving with velocity v is defined by the Equation

$$\bar{f} = \frac{d}{dt}(m\bar{v}) = \frac{d\bar{P}}{dt} = \dot{\bar{P}}$$

A 4- force is defined as

$$\bar{F} = \frac{d\bar{P}}{dT}$$ where $\bar{P}$ is 4- momentum vector

$$\bar{F} = \frac{d}{dT}(m_0\bar{V}) = m_0\frac{d\bar{V}}{dT}$$

The Relation between f and F

$$F = \frac{d}{dT}(\bar{P}, imc)$$

$$= \frac{d}{dt}(\bar{P}, imc)\frac{dt}{dT}$$

$$= (\dot{\bar{P}}, i\dot{m}c)\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$\vec{F} = \left(1-\frac{v^2}{c^2}\right)^{-1/2}(\bar{f}, i\dot{m}c)$$

$$= (1-v^2/c^2)^{-1/2}(f_x, f_y, f_z, i\dot{m}c)$$

$\Rightarrow$ $F_1 = (1-v^2/c^2)^{-1/2}f_x$, $F_2 = (1-v^2/c^2)^{-1/2}f_y$ $F_3 = \left(1-\frac{v^2}{c^2}\right)^{-1/2}f_z$

$$F_4 = (1-v^2/c^2)^{-1/2}\, i\dot{m}c$$

The 4- velocity and 4- Force are Orthogonal.

Since $\bar{V} = (1-v^2/c^2)^{-1/2}(\bar{v}, ic)$

$$\bar{V}\cdot\bar{V} = (1-v^2/c^2)^{-1/2}(\bar{v}, ic)\cdot(1-v^2/c^2)^{-1/2}(\bar{v}, ic)$$

$$V^2 = (1-v^2/c^2)^{-1}(v^2-c^2) = \frac{1}{(1-v^2/c^2)}(v^2-c^2)$$

$$V^2 = \frac{c^2}{(c^2-v^2)}(v^2-c^2) = -\frac{c^2\cancel{(c^2-v^2)}}{\cancel{(c^2-v^2)}}$$

$$\Rightarrow \quad V^2 = -c^2$$

$$\frac{d}{dT}V^2 = 0 \quad \Rightarrow \quad 2\bar{V}\cdot\frac{d\bar{V}}{dT} = 0$$

$$\Rightarrow \quad m_0\,\bar{V}\cdot\left(\frac{d}{dT}\bar{V}\right) = 0$$

$$\Rightarrow \quad \bar{V}\cdot\left(m_0\frac{d\bar{V}}{dT}\right) = 0 \quad \Rightarrow \quad \bar{V}\cdot\frac{d}{dT}(m_0 V) = 0$$

$$\bar{V}\cdot\bar{F} = 0$$

The Kinetic Energy:-

We know that

$$\bar{V}\cdot\bar{F} = 0$$

$$\Rightarrow \quad (1-v^2/c^2)^{-1/2}(\bar{v}, ic)\cdot(1-v^2/c^2)^{-1/2}(\bar{f}, i\dot{m}c) = 0$$

$$\Rightarrow \quad (1-v^2/c^2)^{-1}(\bar{v}, ic)\cdot(\bar{f}, i\dot{m}c) = 0$$

$$\Rightarrow \quad (\bar{v}\cdot\bar{f} - c^2\dot{m}) = 0$$

$$\Rightarrow \quad \bar{v}\cdot\bar{f} = c^2\dot{m}$$

$$\int\bar{v}\cdot\bar{f}\,dt = c^2\int\frac{dm}{dt}\,dt$$

$$\Rightarrow \quad \int\bar{v}\cdot\bar{f}\,dt = c^2 m + A$$

$$\text{So} \quad \int\bar{v}\cdot\bar{f}\,dt = T = c^2 m + A$$

Let $T = c^2 m + A \rightarrow (1)$ where T is the K.E.

The particle where $\bar{v} = 0$

$$m = m_0, \quad T = 0$$

$$(1) \Rightarrow \quad 0 = c^2 m_0 + A \quad \Rightarrow \quad A = -m_0 c^2$$

$$\therefore \quad T = mc^2 - m_0c^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 - m_0c^2$$

$$T = m_0c^2\left((1-v^2/c^2)^{-1/2} - 1\right)$$

$$= m_0c^2\left(\cancel{1} + \frac{1}{2}\frac{v^2}{c^2} - \cancel{1}\right) \qquad v \text{ is Small as Compared with } c$$

$$T = \frac{1}{2} m_0 v^2$$

① shows that if the K.E. Increases, then so does the mass. A Particle of mass m is Supposed to posses total Energy E given by

$$E = mc^2$$

ie ① $\Rightarrow$ $T = mc^2 - m_0c^2$ — by using the value of Ⓐ

$\Rightarrow$ $m_0c^2 + T = mc^2$

$m_0c^2 + T = E$

where m_0c^2 is the internal Energy of the Particle

A Particle of rest mass has the Energy

$$E_0 = m_0c^2$$

$\Rightarrow$ if a particle is Converted into electromagnetic radiation, the Energy released would be m_0c^2. This is a Source of atomic Energy

TRANSFORMATION Equation For the FORCE COMPONENTS

$$\text{Since} \quad \bar{f}\cdot\bar{v} = c^2\dot{m} \quad \rightarrow ①$$

① is written as $i\,\dfrac{\bar{f}\cdot\bar{v}}{c} = ic\dot{m}$

Since $\vec{F} = (1-v^2/c^2)^{-1/2}\,(\bar{f},\ i\dot{m}c)$

$$\vec{F} = (1-v^2/c^2)^{-1/2}\left(\bar{f},\ \frac{i}{c}\bar{f}\cdot\bar{v}\right)$$

So $F_1 = (1-v^2/c^2)^{-1/2} f_x \quad F_2 = (1-v^2/c^2)^{-1/2} f_y \quad F_3 = (1-v^2/c^2)^{-1/2} f_z$

$$F_4 = (1-v^2/c^2)^{-1/2}\ \frac{i}{c}\,\bar{f}\cdot\bar{v}$$

The Transformation Equations are

$$\bar{F}_1 = F_1\cos\alpha + F_4\sin\alpha \quad \bar{F}_2 = F_2 \quad \bar{F}_3 = F_3 \quad \bar{F}_4 = -F_1\sin\alpha + F_4\cos\alpha \rightarrow (1)$$

Since $\cos\alpha = \dfrac{1}{\sqrt{1-u^2/c^2}}$ $\quad \sin\alpha = \dfrac{iu/c}{\sqrt{1-u^2/c^2}}$

and $\bar{F}_1 = \bar{f}_x\left(1-\bar{v}^2/c^2\right)^{-1/2}$ $\quad \bar{F}_2 = \bar{f}_y\left(1-\dfrac{\bar{v}^2}{c^2}\right)^{-1/2}$ $\quad \bar{F}_3 = \bar{f}_z\left(1-\dfrac{\bar{v}^2}{c^2}\right)^{-1/2}$

$F_1 = f_x\left(1-v^2/c^2\right)^{-1/2}$ $\quad F_2 = f_y\left(1-v^2/c^2\right)^{-1/2}$ $\quad F_3 = f_z\left(1-\dfrac{v^2}{c^2}\right)^{-1/2}$

$\bar{F}_4 = i\left(1-\dfrac{\bar{v}^2}{c^2}\right)^{-1/2}\left(\dfrac{\bar{f}\cdot\bar{v}}{c}\right)$ $\quad F_4 = i/c\left(1-v^2/c^2\right)^{-1/2}\bar{f}\cdot\bar{v}$

$(1) \Rightarrow \quad \bar{F}_1 = F_1\cos\alpha + F_4\sin\alpha$

$$\Rightarrow \left(1-\bar{v}^2/c^2\right)^{-1/2}\bar{f}_x = \frac{f_x\left(1-v^2/c^2\right)^{-1/2}}{\left(1-u^2/c^2\right)^{1/2}} + i\left(\frac{f\cdot v}{c}\right)\frac{\left(1-v^2/c^2\right)^{-1/2}\frac{iu}{c}}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \bar{f}_x = \frac{\left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2}}{\left(1-u^2/c^2\right)^{1/2}\left(1-v^2/c^2\right)^{1/2}}\left(f_x - \frac{u}{c^2}\,\bar{f}\cdot\bar{v}\right)$$

$$\Rightarrow \bar{f}_x = \lambda\left(f_x - \frac{u}{c^2}(\bar{f}\cdot\bar{v})\right) \rightarrow (a)$$

$$\bar{F}_2 = F_2 \Rightarrow \left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2}\bar{f}_y = \left(1-v^2/c^2\right)^{-1/2}f_y$$

$$\Rightarrow \bar{f}_y = \left[\frac{\left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2}}{\left(1-v^2/c^2\right)^{1/2}\left(1-u^2/c^2\right)}\right]\left(1-u^2/c^2\right)^{1/2}f_y = \lambda\left(1-\frac{u^2}{c^2}\right)^{1/2}f_y \rightarrow (b)$$

Similarly $\bar{f}_z = \lambda\left(1-u^2/c^2\right)^{1/2}f_z \rightarrow (c)$ and $1 = \lambda\left(1-\dfrac{uv_x}{c^2}\right) \rightarrow (d)$

dividing a, b, c by (d) $\quad \bar{f}_x = \dfrac{1}{\left(1-\frac{uv_x}{c^2}\right)}\left[f_x - \dfrac{u}{c^2}\left(f_xv_x + f_yv_y + f_zv_z\right)\right]$

$$\Rightarrow \bar{f}_x = \frac{1}{\left(1-\frac{uv_x}{c^2}\right)}\left[\left(1-\frac{uv_x}{c^2}\right)f_x - \frac{u}{c^2}\left(v_yf_y + v_zf_z\right)\right] = f_x - \frac{u}{c^2}\frac{\left(v_yf_y+v_zf_z\right)}{\left(1-\frac{uv_x}{c^2}\right)}$$

$$\bar{f}_y = \frac{\left(1-u^2/c^2\right)^{1/2}}{\left(1-\frac{uv_x}{c^2}\right)}f_y \qquad \bar{f}_z = \frac{\left(1-u^2/c^2\right)^{1/2}}{\left(1-\frac{uv_x}{c^2}\right)}f_z$$

If u, v are small as compared to c $\quad \bar{f}_x = f_x,\ \bar{f}_y = f_y,\ \bar{f}_z = f_z$

Motion with Variable Proper Mass

Let m_0 varies due to some cause for instance heating or cooling or due to receipt of energy from an external source

$$\bar{F} = \frac{d}{dT}(m_0\bar{V})$$

$$\bar{F} = m_0\frac{d\bar{V}}{dT} + V\frac{dm_0}{dT} \rightarrow (1)$$

Since $V^2 = -c^2$

$$\Rightarrow 2\bar{V}\cdot\frac{d\bar{V}}{dT} = 0 \Rightarrow \bar{V}\cdot\frac{d\bar{V}}{dT} = 0 \rightarrow (2)$$

$$(1) \Rightarrow \bar{V}\cdot\bar{F} = m_0\,\bar{V}\cdot\frac{d\bar{V}}{dT} + \bar{V}\cdot\bar{V}\frac{dm_0}{dT}$$

$$\bar{V}\cdot\bar{F} = m_0(0) + V^2\frac{dm_0}{dT}$$

$$\Rightarrow \bar{V}\cdot\bar{F} = -c^2\frac{dm_0}{dT} \rightarrow (3)$$

$\Rightarrow$ $\bar{V}$ and $\bar{F}$ are no longer orthogonal

Now $\bar{V} = (1-v^2/c^2)^{-1/2}(v, ic)$

$$\bar{F} = (1-v^2/c^2)^{-1/2}(\bar{f}, i\dot{m}c)$$

$$\Rightarrow \bar{V}\cdot\bar{F} = (1-v^2/c^2)^{-1}\left((v,ic)\cdot(\bar{f}, i\dot{m}c)\right)$$

$$\bar{V}\cdot\bar{F} = (1-v^2/c^2)^{-1}(\bar{v}\cdot\bar{f} - c^2\dot{m}) \rightarrow (4)$$

$$(3)\ \&\ (4) \Rightarrow -c^2\frac{dm_0}{dT} = (1-v^2/c^2)^{-1}(\bar{v}\cdot\bar{f} - c^2\dot{m}) \rightarrow (5)$$

Since $E = mc^2$

$$\dot{E} = \frac{dE}{dt} = \dot{m}c^2 \rightarrow (6)$$

$$(5) \Rightarrow -c^2\frac{dm_0}{dT} = (1-v^2/c^2)^{-1}\left(\bar{v}\cdot f - \frac{dE}{dt}\right) \quad \text{using } (6)$$

$$\Rightarrow -(1-v^2/c^2)c^2\frac{dm_0}{dT} = \bar{v}\cdot\bar{f} - \frac{dE}{dt}$$

$$\frac{dE}{dt} = \bar{v}\cdot\bar{f} + \cancel{c^2}\left(\frac{c^2-v^2}{\cancel{c^2}}\right)\frac{dm_0}{dT}$$

$$\frac{dE}{dt} = \vec{v}\cdot\vec{f} + (c^2 - v^2)\frac{dm_0}{dT}$$

Hence the rate of Increase of Energy due to External Source $\frac{dE}{dt} = (c^2 - v^2)\frac{dm_0}{dT}$

LANGRANGES EQUATION OF MOTION

Lagranges Equation for a Single particle in the classical form

$$\left.\begin{aligned} \frac{d}{dt}(m\dot{x}) &= -\frac{\partial V}{\partial x} \\ \frac{d}{dt}(m\dot{y}) = -\frac{\partial V}{\partial y} \quad & \frac{d}{dt}(m\dot{z}) = -\frac{\partial V}{\partial z} \end{aligned}\right] \rightarrow (1)$$

Hence $\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-v^2/c^2}}\dot{x}\right) = -\frac{\partial V}{\partial x}$ $\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-v^2/c^2}}\dot{y}\right) = -\frac{\partial V}{\partial y}$ $\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-v^2/c^2}}\dot{z}\right) = -\frac{\partial V}{\partial z}$ $\Big] \rightarrow (2)$

This should be of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x},\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y},\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \Big] \rightarrow (3)$$

we define L Such That

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= \frac{m_0\dot{x}}{\sqrt{1-v^2/c^2}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{y}} = \frac{m_0\dot{y}}{\sqrt{1-v^2/c^2}} = m\dot{y} \\ \frac{\partial L}{\partial \dot{z}} &= \frac{m_0\dot{z}}{\sqrt{1-v^2/c^2}} = m\dot{z} \end{aligned}\right] \rightarrow (4)$$

$$\therefore \quad \frac{\partial L}{\partial \dot{x}} = -\frac{\partial V}{\partial x},\quad \frac{\partial L}{\partial \dot{y}} = -\frac{\partial V}{\partial y} \quad \frac{\partial L}{\partial \dot{z}} = -\frac{\partial V}{\partial z} \Big] \rightarrow (5)$$

(4) and (5) are Satisfied

if $L = -m_0c^2(1-v^2/c^2)^{1/2} - V$ where V is Potential

Q Obtain the transformation Equation for component of V directly by differenciating the Lorentz transformation

Sol:- Let the transformation from S to $\bar{S}$ be

$$\bar{x} = \frac{x-ut}{\sqrt{1-u^2/c^2}} \to (i) \quad \bar{y} = y \to (ii) \quad \bar{z} = z \to (iii) \quad \bar{t} = \frac{t - \frac{ux}{c^2}}{\sqrt{1-u^2/c^2}} \to (iv)$$

$$(i) \Rightarrow \bar{V}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{d}{d\bar{t}}\left(\frac{x-ut}{\sqrt{1-u^2/c^2}}\right)$$

$$= \frac{d}{dt}\left(\frac{x-ut}{\sqrt{1-u^2/c^2}}\right)\frac{dt}{d\bar{t}} = \left(\frac{v_x-u}{\sqrt{1-u^2/c^2}}\right)\frac{dt}{d\bar{t}} \to (v)$$

$$(iv) \Rightarrow \frac{d\bar{t}}{dt} = \frac{d}{dt}\left(\frac{t-\frac{ux}{c^2}}{\sqrt{1-u^2/c^2}}\right) = \frac{1}{\sqrt{1-u^2/c^2}}\left(1-\frac{u}{c^2}v_x\right) \to (vi)$$

using (vi) in (v)

$$\bar{V}_x = \left(\frac{V_x-u}{\sqrt{1-u^2/c^2}}\right) \Big/ \frac{1}{\sqrt{1-u^2/c^2}}\left(1-\frac{u}{c^2}v_x\right)$$

$$\Rightarrow \bar{V}_x = \frac{V_x-u}{\cancel{\sqrt{1-u^2/c^2}}} \cdot \frac{\cancel{(1-u^2/c^2)^{1/2}}}{\left(1-\frac{v_x u}{c^2}\right)} = \frac{V_x-u}{\left(1-\frac{uV_x}{c^2}\right)}$$

$$(ii) \Rightarrow \bar{V}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{dy}{d\bar{t}} = \frac{dy}{dt}\cdot\frac{dt}{d\bar{t}}$$

$$\bar{V}_y = v_y \cdot \frac{\sqrt{1-u^2/c^2}}{\left(1-\frac{uV_x}{c^2}\right)}$$

Similarly
$$\bar{V}_z = \frac{V_z\left(1-u^2/c^2\right)^{1/2}}{\left(1-\frac{uV_x}{c^2}\right)}$$

Obtain the transformation Equation for the Component of acceleration

Sol We already Shown in Q1

$$\bar{v}_x = \frac{v_x - u}{\left(1 - \frac{u v_x}{c^2}\right)} \qquad \bar{v}_y = \frac{v_y\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1 - \frac{u v_x}{c^2}\right)} \qquad \bar{v}_z = \frac{v_z\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1 - \frac{u v_x}{c^2}\right)}$$

$$\bar{a}_x = \frac{d\bar{v}_x}{d\bar{t}} = \frac{d}{d\bar{t}}\left(\frac{v_x - u}{1 - \frac{u v_x}{c^2}}\right) = \frac{d}{dt}\left(\frac{v_x - u}{1 - \frac{u v_x}{c^2}}\right)\frac{dt}{d\bar{t}}$$

$$= \left(\frac{\left(1-\frac{u v_x}{c^2}\right)\frac{d}{dt}(v_x - u) - (v_x - u)\frac{d}{dt}\left(1 - \frac{u v_x}{c^2}\right)}{\left(1 - \frac{u v_x}{c^2}\right)^2}\right)\frac{\sqrt{1-\frac{u^2}{c^2}}}{\left(1-\frac{u v_x}{c^2}\right)}$$

$$= \left[\frac{\left(1-\frac{u v_x}{c^2}\right)(a_x - 0) - (v_x - u)\left(0 - \frac{u}{c^2}a_x\right)}{\left(1-\frac{u v_x}{c^2}\right)^3}\right]\left(1-\frac{u^2}{c^2}\right)^{1/2}$$

$$= \frac{a_x\left(1 - \cancel{\frac{u v_x}{c^2}} + \cancel{\frac{u}{c^2}v_x} - \frac{u^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1-\frac{u v_x}{c^2}\right)^3}.$$

$$= \frac{a_x\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1-\frac{u v_x}{c^2}\right)^3} = \frac{a_x\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\left(1-\frac{u v_x}{c^2}\right)^3}$$

$$\bar{v}_y = \frac{\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1-\frac{u v_x}{c^2}\right)} v_y$$

$$\bar{j} = \frac{d\bar{v}_y}{d\bar{t}} = \left(1-\frac{u^2}{c^2}\right)^{1/2}\frac{d}{dt}\left(\frac{v_y}{1-\frac{u v_x}{c^2}}\right)\frac{dt}{d\bar{t}}$$

$$= \left(1-\frac{u^2}{c^2}\right)^{1/2}\left[\frac{\left(1-\frac{u v_x}{c^2}\right)a_y - v_y\left(0 - \frac{u}{c^2}a_x\right)}{\left(1-\frac{u v_x}{c^2}\right)^2}\right]\frac{\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1-\frac{u v_x}{c^2}\right)}$$

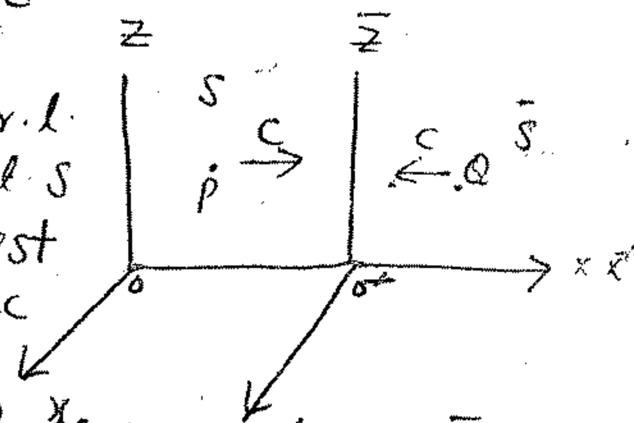
$$\bar{a}_y = \frac{\left(1-\frac{u^2}{c^2}\right)}{\left(1-\frac{uv_x}{c^2}\right)^3}\left[\left(1-\frac{uv_x}{c^2}\right)a_y + \frac{u}{c^2}a_x v_y\right]$$

$$\bar{a}_z = \frac{d}{d\bar{t}}(\bar{V}_z) = \frac{d}{d\bar{t}}\left(\frac{V_z(1-u^2/c^2)^{1/2}}{\left(1-\frac{uv_x}{c^2}\right)}\right) = \frac{d}{dt}\left(\frac{V_z(1-u^2/c^2)^{1/2}}{\left(1-\frac{uv_x}{c^2}\right)}\right)\frac{dt}{d\bar{t}}$$

$$= (1-u^2/c^2)^{1/2}\left[\frac{\left(1-\frac{uv_x}{c^2}\right)a_z - V_z\left(0-\frac{u}{c^2}a_x\right)}{\left(1-\frac{uv_x}{c^2}\right)^2}\right]\frac{\left(1-\frac{u^2}{c^2}\right)^{1/2}}{\left(1-\frac{uv_x}{c^2}\right)}$$

$$= \frac{(1-u^2/c^2)}{\left(1-\frac{uv_x}{c^2}\right)^3}\left[\left(1-\frac{uv_x}{c^2}\right)a_z + \frac{u}{c^2}a_x V_z\right]$$

Q Two Points P and Q are moving in opposite direction each speed c relative to frame S. Show that relative velocity is c



Sol:- Let the speed of P be c w.r.t. S. Let the speed of $\bar{S}$ w.r.t. S be $-c$, so that Q is at rest w.r.t. $\bar{S}$. In this case $U=-c$

velocity of P relative to Q = velocity P relative to $\bar{S}$
Let $(\bar{v}_x, \bar{v}_y, \bar{v}_z)$ be the component of P relative to $\bar{S}$ $\quad V_x = c \quad V_y = 0 \quad V_z = 0$

$$\bar{V}_x = \frac{V_x - U}{1-\frac{UV_x}{c^2}} = \frac{c-(-c)}{1-\frac{(-c)(c)}{c^2}} = \frac{2c}{1+\frac{c^2}{c^2}} = \frac{2c}{2} = c$$

$\bar{v}_y = 0 \quad \bar{V}_z = 0 \quad \therefore$ the velocity of P relative to Q is c which is the required result

Q Show that 4- velocity $\frac{V}{ic}$ is of Constant magnitude

Sol.-

$$\bar{V} = (1-v^2/c^2)^{-1/2}(\bar{v}, ic)$$

$$V \cdot V = (1-v^2/c^2)^{-1/2}(\bar{v}, ic) \cdot (1-v^2/c^2)^{-1/2}(v, ic)$$
$$= (1-v^2/c^2)^{-1}(v^2-c^2)$$
$$= \frac{c^2}{(c^2-v^2)}(v^2-c^2) = -c^2$$

$$\Rightarrow \quad V^2 = -c^2 \Rightarrow V^2 = (ic)^2$$
$$|V| = ic$$

Q If $\bar{S}$ has velocity c relative to S. Show That all Points relative to S with velocities less than C has a velocity c relative to $\bar{S}$

Sol:- Here $u = c$ (along x-axis) ie velocity of $\bar{S}$ relative to S
Let V be the velocity of a Point P w.r.t. S
Let $\bar{V}_x, \bar{V}_y, \bar{V}_z$ be the Component of the velocity of P relative to $\bar{S}$

$$\bar{V}_x = \frac{V_x - U}{1-\frac{UV_x}{c^2}} = \frac{V_x - C}{1-\frac{cV_x}{c^2}} = \frac{(V_x - C)}{\frac{1}{c}(c-V_x)}$$

$$\bar{V}_x = -C$$

$$\bar{V}_y = \frac{\sqrt{1-u^2/c^2}}{\left(1-\frac{uV_x}{c^2}\right)} V_y = \frac{\sqrt{1-c^2/c^2}}{\left(1-\frac{uV_x}{c^2}\right)} V_y = 0$$

Similarly $\bar{V}_z = 0$

Rule of Cancellation of dot (D-Delta Rule)

Let q_1, q_2 be some generalized co-ordinate of the particle whose Cartesian co-ordinate are x, y

$$x = x(q_1, q_2) \qquad y = y(q_1, q_2)$$

then
$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 \rightarrow (1) \qquad \dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2 \rightarrow (2)$$

$\Rightarrow$ $\dot{x}$ and $\dot{y}$ are functions of both four independent variable $q_1, q_2, \dot{q}_1, \dot{q}_2$

from (1)
$$\left.\begin{aligned}\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1} \qquad \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2}\end{aligned}\right] \rightarrow (3)$$
$$\left.\begin{aligned}\frac{\partial \dot{y}}{\partial \dot{q}_1} = \frac{\partial y}{\partial q_1} \qquad \frac{\partial \dot{y}}{\partial \dot{q}_2} = \frac{\partial y}{\partial q_2}\end{aligned}\right] \rightarrow (4)$$

Interchange of d and ∂

from (1)
$$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2 \rightarrow (5)$$
$$\frac{\partial \dot{x}}{\partial q_2} = \frac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2^2}\dot{q}_2 \rightarrow (6)$$

These derivative $\frac{\partial x}{\partial q_1}$ and $\frac{\partial x}{\partial q_2}$ are function of q_1 and q_2 and are also implicit function of the time variable t

$$(5) \Rightarrow \frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2 \rightarrow (7)$$
$$(6) \Rightarrow \frac{d}{dt}\left(\frac{\partial x}{\partial q_2}\right) = \frac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2^2}\dot{q}_2 \rightarrow (8)$$

from (5) and (7) $\Rightarrow \frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial \dot{x}}{\partial q_1}$

from (6) and (8) $\Rightarrow \frac{d}{dt}\left(\frac{\partial x}{\partial q_2}\right) = \frac{\partial \dot{x}}{\partial q_2}$

$\Rightarrow$ d and ∂ can be interchange

$f(x,y) = 0$ is implicit function.

In three dimensions $x = x(q_1, q_2, q_3)$ $y = y(q_1, q_2, q_3)$ $z = z(q_1, q_2, q_3)$
where q_i $(i = 1, 2, 3)$ are the generalized Co-ordinate
or $\quad \bar{r} = \bar{r}(q_1, q_2, q_3)$

$$\text{then} \quad \frac{\partial \dot{\bar{r}}}{\partial \dot{q}_i} = \frac{\partial r}{\partial q_i} \qquad (\text{from } (3))$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial r}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\dot{r} \qquad (\text{from } \underline{\text{Afler 8}})$$

<u>For a System of N-particles</u>

Let $\bar{r}_\alpha$ be the position vector of αth particle $\alpha = 1, 2, 3 - N$
Let the Corresponding generalized Co-ordinate be denoted by q_i $(i = 1, 2, \ldots n)$

<u>generalized Co-ordinate</u> :- (A Set of n parameters $q_1, q_2, \ldots q_n$ are Called generalized Co-ordinate for the System of n particle if the position of particles Can be Specified by these parameters)

N particles $\quad r_i \quad i = 1, 2 \ldots N$
there will be 3N Cartesian Co-ordinates
$\therefore$ No. of generalized Co-ordinate $= 3N$

Concellation of Dots of a System of N particles

$$\frac{\partial \dot{r}_\alpha}{\partial \dot{q}_i} = \frac{\partial r_\alpha}{\partial q_i} \qquad \frac{d}{dt}\left(\frac{\partial r_\alpha}{\partial q_i}\right) = \frac{\partial}{\partial q_i}(\dot{r}_\alpha)$$

$$\alpha = 1, 2, \ldots N$$
$$i = 1, 2, \ldots n$$

CONSTRAINTS FOR A PHYSICAL SYSTEM

A Constraint is a restriction on the motion of a Physical System which Imposes a Condition on the Co-ordinates of the particle of the System

Degree OF Freedom

The Degree of freedom of a System is the number of independent Co-ordinate required to define the Configuration of the System

e.g) Let a Particle be restricted to move on the Surface of the Sphere of radius a. Its Equation of motion

$$x^2+y^2+z^2=a^2$$

three Co-ordinate are Involved out of which two Co-ordinates are independent

So Degree of freedom of this System will be two

Variable :- 3

Equation :- 1 $\quad$ D.F. $= 3-1 = 2$

Consider the particle subjected to the Constraints

$$a_1x+b_1y+c_1z=d_1$$

$$a_2x+b_2y+c_2z=d_2$$

one of the Co-ordinate is independent. So D.F. is one.

Number of variables = - 3

Equation :- 2

D.F. $= 3-2 = 1$

In Spherical polar Co-ordinate

$$x = \rho \cancel{\cos\phi \sin\theta} \sin\phi\cos\theta$$

$$y = \rho \cancel{\cos\phi \cos\theta} \sin\phi\sin\theta$$

$$z = \rho \cancel{\sin\phi} \cos\phi$$

No. of variable :- 5

No. of Equation :- 3

D. F. $= 5-3 = 2$.

CLASSIFICATION OF DYNAMICAL SYSTEM

1 ~~Set~~ Sclernomic OR Rheonomic System

If the Configuration of the System can be described by a function of Co-ordinate q_i $(i=1,2,\dots n)$ only and there is no explicit dependence on time t then the System is Called Sclernomic ie

$$r_\alpha = f_\alpha(q_1, q_2 \dots q_n) \qquad \alpha = 1,2,\dots N$$

If $\bar{r}_\alpha = f_\alpha(q_1, q_2 \dots q_n, t)$ then the System is Called Rheonomic.

Holonomic OR Non Holonomic System

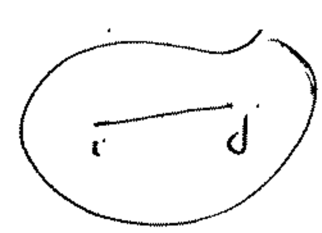
If the Conditions of Constraints Can be expressed by the relation of the form

$$f(q_1, \dots, q_n, t) = \text{Constt}$$

the System is Holonomic otherwise it is Called Non Holonomic System

(e.g) A real body is Holonomic System

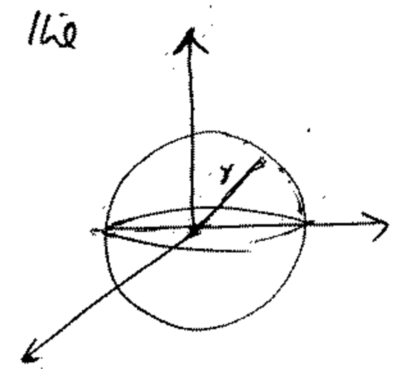
$$(r_i - r_j)^2 = \text{Constt} = C_{ij}$$



ie $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = C_{ij}^2$

A particle is moving Constrained on the Surface of the Sphere

$$x^2 + y^2 + z^2 = a^2$$



$\Rightarrow$ Constrant is Holonomic

" For a rigid body

$$|r_i - r_j| = \text{Constt} = C_{ij}$$

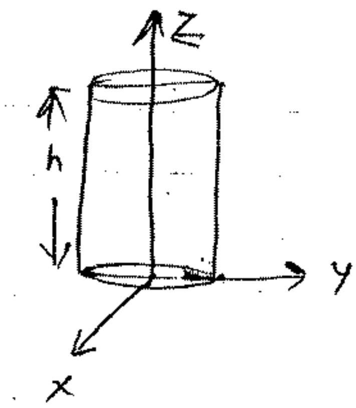
$\Rightarrow$ $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = C_{ij}^2$.

$\Rightarrow$ rigid body is Holonomic "

(5)

For a non Holonomic System :— Consider the motion of the particle within a Cylindre of radius a and height h

In Cylindrical Co-ordinate (r, θ, z)



The Condition of Constraints are

$0 \leq r \leq a \quad 0 \leq \theta \leq 2\pi$

$0 \leq z \leq h$. These Constraints are of non holonomic

Conservative AND NON Conservative System

If F is the total force on the system and it can be derived from a Scalar point function V

i.e. if $F = -\nabla V$

or $\text{Curl } F = 0$ (curl grad $= 0$)

then the field F is Called Conservative and the System is also Called Conservative. the Scalar field V is Called Potential or Potential Energy.

D- Alembert's Principle

According to this Principle, a dynamical problem Can be Considered as a Statical problem.

Let us Consider a system of N particle whose position vector at any time t, is defined as

$$r_i = r_i(q_1, q_2 \ldots q_n, t) \rightarrow (1)$$

where q_i's are the generalized Co-ordinate

Let F_i be the externally applied force on the ith particle then the work done by F_i in the displacement δr_i is given by

$$F_i \cdot \delta r_i = 0 \longrightarrow (2)$$

from (1) $\delta r_i = \frac{\partial r_i}{\partial q_j} \delta q_j$

$$\therefore \quad F \cdot \frac{\partial r_i}{\partial q_j} \delta q_j = 0$$

$$Q_j \, \delta q_j = 0 \quad \text{where } Q_j = F \cdot \frac{\partial r_i}{\partial q_j}$$

Q_j is called generalized force

We know that $F = \dot{p}$

$$\Rightarrow \quad \bar{F} - \dot{p} = 0$$

$$(\bar{F} - \dot{p}) \cdot \delta r_i = 0$$

$$\Rightarrow \quad (\bar{F} - m\ddot{\underline{r}}_i) \cdot \delta r_i = 0$$

which is called D Alembert's Principle.

sir notes

• LAGRANGE EQUATION OF MOTION (DERIVATION from D-Alembert's Principle)

We consider the motion of a system of N-particles with position vectors $r_1, r_2, \ldots r_N$ moving under no constraints

The D-Alembert's Principle states that

$$\sum_{i=1}^{N} \left(F_i^{(a)} - \dot{p}_i \right) \cdot \delta r_i = 0 \longrightarrow (1)$$

$i = 1, 2, \ldots$

We introduce the generalized co-ordinate $q_1, q_2 \ldots q_n$ $(n = 3N = N_0)$ which are all independent

(1)

where $\bar{r}_i = \bar{r}_i(q_1, q_2, \ldots q_n\, t) \rightarrow (2)$

$= r_i(q, t)$

Now $\underline{v_i} = \frac{d\bar{r}_i}{dt} = \frac{\partial r_i}{\partial t} + \frac{\partial r_i}{\partial q_1}\dot{q}_1 + \frac{\partial r_i}{\partial q_2}\dot{q}_2 + \cdots$

$\cdots + \frac{\partial r_i}{\partial q_n}\dot{q}_n$

$$\dot{r}_i = \frac{\partial r_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\dot{q}_j \quad i = 1, 2, \ldots N \rightarrow (3)$$

$(3) \Rightarrow \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$. (Cancellation of dot)

from (1) $\sum_i \dot{p}_i \cdot \delta r_i = \sum_i F_i^a \cdot \delta r_i \rightarrow (4)$

R.H.S. of (4) $\Rightarrow \sum_i F_i^{(a)} \cdot \delta r_i = \sum_i F_i \cdot \delta r_i \rightarrow (5)$

(Fi is the total force)

from (2) $\Rightarrow \delta \bar{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\delta q_j + \frac{\partial r_i}{\partial t}\delta t$

in virtual displacement

$= \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j$

on Substitution in (5)

$$\sum_i F_i^a \cdot \delta r_i = \sum_i F_i \cdot \sum_{j=1} \frac{\partial r_i}{\partial q_j}\delta q_j$$

$$= \sum_i \sum_j \left(F_i \cdot \frac{\partial r_i}{\partial q_j}\right)\delta q_j$$

$$= \sum_j \left(\sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}\right)\delta q_j$$

$$= \sum_j Q_j \, \delta q_j \rightarrow (6)$$

where $Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \rightarrow (7)$

L.H.S. (4) $\sum_i \dot{p}_i \cdot \delta \bar{r}_i = \sum_i \dot{p}_i \cdot \sum_j \frac{\partial r_i}{\partial q_j}\,\delta q_j$

$$= \sum_i \sum_j \left(\frac{d}{dt}(m_i v_i) \cdot \frac{\partial r_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_i \sum_j \left(\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i v_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \right) \delta q_j$$

$$\left(\begin{array}{l} \frac{d}{dt}\left(m_i v_i \cdot \frac{\partial r_i}{\partial q_j}\right) = \frac{d}{dt}(m_i v_i) \cdot \frac{\partial r_i}{\partial q_j} + m_i v_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \\ \frac{d}{dt}\left(m_i v_i \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i v_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{d}{dt}(m_i v_i) \cdot \frac{\partial r_i}{\partial q_j} \end{array} \right)$$

$$= \sum_{i,j} \left(\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i v_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{i,j} \left(\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - m_i v_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{i,j} \left(\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j}\right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_j} \right) \delta q_j \qquad \left(\because \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}\right)$$

$$= \sum_j \sum_i \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) \right) \delta q_j$$

$$= \left(\sum_j \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} T\right) - \frac{\partial}{\partial q_j} T \right) \delta q_j \rightarrow (8)$$

from (6) & (7) $\Rightarrow$ $\sum_j \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right) \delta q_j = \sum_j Q_j \delta q_j$

$$\Rightarrow \sum_j \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$$

Since δq_j is arbitrary & independent

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \rightarrow (9) \quad j = 1, 2, \ldots n$$

These Equations related to Lagranges Equation of Motion

IF THE SYSTEM IS CONSERVATIVE

then

$$F_i = -\nabla_i V$$

$$= -\left(\frac{\partial V}{\partial x_i} i + \frac{\partial V}{\partial y_i} j + \frac{\partial V}{\partial z_i} K\right)$$

then $$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}$$

$$= -\sum_i \left(\frac{\partial V}{\partial x_i} i + \frac{\partial V}{\partial y_i} j + \frac{\partial V}{\partial z_i} K\right) \cdot \left(\frac{\partial x_i}{\partial q_j} i + \frac{\partial y_i}{\partial q_j} j + \frac{\partial z_i}{\partial q_j} K\right)$$

$$= -\sum_i \left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right)$$

$$= -\frac{\partial V}{\partial q_j} \rightarrow (10)$$

Let $L = T - V$ (For a Conservative System)

T is a function of q_i and $\dot{q}_i$, V is a function of q_i only

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \rightarrow (11)$$ ($\because$ V is not dependent function on $\dot{q}$)

$$\frac{\partial L}{\partial q_j} = \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \rightarrow (12)$$

(9) $\Rightarrow$ $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

or $$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial q_j}(T-V) = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

This is called Lagrange Equation of motion. L is Lagrangian

EQUATION OF MOTION OF PARTICLE IN SPACE UNDER GRAVITY

Using Cartesian Co-ordinate (x,y,z), in this case, Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

using the Cartesian Co-ordinate, the Equation of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \rightarrow (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \rightarrow (2) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \rightarrow (3)$$

Now $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(x,y,z)$

on substituting (1) $\Rightarrow$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x,y,z)\right) - \frac{\partial}{\partial x}\left(\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + V(x,y,z)\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{\cancel{2}}m\,\cancel{2}\dot{x}\right) - \frac{\partial V}{\partial x} = 0$$

$$\Rightarrow m\ddot{x} - F_x = 0 \Rightarrow F_x = m\ddot{x}$$

Similarly $F_y = m\ddot{y}$ $\quad F_z = m\ddot{z}$

Using polar Co-ordinate (r,θ)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \rightarrow (1) \qquad (q_1, q_2) \overset{i=1,2}{=} (r,\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \rightarrow (2)$$

Now $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r,\theta)$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{\cancel{2}}\cancel{2}m\dot{r} = m\dot{r} \qquad \frac{\partial L}{\partial r} = \frac{1}{2}m\,2r\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$\frac{d}{dt}\left(m\dot{r}\right) - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) = 0$$

$$\frac{d}{dt}(m\dot{r}) + \frac{\partial V}{\partial r} - mr\dot{\theta}^2 = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r}$$

$$m\ddot{r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{1}{2}m\, r^2\, 2\dot{\theta}\right) - \left(-\frac{\partial V}{\partial \theta}\right) = 0$$

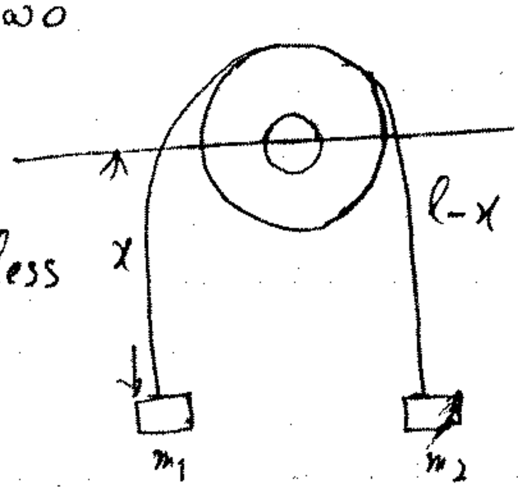
$$\Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) + \frac{\partial V}{\partial \theta} = 0$$

$$\Rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\frac{\partial V}{\partial \theta}$$

1-11-99

MOTION OF A PARTICLE IN STANDARD

Standard consists of two particles fixed to the ends of a string of length 'l' and moving over a frictionless pulley of very very small diameter



$V =$ P.E. of the System

$= -m_1 g x - m_2 g(l-x)$

$$T = K.E. = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2$$

Same velocity for both

$$L = T - V = \frac{1}{2}(m_1+m_2)\dot{x}^2 - m_1 g x - m_2 g(l-x)$$

Lagrange Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{1}{\cancel{2}}(m_1+m_2)\cancel{2}\dot{x}\right)-\left(-m_1g-m_2g(0-1)\right)=0$$

$$\Rightarrow (m_1+m_2)\ddot{x}-\left(-m_1g+m_2g=0\right)$$

$$\Rightarrow (m_1+m_2)\ddot{x}+m_1g-m_2g=0$$

$$\Rightarrow (m_1+m_2)\ddot{x}=(m_2-m_1)g$$

$$\Rightarrow \ddot{x}=\left(\frac{m_2-m_1}{m_1+m_2}\right)g$$

LAGRANGE EQUATIONS IN SPHERICAL Polar Co-Ordinates

$$(r,\theta,\varphi)=(q_1,q_2,q_3)$$

Lagrange Equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0$

$$L=T-V$$

$$T=\frac{1}{2}mv^2=\frac{1}{2}m\left(\dot{r}^2+(r\dot{\theta})^2+(r\sin\theta\,\dot{\varphi})^2\right)$$

$$V=V(r,\theta,\varphi)$$

$v=\dot{r}e_r+r\dot{\theta}e_\theta+r\dot{\varphi}\sin\theta\, e_\varphi$
Book dynamics by D.T Greenwood

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\varphi}^2\right)-V(r,\theta,\varphi)$$

$$\frac{\partial L}{\partial \dot{r}}=m\dot{r}\qquad \frac{\partial L}{\partial r}=\frac{1}{2}m\left(0+2r\dot{\theta}^2+2r\sin^2\theta\,\dot{\varphi}^2\right)-\frac{\partial V}{\partial r}$$

$$\Rightarrow \frac{\partial L}{\partial r}=mr\dot{\theta}^2+mr\sin^2\theta\,\dot{\varphi}^2-\frac{\partial V}{\partial r}$$

$$\frac{d}{dt}(m\dot{r})-\left(mr\dot{\theta}^2+mr\sin^2\theta\,\dot{\varphi}^2-\frac{\partial V}{\partial r}\right)=0$$

$$m\ddot{r}-mr\dot{\theta}^2-mr\sin^2\theta\,\dot{\varphi}^2=-\frac{\partial V}{\partial r}$$

19-10-04

DERIVATION OF LAGRANGES EQUATION OF MOTION
(By Variational Method)
(Conservative System without Constraints)

For a Single particle $K.E. = T = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$

$P.E. = V = V(q_1, q_2, q_3)$ $\therefore L = T - V = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$

The equation of motions are derived by an appeal to the principle of least action (Hamilton Principle)

Principle of Least ACTION : The actual path travel by a particle during the time interval (t_1, t_2) in a field of force derived by Lagrangian L Corresponds to the least value of action of the integral

$$\int_{t_1}^{t_2} L\, dt$$

$\therefore$ According to this principle, we must minimize the integral $\int_{t_1}^{t_2} L\, dt$ to obtain the Lagrange Equation of Motion

For a Conservative field

$$L = T - V = L(q_1, q_2, \cdots q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n, t)$$

for a Single particle

~~L = T+V = L(q, ...)~~

$$L = T - V = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$$

Here $y_i \longrightarrow q_i$ $\quad i = 1, 2, 3$

$x \longrightarrow t$

$F \longrightarrow L$

The External Force are given by $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = 0$

$i = 1, 2, 3$

Detailed Derivation

$$I(q_1, q_2, q_3) = \int_{t_1}^{t_2} L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)\, dt \longrightarrow (1)$$

According to the principle of least action, the actual path of motion corresponds to the least value of I

In that case

$$\delta I = \text{Variation in } I = 0$$

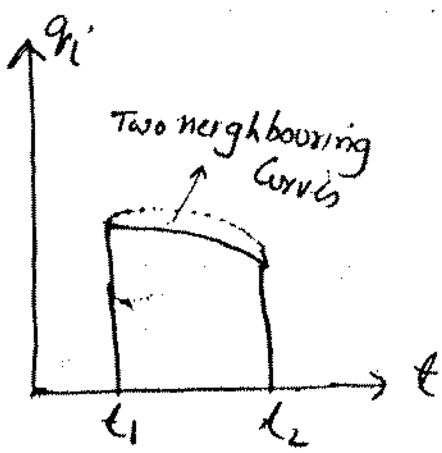
From (1)

$$\delta I = \delta \int_{t_1}^{t_2} L(\underline{q}, \underline{\dot{q}}, t)\, dt = 0$$

$$0 = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t)\, dt$$

$$0 = \int_{t_1}^{t_2} \sum_{i=1}^{3} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \rightarrow (2)$$

now ~~$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\delta q_i\right)$~~



now $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i = \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$(2) \Rightarrow \quad 0 = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \right) dt$$

$$0 = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) \delta q_i\, dt + \int_{t_1}^{t_2} \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) dt$$

(now $$\int_{t_1}^{t_2} \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) dt = \sum_i \left| \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} = 0$$

As $\delta q_i = 0$ at $t = t_1$, $t = t_2$

Hence $$0 = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) \delta q_i\, dt$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1,2,3$$

27-10

(Generalization to Non Conservative System Generalized Hamilton Principle)

For a Conservative System, we state the principle as

$$I = \int_{t_1}^{t_2} (T + W)\, dt$$

$$\text{or} \quad \delta I = \delta \int_{t_1}^{t_2} T\, dt + \delta \int_{t_1}^{t_2} W\, dt = 0 \quad \rightarrow (1)$$

For non Conservative System, the Equations similar to Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad \longrightarrow (2) \qquad i = 1,2,3, \ldots n$$

where $\sum_i F_i \cdot \delta r_i = \sum_j Q_j\, \delta q_j \longrightarrow (3)$

we derive Equation (2) by making an appeal to generalized Hamilton principle (1)

where δW = work done during a virtual displacement of Co-ordinate

$$= \sum_i F_i \cdot \delta r_i = \sum_j Q_j\, \delta q_j \quad (\text{from } (3))$$

$$(1) \Rightarrow \int_{t_1}^{t_2} \delta T\, dt + \int_{t_1}^{t_2} \sum_j Q_j\, \delta q_j\, dt = 0 \rightarrow (4)$$

$$T = T(q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, t)$$

δt is zero for virtual displacement

$$\delta T = \sum_{j=1}^{n} \frac{\partial T}{\partial q_j}\, \delta q_j + \sum_j \frac{\partial T}{\partial \dot{q}_j}\, \delta \dot{q}_j$$

using in (4)

$$\Rightarrow \int_{t_1}^{t_2} \sum_j \left(\frac{\partial T}{\partial q_j}\, \delta q_j + \frac{\partial T}{\partial \dot{q}_j}\, \delta \dot{q}_j\right) dt + \int_{t_1}^{t_2} \sum_j Q_j\, \delta q_j\, dt = 0 \rightarrow (5)$$

$$\text{Now} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\, \delta q_j\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \delta q_j + \frac{\partial T}{\partial \dot{q}_j}\, \delta \dot{q}_j$$

$$\cancel{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j = \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j}$$

$$\int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j}\delta q_j + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j\right)dt$$
$$+\int_{t_1}^{t_2}\sum_j Q_j\,\delta q_j\,dt = 0$$

$$\int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j\right)\delta q_j\,dt$$
$$+\int_{t_1}^{t_2}\sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right)dt = 0$$

$$\left(\int_{t_1}^{t_2}\sum_j\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right)dt = \sum_i \left|\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right|_{t_1}^{t_2}\right.$$

$\delta q_j = 0$ At $t = t_1$, $t = t_2$

$$\therefore \int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j\right)\delta q_j\,dt = 0 \quad \rightarrow (6)$$

(6) is true for both Holonomic and non Holonomic Constraints.
However if the Constraint is Holonomic and $q_1, q_2 \ldots q_n$ are independent Co-ordinate
(6) $\Rightarrow$ it follows that

$$\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j = 0$$

$$\Rightarrow -\frac{\partial T}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = Q_j$$

$$\Rightarrow Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

• LAGRANGE EQUATION FOR NON HOLONOMIC SYSTEM

In non holonomic System, the generalized Co-ordinate are not related together by an Equation of the form

$$f_i(q_1, q_2, \dots q_n, t) = 0 \quad \rightarrow (1) \quad i = 1, 2, \dots$$

Because of this we Can not eliminate extra Co-ordinates and obtain a set of independent Co-ordinate we Can still treat non holonomic System provided the Equation of the Constraints Can be in the form

$$\sum_{k=1}^{n} a_{lk}\, dq_k + a_{lt}\, dt = 0 \quad \rightarrow (2) \quad l = 1, 2, 3$$

we Use generalized Hamilton's principle to obtain lagrange Equation in the Case of non holonomic for which (2) hold

the variation proces in Hamilton's principle is one in which the time is kept fixed.
Virtual displacement of Co-ordinate Satisfies

$$\sum_{k=1}^{n} a_{lk}\, \delta q_k = 0 \quad \rightarrow (3) \quad l = 1, 2 \dots m$$

(3) utilize to reduce the number of virtual displacement to independent ones. The method of Lagrange multipliers is Used to eliminate extra virtual displacement (3) $\Rightarrow \lambda_l \sum_{k=1}^{n} a_{lk}\, \delta q_k = 0$

$$\Rightarrow \sum_k \lambda_l\, a_{lk}\, \delta q_k = 0 \rightarrow (4)$$

where λ_l $(l = 1, 2, \dots m)$ are Undetermined Coefficient

we will Combine (4) with

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - Q_k \right) \delta q_k\, dt = 0 \rightarrow (5)$$

which Can be derived for Holonomic or non Holonomic Constraints from Hamilton principle for n Co-ordinate $q_1, q_2, \dots q_n$. in This Co-ordinate q_k may not independent for a Conservative System (5)

$$\Rightarrow \int_{t_1}^{t_2} \sum_{k=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \right) \delta q_k\, dt = 0 \rightarrow (6)$$

(4) $\Rightarrow$ integrating w.r.t. t b/n t_1 & t_2 & on summing over l

$$\int_{t_1}^{t_2} \sum_{l} \sum_{k} \lambda_l a_{lk} \delta q_k \, dt = 0 \rightarrow (7)$$

(Adding (6) & (7))

$$\int_{t_1}^{t_2} \left(\sum_{k} \left[\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \right) + \sum_{l} \lambda_l a_{lk} \right] \right) \delta q_k \, dt = 0 \rightarrow (8)$$

The displacement δq_k are not independent $k = 1, 2, \dots n$

Only $n-m$ of them are independent

Now we choose λ_l in such a way

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \sum_{l} \lambda_l a_{lk} = 0 \rightarrow (9)$$

$$(8) \Rightarrow \int_{t_1}^{t_2} \sum_{k=1}^{n-m} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \sum_{l} \lambda_l a_{lk} \right) \delta q_k \, dt = 0$$

Hence virtual displacement δq_k $k = 1, 2, \dots n-m$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \sum_{l} a_{lk} \lambda_l = 0 \rightarrow (10)$$

$$(9) \,\&\, (10) \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \sum_{l} a_{lk} \lambda_l = 0$$

which are the required Lagrange Equation $k = 1, 2, \dots n$

where the non Holonomic Constraints are of the form

$$\sum_{k} a_{lk} \, dq_k + a_{lt} \, dt = 0 \longrightarrow (11)$$

There are new unknown $q_1, q_2, \dots q_n, \lambda_1, \lambda_2, \dots \lambda_m$

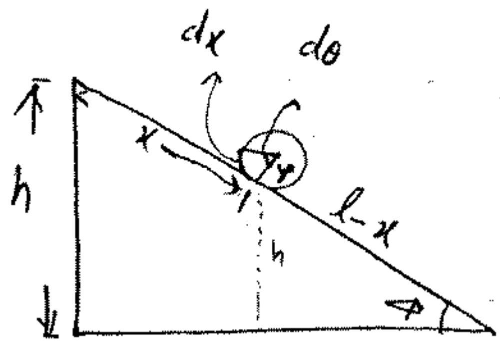
There are (i) n D-E

ii) m D-E.

$$(11) \Rightarrow \sum_{k} a_{lk} \frac{dq_k}{dt} + a_{lt} = 0 \quad l = 1, 2, \dots m$$

Example :- A loop of radius r is rolling (without slipping) down an inclined plane of angle φ. Use appropriate Lagrange Equation to find the acceleration and forces of constraints

Sol - Let x be the distance down the incline when the loop rotated through an angle θ at t. x and θ are two variable here
$q_1 = \theta$, $q_2 = x$



Let dx be the displacement along the incline when the loop has rotated ~~through~~ $d\theta$

$$\frac{dx}{r} = d\theta \Rightarrow dx = r\,d\theta$$

$l = r\theta$, $dl = r\,d\theta$

$$\Rightarrow r\,d\theta - dx = 0 \longrightarrow (1)$$

Comparing with Standard equation of Constraints

$$\sum_k a_{lk}\, dq_k + a_{lt}\, dt = 0 \longrightarrow (2)$$

Here $l = 1$, only one constraints equation $\therefore$ only one λ
Since $q_1 = \theta$ $q_2 = x$

$$(2) \Rightarrow \sum_{k=1}^{2} a_k\, dq_k = 0 \Rightarrow a_1\, dq_1 + a_2\, dq_2 = 0$$
$$\Rightarrow a_\theta\, d\theta + a_x\, dx = 0 \rightarrow (3)$$

from (1) & (3) $\Rightarrow a_\theta = r$, $a_x = -1$

Two Lagrange Equations are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = \lambda a_\theta \rightarrow (4)$ ($\lambda_l a_{lk}$, $l=1$, $k=0$, λa_θ)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = \lambda a_x \rightarrow (5)$$

where $L = T - V$ $\qquad T =$ Translational K.E. + Rotational K.E.

$$= \frac{1}{2} m\dot x^2 + \frac{1}{2} m (r\dot\theta)^2$$

$$T = \frac{1}{2} m \left(\dot x^2 + r^2\dot\theta^2\right)$$
$$V = P.E = mgh = mg(l-x)\sin\varphi$$

$\frac{h}{l-x} = \sin\varphi$, $h = (l-x)\sin\varphi$

$v = r\omega = r\dot\theta$

$$\Rightarrow L = T - V = \frac{1}{2} m(\dot x^2 + r^2\dot\theta^2) - mg(l-x)\sin\varphi$$

$$\frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta \quad \frac{\partial L}{\partial \theta} = 0 \quad \frac{\partial L}{\partial \dot x} = m\dot x \quad \frac{\partial L}{\partial x} = mg\sin\varphi$$

$$(4) \Rightarrow \frac{d}{dt}(mr^2\dot\theta) - 0 = \lambda(r) \rightarrow (6) \Rightarrow mr^2\ddot\theta = \lambda r \Rightarrow mr\ddot\theta = \lambda \text{ --- } (6)$$

$$(5) \Rightarrow \frac{d}{dt}(m\dot x) - mg\sin\varphi = \lambda(-1) \rightarrow (7) \Rightarrow m\ddot x - mg\sin\varphi = -\lambda \text{ --- } (7)$$

along with Equation of Constraint $r\dot{\theta} - \dot{x} = 0 \rightarrow (8)$

λ, θ, x are to be determine from (6) (7) & (8)

$(6) \Rightarrow \quad mr^2\ddot{\theta} = r\lambda \rightarrow (6a)$

$m\ddot{x} - mg\sin\varphi = -\lambda \quad (7a) \qquad r\dot{\theta} = \dot{x} \quad \rightarrow (8a)$

from $(8a) \Rightarrow \quad \ddot{\theta} = \frac{\ddot{x}}{r} \qquad (6a) \Rightarrow mr^2\left(\frac{\ddot{x}}{r}\right) = r\lambda \Rightarrow m\ddot{x} = \lambda \rightarrow (9)$

using (9) in (7a) $\quad m\left(\frac{\lambda}{m}\right) - mg\sin\varphi = -\lambda$

$\Rightarrow \quad \lambda - mg\sin\varphi + \lambda = 0 \Rightarrow \lambda = \frac{1}{2}mg\sin\varphi$

$\therefore (9) \Rightarrow \quad m\ddot{x} = \frac{1}{2}mg\sin\varphi \Rightarrow \ddot{x} = \frac{1}{2}g\sin\varphi$

Also $\quad \ddot{\theta} = \frac{\ddot{x}}{r} = \frac{1}{2r}g\sin\varphi \quad$ (using $\ddot{x}$)

Canonical or Conjugate Momentum

$T = \frac{1}{2}\sum_i m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \qquad V = V(x_i, y_i, z_i)$

$L = T - V$

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = m_i\dot{x}_i \quad i = 1, 2, \cdots$$
$$= p_x^i$$

Similarly $\frac{\partial L}{\partial \dot{y}_i} = p_y^i \qquad \frac{\partial L}{\partial \dot{z}_i} = p_z^i$

In general we define $\left(L = L(q_1, q_2 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n, t)\right)$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i = 1, 2, \cdots)$$ and p_i is called

Canonical or Conjugate momentum Corresponding to generalize Co-ordinate q_i

Cyclic or Ignorable Co-ordinate:- If the ~~Lagra~~ Langrangian L does not depend on a particular generalized Co-ordinate q_k (k-fixed), then q_k is called a Cyclic or Ignorable Co-ordinate

Conservation of Conjugate Momentum:- Let ~~..~~ a Co-ordinate q_j be Cyclic for the Lagrangian L then

$$\frac{\partial L}{\partial q_j} = 0 \rightarrow (1)$$

Lagrange Equation of motion Corresponding to q_j

(4) ... Lagrangian

is such that dq_j corresponds to rotation of the system around some axis, then the conservation of its conjugate momentum corresponds to conservation of an angular momentum.

By the same argument as used above, T cannot depend on q_j & $\frac{\partial T}{\partial q_j} = 0$. Also V is independent of $\dot{q}_j$

& $\dot{p}_j = Q_j = -\frac{\partial V}{\partial q_j}$.

Now we will show that when q_j denotes a rotational co-ordinate for Q_j is the component of the total applied torque about the axis of rotation and p_j is the component of the total angular momentum along the same axis.

Now $Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}$ Now $\frac{\partial r_i}{\partial q_j}$ is different meaning

the change dq_j in q_j corresponds to an infinitesimal rotation of the vector r_i, keeping the magnitude of vector constant

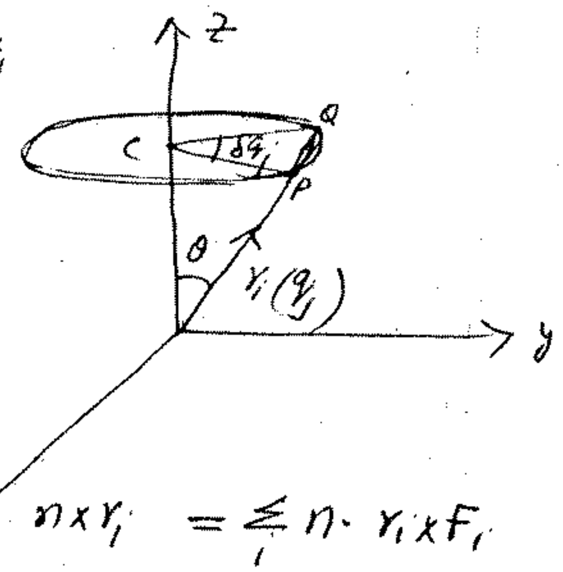
$\overline{CP} = \overline{CQ}$ $\widehat{PQ} \cong \overline{PQ}$

$|\delta r_i| \cong PQ \cong \overline{CP}\,\delta q_j \cong r\sin\theta\,\delta q_j$

$\left|\frac{\delta r_i}{\delta q_j}\right| \cong r\sin\theta$

$\Rightarrow \left|\frac{\partial r_i}{\partial q_j}\right| = r\sin\theta$

the direction of $\frac{\partial r_i}{\partial q_j}$ is $\perp$ to r_i & n



$\Rightarrow \frac{\partial r_i}{\partial q_j} = n \times r_i$

$\therefore Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i F_i \cdot n \times r_i = \sum_i n \cdot r_i \times F_i$

$= n \cdot \sum r_i \times F_i = n \cdot \sum N_i = n \cdot N$

$N = \sum N_i$ is the total torque in the direction of axis of rotation. Also $p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum m_i \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \sum m \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}$

$= \sum m_i v_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i m_i v_i \cdot n \times r_i = n \cdot \sum m_i r_i \times v_i$

$= n \cdot \sum_i r_i \times (m_i v_i) = n \cdot \sum_i L_i$ $(L_i = r_i \times (m_i v_i))$

$\Rightarrow p_j = n \cdot L$ where L is the total Angular Momentum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)-\frac{\partial L}{\partial q_j}=0 \rightarrow (2) \quad \text{using (1) (2)} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)=0$$

$$\Rightarrow \frac{d}{dt}(p_j)=0 \quad \text{(by Def. of Conjugate momentum)}$$

$\Rightarrow \quad p_j = \text{const}$

$\Rightarrow$ momentum Conjugate to a Cyclic Co-ordinate is Conserved

CONSERVATION OF MOMENTUM AND DISPLACEMENT OF THE System

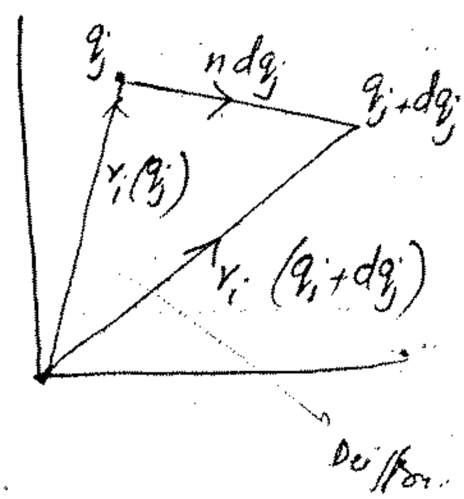
Let q_i be the generalized Co-ordinate Similar to the Cartesian Co-ordinate x, y, z. Let dq_j represents a translation of the System as a whole in some given direction. The K.E. T will not depend on q_j $\therefore \frac{\partial T}{\partial q_j}=0$

Now Consider the System for which V is not velocity dependent (Conservative System). Lagrange Equation will be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)-\frac{\partial L}{\partial q_j}=0 \rightarrow (1) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j}\right)-\frac{\partial T}{\partial q_j}=Q_j$$

$$\Rightarrow \frac{d}{dt}(p_j)-0=Q_j \Rightarrow \dot p_j = Q_j = -\frac{\partial V}{\partial q_j} \rightarrow (2)$$

now we Show that (2) represent Equation of motion for the total linear momentum



Since dq_j Corresponds to a translation of the System along some direction $\bar n$. more in the Case $\bar n$ is the Unit vector in the direction of translation

$$\therefore \frac{\partial r_i}{\partial q_j} = \lim_{\delta q_j \to 0} \frac{r_i(q_j+\delta q_j)-r_i(q_j)}{\delta q_j}$$

$$= \lim \frac{n\,\delta q_j}{\delta q_j} = \bar n$$

$\therefore Q_j = \sum_i F_i \cdot n = n\cdot\left(\sum F_i\right) = n\cdot F$ = (Component of total force along n)

To prove Similar result for momentum we have

$$T=\frac{1}{2}\sum_i m_i \dot r_i^2 \Rightarrow p_j = \frac{\partial L}{\partial \dot q_j} = \frac{\partial T}{\partial \dot q_j} = \sum_i m_i \dot r_i \cdot \frac{\partial \dot r_i}{\partial \dot q_j} \quad \text{(for Conservative System)}$$

$$= \sum m_i \dot r_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i m_i v_i \cdot n$$

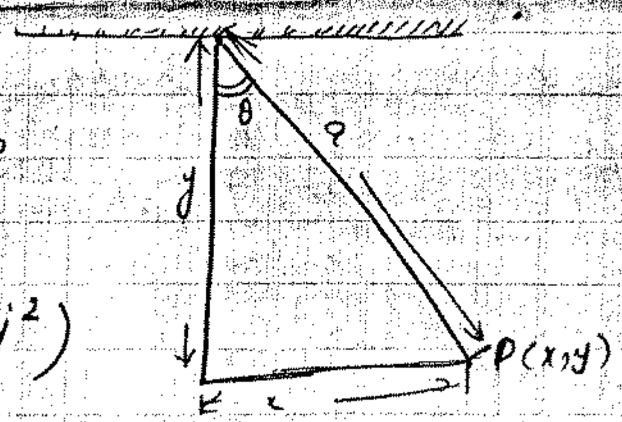
$\Longrightarrow \quad p_j = p\cdot n, \quad p = \text{Total momentum}$

In a similar way it Can be proved of a Cyclic Co-ordinate q_j

Q No. 1

Find the Lagrange Equation of motion for the motion of Simple Pendulum.

Let bob of mass m fastened by a thread of length a is at the point $P(x,y)$ making an angle θ with the vertical



$$K.E. = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$P.E = V = -mgy$$

Put $x = a\sin\theta$ $\quad y = a\cos\theta$

$$\dot{x} = \frac{dx}{dt} = +a\cos\theta\,\dot{\theta} \qquad \dot{y} = \frac{dy}{dt} = -a\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(a^2\sin^2\theta\,\dot{\theta}^2 + a^2\cos^2\theta\,\dot{\theta}^2)$$

$$T = \frac{1}{2}ma^2\dot{\theta}^2$$

$$P.E = -mg(a\cos\theta)$$

$$L = T - V = \frac{ma^2\dot{\theta}^2}{2} - (-mga\cos\theta) = \frac{ma^2\dot{\theta}^2}{2} + mga\cos\theta$$

Lagrange Equation of motion is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \qquad s = 1,2,\ldots n$$

Here the generalized co-ordinate is θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{2ma^2\dot{\theta}}{2}\right) - \left(mg(-a\sin\theta)\right) = 0$$

$$\Rightarrow \quad ma^2\ddot{\theta} + mga\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{a}\sin\theta \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

As θ is very small then $\sin\theta \simeq \theta$ $\qquad \sin\theta \simeq \theta$

$$\ddot{\theta} = -\frac{g}{a}\theta \Rightarrow \ddot{\theta} + \frac{g}{a}\theta = 0$$

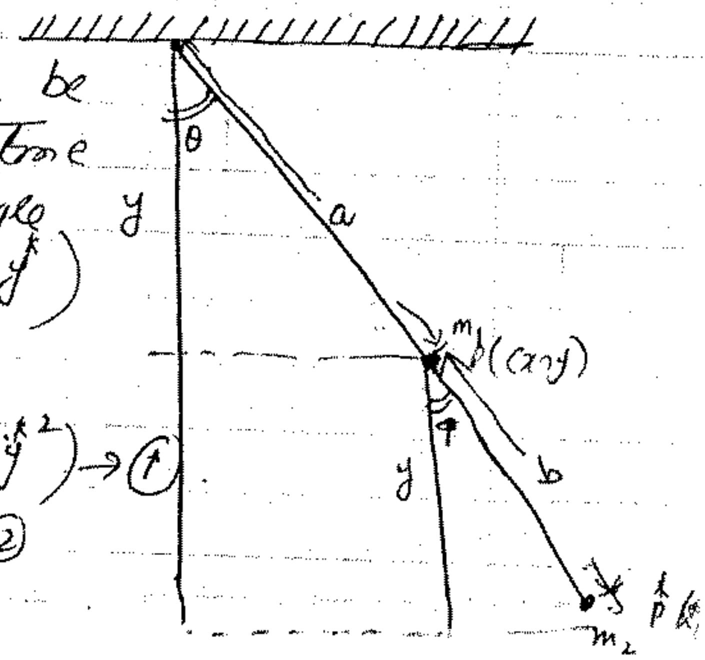
$$\Rightarrow \ddot{\theta} + \omega^2\theta = 0 \qquad \omega^2 = g/a$$

Find the Lagrange Equation of motion for Double (Compound) Pendulum.

Let the Particles of mass m_1 and m_2 be fastened by thread and supported from the upper end by a rigid support,

While mass m_1 makes an angle θ with the vertical be at Position $P(x, y)$ at any time

the mass m_2 makes an angle φ with vertical at $P^*(x^*, y^*)$



K.E.

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 (\dot{x}^{*2} + \dot{y}^{*2}) \rightarrow (1)$$

$$V = -m_1 g y - m_2 g y^* \rightarrow (2)$$

where

$x = a \sin\theta \qquad y = a\cos\theta$

$$x^* = a\sin\theta + b\sin\varphi$$

$$y^* = a\cos\theta + b\cos\varphi$$

$$\dot{x} = +a\cos\theta\,\dot{\theta} \qquad \dot{y} = -a\sin\theta\,\dot{\theta}$$

$$\dot{x}^* = a\cos\theta\,\dot{\theta} + b\cos\varphi\,\dot{\varphi}$$

$$\dot{y}^* = -a\sin\theta\,\dot{\theta} - b\sin\varphi\,\dot{\varphi}$$

$(1) \Rightarrow$

$$T = \frac{1}{2} m_1 (a^2\dot{\theta}^2\cos^2\theta + a^2\dot{\theta}^2\sin^2\theta) + \frac{1}{2} m_2 (a^2\cos^2\theta\,\dot{\theta}^2 + b^2\cos^2\varphi\,\dot{\varphi}^2 + 2ab\cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + a^2\sin^2\theta\,\dot{\theta}^2 + b^2\sin^2\varphi\,\dot{\varphi}^2 + 2ab\sin\theta\sin\varphi\,\dot{\theta}\dot{\varphi})$$

$$= \frac{1}{2} m_1 (a^2\dot{\theta}^2) + \frac{1}{2} m_2 (a^2\dot{\theta}^2 + b^2\dot{\varphi}^2 + 2ab\dot{\theta}\dot{\varphi}(\cos\theta\cos\varphi + \sin\theta\sin\varphi))$$

$$= \frac{1}{2} m_1 a^2\dot{\theta}^2 + \frac{1}{2} m_2 (a^2\dot{\theta}^2 + b^2\dot{\varphi}^2 + 2ab\dot{\theta}\dot{\varphi}\cos(\theta - \varphi))$$

$(2) \Rightarrow \quad V = -m_1 g (a\cos\theta) - m_2 g (a\cos\theta + b\cos\varphi)$

$$L = T - V$$

$$L = \frac{1}{2} m_1 a^2 \dot\theta^2 + \frac{1}{2} m_2 (a^2\dot\theta^2 + b^2\dot\varphi^2 + 2ab\,\dot\theta\dot\varphi \cos(\theta-\varphi))$$
$$+ m_1 g a \cos\theta + m_2 g a \cos\theta + m_2 g b \cos\varphi$$

now there are two generalized co-ordinate θ & φ.

$\therefore$ Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \rightarrow (3)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = 0 \rightarrow (4)$$

$$(1) \Rightarrow \frac{d}{dt}\left(\frac{m_1 a^2 \cancel{2}\dot\theta}{\cancel{2}} + \frac{1}{2} m_2 (a^2 2\dot\theta + 0 + 2ab\,\dot\varphi\,\theta \cos(\theta-\varphi\right.$$
$$- \left[(0 + \frac{1}{2} m_2 (0 + 0 + 2ab\,\dot\theta\dot\varphi\,(-\sin(\theta-\varphi))) + (-m_1 g a \sin\theta - m_2 g a \sin\theta)\right) = 0$$
$$m_1 a^2 \ddot\theta + m_2 a^2 \ddot\theta$$

$$\frac{\partial L}{\partial \dot\theta} = m_1 a^2 \dot\theta + \frac{1}{2} m_2 (a^2 2\dot\theta + 0 + 2ab\,\dot\varphi \cos(\theta-\varphi))$$
$$= m_1 a^2 \dot\theta + m_2 a^2 \dot\theta + m_2 ab\,\dot\varphi \cos(\theta-\varphi)$$
$$\frac{\partial L}{\partial \theta} = \frac{1}{2}(0) + \frac{1}{2} m_2 (0 + 0 + 2ab\,\dot\theta\dot\varphi\,(-\sin(\theta-\varphi)))$$
$$- m_1 g a \sin\theta - m_2 g a \sin\theta.$$

$$(1) \Rightarrow \frac{d}{dt}\left(m_1 a^2 \dot\theta + m_2 a^2 \dot\theta + m_2 ab\,\dot\varphi \cos(\theta-\varphi)\right)$$
$$- \left(-m_2 ab\,\dot\theta\dot\varphi \sin(\theta-\varphi) - m_1 g a \sin\theta - m_2 g a \sin\theta\right)$$

$$\Rightarrow m_1 a^2 \ddot\theta + m_2 a^2 \ddot\theta + m_2 ab\left(\ddot\varphi \cos(\theta-\varphi) + \dot\varphi(-\sin(\theta-\varphi))(\dot\theta - \dot\varphi)\right)$$
$$+ m_2 ab\,\dot\theta\dot\varphi \sin(\theta-\varphi) + m_1 g a \sin\theta + m_2 g a \sin\theta = 0$$

$$m_1 a^2 \ddot\theta + m_2 a^2 \ddot\theta + m_2 ab\,\ddot\varphi \cos(\theta-\varphi) \cancel{+ m_2 ab\,\dot\varphi\dot\theta \sin(\theta-\varphi)} + m_1 ab\,\dot\varphi^2 \sin(\theta-\varphi)$$
$$+ m_2 ab\,\dot\theta\dot\varphi \cancel{\sin(\theta-\varphi)} + m_1 g a \sin\theta + m_2 g a \sin\theta = 0$$

$$m_1 a^2 \ddot\theta + m_2 a^2 \ddot\theta + m_2 ab\,\ddot\varphi \cos(\theta-\varphi) + m_2 ab\,\dot\varphi^2 \sin(\theta$$
$$+ m_1 g a \sin\theta + m_2 g a \sin\theta = 0$$

$$\cos(\theta-\varphi) = 1$$
$$\sin(\theta-\varphi) = 0 \qquad \sin\theta \approx \theta$$

$$m_1 a^2 \ddot{\theta} + m_2 a^2 \ddot{\theta} + m_2 ab\ddot{\varphi} + m_1 g a\theta + m_2 g a\theta = 0$$

$$(m_1+m_2)\, a^2\ddot{\theta} + m_2 ab\ddot{\varphi} + ga\theta\,(m_1+m_2) = 0$$

$$(m_1+m_2)\,(a^2\ddot{\theta} + ga\theta) + m_2 ab\ddot{\varphi} = 0$$

$$\Rightarrow \quad a^2\ddot{\theta} + ga\theta = -\frac{m_2}{(m_1+m_2)}\, ab\ddot{\varphi}$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{a}\theta = -\frac{m_2}{(m_1+m_2)}\left(\frac{b}{a}\right)\ddot{\varphi} \longrightarrow (1)$$

for the generalized co-ordinate φ

$$\frac{d}{dt}\Bigg(\frac{\partial}{\partial\dot{\varphi}}\Big(\tfrac{1}{2}m_1a^2\dot{\theta}^2 + \tfrac{1}{2}m_2(a^2\dot{\theta}^2 + b^2\dot{\varphi}^2 + 2ab\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)) + m_1ga\cos\theta + m_2ga\cos\theta + m_2gb\cos\varphi\Big)\Bigg)$$

$$-\frac{\partial}{\partial\varphi}\Big(\tfrac{1}{2}m_1a^2\dot{\theta}^2 + \tfrac{1}{2}m_2(a^2\dot{\theta}^2 + b^2\dot{\varphi}^2 + 2ab\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)) + m_1ga\cos\theta + m_2ga\cos\theta + m_2gb\cos\varphi\Big) = 0$$

$$\frac{d}{dt}\Big(0 + \tfrac{1}{2}m_2(0 + 2b^2\dot{\varphi} + 2ab\dot{\theta}\cos(\theta-\varphi))\Big)$$

$$-\Big(0 + \tfrac{1}{2}m_2(0 + 0 + 2ab\dot{\theta}\dot{\varphi}(-\sin(\theta-\varphi))) + 0 + 0 + m_2gb(-\sin\varphi)\Big) = 0$$

$$\Rightarrow \tfrac{1}{\cancel{2}}m_2\cdot\cancel{2}b^2(\ddot{\varphi}) + \cancel{2}ab\cdot\tfrac{1}{\cancel{2}}m_2\big(\ddot{\theta}\cos(\theta-\varphi) + \dot{\theta}(-\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi}))\big)$$

$$+ m_2ab\dot{\theta}\dot{\varphi}\sin(\theta-\varphi) + m_2gb\sin\varphi = 0$$

$$m_2b^2\ddot{\varphi} + abm_2\ddot{\theta}\cos(\theta-\varphi) - abm_2\dot{\theta}^2\sin(\theta-\varphi)$$

$$\cancel{+\dot{\varphi}\sin(\theta-\varphi)\dot{\theta}\,abm_2} + \cancel{m_2gb\sin\varphi}$$

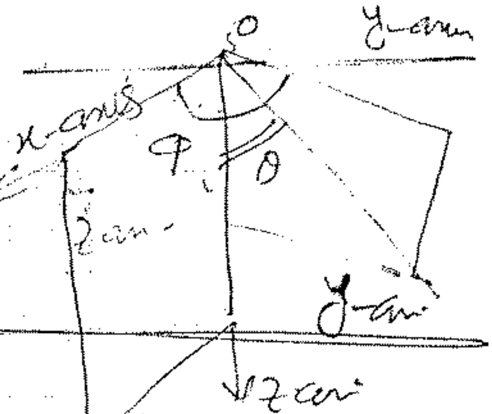
$$+ m_2ab\cancel{\dot{\theta}\dot{\varphi}\sin(\theta-\varphi)} + m_2gb\sin\varphi = 0$$

$$m_2b^2\ddot{\varphi} - m_2ab\dot{\theta}^2\sin(\theta-\varphi) + m_2ab\ddot{\theta}\cos(\theta-\varphi)$$

$$+ m_2gb\sin\varphi = 0$$

Imposing condition $\cos(\theta-\varphi) = 1 \quad \sin(\theta-\varphi) = 0$

$$\theta \approx \varphi$$



$$m_2 b^2 \ddot{\varphi} + m_2 ab\, \ddot{\theta} + m_2 g b \varphi = 0$$

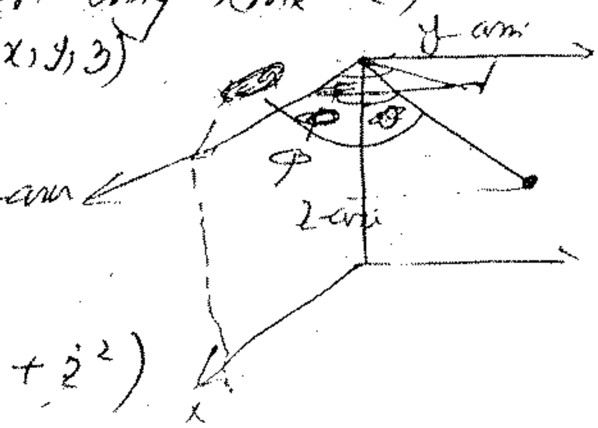
$$\ddot{\varphi} + \frac{g}{b}\varphi + \frac{a}{b}\ddot{\theta} = 0$$

$$\Rightarrow \ddot{\varphi} + \frac{g}{b}\varphi = -\frac{a}{b}\ddot{\theta} \longrightarrow (B)$$

A and (B) are the generalized Lagrange Equation of Motion

Find the Lagrange Equation of motion for a Particle moving in a spherical equilaterial i.e spherical pendulum and also solve them.

Sol.- Consider a Particle mass m is attached to a fixed Point by a rod of length 'a' and oscillates under the action of gravity. Since the Particle is constrained to move on a sphere, this is spherical Pendulum. At any time t, let the Particle is at $P(x,y,z)$



then
$$x = a\sin\theta\cos\varphi$$
$$y = a\sin\theta\sin\varphi$$
$$z = a\cos\theta$$

So

$$K.E = T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2} m \Big[\big(a\sin\theta(-\sin\varphi)\dot{\varphi} + a\cos\theta\,\dot{\theta}\cos\varphi \big)^2 + \big(a\sin\theta\cos\varphi\,\dot{\varphi} + \sin\varphi\cos\theta\,\dot{\theta}\,a \big)^2 + (-a\sin\theta\,\dot{\theta})^2 \Big]$$

$$T = \frac{m}{2}\Big(a^2\dot{\varphi}^2\sin^2\theta\sin^2\varphi + a^2\dot{\theta}^2\cos^2\theta\cos^2\varphi \cancel{- 2a^2\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi\sin\theta\sin\varphi} + a^2\dot{\varphi}^2\sin^2\theta\cos^2\varphi + \dot{\theta}^2 a^2\sin^2\varphi\cos^2\theta \cancel{+ 2a^2\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi\sin\theta\sin\varphi} + a^2\dot{\theta}^2\sin^2\theta \Big)$$

$$= \frac{m}{2}\Big[a^2\dot{\varphi}^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi) + a^2\dot{\theta}^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + a^2\dot{\theta}^2\sin^2\theta$$

$$T = \frac{m}{2}\left\{ a^2\dot\varphi^2 \sin^2\theta + a^2\dot\theta^2\cos^2\theta + a^2\dot\theta^2 \sin^2\theta \right.$$

$$= \frac{m}{2}\left[a^2\dot\theta^2(\cos^2\theta + \sin^2\theta) + a^2\dot\varphi^2\sin^2\theta \right]$$

$$= \frac{m}{2}\left[a^2\dot\theta^2 + a^2\dot\varphi^2\sin^2\theta \right)$$

$$= \frac{ma^2}{2}\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right)$$

$$V = -mga\cos\theta$$

$$L = T - V = \frac{ma^2}{2}\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + mga\cos\theta$$

where θ & φ are the generalized co-ordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \rightarrow (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = 0 \rightarrow (2)$$

$$(1) \Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial\dot\theta}\left(\left(\frac{ma^2}{2}\dot\theta^2 + \frac{ma^2}{2}\dot\varphi^2\sin^2\theta\right) + mga\cos\theta\right)\right)$$
$$- \frac{\partial}{\partial\theta}\left(\frac{ma^2}{2}\dot\theta^2 + \frac{ma^2}{2}\dot\varphi^2\sin^2\theta + mga\cos\theta\right) = 0$$

$$\frac{d}{dt}\left(\frac{ma^2}{\cancel{2}}\cancel{2}\dot\theta + 0 + 0\right) - \left(0 + \frac{ma^2}{2}\dot\varphi^2\, 2\sin\theta\cos\theta - mga\sin\theta\right) = 0$$

$$ma^2\ddot\theta - ma^2\dot\varphi^2\sin\theta\cos\theta + mga\sin\theta = 0$$

$$\Rightarrow \ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 + \frac{g}{a}\sin\theta = 0$$

$$\Rightarrow \ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = -\frac{g}{a}\sin\theta \longrightarrow (A)$$

$$(2) \Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial\dot\varphi}\left(\frac{ma^2}{2}\dot\theta^2 + \frac{ma^2}{2}\dot\varphi^2\sin^2\theta + mga\cos\theta\right)\right.$$
$$- \frac{\partial}{\partial\varphi}\left(\frac{ma^2}{2}\dot\theta^2 + \frac{ma^2}{2}\dot\varphi^2\sin^2\theta + mga\right.$$

$$\frac{d}{dt}\left(0 + \frac{ma^2}{2}\sin^2\theta\; 2\dot\varphi + 0\right)$$
$$- \left(0 + \frac{ma^2}{2}\, 0 + 0\right) = 0$$

$$\frac{d}{dt} = \frac{d}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{d}{d\varphi}\frac{h}{\sin^2\theta}$$

(29)

$$\Rightarrow \quad \frac{d}{dt} = \frac{h}{\sin^2\theta}\frac{d}{d\varphi}$$

$$\frac{d}{dt}\left(ma^2\sin^2\theta\ \dot\varphi\right.$$

$$\frac{d}{dt}\left(\sin^2\theta\ \dot\varphi\right) = 0$$

$$\Rightarrow \int d\left(\sin^2\theta\ \dot\varphi\right) = 0\int dt$$

$$\sin^2\theta\ \dot\varphi = h$$

$$\Rightarrow \dot\varphi = \frac{h}{\sin^2\theta} \to B$$

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{h}{\sin^2\theta}\frac{d}{d\varphi}\cdot\frac{h}{\sin^2\theta}\frac{d\theta}{d\varphi}$$

$$= \frac{h^2}{\sin^2\theta}\frac{d^2}{d\varphi^2}(\operatorname{cosec}^2\theta)$$

$$= -\frac{\cot\theta\, h^2}{\sin^2\theta}\frac{d^2}{d\varphi^2}$$

$$\int \operatorname{cosec}^2\theta\, d\theta$$

Using B in (A)

$$\ddot\theta - \sin\theta\cos\theta\cdot\frac{h^2}{\sin^4\theta} = -\frac{g}{a}\sin\theta$$

$$\ddot\theta - \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin^2\theta}h^2 = -\frac{g}{a}\sin\theta$$

$$\Rightarrow \ddot\theta - \cot\theta\operatorname{cosec}^2\theta\, h^2 + \frac{g}{a}\sin\theta = 0 \to \text{(C)}$$

$$(D^2+1)\cot\theta = 0$$

A.E. $D^2+1 = 0$, $D = \pm i$

$$\cot\theta = C_1\cos\varphi + C_2\sin\varphi$$

$$= A\cos B\cos\varphi + A\sin B\sin\varphi$$

$$= A\cos(B+\varphi)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{d}{d\varphi}\left(\frac{h}{\sin^2\theta}\right) = \dot\varphi\frac{d\theta}{d\varphi} = \frac{h}{\sin^2\theta}\frac{d\theta}{d\varphi}$$

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{d\varphi}\left(\frac{d\theta}{dt}\right)\frac{d\varphi}{dt}$$

$$= \dot\varphi\frac{d}{d\varphi}\left(\frac{d\theta}{dt}\right) = \frac{h}{\sin^2\theta}\frac{d}{d\varphi}\left(\frac{d\theta}{d\varphi}\frac{d\varphi}{dt}\right)$$

$$= \dot\varphi\frac{d}{d\varphi}\left(\frac{d\theta}{d\varphi}\cdot\frac{d\varphi}{dt}\right) = \dot\varphi\frac{d}{d\varphi}\left(\dot\varphi\frac{d\theta}{d\varphi}\right)$$

$$= \frac{h}{\sin^2\theta}\frac{d}{d\varphi}\left(\frac{h}{\sin^2\theta}\frac{d\theta}{d\varphi}\right)$$

$$= \frac{h}{\sin^2\theta}\frac{d}{d\varphi}\left(\frac{h}{\sin^2\theta}\cdot\frac{d\theta}{d\varphi}\right)$$

$$= -\frac{h^2}{\sin^2\theta}\frac{d^2}{d\varphi^2}(\cancel{\operatorname{cosec}}\cot\theta)$$

$$\frac{d^2\theta}{dt^2} = \frac{h}{\sin^2\theta}\frac{d}{dt}\left(\frac{d\theta}{d\varphi}\right)$$

$$= \frac{h}{\sin^2\theta}\frac{d}{d\varphi}\left(\frac{d\theta}{d\varphi}\right)\frac{d\varphi}{dt}$$

$$= \frac{h^2}{\sin^2\theta}\frac{d}{d\varphi^2}\ d\int \operatorname{cosec}^2\theta\, d\theta$$

$$\because \text{(C)} = -\frac{h^2}{\sin^2\theta}\frac{d^2}{d\varphi^2}\cot\theta - \cot\theta\operatorname{cosec}^2\theta\, h^2 + \frac{g}{a}\sin\theta = 0$$

$$\frac{d^2}{d\varphi^2}(\cot\theta) + \cot\theta - \frac{g\sin^3\theta}{ah^2} = 0$$

$$= \frac{h^2}{\sin^2\theta}\frac{d^2}{d\varphi^2}(\cot\theta$$

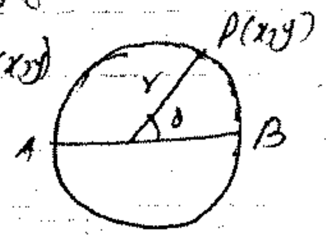
If θ is small $\sin^3\theta = 0$

$$\Rightarrow \frac{d^2}{d\varphi^2}(\cot\theta) + \cot\theta = 0 \quad \Rightarrow \cot\theta = A\cos(\varphi+B)$$

$$\theta = \cot^{-1}A\cos(\varphi+B)$$

NO: 4 Find the Lagrange Equation of motion of Circular orbit

Let a particle of mass m moves in a circular orbit of radius r making an angle θ when it is at P(x,y)



$x = r\cos\theta \qquad y = r\sin\theta$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(\dot{r}\cos\theta + r\sin\theta)^2$
$\qquad + \frac{1}{2} m(\dot{r}\sin\theta + r\cos\theta)^2$

$T = \frac{1}{2} m\left(\dot{r}^2(\cos^2\theta + \sin^2\theta) + r^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) - \cancel{2r\dot{r}\dot{\theta}\cos\theta} + 2r\dot{r}\dot{\theta}\cos\theta\right)$

$T = \frac{1}{2} m(\dot{r}^2 + r\dot{\theta}^2)$

$P.E = mgr\sin\theta.$

$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$

where θ & r are the Lagrangian co-ordinate

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \rightarrow (1) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

$(1) \Rightarrow \frac{d}{dt}\left(\frac{1}{2}m(0 + 2r^2\dot{\theta}) - 0\right) - \left(\frac{1}{2}m(0) - mgr\cos\theta\right) = 0$

$\frac{d}{dt}(mr^2\dot{\theta}) + mgr\cos\theta = 0$

$mr^2\ddot{\theta} + mgr\cos\theta = 0$

$\Rightarrow \ddot{\theta} + \frac{g}{r}\cos\theta = 0 \rightarrow (1)$

$\cancel{(2)\ \frac{d}{dt}\left(\left(\frac{1}{2}m(0+0) - 0\right) - \left(\frac{1}{2}m(2r + 2r\dot{\theta}^2) - mg\sin\theta\right)\right)} =$

$\Rightarrow \cancel{m\ddot{r} - r\dot{\theta}^2 -}$

$\frac{d}{dt}\left(\frac{1}{2}m 2\dot{r}\right) - \left(\frac{1}{2}m\cdot(0 + 2r\dot{\theta}^2) - mg\sin\theta\right)$

$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0$

$\ddot{r} - r\dot{\theta}^2 + g\sin\theta$

for θ small $\sin\theta \approx \theta$

$\Rightarrow \ddot{r} - r\dot{\theta}^2 + g\theta \rightarrow (2)$

(1) and (2) are the Lagrange Equation of motion

HAMILTONIAN FUNCTION AND TOTAL Energy

Consider a Conservative System for which

$$F = -\nabla V$$

we further assume that the Constraints do not explicitly depend on time ie. the System is Scleronomic

$\Rightarrow$ that the L does not depend on t explicitly

ie $L = L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n)$

$$\therefore \quad \frac{dL}{dt} = \sum_{j=1}^{n}\left(\frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right) \rightarrow (1)$$

By Lagrange Equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$ $\quad j = 1, 2, \cdots$

$$\therefore (1) \Rightarrow \quad \frac{dL}{dt} = \sum_j\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right)$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\Rightarrow \quad \frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(L - \sum_j\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right)\right) = 0$$

$$\therefore \quad L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \text{constt.}$$

we take this Constant of motion as $-H$

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = -H$$

$$\Rightarrow \quad H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

Now using the definition of Conjugate momentum we have

$$H = \sum_j \dot{q}_j p_j - L \rightarrow (2) \quad \because p_j = \frac{\partial L}{\partial \dot{q}_j}$$

is the Equation of Hamiltian

For a Conservative System $p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$

$\therefore$ (2) $\Rightarrow$ $H = \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L \rightarrow$ (3)

from Euler's theorem for homogenous function $f(x_1, x_2 \dots x_n)$

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x_1 x_2 \dots x_n) = d f(x_1, x_2 \dots x_n) \rightarrow (4)$$

where d, (degree of function)

Now when the field is Conservative and there is no explicit time dependence of T on t then T is homogenous function in $\dot{q}_1 \dot{q}_2 \dots \dot{q}_n$ of degree 2

using (4)

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \rightarrow (5)$$

using in (3) $H = 2T - L = 2T - (T - V)$

$= T + V$

ie $H = T + V$ Total Energy for a Conservative Scleronomic System

HAMILTONIAN FORMULATION OF MECHANICS

In this formulation, we use Hamiltonian H as above quantity rather than Lagrangian L

In Lagrangian formulation $q_i, \dot{q}_i, t$ $i = 1, 2, 3 \ldots$
In Hamiltonian formulation q_i, P_i, t $i = 1, 2, 3, \ldots$

$\Rightarrow$ i) $L = L(q, \dot{q}, t)$
ii) $H = H(q, p, t)$

To change the basis $(q, \dot{q}, t)$ to (q, p, t) we use Legendre transformation

Let $f = f(x, y)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy$$

or $$df = u\,dx + v\,dy \rightarrow (1)$$

Let $g = g(u, y)$ s.t

$$g = f - ux$$

$$dg = df - d(ux) = df - u\,dx - x\,du$$

using (1)

$$dg = u\cancel{dx} + v\,dy - u\cancel{dx} - x\,du$$

$$dg = v\,dy - x\,du \rightarrow (2)$$

Also $$dg = \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du = g_y dy + g_u du \rightarrow (3)$$

for (3) & (4) $\Rightarrow$ $\left.\begin{array}{l} g_y = v \quad g_u = -x \end{array}\right] \rightarrow (4)$

This transformation is called Legendre transformation

HAMILTON'S EQUATION OF MOTION

Now $H(q,p,t) = \sum_i \dot{q}_i p_i - L(q,\dot{q},t) \rightarrow (1)$

$$dH = \sum_i \dot{q}_i\, dp_i + \sum_i p_i\, d\dot{q}_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \rightarrow (2)$$

Also $\because$ H is a function of q's, p's and t,
we have

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \rightarrow (3)$$

~~in (2) and (3)~~

from (2) $\sum_i p_i\, d\dot{q}_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i = \sum_i d\dot{q}_i \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right)$

$$= \sum_i d\dot{q}_i (p_i - p_i) = 0$$

$$\Rightarrow \quad dH = \sum_i \dot{q}_i\, dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \rightarrow (4)$$

from Lagrange Equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$

$$\Rightarrow \quad \frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i} \Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i}$$

(4) $\Rightarrow \quad dH = \sum_i (\dot{q}_i\, dp_i - \dot{p}_i\, dq_i) - \frac{\partial L}{\partial t} dt$

from (3) and (4)

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Routh's Procedure

This is a method for solving equation of motion which combine the advantages of Hamiltonian formulation with those Lagranges formulation.

Let $q_1, q_2 \ldots q_s$ be cyclic and the remaing $n-s$ ie $q_{s+1}, q_{s+2} \ldots q_n$

we introduce a function R.

$$R(q_1, q_2 \ldots q_n, p_1, p_2 \ldots p_s, \dot{q}_{s+1} \ldots \dot{q}_n, t) = \sum_{i=1}^{s} p_i \dot{q}_i - L(q, \dot{q}, t)$$

This function is called Routhain

$$dR = d\left(\sum_{i=1}^{s} p_i \dot{q}_i - L\right)$$

$$= \sum_{i=1}^{s} p_i d\dot{q}_i + \sum_{i=1}^{s} \dot{q}_i dp_i - \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) - \frac{\partial L}{\partial t}$$

$$= \sum_{i=1}^{s} p_i d\dot{q}_i + \sum_{i=1}^{s} \dot{q}_i dp_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i - \sum_{i=1}^{n} p_i d\dot{q}_i - \frac{\partial L}{\partial t} dt$$

$$= \cancel{\sum_{i=1}^{s} p_i d\dot{q}_i} + \sum_{i=1}^{s} \dot{q}_i dp_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i - \cancel{\sum_{i=1}^{s} p_i d\dot{q}_i} - \sum_{i=s+1}^{n} p_i d\dot{q}_i - \frac{\partial L}{\partial t} dt$$

$\cancel{dR \neq \sum_{i=1}^{s} p_i d\dot{q}_i}$

$$dR = \sum_{i=1}^{s} \dot{q}_i dp_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i - \sum_{i=s+1}^{n} p_i d\dot{q}_i - \frac{\partial L}{\partial t} dt \rightarrow (1)$$

where $R = R(q_1, q_2 \ldots q_n, p_1, p_2 \ldots p_s, \dot{q}_{s+1} \ldots \dot{q}_n, t)$

$$dR = \sum_{i=1}^{n} \frac{\partial R}{\partial q_i} dq_i + \sum_{i=1}^{s} \frac{\partial R}{\partial p_i} dp_i + \sum_{i=s+1}^{n} \frac{\partial R}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial R}{\partial t} dt \rightarrow (2)$$

fm (1) and (2) $\Rightarrow$

$$\frac{\partial R}{\partial q_i} = -\frac{\partial L}{\partial q_i} \quad i = 1, 2, \ldots n \rightarrow (i) \qquad \frac{\partial R}{\partial p_i} = \dot{q}_i \quad i = 1, 2, \ldots s \rightarrow (ii)$$

$$\frac{\partial R}{\partial \dot{q}_i} = -p_i \quad i = s+1 \ldots n \rightarrow (iii) \qquad \frac{\partial R}{\partial t} = -\frac{\partial L}{\partial t} \rightarrow (iv)$$

We write these equations

(ii) $\dfrac{\partial R}{\partial p_i} = \dot{q}_i \quad i = 1,2,3,\ldots s$

(i) $\Rightarrow \dfrac{\partial R}{\partial q_i} = -\dfrac{\partial L}{\partial q_i} \quad i = 1,2,3,\ldots s \qquad \rightarrow (3)$

3 (b) $\Rightarrow \dfrac{\partial R}{\partial q_i} = -\dfrac{\partial L}{\partial q_i} \quad i = s+1 \ldots n$

(iii) $\Rightarrow \dfrac{\partial R}{\partial \dot{q}_i} = -\dfrac{\partial L}{\partial \dot{q}_i} \quad i = s+1 \ldots n \qquad \dfrac{\partial L}{\partial \dot{q}_i} = p_i \qquad \rightarrow (4)$

$\Rightarrow$ are s 1st order equation similar to Hamilton's equation (R act the role of H)

(4) $\Rightarrow$ R satisfies the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0 \quad i = s+1 \ldots n$$

$$\frac{dH}{dt} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t}$$

by Hamilton's equation

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\frac{dH}{dt} = \sum_{i=1}^{n}\left(-\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i\right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

In the present situation $\dfrac{\partial H}{\partial t} = 0 \Rightarrow \dfrac{dH}{dt} = 0$

$$\Rightarrow H = \text{Const}$$

Application of Hamilton's Equation To Central Force Problem

Central force :- A force which is directed towards or away from a fixed point is called the central force.

general form is $F(r) = f(r)\,\hat{r}$

For a central force $P.E. = V = V(r)$

$K.E. = T = \frac{1}{2} m v^2$

$= \frac{1}{2} m \left(v_r^2 + v_\theta^2\right)$

Here $q_1 = r$ $\quad q_2 = \theta$

$p_1 = p_r = p \quad p_2 = p_\theta = p$

Hamiltonian function $= H = T + V = \frac{1}{2} m \left(v_r^2 + v_\theta^2\right) + V(r)$

For writing Hamilton's Equations, we need to express H in terms of q_1, q_2, p_1, p_2 or r, θ, p_r, p_θ

$p_r = m v_r = m \dot{r}$ (Conjugate momentum for r)

$p_\theta =$ Angular momentum component in the direction in which θ increases

$= p_\theta \times r = r p_\theta = r(m v_\theta) = m r (r\dot{\theta}) = m r^2 \dot{\theta}$

$$H = \frac{1}{2} m \left(\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2} \right) + V(r)$$

p_θ is onto radius vector

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

$$= \frac{\partial H}{\partial p_r} = p_r/m = \frac{m\dot{r}}{m} = \dot{r} \qquad , \quad v_\theta = \frac{p_\theta}{rm}$$

$$\frac{\partial H}{\partial p_\theta} = \frac{2p_\theta}{2mr^2} = \frac{mr^2\dot{\theta}}{mr^2} = \dot{\theta}$$

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -m\ddot{r}$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta}$$

DERIVATION OF HAMILTONIAN EQUATIONS FROM VARIATIONAL PRINCIPLE

(ie Principle of least Action)

the Hamiltonian is defined $H = \sum_i p_i \dot{q}_i - L$

$\therefore \quad L = \sum p_i \dot{q}_i - H \rightarrow ①$

ow Action integral $\quad I = \int_{t_1}^{t_2} L\, dt$

$\therefore \delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0 \rightarrow ②$

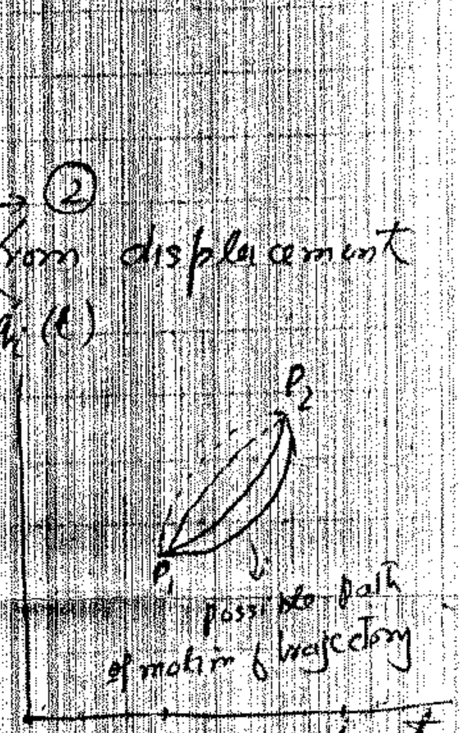
where δ denote variational resulting from displacement of trajectories

variation of trajectories at t_1 and t_2 is zero

ie $\delta q_i = 0$

Similarly $\delta q_1 = 0 = \delta q_2$

& $\delta p_1 = 0 = \delta p_2$ at t_1 & t_2



ing ① in ② $\quad \delta \int_{t_1}^{t_2} \left(\sum p_i \dot{q}_i - H(p, q, t) \right) dt = 0 \rightarrow ③$

o Calculate the variation of the integral in ③ we have introduce a parameters α to label different trajectories. All these trajectories pass through the fixed points P_1 and P_2.

we write $\quad p = p(\alpha, t)$ & $q = q(\alpha, t)$

$$\Rightarrow \delta I = \delta \left(\int_{t_1}^{t_2} \sum_i p_i(\alpha, t)\, \dot{q}_i(\alpha, t) - H\left(p(\alpha, t), q(\alpha, t)\right) \right) = 0$$

$$\int_{t_1}^{t_2} \left(\sum_i \left(\frac{\partial p_i}{\partial \alpha} \dot{q}_i + p_i \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right) \right) - \sum_i \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} - \sum_i \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \alpha} \right) \delta\alpha\, dt = 0$$

$$\int_{t_1}^{t_2} \sum_i \left(\frac{\partial p_i}{\partial \alpha} \dot{q}_i + p_i \frac{\partial \dot{q}_i}{\partial \alpha} - \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} - \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \alpha} \right) \delta\alpha\, dt = 0 \rightarrow ④$$

$$\int p_i \frac{\partial q_i}{\partial \alpha}\delta\alpha\, dt \quad \delta\alpha \int p_i \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right)dt$$

$$= \delta\alpha\left(p_i \int \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right)dt - \int\left(\int \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right)dt\right)\dot{p}_i\, dt\right)$$

$$= \delta\alpha\; p_i \frac{\partial q_i}{\partial \alpha} - \int \frac{\partial q_i}{\partial \alpha}\dot{p}_i\,\delta\alpha\, dt$$

$$\frac{\partial q_i}{\partial \alpha}\delta\alpha = \left.\delta q_i\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial q_i}{\partial \alpha}\dot{p}_i\,\delta\alpha\, dt$$

note $\delta\alpha \frac{\partial q_i}{\partial \alpha} = \delta q_i$

$\delta\alpha \frac{\delta q_i}{\delta \alpha} = \delta q_i$

$$= \left(\delta q_i(t_2) - \delta q_i(t_1)\right) - \int_{t_1}^{t_2}\delta q_i\,\dot{p}_i\, dt$$

$$= 0 - \int_{t_1}^{t_2}\delta q_i\,\dot{p}_i\, dt$$

$$\Rightarrow \int_{t_1}^{t_2}\sum_i\left\{\delta p_i\,\dot{q}_i - \delta q_i\,\dot{p}_i - \frac{\partial H}{\partial p_i}\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i\right\}dt = 0$$

$$\int_{t_1}^{t_2}\sum_i\left(\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right)dt = 0$$

Since δp_i and δq_i are arbitrary

$$\therefore \quad \dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \qquad \dot{p}_i + \frac{\partial H}{\partial q_i} = 0$$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

which are Hamilton Equation

HAMILTON'S PRINCIPLE OF LEAST ACTION in (Extended Form)

ACTION :- The general definition of a action A is

$$A = \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i \, dt \right)$$

Statement OF Extended Principle of least ACTION

For a System in which H is a Constant of motion

$$\Delta A = \Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt = 0 \quad \rightarrow (1)$$

for the path along which the motion takes place.

Proof :- Before to Prove Principle, we have to know that how to Calculate the total variation ΔA. We label the various trajectories by the parameter α, then

$$q_i = q_i(\alpha, t) \qquad t = t(\alpha)$$

$$\Delta q = \text{total variation of } q = \frac{dq}{d\alpha} d\alpha$$

$$= d\alpha \left(\frac{\partial q}{\partial \alpha} + \frac{\partial q}{\partial t} \frac{dt}{d\alpha} \right)$$

$$= \frac{\partial q}{\partial \alpha} d\alpha + \dot{q} \frac{dt}{d\alpha} d\alpha$$

$$= \delta q + \dot{q} \Delta t \quad \rightarrow (2)$$

$$\left(\frac{\partial q}{\partial \alpha} d\alpha = \delta q \qquad \frac{dt}{d\alpha} d\alpha = \Delta t \right)$$

Similarly for any function $\Delta f = \delta f + \dot{f} \Delta t \rightarrow (3)$

$$\left(f = f(q, t) \right.$$

$$\Delta f = \sum_i \frac{\partial f}{\partial q_i} \Delta q_i + \frac{\partial f}{\partial t} \Delta t$$

$$= \sum_i \frac{\partial f}{\partial q_i} (\delta q_i + \dot{q}_i \Delta t) + \frac{\partial f}{\partial t} \Delta t$$

$$= \sum_i \frac{\partial f}{\partial q_i} \delta q_i + \left(\sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t} \right) \Delta t$$

$$= \delta f + \frac{df}{dt} \Delta t = \delta f + \dot{f} \Delta t \left. \right)$$

$$f = f(q, t), \quad \frac{df}{dt} = \frac{\partial f}{\partial q} \frac{dq}{dt} + \frac{\partial f}{\partial t}, \quad \dot{f} = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial t}$$

we use 3 instead of the Parameters Equation to Calculate the total variation of a function as a trajectory displacement

As $$A = \int_{t_1}^{t_2} \sum p_i \dot{q}_i \, dt$$

$$= \int_{t_1}^{t_2} (H+L)\, dt$$ As $H = \sum p_i \dot{q}_i - L$

$$\Delta A = \Delta \int_{t_1}^{t_2} (H+L)\, dt$$

$$= \Delta \int_{t_1}^{t_2} L\, dt + \Delta \int_{t_1}^{t_2} H\, dt$$

$$= \Delta \int_{t_1}^{t_2} L\, dt + H \Delta \int_{t_1}^{t_2} dt \quad (\text{H const H})$$

$$= \Delta \int_{t_1}^{t_2} L\, dt + H\Delta\left((t_2 - t_1)\right)$$

$$= \Delta \int_{t_1}^{t_2} L\, dt + H(\Delta t_2 - \Delta t_1) \rightarrow (4)$$

now $$\int_{t_1}^{t_2} L\, dt = \left| I(t) \right|_{t_1}^{t_2} = I(t_2) - I(t_1)$$

$$\Delta \int_{t_1}^{t_2} L\, dt = \Delta I(t_2) - \Delta I(t_1)$$

$\delta \int L\,dt = \delta I(t)$

$$\Delta \int_{t_1}^{t_2} L\, dt = \delta I(t_2) - \delta I(t_1) + \dot{I}(t_2)\Delta t_2 - \dot{I}(t_1)\Delta t_1 \rightarrow (5)$$

$$= \delta I(t) \Big|_{t_1}^{t_2} + L\Delta(t)\Big|_{t_1}^{t_2}$$

~~the~~ (add)

$\int L\, dt = I(t)$

$\frac{d}{dt}\int L\,dt = \dot{I}(t)$

$L = \dot{I}(t)$

$L\,\Delta t = \dot{I}(t)\,\Delta t$

End Point

$\delta I(t_1) = \delta I(t_2) = 0$

$$\Delta\int_{t_1}^{t_2} L\,dt = \delta\int_{t_1}^{t_2} L\,dt + L\Delta t\Big|_{t_1}^{t_2} \longrightarrow (6)$$

But $\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\delta L\,dt$

$$= \int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot q_i}\delta \dot q_i\right)dt$$

But $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)$

$$\therefore\ \delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_i\left(\delta q_i\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) + \frac{\partial L}{\partial \dot q_i}\delta\dot q_i\right)dt$$

$$= \int_{t_1}^{t_2}\sum_i\frac{d}{dt}\left(\delta q_i\frac{\partial L}{\partial \dot q_i}\right)dt$$

$$= \sum_i\delta q_i\left(\frac{\partial L}{\partial \dot q_i}\right)\Big|_{t_1}^{t_2}$$

$$= \sum\delta q_i\,p_i\Big|_{t_1}^{t_2}$$

$$\therefore\ (6) \Rightarrow \Delta\int_{t_1}^{t_2} L\,dt = \sum_i p_i\delta q_i\Big|_{t_1}^{t_2} + L\Delta t\Big|_{t_1}^{t_2}.$$

$$(4) \Rightarrow \Delta A = \sum_i p_i\delta q_i\Big|_{t_1}^{t_2} + L\Delta t\Big|_{t_1}^{t_2} + H(\Delta t_2 - \Delta t_1)$$

$$= \sum_i p_i\delta q_i\Big|_{t_1}^{t_2} + L\Delta t\Big|_{t_1}^{t_2} + H\Delta t\Big|_{t_1}^{t_2} \longrightarrow (7)$$

$$(3) \Rightarrow \Delta q_i = \delta q_i + \dot q_i\Delta t$$

$$\delta q_i = \Delta q_i - \dot q_i\Delta t$$

$(7) \Rightarrow$

$$\Delta A = \sum_i p_i(\Delta q_i - \dot q_i\Delta t)\Big|_{t_1}^{t_2} + (L+H)\Delta t\Big|_{t_1}^{t_2}$$

$$= \sum_i p_i\Delta q_i\Big|_{t_1}^{t_2} - \sum_i p_i\dot q_i\Delta t\Big|_{t_1}^{t_2} + (L+H)\Delta t\Big|_{t_1}^{t_2}$$

$$= \sum_i p_i \Delta q_i \Big|_{t_1}^{t_2} + \left(H + L - \sum_i p_i \dot{q}_i \Delta t\right) \Big|_{t_1}^{t_2}$$

As $H = -L + \sum_i p_i \dot{q}_i$

$$\therefore \Delta A = \sum_i p_i \Delta q_i \Big|_{t_1}^{t_2} + 0$$

Because the total variation of t_1 and t_2 at the end points remain same

So $\Delta q_i = 0$ $\qquad q_i = \text{const}$

$\Delta A = 0$

Different Forms of Principle of least ACTION

$$\Delta A = \Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt = 0$$

for a System in which H is a Constant of motion

1) In terms of K.E.

When Equations of transformations $r_i = r_i(q_1, q_2 \ldots q_n, t)$ donot depend on t explicitly $\quad i = 1, 2, \ldots N$

ie $\quad r_i = r_i(q_1, q_2 \ldots q_n) \quad i = 1, 2, \ldots N$

then $\quad H = T + V$

& $\quad \sum p_i \dot{q}_i = H + L$

$= T + V + T - V$

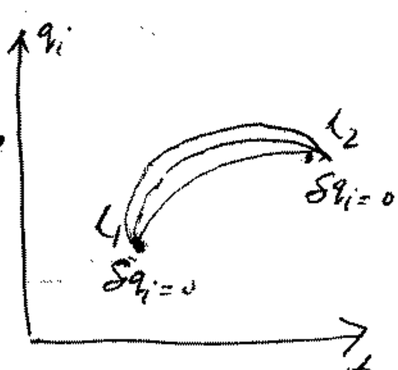
$= 2T$

Principle of least action takes the form

$$\Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt = \Delta \int_{t_1}^{t_2} 2T \, dT = 0$$

$$\Rightarrow \Delta \int_{t_1}^{t_2} T \, dT = 0 \quad \rightarrow (1)$$

If there are no external forces, then the total K.E. is also Conserved along with the total Hamilton H. In this Case the Principle Stated in (1) Takes the form



$$T \Delta \int_{t_1}^{t_2} dt = 0 \Rightarrow \Delta \int_{t_1}^{t_2} dt = 0$$

$$\Rightarrow \Delta(t_2 - t_1) = 0$$

$\Rightarrow$ all the Particles between two points Consistent with the Conservation of Energy. the System will move along that path for which the time of transit is the least or extremum. This form of Principle of least action is Similar to Fermat's principle in geometric optics which States that " the path taken by light in a given medium transporate to minimum value of a Certain quantity

II JACOBI FORM OF Principle of least ACTION

For a Single particle $T = \frac{1}{2} m v^2 = \frac{1}{2} m \left| \frac{d\bar{r}}{dt} \right|^2$

But $\left| \frac{d\bar{r}}{dt} \right| = \sqrt{\frac{2T}{m}}$

$$\Rightarrow \quad dt = \sqrt{\frac{m}{2T}}\, |d\bar{r}| = \sqrt{\frac{m}{2T}}\, ds$$

$\because ds = |d\bar{r}|$

where dt is the small element of time in which the Particles traverses a distance ds

The Principle of least action can be written as

$$\Delta \int_{t_1}^{t_2} 2T\, dt = 0 \Rightarrow \Delta \int_{P}^{Q} 2T \sqrt{\frac{m}{2T}}\, ds = 0$$

$$\Rightarrow \Delta \int \sqrt{2mT}\, ds = 0$$

$$\Delta \int \sqrt{2m(H-V)}\, ds = 0 \qquad \text{(For a Single Particle)}$$

which is a form for the path and does not involve time

This form can be extended to a System of particles for that purpose, we are to express dt in terms of an expression for the arc element corresponding to Configuration Space of n particles when the generalized co-ordinate are q_i and there is no explicit time dependence then K. E. is a homogenous quadratic function of $\dot{q}_i$

$$T = \frac{1}{2} \sum_{i,k} m_{ik}\, \dot{q}_i \dot{q}_j$$

$$= \frac{1}{2} \sum_{i,k} m_{ik} \frac{dq_i}{dt} \frac{dq_j}{dt}$$

If we define

$$(d\rho)^2 = \sum_{i,k} m_{ik}\, dq_i\, dq_j$$

then

$$T = \frac{1}{2} \left(\frac{d\rho}{dt} \right)^2$$

$$(dt)^2 = \frac{(d\rho)^2}{2T} \Rightarrow dt = \frac{d\rho}{\sqrt{2T}}$$

using the relation
Principle of least action as

$$\Delta \int_{t_1}^{t_2} 2T\,dt = \Delta \int 2T \frac{d\rho}{\sqrt{2T}} = \Delta \int \sqrt{2T}\,d\rho = 0$$

$$\Rightarrow \quad \Delta \int \sqrt{T}\,d\rho = 0$$

$$\Rightarrow \quad \Delta \int \sqrt{H - V(q)}\,d\rho = 0$$

This is form of Principle of least action is Called Jacobi form

Canonical transformation OR CONTACT TRANSFORMATION

In Hamiltonian formulation, we use Canonical variables; generalized co-ordinates q_i and their conjugate momenta p_i they satisfy

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad i=1,2,\dots n$$

In Calculations Hamiltonian Equations will be more easily solvable if suitable Canonical co-ordinate are used. For this purpose, we may have to make a transformation the co-ordinate p_i, q_i to new co-ordinate Q_i, P_i

$$\left.\begin{aligned} Q_i &= Q_i(q, p, t) \\ P_i &= P_i(q, p, t) \end{aligned}\right] \to (1)$$

where the new Canonical co-ordinate satisfy

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i \qquad \frac{\partial K}{\partial Q_i} = -\dot{P}_i \to (2)$$

where $K = K(Q, P, t)$ is the new Hamiltonian. Hamilton Equation in terms of (p, q, t) or (P, Q, t) follow from principle of least ACTION which may write be as

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

now $H = \sum_i p_i \dot{q}_i - L \Rightarrow K = \sum_i P_i \dot{Q}_i - L$

So $$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H\right) dt = 0 \to (3)$$

and $$\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K\right) dt = 0 \to (4)$$

from (3) & (4) it follows that integrands may differ by the total time derivative $\frac{dF}{dt}$ of a function F

$$\int_{t_1}^{t_2} \frac{dF}{dt}\, dt = F \Big|_{t_1}^{t_2} = F(t_2) - F(t_1) \quad (3)$$

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \delta\big(F(t_2) - F(t_1)\big) \quad (4)$$

⇒ the function F is called a generation function of transformation (1)

Q. of which variable is F a function. In general F well be function of p's, q's, P's, Q's and t ie it well be a function of 4n Canonical variables, Out of these 4n variables, only 2n are Independent There are four types of generating function

i) $F_1(q, Q, t)$ (ii) $F_2(q, P, t)$ $F_3(p, Q, t)$, $F_4(p, P, t)$

i) $F = F(q, Q, t)$

(3) & (4) ⇒

(A Condition for a transformation to a Canonical transformation
A transformation is said to be Canonical transformation if

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$$

$$p_i \dot{q}_i - H + K - P_i \dot{Q}_i = \frac{dF}{dt}$$

$$p_i \frac{dq_i}{dt} - P_i \frac{dQ_i}{dt} - H + K = \frac{dF}{dt}$$

$$\Rightarrow p_i\, dq_i - P_i\, dQ_i - (H-K)dt = dF$$

$$p_i\, \delta q_i - P_i\, \delta Q_i - (H-K)\, dt = dF \;)$$

Case I When $F = F_1(q, Q, t)$
where

$$\frac{dF_1}{dt}(q,Q,t) = \frac{\partial F}{\partial t} + \sum_i \left(\frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i \right)$$

Now $\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \sum_i \left(\frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q} \dot{Q}_i \right)$$

$$\sum_i \left(p_i - \frac{\partial F_1}{\partial q_i} \right) \dot{q}_i - \sum_i \left(P_i + \frac{\partial F_1}{\partial Q_i} \right) \dot{Q}_i + \left(K - H - \frac{\partial F_1}{\partial t} \right) = 0$$

where $\dot{q}_i$, $\dot{Q}_i$ are independent variables ∴ we must have

$$p_i = \frac{\partial F_1}{\partial q_i} \rightarrow (i) \quad P_i = -\frac{\partial F_1}{\partial Q_i} \rightarrow (ii)$$

$$K_1 = H + \frac{\partial F_1}{\partial t} \rightarrow (iii)$$

(i) and (ii) are Canonical transformation Equation
if Hamiltonian is independent of t,
(iii) $\Rightarrow$ $H - K = 0 \Rightarrow H = K_1$

Case II $F = F_2(q, P, t)$
we can express F_2 in terms of F_1 and then proceed as before using (ii)
∴ we write

$$F_2 = F_1 + \sum_i P_i Q_i \rightarrow (7)$$

from (7) we can prove that F_2 is a function of q and P and not of Q

$$\frac{\partial F_2}{\partial Q_i} = \frac{\partial F_1}{\partial Q_i} + P_i = -P_i + P_i = 0$$

using (ii)

$$\sum p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$$

$$= \sum_i P_i \dot{Q}_i - K + \frac{d}{dt}\left(F_2 - \sum_i P_i Q_i \right)$$

~~$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{\partial F_2}{\partial t}$~~

$$+ \sum_i \left(\frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial Q_i} \dot{Q}_i \right)$$

$$- \sum_i \left(P_i \dot{Q}_i + \dot{P}_i Q_i \right)$$

$$\sum p_i \dot{q}_i - H = \cancel{\sum_i P_i \dot{Q}_i} - K + \frac{\partial F_2}{\partial t} + \sum_i \left(\frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial P_i} \dot{P}_i \right)$$

$$- \cancel{\sum_i P_i \dot{Q}_i} - \sum_i \dot{P}_i Q_i$$

$$\sum_i \left(p_i - \frac{\partial F_2}{\partial q_i} \right) \dot{q}_i - \sum_i \left(\frac{\partial F_2}{\partial P_i} - Q_i \right) \dot{P}_i + \left(K - H - \frac{\partial F_2}{\partial t} \right) = 0$$

where $\dot{q}_i$ and $\dot{P}$ are independant variable

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$K = H + \frac{\partial F_2}{\partial t}$$

Case III $\quad F = F_3 (p, Q, t)$

$$F_3 (p, Q, t) = F(p, Q, t) - \sum_i p_i q_i$$

now $\quad \sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{d}{dt} F_3(p, Q, t) + \frac{d}{dt}\left(\sum p_i q_i\right)$$

$$\sum_i \cancel{p_i \dot{q}_i} - H = \sum_i P_i \dot{Q}_i - K + \frac{\partial F_3}{\partial t} + \sum_i \left(\frac{\partial F_3}{\partial p_i} \dot{p}_i + \frac{\partial F_3}{\partial Q_i} \dot{Q}_i \right)$$

$$+ \cancel{\sum p_i \dot{q}_i} + \sum \dot{p}_i q_i$$

$$\Rightarrow \quad \sum_i \left(P_i + \frac{\partial F_3}{\partial Q_i} \right) \dot{Q}_i + \sum_i \left(\frac{\partial F_3}{\partial p_i} + q_i \right) \dot{p}_i + H - K + \frac{\partial F_3}{\partial t}$$

where $\dot{p}_i$ and $\dot{Q}$ are independent

$$\therefore \quad q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$K = H + \frac{\partial F_3}{\partial t}$$

Case 4 when $F = F_4(p, P, t)$

we write $F_4(p,P,t) = F_0 + \sum_i P_i Q_i - \sum_i p_i q_i$

now $\sum_i p_i \dot q_i - H = \sum_i P_i \dot Q_i - K + \frac{dF}{dt}$

$$\sum_i p_i \dot q_i - H = \sum_i P_i \dot Q_i - K + \frac{d}{dt}\left(F_4(p,P,t) - \sum_i P_i Q_i + \sum_i p_i q_i\right) = 0$$

$$\sum_i \cancel{p_i \dot q_i} - H = \sum \cancel{P_i \dot Q_i} - K + \frac{\partial F_4}{\partial t} + \sum_i \left(\frac{\partial F_4}{\partial p_i}\dot p_i + \frac{\partial F_4}{\partial P_i}\dot P_i\right) - \sum_i \cancel{P_i \dot Q_i} - \sum \dot P_i Q_i + \sum \cancel{p_i \dot q_i} + \sum \dot p_i q_i = 0$$

$$\sum_i \left(\frac{\partial F_4}{\partial p_i} + q_i\right)\dot p_i + \sum_i \left(\frac{\partial F_4}{\partial P_i} - Q_i\right)\dot P_i + H - K + \frac{\partial F_4}{\partial t} = 0$$

Since $\dot p, \dot P$ are independent variables

$$\therefore \quad -\frac{\partial F_4}{\partial p_i} = q_i \qquad \frac{\partial F_4}{\partial P_i} = Q_i \qquad H - K = -\frac{\partial F_4}{\partial t}$$

EQUATION OF MOTIONS in the 4 Cases

1/ $p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad K = H + \frac{\partial F_1}{\partial t}$

2/ $p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = H + \frac{\partial F_2}{\partial t}$

where $F = F_2(q,P,t) \doteq F + \sum_i P_i Q_i$

3/ $F = F_3(p,Q,t) = F - \sum_i p_i q_i$

$q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = \frac{\partial F_3}{\partial Q_i} \qquad K =$

4/ $F = F_4(p,P,t) = F + \sum_i P_i Q_i$

$q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i} \qquad K = H$

Examples of Canonical transformation

let $F_2 = F_2(q,P,t) = \sum_i q_i P_i \rightarrow (A)$

$\therefore \quad p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial}{\partial q_i} \sum_i q_i P_i = P_i$

and $Q_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial}{\partial P_i} \sum_i (q_i P_i) = q_i$

ie $Q_i = q_i \quad p_i = P_i$

which is an identity transformation

More general form of Equation A

let $F_2 = \sum_i f_i(q_i, t) P_i \rightarrow A'$

where $f_i(q,t)$ are known function

$p_i = \frac{\partial F_2}{\partial q_i} = \sum_i \frac{\partial f_i}{\partial q_i} P_i$

$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q,t)$

which shows that the new co-ordinates Q_i depends only on q_i and t and not only on the Canonical momenta

Such a transformation is Called Point TRANSFORMATION

All Point transformation are Canonical transformation

Application to HORMONIC OSCILATOR

A particle which performs S.H.M. is called Hormonic Oscilator.

$$F \propto x \Rightarrow F = -kx$$

$$\Rightarrow m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x \Rightarrow \ddot{x} + \frac{k}{m}x$$

$$\Rightarrow \ddot{x} + \omega^2 x = 0$$

$$(D^2 + \omega^2)x = 0 \Rightarrow D = \pm i\omega$$

$$\therefore x = a\cos\omega t + b\sin\omega t$$

Let $b = r\cos\alpha$ $a = r\sin\alpha$

$$\therefore x = r\sin(\omega t + \alpha)$$

P.E. $= V$ & $F = -\nabla V = -\frac{dv}{dx}$

$$\therefore -\frac{dv}{dx} = -kx \Rightarrow \frac{dv}{dx} = kx \Rightarrow V = \frac{kx^2}{2}$$

$$V = \frac{x^2}{2}(m\omega^2)$$

$$\left(\because \frac{k}{m} = \omega^2,\ k = m\omega^2\right)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$\text{So T.E.} = H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2 \rightarrow (A')$$

Next we apply the Hamiltonian formulation and the Concept of generating function to Study the motion of Hormonic Oscilator

Hamiltonian FORMULISM

$$(x, \dot{x}) \longrightarrow (q, p) \qquad (p = m\dot{x} = m\dot{q})$$

$$F_1 = \frac{m}{2}\omega q^2 \cot Q$$

$$p = \frac{\partial F_1}{\partial q} = m\omega q\cot Q \rightarrow (1)$$

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{m}{2}\omega q^2(-\csc^2 Q) = \frac{m}{2}\omega q^2\csc^2 Q \rightarrow (2)$$

(in (2) $\quad q^2 = \frac{2P}{m\omega}\sin^2 Q \Rightarrow q = \sqrt{\frac{2P}{m\omega}}\sin Q \rightarrow (3)$

$\therefore$ (1) $\Rightarrow$ $\quad p = m\omega\sqrt{\frac{2P}{m\omega}}\sin Q\cot Q = \sqrt{2m\omega P}\cos Q \rightarrow (4)$

(3) and (4) gives us old Co-ordinate interms of new Co-ordinate

$$\text{now} \quad K = H + \frac{\partial F_1}{\partial t} = H + 0$$

$$\text{or} \quad K(P, Q) = H(p, q)$$

now $\cancel{H = \frac{1}{2}m\,(2m\omega P)\cos^2 Q + \frac{1}{2}m\omega^2 \frac{2P}{m\omega}\sin^2 Q}$

now (m

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{1}{2}m\left(\frac{p}{m}\right)^2$$

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m}$$

$(\because (x, \dot{x}) \rightarrow (q, p) \quad \therefore x = q$

Substituting (3) & (4) of $K = H$

$$K = H = \frac{1}{2}m\omega^2 \cdot \frac{2P}{m\omega}\sin^2 Q + \frac{1}{2m}2m\omega P\cos^2 Q$$

$$= \omega P\sin^2 Q + \omega P\cos^2 Q$$

$$K = H(P, Q) = \omega P$$

$\Rightarrow$ Hamiltonian function is cyclic w.r.t. Co-ordinate Q..

$\therefore$ Conjugate momentum P is a Constant of Motion

$$P = H/\omega$$

But in this case $H = E = $ const $\Rightarrow P = E/\omega$

The Equation of motion for Q reduces to

$$P = \text{Const} = E/\omega$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$

$$\Rightarrow Q = \omega t + \alpha$$

$$\sin Q = \sin(\omega t + \alpha)$$

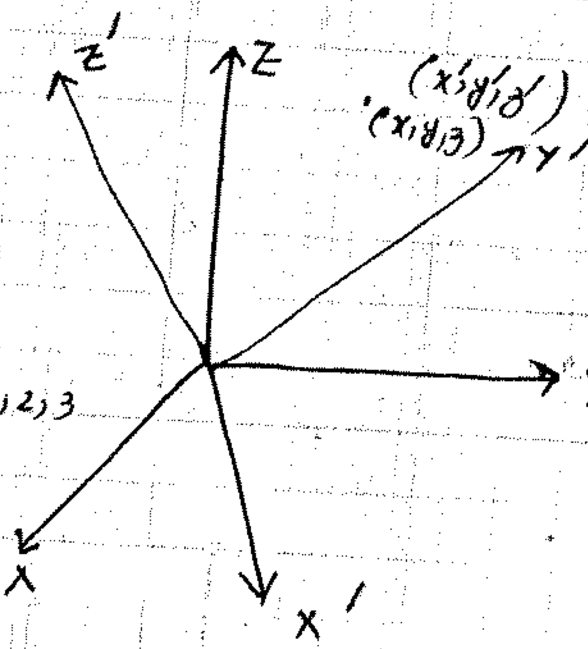
$$\underset{(3)}{\Rightarrow} \quad q\sqrt{\frac{m\omega}{2P}} = \sin(\omega t + \alpha)$$

$$\Rightarrow q = \sqrt{\frac{2P}{m\omega}}\sin(\omega t + \alpha)$$

$$= \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha)$$

Orthogonal Point Transformation

$x' =$ linear function of x, y, z
$y' =$ " " " x, y, z
$z' =$ " " " x, y, z



Change of NOTATIONS

$(x, y, z) = (x_1, x_2, x_3) = x_i \quad i = 1, 2, 3$
$(x', y', z') = x_i' \quad i = 1, 2, 3$

The transformation Equations can be written as

$$x_i' = \sum_j a_{ij} x_j \rightarrow (1)$$

where a_{ij} are real constants which are the DC's of the new Co-ordinate axes w.r.t. old ones.
They satisfy

$$\sum_i a_{ij} a_{ik} = \delta_{jk} = \sum_i a_{ji} a_{ki} \rightarrow (2)$$

By using the orthogonality relation of (2) (for orthogonal relat.) we can obtain the Inverse relation of (1)

$$x_i = \sum_j a_{ji} x_j'$$

In matrix form (1) $\Rightarrow \quad X' = AX \rightarrow (3)$
" " $1' \Rightarrow \quad X = A^t X' \rightarrow (3')$

where $A = a_{ij}$ is an orthogonal matrix. If $A^t = A^{-1}$ then A is orthogonal.

orthogonal point transformation

we know $F_2 = \sum_i f_i(q,t)\, P_i$

Suppose $f_i = Q_i = \sum_k a_{ik} q_k$

then the generating function F_2

$$F_2(q,P,t) = \sum_i f_i P_i = \sum_i \sum_k a_{ik} q_k P_i$$

$$= \sum_{i,k} a_{ik} q_k P_i$$

now

$$p_k = \frac{\partial F_2}{\partial q_k} = \frac{\partial}{\partial q_k} \sum_{i,k} a_{ik} q_k P_i$$

$$p_k = \sum_i a_{ik} P_i \longrightarrow A$$

If the transformation $Q_i = \sum_k a_{ik} q_k$ is orthogonal

then $\sum_i a_{ij} a_{ik} = \sum_i a_{ji} a_{ki} = \delta_{jk}$

from A, we find P in terms of p, Multiplying both sides of A with a_{jk}

$$\sum_k a_{jk} p_k = \sum_k \sum_i a_{jk} a_{ik} P_i$$

$$= \sum_i \left(\sum_k a_{ik} a_{jk}\right) P_i$$

$$= \sum_i \delta_{ij} P_i = P_j$$

now $P_j = \sum_k a_{jk} p_k$

or $P_i = \sum_k a_{ik} p_k \longrightarrow B$

Both (A) and (B) are orthogonal point transformation

THE POINCARE THEOREM

We Consider 2n-dimensional Cartesian Space in which a point has Co-ordinates q_i, p_i $(i=1, \dots n)$
This Space is Called Phase Space
A Mechanical System will be Completely Specified dynamically by a point in this Space
The Poincare Theorem States that the integral

$$J_1 = \iint_S \sum_i dq_i \, dp_i \rightarrow (1)$$

is an invariant Under Canonical transformation where S is an arbitrary two dimensional Surface in phase Space

The Co-ordinate of any point in phase Space and on the Surface S Can be expressed interms of two parameters u, v

$$q_i = q_i(u,v) \quad p_i = p_i(u,v)$$

If Q_K, P_K $(K=1, \dots n)$ are the Co-ordinate of the Same point Under a Canonical Transformation, then we have to prove that

$$J_1 = \iint_S \sum_i dq_i \, dp_i = \iint_S \sum_K dQ_K \, dP_K \rightarrow (4)$$

where Q_K, P_K Can also be expressed interms of u, v

$$dq_i \, dp_i = \frac{\partial(q_i, p_i)}{\partial(u,v)} du \, dv \rightarrow (2)$$

where $\frac{\partial(q_i, p_i)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} \rightarrow (3)$

(A) $\Rightarrow$

$$J_1 = \iint_S \sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)}\, du\, dv = \iint_S \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}\, du\, dv$$

$$= \iint_S \left[\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} - \sum_k \frac{\partial(Q_K, P_K)}{\partial(u,v)} \right] du\, dv = 0$$

$$\Rightarrow \sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)} \longrightarrow (5)$$

The proof Poincare theorem reduces to prove (5)

$$J_1 = \sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}$$

$$= \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} = \sum_k \begin{vmatrix} \frac{\partial Q_k}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial Q_k}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} \longrightarrow (6)$$

To prove that : Det. on the L.H.S. of (6) is equal to the determinant on the R.H.S.

For this, we use the generating function $F_2(q, P, t)$

$$\frac{\partial p_i}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial q_i}\right)$$

$$\frac{\partial p_i}{\partial u} = \sum_k \frac{\partial^2 F_2}{\partial q_k \partial q_i} \frac{\partial q_k}{\partial u} + \sum_k \frac{\partial^2 F_2}{\partial P_k \partial q_i} \frac{\partial P_k}{\partial u}.$$

Similarly

$$\frac{\partial p_i}{\partial v} = \sum_k \frac{\partial^2 F_2}{\partial q_k \partial q_i} \frac{\partial q_k}{\partial v} + \sum_k \frac{\partial^2 F_2}{\partial P_k \partial q_i} \frac{\partial P_k}{\partial v}$$

$\because F_2(q, P)$

$$\frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial q_i}\right) = \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial F_2}{\partial q_i}\right)\frac{\partial q_k}{\partial u} + \sum_k \frac{\partial}{\partial P_k}\left(\frac{\partial F_2}{\partial q_i}\right)\frac{\partial P_k}{\partial u}$$

L.H.S. of 6

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \sum_k \frac{\partial^2 F_2}{\partial q_k \partial q_i}\frac{\partial q_k}{\partial u} + \sum_k \frac{\partial^2 F_2}{\partial p_k \partial q_i}\frac{\partial p_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \sum_k \frac{\partial^2 F_2}{\partial q_k \partial q_i}\frac{\partial q_k}{\partial v} + \sum_k \frac{\partial^2 F_2}{\partial p_k \partial q_i}\frac{\partial p_k}{\partial v} \end{vmatrix}$$

$$= \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \sum_k \frac{\partial^2 F_2}{\partial q_k \partial q_i}\frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \sum_k \frac{\partial^2 F_2}{\partial q_k \partial q_i}\frac{\partial q_k}{\partial v} \end{vmatrix} + \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \sum_k \frac{\partial^2 F_2}{\partial p_k \partial q_i}\frac{\partial p_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \sum_k \frac{\partial^2 F_2}{\partial p_k \partial q_i}\frac{\partial p_k}{\partial v} \end{vmatrix}$$

$$= \sum_{i,k} \frac{\partial^2 F_2}{\partial q_k \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} + \sum_{i,k} \frac{\partial^2 F_2}{\partial p_k \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_k}{\partial v} \end{vmatrix} \longrightarrow (7)$$

The first term of R.H.S. of (7) is zero (to prove)

now $D_1 = \sum_{i,k} \frac{\partial^2 F_2}{\partial q_i \partial q_k} \begin{vmatrix} \frac{\partial q_k}{\partial u} & \frac{\partial q_i}{\partial u} \\ \frac{\partial q_k}{\partial v} & \frac{\partial q_i}{\partial v} \end{vmatrix}$

$$\frac{\partial^2 F_2}{\partial q_i \partial q_k} = \frac{\partial^2 F_2}{\partial q_k \partial q_i}$$

$\because$ of Continuity

$$= -\sum_{i,k} \frac{\partial^2 F_2}{\partial q_k \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix}$$

$$= -D_1$$

$\Rightarrow \quad 2D_1 = 0 \Rightarrow D_1 = 0$

$(7) \Rightarrow$

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u,v)} = 0 + \sum_{i,k} \frac{\partial^2 F_2}{\partial p_k \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_k}{\partial v} \end{vmatrix}$$

we can also be written as

$$\Rightarrow \frac{\partial(q_i, p_i)}{\partial(u,v)} = \sum_{i,k} \frac{\partial^2 F_2}{\partial p_k \partial p_i} \begin{vmatrix} \frac{\partial p_i}{\partial u} & \frac{\partial p_k}{\partial u} \\ \frac{\partial p_i}{\partial v} & \frac{\partial p_k}{\partial v} \end{vmatrix} \longrightarrow (\because \text{ of zero})$$

$$+ \sum_{i,k} \frac{\partial^2 F_2}{\partial p_k \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_k}{\partial v} \end{vmatrix}$$

$$= \sum_k \begin{vmatrix} \sum_i \frac{\partial^2 F_2}{\partial P_k \partial P_i} \frac{\partial P_i}{\partial u} + \sum_i \frac{\partial^2 F_2}{\partial P_k \partial q_i} \frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \sum_i \frac{\partial^2 F_2}{\partial P_k \partial P_i} \frac{\partial P_i}{\partial v} + \sum_i \frac{\partial^2 F_2}{\partial P_k \partial q_i} \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix}$$

$$= \sum_k \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial P_k}\right) & \frac{\partial P_k}{\partial u} \\ \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_k}\right) & \frac{\partial P_k}{\partial v} \end{vmatrix}$$

the first element of the first column

$$\frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial P_k}\right) = \frac{\partial}{\partial u} Q_k \quad \cancel{\frac{\partial F_2}{\partial P_k} \Big| \frac{\partial F_2}{\partial P_k}(q_i, P_i)}$$

the 2nd element $\frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_k}\right) = \frac{\partial}{\partial v} Q_k \qquad \because \quad \frac{\partial F_2}{\partial P_k} = Q_k$.

$$(5) \Rightarrow \quad \sum_i \frac{\partial(q_i, P_i)}{\partial(u,v)} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}$$

$$\sum_i \frac{\partial(q_i, P_i)}{\partial(u,v)} = \sum_k \begin{vmatrix} \frac{\partial Q_k}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial Q_k}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}$$

which proves the invariance of J

LAGRANGE BRACKET

This is defined as $\{u,v\}_{q,p} = \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}\right)$

This is the form of Lagrange Bracket

$$= \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} = \sum_i \frac{\partial(q_i,p_i)}{\partial(u,v)}$$

It has been proved to an invariant under canonical transformation and it is in fact J

$$\{u,v\}_{q,p} = \{u,v\}_{Q,P}$$

$$\{u,v\} = -\{v,u\}$$

Fundamental Lagrange Bracket

Lagrange bracket can be evaluated w.r.t. any canonical Co-ordinates

i) $\{u,v\} = -\{v,u\}$

$$\{u,v\} = \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}\right)$$

$$= -\left(\frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v} - \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v}\right)$$

$$= -\left(\frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u} - \frac{\partial p_i}{\partial v}\frac{\partial q_i}{\partial u}\right)$$

$$= -\{v,u\}$$

II $\{q_i, q_j\} = \left(\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \frac{\partial p_k}{\partial q_i}\frac{\partial q_k}{\partial q_j}\right)$

$$= \delta_{ki}(0) - (0)\delta_{kj} = 0$$

III $\{p_i, p_j\} = \frac{\partial q_k}{\partial p_i}\frac{\partial p_k}{\partial p_j} - \frac{\partial p_k}{\partial p_i}\frac{\partial q_k}{\partial p_j}$

$$= (0)\delta_{kj} - \delta_{ki}(0)$$

$$= 0$$

$$\{q_i, p_j\} = \frac{\partial q_h}{\partial q_i}\frac{\partial p_h}{\partial p_j} - \frac{\partial p_h}{\partial q_i}\frac{\partial q_h}{\partial p_j}$$
$$= \delta_{ki}\,\delta_{hj} - (0)(0)$$
$$= \delta_{ij}$$

The Poisson Bracket

$$[u, v] = \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}\right)$$

) $[u, v] = -[v, u]$

$$[u, v] = \sum_k \left(\frac{\partial u}{\partial q_h}\frac{\partial v}{\partial p_h} - \frac{\partial u}{\partial q_h}\frac{\partial v}{q_h}\right)$$
$$= -\sum_k \left(\frac{\partial u}{\partial p_h}\frac{\partial v}{\partial q_h} - \frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_h}\right)$$
$$= -\sum_k \left(\frac{\partial v}{\partial q_h}\frac{\partial u}{\partial p_k} - \frac{\partial v}{\partial p_h}\frac{\partial u}{\partial q_h}\right) = -[v, u]$$

$[u, u] = [v, v] = 0$ + $[u, c] = 0$
where c const

$$[u, u] = \sum_k \left(\frac{\partial u}{\partial q_h}\frac{\partial u}{\partial p_h} - \frac{\partial u}{\partial q_h}\frac{\partial u}{\partial p_h}\right) = 0$$
$$[v, v] = \sum_h \left(\frac{\partial v}{\partial q_h}\frac{\partial u}{\partial p_h} - \frac{\partial u}{\partial q_h}\frac{\partial v}{\partial p_h}\right) = 0$$
$$[u, c] = \sum_h \left(\frac{\partial u}{\partial q_h}\frac{\partial c}{\partial p_h} - \frac{\partial c}{\partial q_h}\frac{\partial u}{\partial p_h}\right)$$
$$= \sum_h \left(\frac{\partial u}{\partial q_h}(0) - (0)\frac{\partial u}{\partial p_k}\right) = 0$$

iii) $$[u, v+w] = \sum_h \left(\frac{\partial u}{\partial q_h} \frac{\partial}{\partial p_h}(v+w) - \frac{\partial}{\partial q_h}(v+w) \frac{\partial u}{\partial p_h} \right)$$

$$= \sum_h \left(\frac{\partial u}{\partial q_h} \left(\frac{\partial v}{\partial p_h} + \frac{\partial w}{\partial p_h} \right) - \left(\frac{\partial v}{\partial q_h} + \frac{\partial w}{\partial q_h} \right) \frac{\partial u}{\partial p_h} \right)$$

$$= \sum_h \left(\frac{\partial u}{\partial q_h} \frac{\partial v}{\partial p_h} + \frac{\partial u}{\partial q_h} \frac{\partial w}{\partial p_h} - \frac{\partial v}{\partial q_h} \frac{\partial u}{\partial p_h} - \frac{\partial w}{\partial q_h} \frac{\partial u}{\partial p_h} \right)$$

$$= \sum_h \left(\frac{\partial u}{\partial q_h} \frac{\partial v}{\partial p_h} - \frac{\partial v}{\partial q_h} \frac{\partial u}{\partial p_h} \right) + \sum_h \left(\frac{\partial u}{\partial q_h} \frac{\partial w}{\partial p_h} - \frac{\partial w}{\partial q_h} \frac{\partial u}{\partial p_h} \right)$$

$$= [u, v] + [u, w]$$

iv) $$\frac{\partial}{\partial t}[u, v] = \left[\frac{\partial u}{\partial t}, v \right] + \left[u, \frac{\partial v}{\partial t} \right]$$

$$\frac{\partial}{\partial t}[u, v] = \sum_h \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial q_h} \frac{\partial v}{\partial p_h} - \frac{\partial u}{\partial p_h} \frac{\partial v}{\partial q_h} \right)$$

$$= \sum_h \left(\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial q_h} \right) \frac{\partial v}{\partial p_h} + \frac{\partial u}{\partial q_h} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p_h} \right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p_h} \right) \frac{\partial v}{\partial q_h} - \frac{\partial u}{\partial p_h} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial q_h} \right) \right)$$

$$= \sum_h \left(\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial q_h} \right) \frac{\partial v}{\partial p_h} - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p_h} \right) \frac{\partial v}{\partial q_h} \right) + \sum_h \left(\frac{\partial u}{\partial q_h} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p_h} \right) - \frac{\partial u}{\partial p_h} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial q_h} \right) \right)$$

$$= \left[\frac{\partial u}{\partial t}, v \right] + \left[u, \frac{\partial v}{\partial t} \right]$$

v) $$[q_i, q_j] = 0$$

$$[q_i, q_j] = \sum_h \left(\frac{\partial q_i}{\partial q_h} \frac{\partial q_j}{\partial p_h} - \frac{\partial q_i}{\partial p_h} \frac{\partial q_j}{\partial q_h} \right)$$

$$= \sum_h \left(\delta_{ih}(0) - 0\,\delta_{jh} \right) = 0$$

vi) $$[p_i, p_j] = \sum_h \left(\frac{\partial p_i}{\partial q_h} \frac{\partial p_j}{\partial p_h} - \frac{\partial p_i}{\partial p_h} \frac{\partial p_j}{\partial q_h} \right)$$

$$= \sum_h \left(0\,\delta_{jh} - \delta_{ih}(0) \right)$$

$$= 0$$

64.

$$[q_i, p_j] = \delta_{ij}$$

$$[q_i, p_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right)$$

$$= \sum_k \left(\delta_{ik}\delta_{jk} - (0)(0) \right) = \delta_{ij}$$

(i) $[q_i, H] = \dot{q}_i$ (ii) $[p_i, H] = \dot{p}_i$

$$[q_i, H] = \frac{\partial q_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial H}{\partial q_k}$$

$$= \delta_{ik}(\dot{q}_k) - (0)\dot{p}_k$$

$$= \dot{q}_k$$

$\delta_{ik} = 1$
As $i = k$

$$[p_k, H] = \dot{p}_k$$

$$[p_k, H] = \frac{\partial p_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial H}{\partial q_k}$$

$$= 0 \cdot \frac{\partial H}{\partial p_k} - \delta_{ik}\frac{\partial H}{\partial q_k}$$

$$= 0 - (-\dot{p}_k)$$

$$[p_k, H] = \dot{p}_k$$

$\delta_{ik} = 1$
$\delta\ i = k$

Show that Lagrange Poisson Bracket are reciprocal to each other

Theorem Let u_l, $l = 1, 2, \ldots 2n$ be a function of $2n$ variables $q_1, \ldots q_n; p_1, p_2 \ldots p_n$

then

$$\sum_{l=1}^{2n} \{u_l, u_i\} [u_l, u_j] = \delta_{ij}$$

Proof:- Using definition of the two lagrange bracket

$$\sum_{l=1}^{2n} \{u_l, u_i\} [u_l, u_j] = \sum_{l=1}^{2n} \left(\sum_{h=1}^{n} \left(\frac{\partial q_h}{\partial u_l} \frac{\partial p_h}{\partial u_i} - \frac{\partial q_h}{\partial u_i} \frac{\partial p_h}{\partial u_l} \right) \left(\sum_{m=1}^{n} \left(\frac{\partial u_l}{\partial q_m} \frac{\partial u_j}{\partial p_m} - \frac{\partial u_l}{\partial p_m} \frac{\partial u_j}{\partial q_m} \right) \right) \right)$$

$$= \sum_{l=1}^{2n} \sum_{h,m=1}^{n} \left(\frac{\partial q_h}{\partial u_l} \frac{\partial p_h}{\partial u_i} \frac{\partial u_l}{\partial q_m} \frac{\partial u_j}{\partial p_m} - \frac{\partial q_h}{\partial u_l} \frac{\partial p_h}{\partial u_i} \frac{\partial u_l}{\partial p_m} \frac{\partial u_j}{\partial q_m} - \frac{\partial q_h}{\partial u_i} \frac{\partial p_h}{\partial u_l} \frac{\partial u_l}{\partial q_m} \frac{\partial u_j}{\partial p_m} + \frac{\partial q_h}{\partial u_i} \frac{\partial p_h}{\partial u_l} \frac{\partial u_l}{\partial p_m} \frac{\partial u_j}{\partial q_m} \right)$$

First term on R.H.S

$$\sum_{h=m=1}^{n} \left(\sum_{l=1}^{2n} \frac{\partial q_h}{\partial u_l} \frac{\partial u_l}{\partial q_m} \right) \left(\frac{\partial p_h}{\partial u_i} \frac{\partial u_j}{\partial p_m} \right) = \sum_{h=m} \frac{\partial q_h}{\partial q_m} \frac{\partial p_h}{\partial u_i} \frac{\partial u_j}{\partial p_m}$$

$$= \sum_{h} \delta_{hm} \frac{\partial p_h}{\partial u_i} \frac{\partial u_j}{\partial p_m}$$

$$= \sum_{k} \frac{\partial p_h}{\partial u_i} \frac{\partial u_j}{\partial p_h} \qquad (\delta_{hh} = 1$$

$$= \sum_{h} \frac{\partial u_j}{\partial p_h} \frac{\partial p_h}{\partial u_i} \rightarrow (1)$$

last term on the R.H.S.

$$\sum_{h=m=1}^{n} \left(\sum_{l=1}^{2n} \frac{\partial p_h}{\partial u_l} \frac{\partial u_l}{\partial p_m} \right) \frac{\partial q_h}{\partial u_i} \frac{\partial u_j}{\partial q_m} = \sum_{k,m} \frac{\partial p_h}{\partial p_m} \frac{\partial q_h}{\partial u_i} \frac{\partial u_j}{\partial q_m}$$

$$= \sum_{h,m} \delta_{km} \frac{\partial q_h}{\partial u_i} \frac{\partial u_j}{\partial q_m} = \sum_{h} \frac{\partial q_h}{\partial u_i} \frac{\partial u_j}{\partial q_h} \rightarrow (2)$$

Second term on the R.H.S

$$\sum_{h,m}\left(\sum_{l=1}^{2n}\frac{\partial q_h}{\partial u_l}\frac{\partial u_l}{\partial p_m}\right)\frac{\partial p_h}{\partial u_i}\frac{\partial u_j}{\partial q_m}$$

$$=\sum_{h,m}\frac{\partial q_h}{\partial p_m}\left(\frac{\partial p_h}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right)$$

$= 0$ $\quad$ p, q are independent

Similarly 3rd term on the R.H.S.

$$\sum_{h,m}\left(\sum_{l=1}^{2n}\left(\frac{\partial p_h}{\partial u_l}\cdot\frac{\partial u_l}{\partial q_m}\right)\frac{\partial q_h}{\partial u_i}\frac{\partial u_j}{\partial p_m}\right.$$

$$=\sum_{h,m}\frac{\partial p_h}{\partial q_m}\frac{\partial q_h}{\partial u_i}\frac{\partial u_j}{\partial p_m}=0$$

R.H.S.

$$=\sum_h\frac{\partial u_j}{\partial p_h}\frac{\partial p_h}{\partial u_i}-0-0+\sum_h\frac{\partial q_h}{\partial u_i}\frac{\partial u_j}{\partial q_h}$$

$$=\sum_k\left(\frac{\partial u_j}{\partial p_h}\frac{\partial p_h}{\partial u_i}+\frac{\partial u_j}{\partial q_h}\cdot\frac{\partial q_h}{\partial u_i}\right)$$

$$=\frac{\partial u_j}{\partial u_i}=\delta_{ij}$$ by Chain Rule.

Poission Brackets of Arbitrary Funtion

Prove that Poission Bracket are invariant under the Canonical transformation
ie, we have to Prove

$$[F, G]_{q,p} = [F, G]_{Q,P}$$

Proof:-

$$[F, G]_{q,p} = \sum_j \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right) \rightarrow (1)$$

Let that q_j, p_j are related to Canonical Co-ordinates Q_h, P_h, then

$$[F, G]_{q,p} = \sum_j \frac{\partial F}{\partial q_j} \sum_k \left(\frac{\partial G}{\partial Q_h} \frac{\partial Q_h}{\partial p_j} + \frac{\partial G}{\partial P_h} \cdot \frac{\partial P_h}{\partial p_j} \right)$$
$$- \sum_j \frac{\partial F}{\partial p_j} \sum_h \left(\frac{\partial G}{\partial Q_h} \frac{\partial Q_h}{\partial q_j} + \frac{\partial G}{\partial P_h} \frac{\partial P_h}{\partial q_j} \right)$$

$$= \sum_{j,h} \left(\frac{\partial G}{\partial Q_h} \left(\frac{\partial F}{\partial q_j} \frac{\partial Q_h}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial Q_h}{\partial q_j} \right) + \frac{\partial G}{\partial P_h} \left(\frac{\partial P_h}{\partial p_j} \frac{\partial F}{\partial q_j} - \frac{\partial F}{\partial p_j} \frac{\partial P_h}{\partial q_j} \right) \right)$$

$$= \sum_h \left(\frac{\partial G}{\partial Q_h} [F, Q_h]_{q,p} + \frac{\partial G}{\partial P_h} [F, P_h]_{q,p} \right) \rightarrow (2)$$

$$[F, G]_{q,p} = \sum_j \left(\frac{\partial G}{\partial Q_j} [F, Q_j] + \frac{\partial G}{\partial P_j} [F, P_j] \right) \longrightarrow 2'$$

We will use relation (2) to Calculate the Poission brackets on the R.H.S. of (2) Using Fundamental brackets

$$[F, Q_h] = ?$$
$$[F, P_h] = ?$$

First we Calculate $[Q_h, F]$ by making $F \rightarrow Q_h$, $G \rightarrow F$

$$[Q_h, F] = \sum_j \left(\frac{\partial F}{\partial Q_j} [Q_h, Q_j] + \frac{\partial F}{\partial P_j} [Q_h, P_j] \right)$$

$$= \sum_j \left(\frac{\partial F}{\partial Q_j} (0) + \frac{\partial F}{\partial P_j} \delta_{hj} \right)$$

$$= \frac{\partial F}{\partial P_h} \qquad \delta_{hh} = 1$$

To Calculate $[P_h, F]$, Replacing $F \to P_h$, $G \to F$ in ②

$$[P_h, F] = \sum_j \left(\frac{\partial F}{\partial Q_j} [P_h, Q_j] + \frac{\partial F}{\partial P_j} [P_h, P_j] \right)$$

$$= \sum_j \frac{\partial F}{\partial Q_j} (-\delta_{hj}) + 0$$

$$= -\frac{\partial F}{\partial Q_h}$$

$$[P_h, F] = -\frac{\partial F}{\partial Q_h} \to [F, P_h] = \frac{\partial F}{\partial Q_h} \to ③$$

$$[F, Q_h] = -\frac{\partial F}{\partial P_h} \to ④$$

using ③ & ④ in ②

$$[F, G]_{q,p} = \sum_h \left(\frac{\partial G}{\partial Q_h} \left(\frac{-\partial F}{\partial P_h} \right) + \frac{\partial G}{\partial P_h} \frac{\partial F}{\partial Q_h} \right)$$

$$= \sum_h \left(\frac{\partial F}{\partial Q_h} \frac{\partial G}{\partial P_h} - \frac{\partial G}{\partial Q_h} \frac{\partial F}{\partial P_h} \right)$$

$$= [F, G]_{Q,P}$$

EQUATION OF MOTION AND POISSION BRACKET

$$[Q_h, F] = \frac{\partial F}{\partial P_h} \qquad [P_h, F] = -\frac{\partial F}{\partial Q_h}$$

Let H is the Hamiltonian function

$$[q_k, H] = \frac{\partial H}{\partial p_h} \rightarrow (1) \quad [p_h, H] = -\frac{\partial H}{\partial q_h} \rightarrow (2)$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i \rightarrow (3)$$

Subsituting from (1) and (2) into (3)

$$[q_i, H] = \dot{q}_i \qquad [p_i, H] = \dot{p}_i$$

Time Rate of Change of an ARBITRARY FUNCTION

$$u = u(q, p, t)$$

$$\frac{du}{dt} = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial u}{\partial p_i}\frac{\partial p_i}{\partial t} \right) + \frac{\partial u}{\partial t}$$

$$= \sum_i \left(\frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i \right) + \frac{\partial u}{\partial t}$$

from Hamilton's equation of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\frac{du}{dt} = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial u}{\partial t}$$

$$= [u, H] + \frac{\partial u}{\partial t}$$

If the relation is no explicit dependence on t then $\frac{du}{dt} = [u, H]$

If $[u, H] = 0 \implies \frac{du}{dt} = 0$

$\implies u =$ a constant of motion

Conclusion If a dynamically quantity u is such that

i) does not depend on t explicitly

ii) Poission brackets with Hamiltonian H is zero then it is a constant of motion

Hamiltonian as a constant of Motion

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

when $u = H$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \qquad \text{if } \frac{\partial H}{\partial t} = 0$$

then $\frac{dH}{dt} = 0$

$\Rightarrow H =$ a constant of motion

Infinitesimal Contact TRANSFORMATION

Such transformations can be written as

$$\left.\begin{aligned} Q_i &= q_i + \delta q_i \\ P_i &= p_i + \delta p_i \end{aligned}\right\} \longrightarrow (1)$$

where δq_i, δp_i represent infinitesimal increments in q_i and p_i

Now the generating function for an identity transformation is

$$F_2(\underline{q}, \underline{P}) = \sum_i q_i P_i \longrightarrow (2)$$

therefore the generating function corresponding to the infinitesimal transformation (1) also differs only by an infinitesimal amount from that corresponding to the identity transformation

Hence from (2)

$$F_2(q, P) = \sum_i q_i P_i + \varepsilon\, G(q, P) \longrightarrow (3)$$

where ε is the some infinitesimal (ε is parameter of the transformation)

From (3) we will have transformation equations

The transformation equation for F_2 are

$$\frac{\partial F_2}{\partial q_i} = p_i \qquad \frac{\partial F_2}{\partial P_i} = Q_i$$

(3) $\Rightarrow$

$$p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\sum_i q_i P_i + \varepsilon G(\underline{q}, \underline{P})\right)$$

$$p_i = P_i + \varepsilon \frac{\partial G}{\partial q_i} \longrightarrow (i)$$

Also

$$Q_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial}{\partial P_i}\left(\sum_i q_i P_i + \varepsilon G(\underline{q}, \underline{P})\right)$$

$$Q_i = q_i + \varepsilon \frac{\partial G}{\partial P_i} \longrightarrow (ii)$$

i) $\Rightarrow$ $P_i - p_i = \delta p_i = -\varepsilon \dfrac{\partial G}{\partial q_i}$

ii) $\Rightarrow$ $Q_i - q_i = \delta q_i = \varepsilon \dfrac{\partial G}{\partial P_i}$

i.e. $\delta q_i = \epsilon \dfrac{\partial G}{\partial P_i}$

$$\left.\begin{aligned} \delta q_i &= \epsilon \frac{\partial G}{\partial P_i} \\ \delta p_i &= -\epsilon \frac{\partial G}{\partial q_i} \end{aligned}\right] \rightarrow (4)$$

Since P_i and p_i differ by an infinitesimal it will be correct to the first order to replace P_i by p_i in the derivative $\dfrac{\partial G}{\partial P_i}$

$$\frac{\partial G}{\partial P_i} \longrightarrow \frac{\partial G}{\partial p_i}$$

$$(4) \Rightarrow \quad \left.\begin{aligned} \delta p_i &= -\epsilon \frac{\partial G}{\partial q_i} \\ \delta q_i &= \epsilon \frac{\partial G}{\partial p_i} \end{aligned}\right] \rightarrow (5)$$

where $G = G(\underline{q}, \underline{p})$ is also called the generating function of the infinitesimal transformation

Infinitesimal Canonical Transformation and Poission Brackets

Let $u = u(\underline{q}, \underline{p})$ be a function of generalised co-ordinate and momenta

$$\delta u = u(q+\delta q, p+\delta p) - u(q,p)$$

$$\delta u = u(q,p) + \sum_i \left(\frac{\partial u}{\partial q_i}\delta q_i + \frac{\partial u}{\partial p_i}\delta p_i\right) - u(q,p)$$

$$\delta u = \sum_i \left(\frac{\partial u}{\partial q_i}\delta q_i + \frac{\partial u}{\partial p_i}\delta p_i\right)$$

now the Equation of transformation for an infinitesimal transformation are given by

$$\delta p_i = -\varepsilon\frac{\partial G}{\partial q_i} \qquad \delta q_i = \varepsilon\frac{\partial G}{\partial p_i}$$

$$\therefore \delta u = \sum_i \left(\frac{\partial u}{\partial q_i}\varepsilon\frac{\partial G}{\partial p_i} + \frac{\partial u}{\partial p_i}\left(-\varepsilon\frac{\partial G}{\partial q_i}\right)\right)$$

$$= \varepsilon\sum_i\left(\frac{\partial u}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$= \varepsilon\,[U, G]$$

When $u = H$, the Hamiltonian

then $\delta H = \varepsilon[H, G]$

which gives the Incremental Change in the Hamiltonian Under an infinitesimal transformation by G

If the infinitesimal Canonical transformation leads the Hamiltonian Unchanged,

then $\delta H = 0$

$$\therefore [H, G] = 0$$

Since the Poission Bracket of Constant with H is zero $\Rightarrow$ G is a Constant of motion

with Hamiltonian Invariance

> the Constant of motion of generating function of both infinitesimal Canonical transformation which leaves the Invariant

Jacobi Identity

If $f(q,p,t)$ $g(q,p,t)$ and $h(q,p,t)$ are given function

then $[f[gh]] + [g[hf]] + [h[fg]] = 0$

is called jacobi identity

Proof:-

Consider $[f[gh]] + [g[hf]] = [f[gh]] - [g[fh]] \rightarrow (1)$

now using the definition of Poission Bracket

$$[gh] = \frac{\partial g}{\partial q_1}\frac{\partial h}{\partial p_1} - \frac{\partial g}{\partial p_1}\frac{\partial h}{\partial q_1} = \left(\frac{\partial g}{\partial q_1}\frac{\partial}{\partial p_1} - \frac{\partial g}{\partial p_1}\frac{\partial}{\partial q_1}\right)\cdot h$$

$$= Dg\cdot h$$

where $Dg = \frac{\partial g}{\partial q_1}\frac{\partial}{\partial p_1} - \frac{\partial g}{\partial p_1}\frac{\partial}{\partial q_1}$; $Dg = \alpha_i \frac{\partial}{\partial \xi_i}$ $(i = 1, 2, \dots 2n)$

$\&$ $\alpha_i = \frac{\partial g}{\partial q_1}$ $\&$ $\xi_i = p_i$ for $i = 1, 2, 3 \dots n$

$\alpha_i = -\frac{\partial g}{\partial p_1}$ $\&$ $\xi_i = q_i$ for $i = n+1, \dots 2n$

Similarly $[fh] = Df\cdot h$

where $Df = \beta_j \frac{\partial}{\partial \eta_j}$ $j = 1, 2, \dots 2n$

where $\beta_j = \frac{\partial f}{\partial q_j}$ $\&$ $\eta_j = p_j$ $j = 1, 2, \dots n$

$\beta_j = -\frac{\partial f}{\partial p_j}$ $\&$ $\eta_j = q_j$ $j = n+1 \dots 2n$

(1) $\Rightarrow$ $[f[gh]] - [g[fh]] = Df[gh] - Dg[fh]$

$$= Df\,Dg\cdot h - Dg\,Df\cdot h$$

$$= \beta_j \frac{\partial}{\partial \eta_j}\left(\alpha_i \frac{\partial}{\partial \xi_i}\right)\cdot h - \alpha_i \frac{\partial}{\partial \xi_i}\,\beta_j \frac{\partial}{\partial \eta_j}\cdot h$$

$$= \left(\beta_j \frac{\partial \alpha_i}{\partial \eta_j}\frac{\partial h}{\partial \xi_i} + \beta_j \alpha_i \frac{\partial^2 h}{\partial \xi_i \partial \eta_j}\right) - \left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i}\frac{\partial h}{\partial \eta_j} + \alpha_i \beta_j \frac{\partial^2 h}{\partial \xi_i \partial \eta_j}\right).$$

$$B_j \frac{\partial \alpha_i}{\partial \eta_j} \frac{\partial \rho}{\partial \xi_1} - \alpha_i \frac{\partial B_j}{\partial \xi_1} \frac{\partial \rho}{\partial \eta_j}$$

$$= A_j \frac{\partial \rho}{\partial \xi_1} + B_j \frac{\partial \rho}{\partial \eta_j} \rightarrow (2) \qquad A_j = B_j \frac{\partial \alpha_i}{\partial \eta_j}$$

$$= A_j \frac{\partial h}{\partial p_j} + B_j \frac{\partial \rho}{\partial q_j} \longrightarrow (3) \qquad B_j = -\alpha_i \frac{\partial B_j}{\partial \xi_1}$$

re A_j and B_j are function f and $\xi_1 = p_j$ $\eta_j = q_j$

and are yet to be ~~the~~ determined

d donot involve h

idr the special case where $h = p_i$ which

is not change A_j and B_j

$$[q\, p_i]] - [g [f\, p_i]] = A_j \frac{\partial p_i}{\partial p_j} + B_j \frac{\partial p_i}{\partial q_j}$$

$$= A_j \delta_{ij} + B_j (0)$$

$$= A_j (1) = A_j$$

$$A_j = [f [g\, p_i]] - [g [f\, p_i]]$$

$$= \frac{\partial f}{\partial q_h} \frac{\partial}{\partial p_h} [g, p_i] - \frac{\partial f}{\partial p_h} \frac{\partial}{\partial q_h} [g, p_i] - \left(\frac{\partial g}{\partial q_h} \frac{\partial}{\partial p_h} [f\, p_i] - \frac{\partial g}{\partial p_h} \frac{\partial}{\partial q_h} [f\, p_i] \right)$$

$$= \frac{\partial f}{\partial q_h} \frac{\partial}{\partial p_h} \left(\frac{\partial g}{\partial q_l} \frac{\partial p_i}{\partial p_l} - \frac{\partial g}{\partial p_l} \frac{\partial p_i}{\partial q_l} \right) - \frac{\partial f}{\partial p_h} \frac{\partial}{\partial q_h} \left(\frac{\partial g}{\partial q_l} \frac{\partial p_i}{\partial p_l} - \frac{\partial g}{\partial p_l} \frac{\partial p_i}{\partial q_l} \right)$$

$$- \left(\frac{\partial g}{\partial q_h} \frac{\partial}{\partial p_h} \left(\frac{\partial f}{\partial q_l} \frac{\partial p_i}{\partial p_l} - \frac{\partial f}{\partial p_l} \frac{\partial p_i}{\partial q_l} \right) - \frac{\partial g}{\partial p_h} \frac{\partial}{\partial q_h} \left(\frac{\partial f}{\partial q_l} \frac{\partial p_i}{\partial p_l} - \frac{\partial f}{\partial p_l} \frac{\partial p_i}{\partial q_l} \right) \right)$$

$$= \frac{\partial f}{\partial q_h} \frac{\partial}{\partial p_h} \left(\frac{\partial g}{\partial q_l} \delta_{il} - 0 \right) - \frac{\partial f}{\partial p_h} \frac{\partial}{\partial q_h} \left(\frac{\partial g}{\partial q_l} \delta_{il} - 0 \right)$$

$$- \left(\frac{\partial g}{\partial q_h} \frac{\partial}{\partial p_h} \left(\frac{\partial f}{\partial q_l} \delta_{il} - 0 \right) - \frac{\partial g}{\partial p_h} \frac{\partial}{\partial q_h} \left(\frac{\partial f}{\partial q_l} \delta_{il} - 0 \right) \right)$$

$$= \left(\frac{\partial f}{\partial q_h} \frac{\partial}{\partial p_h} \left(\frac{\partial g}{\partial q_l} \right) - \frac{\partial f}{\partial p_h} \frac{\partial}{\partial q_h} \left(\frac{\partial g}{\partial q_l} \right) \right) - \left[\left(\frac{\partial g}{\partial q_h} \frac{\partial}{\partial p_h} \left(\frac{\partial f}{\partial q_l} \right) + \frac{\partial g}{\partial p_h} \frac{\partial}{\partial q_h} \left(\frac{\partial f}{\partial q_l} \right) \right) \right]$$

$$\left[f\frac{\partial g}{\partial q_l}\right]-\left[g\frac{\partial f}{\partial q_l}\right]=\left[f\frac{\partial g}{\partial q_l}\right]+\left[\frac{\partial f}{\partial q_l}g\right]$$

$$=\frac{\partial}{\partial q_l}[f\,g]\rightarrow(a)$$

$$A_j=\frac{\partial}{\partial q_j}[f\,g]\rightarrow(4)$$

Similarly putting $h=q_1$ in (3) & preeding similar as above

$$B_j=\frac{-\partial}{\partial p_j}[f.g]\rightarrow(5)$$

Putting (4) & (5) in (3)

$$[f[g\,h]]-[g[f\,h]]=\frac{\partial}{\partial q_j}[f\,g]\frac{\partial h}{\partial p_j}-\frac{\partial}{\partial p_j}[f\,g]\frac{\partial h}{\partial q_j}$$

$$[f(g\,h)]+[g(h\,f)]=[(f\,g)\,h]$$

$$[f(g\,h)]+[g\,(h\,f)]=-[h\,[f\,g]]$$

$$\Rightarrow [f[g\,h]]+[g[h\,f]]+[h[f\,g]]=0$$

Hamilton Jacobi theorem. — Statement:— If $S(q_1, q_2, \ldots q_n, \alpha_1, \alpha_2, \ldots \alpha_n, t)$ containing n arbitrary constants be any complete solution (called the complete integral) of the Hamilton's Jacobi Equation

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, \ldots q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots \frac{\partial S}{\partial q_n}, t\right) \to (1) \text{ and}$$

the Equations (Solution of canonical Equations of motion)

$$\frac{\partial S}{\partial \alpha_i} = -\beta_i \to (2) \qquad \frac{\partial S}{\partial q_i} = p_i \to (3)$$

where α_i, β_i are arbitrary constants are used to solve for $q_s = (\alpha's, \beta's, t)$ and $p's = (\alpha s, \beta s, t)$. Then these expression provide the general solution of canonical equation associated with Hamilton $H(q, p, t)$

Proof. — Differenciate (1) w.r.t. α_i

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \frac{\partial H}{\partial q_s}\frac{\partial q_s}{\partial \alpha_i} + \frac{\partial H}{\partial\left(\frac{\partial S}{\partial q_s}\right)} \frac{\partial}{\partial \alpha_i}\left(\frac{\partial S}{\partial q_s}\right) = 0$$

$$\Rightarrow \frac{\partial^2 S}{\partial \alpha_i \partial t} + \frac{\partial H}{\partial q_s}(0) + \frac{\partial H}{\partial p_s}\frac{\partial^2 S}{\partial \alpha_i \partial q_s}$$

($\because q_s$ & α_i are independent & $\frac{\partial S}{\partial q_s} = p_s$ by (3))

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \frac{\partial H}{\partial p_s}\frac{\partial^2 S}{\partial \alpha_i \partial q_s} = 0 \to (4)$$

Differenciate (2) w.r.t. t we have

$$\frac{\partial^2 S}{\partial t \partial \alpha_i} + \frac{\partial^2 S}{\partial q_s \partial \alpha_i}\frac{\partial q_s}{\partial t} = 0$$

$\left(\frac{\partial \beta_i}{\partial \alpha_i} = 0 \quad (\beta_i \text{ \& } \alpha_i \text{ are independent})\right)$

$$\Rightarrow \frac{\partial^2 S}{\partial t \partial \alpha_i} + \frac{\partial^2 S}{\partial q_s \partial \alpha_i}\frac{\partial q_s}{\partial t} = 0$$

$$\frac{\partial^2 S}{\partial t \partial \alpha_i} + \frac{\partial^2 S}{\partial q_s \partial \alpha_i}\dot{q}_s = 0 \to (5)$$

$$(5) - (4) \quad \left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right)\frac{\partial^2 S}{\partial q_s \partial \alpha_i} = 0$$

$$\Rightarrow \dot{q}_s - \frac{\partial H}{\partial p_s} = 0 \Rightarrow \dot{q}_s = \frac{\partial H}{\partial p_s} \longrightarrow (A)$$

$$\frac{\partial^2 S}{\partial q_s \partial \alpha_i} \neq 0$$

which is the first Hamilton's Equation
now Differenciate ① w.r.t q_s

$$\frac{\partial^2 S}{\partial q_s \partial t} + \frac{\partial H}{\partial q_s} + \frac{\partial H}{\partial\left(\frac{\partial S}{\partial q_r}\right)} \frac{\partial}{\partial q_s}\left(\frac{\partial S}{\partial q_r}\right) = 0$$

$$\frac{\partial^2 S}{\partial q_s \partial t} + \frac{\partial H}{\partial q_s} + \frac{\partial H}{\partial p_r} \frac{\partial^2 S}{\partial q_s \partial q_r} = 0 \rightarrow ⑥$$

Differenciate ③ w.r.t t

$$\frac{\partial^2 S}{\partial t \partial q_s} + \frac{\partial^2 S}{\partial q_s \partial q_r} \frac{\partial q_r}{\partial t} = \frac{\partial p_s}{\partial t}$$

$$\Rightarrow \frac{\partial^2 S}{\partial t \partial q_s} + \frac{\partial^2 S}{\partial q_s \partial q_r} \dot{q}_r = \dot{p}_s$$

$$\dot{p}_s - \frac{\partial^2 S}{\partial q_s \partial q_r} \dot{q}_r - \frac{\partial^2 S}{\partial t \partial q_s} = 0 \rightarrow ⑦$$

⑥ − ⑦ $\quad \dot{p}_s + \frac{\partial H}{\partial q_s} + \left(\frac{\partial H}{\partial p_r} - \dot{q}_r\right) \frac{\partial^2 S}{\partial q_s \partial q_r} = 0$

now $\quad \dot{q}_r = \frac{\partial H}{\partial p_r}, \Rightarrow \left(\frac{\partial H}{\partial p_r} - \dot{q}_r\right) = 0$

$$\Rightarrow \quad \dot{p}_s + \frac{\partial H}{\partial q_s} + 0 \cdot \frac{\partial^2 S}{\partial q_s \partial q_r} = 0$$

$$\dot{p}_s + \frac{\partial H}{\partial q_s} = 0 \quad \Rightarrow \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}$$

which is 2nd Canonical Equation

Solve the Harmonic Oscilation by Hamilton Jacobi Method

Sol. – In S.H.M $K.E. = \frac{1}{2} m\dot{x}^2$ $P.E = V = \frac{1}{2} Kx^2$

$$\therefore H = T + V = \frac{p^2}{2m} + \frac{1}{2} Kq^2$$

$x \to q$, $m\dot{x} \to p$ $\left(m\dot{x} = p, \ \dot{x} = p/m, \ \frac{1}{2} m \cdot \frac{p^2}{m^2} \right)$

now the Hamilton Jacobi Equation

$$H(q, p, t) + \frac{\partial S}{\partial t} = 0$$

$$\frac{p^2}{2m} + \frac{1}{2} Kq^2 + \frac{\partial S}{\partial t} = 0$$

But $p = \frac{\partial S}{\partial q}$

$$\therefore \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} Kq^2 + \frac{\partial S}{\partial t} = 0 \to (1)$$

Since explicit time dependence is involved in the last term on L.H.S of (1)

$\therefore$ We put $S = S(q, \alpha, t) = W(q, \alpha) - \alpha t \to (2)$

where α is constant of integration

from (2) in (1) we have $\left(\frac{\partial S}{\partial t} = -\alpha \right.$

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} Kq^2 - \alpha = 0 \to (3)$$

$$\text{(3)} \Rightarrow \left(\frac{\partial w}{\partial q}\right)^2 = 2m\alpha - Kq^2 m = 2m\left(\alpha - \frac{Kq^2}{2}\right)$$

$$= \cancel{2}m\left(\frac{2\alpha - Kq^2}{\cancel{2}}\right) = m(2\alpha - Kq^2)$$

$$\frac{\partial w}{\partial q} = \sqrt{m}\sqrt{(2\alpha - Kq^2)} = \sqrt{m}\sqrt{K}\sqrt{\frac{2\alpha}{K} - q^2}$$

$$w = \sqrt{mK}\int\sqrt{\frac{2\alpha}{K} - q^2}\, dq - \alpha t$$

from Hamilton Jacobi theory

$$\beta = \frac{\partial S}{\partial \alpha}$$

$$\beta = \sqrt{mK}\int \frac{1}{2}\left(\frac{2\alpha}{K} - q^2\right)^{-1/2}\left(\frac{2}{K}\right) dq - t$$

$$= \frac{\sqrt{mK}}{K}\int \frac{dq}{\sqrt{\frac{2\alpha}{K} - q^2}} - t$$

$$\beta = -\sqrt{\frac{m}{K}} \int \frac{-dq}{\sqrt{\left(\frac{2\alpha}{K}\right)^2 - q^2}} - t$$

$$= -\sqrt{\frac{m}{K}} \cos^{-1} \frac{q}{\sqrt{\frac{2\alpha}{K}}} - t$$

$$\Rightarrow \quad \beta + t = -\left(\sqrt{\frac{m}{K}} \cos^{-1} \sqrt{\frac{K}{2\alpha}}\, q\right)$$

$$m\ddot{x} + Kx = 0$$

$$\Rightarrow \quad \ddot{x} + \frac{K}{m} x = 0 \quad \Rightarrow \quad \ddot{x} + \omega^2 x = 0$$

where $\omega^2 = \frac{K}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{K}{m}}$

where $K =$ Spring Const $\quad \omega =$ Angular velocity

$2\pi\omega = \nu \qquad \lambda V = \nu$

$$\therefore \quad \beta + t = -\frac{1}{\omega}\left(\cos^{-1} \sqrt{\frac{K}{2\alpha}}\, q\right)$$

$$\Rightarrow \quad -\omega t - \omega\beta = \cos^{-1}\left(\sqrt{\frac{K}{2\alpha}}\, q\right)$$

$$\Rightarrow \quad \sqrt{\frac{K}{2\alpha}}\, q = \cos\left(-(\omega t + \omega\beta)\right) = \cos\omega(t+\beta)$$

$$\Rightarrow \quad \sqrt{\frac{m\omega^2}{2\alpha}}\, q = \cos\omega(t+\beta)$$

$$q = \frac{1}{\omega}\sqrt{\frac{2\alpha}{m}} \cos(\omega t + \beta) = q_0 \cos(\omega t + \beta)$$

Solve the Problem of motion of a projectile using the Hamilton Jacobi Method.

Sol.– Let that motion takes place in the xy — plane we choose the Co-ordinate s.t.

$x(t=0)=0$ & $y(t=0)=0$

Now $H = T+V$

$= \frac{1}{2}mv^2 + mgy = \frac{1}{2}\frac{m^2v^2}{m} + mgy$

$= \frac{1}{2m}p^2 + mgy$

$= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy$

Here $\left(q_1 = x,\ q_2 = y\ \ p_1 = p_x,\ p_2 = p_y\right)$

H. J Equation is $H(q_1, q_2, p_1, p_2) + \frac{\partial S}{\partial t} = 0$

$\Rightarrow$ $\Rightarrow H(x, y, p_x, p_y) + \frac{\partial S}{\partial t} = 0$

$\Rightarrow \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy + \frac{\partial S}{\partial t} = 0$

To obtain H. J. Equation

we put $p_x = \frac{\partial S}{\partial x}$ $\quad p_y = \frac{\partial S}{\partial y}$ $\quad \because p_i = \frac{\partial S}{\partial q_i}$

$\Rightarrow \frac{1}{2m}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right) + mgy + \frac{\partial S}{\partial t} = 0 \rightarrow (2)$

To Solve (2) we put

$S = W - \alpha_1 t$

$(2) \Rightarrow \frac{1}{2m}\left(\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right) + mgy - \alpha_1 = 0 \rightarrow (4)$

To Solve (4) we take

$W = W_x + W_y$ where $W_x = W_x(x)$
$W_y = W_y(y)$

$\frac{1}{2m}\left[\left(\frac{dW_x}{dx}\right)^2 + \left(\frac{dW_y}{dy}\right)^2\right] + mgy - \alpha_1 = 0$

$\left(\frac{dW_x}{dx}\right)^2 = 2m\alpha_1 - 2m^2gy - \left(\frac{dW_y}{dy}\right)^2$

$\frac{dW_x}{dx} = \sqrt{2m(\alpha_1 - mgy) - \left(\frac{dW_y}{dy}\right)^2} \longrightarrow (5)$

L.H.S. of (5) is a function of x only whereas on the R.H.S. of y only
Equality of (5) will be hold for all values of x and y
This is possible only if each side is const. say α_2

$$\frac{dw_x}{dx} = \alpha_2 \rightarrow (6)$$

& $$\sqrt{2m(\alpha_1 - mgy) - \left(\frac{dw_y}{dy}\right)^2} = \alpha_2 \rightarrow (7)$$

from (6) $\Rightarrow$ $w_x = \alpha_2 x \rightarrow (8)$

& from (7) $\alpha_2^2 = 2m(\alpha_1 - mgy) - \left(\frac{dw_y}{dy}\right)^2$

$$\Rightarrow \left(\frac{dw_y}{dy}\right)^2 = 2m(\alpha_1 - mgy) - \alpha_2^2$$

$$\frac{dw_y}{dy} = \sqrt{2m(\alpha_1 - mgy) - \alpha_2^2}$$

$$w_y = \int (2m\alpha_1 - 2m^2gy - \alpha_2^2)^{1/2} dy$$

$$w_y = \frac{1}{-2m^2g}\int (2m\alpha_1 - 2m^2gy - \alpha_2^2)^{1/2}(-2m^2g)\, dy$$

$$w_y = \frac{-1}{\cancel{2}m^2g}\frac{(2m(\alpha_1 - mgy) - \alpha_2^2)^{3/2}}{3/\cancel{2}} = \frac{-1}{3m^2g}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{3/2} \rightarrow (9)$$

$w = w_x + w_y$

$$S = w - \alpha_1 t = w_x + w_y - \alpha_1 t$$
$$= \alpha_2 x - \frac{1}{3m^2g}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{3/2} - \alpha_1 t$$

now from the Hamilton Jacobi theory

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = 0 - \frac{1}{\cancel{3}m^2g}\cdot\frac{\cancel{3}}{\cancel{2}}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{1/2}\cancel{2m} - t$$

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{-1}{mg}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{1/2} - t \rightarrow (i)$$

and $$\beta_2 = \frac{\partial S}{\partial \alpha_2} = x - \frac{1}{\cancel{3}m^2 g}\frac{\cancel{3}}{\cancel{2}}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{1/2}\cancel{(-2\alpha_2)}$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = x + \frac{\alpha_2}{m^2 g}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{1/2} \rightarrow (ii)$$

(i) $$\beta_1 + t = \frac{-1}{mg}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{1/2} \rightarrow (11a)$$

(ii) $$\beta_2 - x = \frac{\alpha_2}{m^2 g}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)^{1/2} \quad (11b)$$

using (Using 11a & 11b)

$$\beta_2 - x = \frac{\alpha_2}{m^2 g}\left(-mg(\beta_1 + t)\right) = \frac{-\alpha_2}{m}(\beta_1 + t)$$

$$x = \beta_2 + \frac{\alpha_2}{m}(\beta_1 + t) \rightarrow (12a)$$

and $$\dot{x} = \frac{\alpha_2}{m} \rightarrow (12b)$$

To determine the constants, we use initial condition

at $t=0$ $\quad x=0 \quad \dot{x} = v_0 \quad y = 0 \quad \dot{y} = v_0$

from (12a) $\Rightarrow$ $$0 = \beta_2 + \frac{\alpha_2 \beta_1}{m}$$

$$\beta_2 = -\frac{\alpha_2 \beta_1}{m}$$

(12b) $\Rightarrow$ $v_0 = \frac{\alpha_2}{m} \Rightarrow \alpha_2 = v_0 m \rightarrow (13i)$

$\therefore$ $$\beta_2 = -\frac{v_0 \cancel{m} \beta_1}{\cancel{m}} \Rightarrow \beta_2 = -v_0\beta_1 \rightarrow (13)$$

Using (13) in (12a)

$$x = -v_0\beta_1 + \frac{\alpha_2}{m}(\beta_1 + t)$$

Using (13i) in (12b) $$\dot{x} = \frac{v_0 \cancel{m}}{\cancel{m}} = v_0$$

now we find y and $\dot{y}$ using (11a) & (11b)

(11b) $\Rightarrow$

$$(\beta_2 - x)^2 = \frac{\alpha_2^2}{m^4 g^2}\left(2m(\alpha_1 - mgy) - \alpha_2^2\right)$$

$$= \frac{v_0^2 m^2}{m^4 g^2}\left(2m(\alpha_1 - mgy) - v_0^2 m^2\right) \quad \text{using (13i)}$$

$$(\beta_2 - x)^2 = \frac{v_0^2}{m^2 g^2}\left(2m(\alpha_1 - mgy) - v_0^2 m^2\right)$$

Using

$$\left(\text{or} \quad \beta_2 = -\beta_1 v_0 \right) \quad \text{in (13)}$$

$$\left(-\beta_1 v_0 - x\right)^2 = \frac{v_0^2}{m^2 g^2} \left(2(\alpha_1 - mgy) - v_0^2 m\right)$$

$$\Rightarrow m g^2 \left(\beta_1 v_0 + x\right)^2 = v_0^2 \left(2(\alpha_1 - mgy) - v_0^2 m\right) \to (15)$$

Now using the initial condition

$t = 0 \quad y = 0 \quad \dot{y} = v_0 \quad x = 0$

$$m g^2 \left(\beta_1 v_0 + 0\right)^2 = v_0^2 \left(2(\alpha_1 - 0) - m v_0^2\right)$$

$$\therefore \quad m g^2 \beta_1^2 \cancel{v_0^2} = \cancel{v_0^2} \left(2\alpha_1 - m v_0^2\right)$$

$$m g^2 \beta_1^2 + m v_0^2 = 2\alpha_1$$

$$\Rightarrow \alpha_1 = \frac{1}{2}\left(m g^2 \beta_1^2 + m v_0^2\right) \to (16)$$

Also in (15)

$$m g^2 \cdot 2\left(\beta_1 v_0 + x\right)\left(0 + \dot{x}\right) = v_0^2 \left(0 - 2mg\dot{y} - 0\right)$$

$t = 0 \quad x = y = 0 \quad \dot{x} = \dot{y} = v_0$

$$2 m g^2 \left(\beta_1 v_0 + 0\right) v_0 = v_0^2 \left(-2mg\, v_0\right)$$

~~But $\beta_1 = -\frac{v_0}{g}$~~

$$\cancel{2} \cancel{m} g^2 \beta_1 \cancel{v_0^2} = -\cancel{2}\cancel{mg}\, v_0^{\cancel{3}}$$

$$\beta_1 = \frac{-v_0}{g}$$

$$\text{in (16)} \Rightarrow \alpha_1 = \frac{1}{2}\left(m \cancel{g^2} \cdot \frac{v_0^2}{\cancel{g^2}} + m v_0^2\right) = m v_0^2$$

making substitution (15)

$$m g^2 \left(+\frac{v_0^2}{g} - x\right)^2 = v_0^2 \left(2 m v_0^2 - 2mgy - m v_0^2\right)$$

$$\Rightarrow \cancel{m v_0^4} + m g^2 x^2 - 2mg\, v_0^2 x = 2m\cancel{v_0^4} - 2mgy\, v_0^2 - \cancel{m v_0^4}$$

$$g x^2 - 2 v_0^2 x = -2y\, v_0^2$$

$$\Rightarrow \quad y = x - \frac{g x^2}{2 v_0^2}$$

represents a Parabola trajectory

Application of Hamilton Jacobi theory

A System is said to be separable if the complete integral of the corresponding Hamilton Jacobi partial differential equation can be written as the sum of the number of function which depends upon the co-ordinate

$$\frac{\partial S}{\partial t} + H(t, q, \alpha) \qquad S = S(t, q, \alpha)$$

$$S = S_1(q_1) + \cdots + S_n(q_n)$$

$$S_r = S_r(t, q_r, \alpha_1 \alpha_2 \ldots \alpha_n)$$

Necessary and sufficient conditions for a system to be separable is that K.E. and P.E. of the following form

$$T = \sum_{r=1}^{n} \frac{P_r(q_r) p_r^2}{\sum_{s=1}^{n} X_s(q_r)} \rightarrow (i) \quad \& \quad V = \sum_{r=1}^{n} \frac{\xi_r(q_r)}{\sum_{s=1}^{n} X_s}$$

The System is

i) Conservative

ii) Product of different ps does not occur

STOCKET'S Theorem:- Consider a system with K.E. and P.E

$$T = \frac{1}{2} \sum_{r=1}^{n} C_r p_r^2$$

and $V = V(q_1, q_2 \ldots q_n)$ and $C_r = C_r(q_1, q_2 \ldots q_n) \geq 0$

The System is separable iff $\exists$ a regular $m \times n$ numbers U_{rs} and a column matrix $\begin{bmatrix} w_1 \\ w_2 \\ w_n \end{bmatrix}$ where

U_{rs} and w_s depends only upon q_r s.t.

$$\sum_{r=1}^{n} C_r U_{rs} = \delta_{1s} \longrightarrow \mathrm{II}$$

$$\sum_{r=1}^{n} C_r w_r = V \longrightarrow \mathrm{III}$$

As $C_r = \frac{P_r}{\sum X_s}$
by (i) 9 Stockel

Proof:-

As $C_r = \dfrac{P_r}{x_1+x_2+\cdots+x_n}$

$$\sum C_r w_r = \frac{\sum P_r w_r}{x_1+x_2+\cdots+x_n}$$

$$= \frac{\sum \xi_r}{\sum x_s} \quad \text{where } \xi_r = P_r w_r \quad \left(\Rightarrow w_r = \frac{\xi_r}{P_r}\right)$$

$\Rightarrow \sum C_r U_{r1} = 1 \qquad \sum C_r U_{rs} = 0 \quad r \neq s \qquad$ As $\delta_{ii}=1$

$$C_1 U_{11} + C_2 U_{21} + C_3 U_{31} + \cdots + C_n U_{n1} = 1$$
$$C_1 U_{12} + C_2 U_{22} + C_3 U_{32} + \cdots + C_n U_{n2} = 0$$
$$\vdots$$
$$C_1 U_{1n} + C_2 U_{2n} + \cdots + C_n U_{nn} = 0$$

$$\Rightarrow \left[\frac{P_1}{\sum x_s} \quad \frac{P_2}{\sum x_s} \cdots \frac{P_n}{\sum x_s}\right]\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & & & \\ u_{n1} & u_{n2} & \cdots & u_{nn}\end{bmatrix} = (1,0,0\cdots 0)$$

$C_r = \dfrac{P_r}{\sum x_r}$

$\sum C_r U_{r1}$

$= \sum \dfrac{P_r u_{r1}}{\sum x_r}$

$$\Rightarrow \left[\frac{P_1}{\sum x_s} \quad \frac{P_2}{\sum x_s} \cdots \frac{P_n}{\sum x_s}\right]\begin{bmatrix} \frac{x_1}{P_1} & \frac{1}{P_1} & \frac{1}{P_1} & \cdots & \frac{1}{P_1} \\ \frac{x_2}{P_2} & \frac{1}{P_2} & 0 & & \frac{1}{P_2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{x_n}{P_n} & 0 & 0 & & \frac{1}{P_n}\end{bmatrix} = (1,0,0\cdots 0)$$

$$\Rightarrow U_{rs} = \begin{bmatrix} \frac{x_1}{P_1} & \frac{1}{P_1} & & \frac{1}{P_1} \\ \frac{x_2}{P_2} & \frac{1}{P_2} & & \frac{1}{P_2} \\ \frac{x_n}{P_n} & 0 & 0 \cdots & \frac{1}{P_n}\end{bmatrix}$$

Stockel's Theorem :— Necessary Condition:

If S is separable then consider condition II and III are satisfied

Also $\quad \ell = H = T + V \rightarrow$

$\Rightarrow \quad h = \frac{1}{2}\sum_{r=1}^{n} C_r p_r^2 + V \qquad$ (using Stockel theorem Statem)

$$\Rightarrow \quad \frac{1}{2}\sum C_r \left(\frac{\partial K}{\partial q_r}\right)^2 + V = h = \alpha \rightarrow (4)$$

Using def. of separable $\frac{\partial K}{\partial q_r} = p_r \qquad K = K_r(q_r)$

here α is chosen as total energy because of assumed separability

$$K = K_1(q_1) + K_2(q_2) + \cdots + K_n(q_n) \rightarrow (5)$$

where $K_r = K_r(q_r, \alpha_1, \alpha_2, \ldots \alpha_n)$

using (5) in (4), we will get identities in q_s and α_s

differenciate (4) w.r.t $\alpha(\alpha_1, \alpha_2 \ldots \alpha_n)$

ie Ist order w.r.t α_1 and α_s ; $s = 2, \ldots n$

$$\Rightarrow \quad \frac{1}{2}\sum_{r=1}^{n} 2 C_r \left(\frac{\partial K}{\partial q_r}\right) \frac{\partial^2 K}{\partial \alpha_1 \partial q_r} + 0 = 1$$

$$\sum_{r=1}^{n} C_r \frac{\partial K}{\partial q_r} \frac{\partial^2 K}{\partial \alpha_1 \partial q_r} = 1 \rightarrow (6)$$

$$\sum_{r=1}^{n} C_r \frac{\partial K}{\partial q_r} \frac{\partial^2 K}{\partial \alpha_s \partial q_r} = 0 \qquad s = 2, 3, \ldots n$$

$\Rightarrow$ Coefficient of C_r is a function of q only

$$\because \quad K = K_1 + K_2 + \cdots + K_n$$

$$\begin{bmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \vdots & & & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\frac{x_1}{p_1}, \frac{1}{p_1} \quad \frac{1}{p_1}\right] \begin{bmatrix} p_1/\Sigma x_1 \\ \vdots \\ p_n/\Sigma x_s \end{bmatrix} = 1$$

$$\Rightarrow \quad u_{r1} = \frac{\partial K}{\partial q_r} \frac{\partial^2 K}{\partial \alpha_1 \partial q_r} \quad ; \quad u_{rs} = \frac{\partial K}{\partial q_r} \frac{\partial^2 K}{\partial \alpha_s \partial q_r}$$

now to show that U_{rs} is non singular by Def. of Complete integral

$$\left\| \frac{\partial^2 K}{\partial \alpha_i \partial q_j} \right\| \neq 0$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \frac{\partial K}{\partial q_1} \frac{\partial^2 K}{\partial \alpha_1 \partial q_1} & \cdots & \frac{\partial K}{\partial q_n} \frac{\partial^2 K}{\partial \alpha_1 \partial q_r} \\ \frac{\partial K}{\partial q_1} \frac{\partial^2 K}{\partial \alpha_2 \partial q_1} & \cdots & \frac{\partial K}{\partial q_n} \frac{\partial^2 K}{\partial \alpha_2 \partial q_r} \\ \cdots & \cdots & \cdots \\ \frac{\partial K}{\partial q_1} \frac{\partial K}{\partial \alpha_n \partial q_r} & \cdots & \frac{\partial K}{\partial q_n} \frac{\partial^2 K}{\partial \alpha_n \partial q_r} \end{vmatrix}$$

$$\frac{\partial K}{\partial q_1} \frac{\partial K}{\partial q_2} \cdots \frac{\partial K}{\partial q_n} \left\| \frac{\partial^2 K}{\partial \alpha_i \partial q_j} \right\| \neq 0.$$

Sufficient Condition :— To show that the System is separable

$$\sum_{r=1}^{n} C_r u_{rs} = \delta_{1s} \rightarrow (2)$$

$$\sum_{r=1}^{n} C_r w_r = V \rightarrow (3)$$

$$(i) \Rightarrow \quad \frac{1}{2} \sum_{r=1}^{n} C_r \left(\frac{\partial K}{\partial q_r}\right)^2 + V = k$$

$$\Rightarrow \quad \frac{1}{2} \sum_{r=1}^{n} C_r \left(\frac{\partial K}{\partial q_r}\right)^2 + \sum_{r=1}^{n} C_r w_r$$

$$= \alpha_1 \sum C_r u_{r1} + \alpha_2 \sum C_r u_{r2} + \cdots + \alpha_n \sum C_r u_{rn}$$

$$\Rightarrow \quad C_r \left[\frac{1}{2} \left(\frac{\partial K}{\partial q_r}\right)^2 - (\alpha_1 u_{r1} + \alpha_2 u_{r2} + \cdots + \alpha_n u_{rn} - w_r) \right] = 0$$

$$\because \quad K = K_1(q_1) + \cdots + K_n(q_n)$$

$$\left(\frac{\partial K}{\partial q_r}\right)^2 = \left(\frac{dK}{dq}\right)^2 = 2(\alpha_1 u_{r1} + \cdots + \alpha_n u_{rn} - w_r)$$

$$= f_r q_r$$

$$\Rightarrow \quad \frac{dK_r}{dq} = \sqrt{f_r q_r} \quad \Rightarrow \quad dK_r = (f_r q_r)^{1/2} dq_r$$

$$\Rightarrow \quad K_r = \int \sqrt{f_r q_r}\, dq_r$$

$$t - t_0 = \frac{\partial K_1}{\partial \alpha_1} = \int \frac{2 \quad u_{r1} \cdot \frac{1}{2}}{\sqrt{f_r q_r}} \, dq_r$$

$$\Rightarrow \quad -\beta_s = \int \frac{u_{rs}}{\sqrt{f_r q_r}} \, dq_r$$

Since $\quad p_r = \frac{\partial K_1}{\partial q_r} = \sqrt{f_r q_r}$

$$\frac{d^2 K}{\partial \alpha_1 \partial q_r} = \frac{u_{r1}}{\frac{\partial K}{\partial q_r}}$$

$$\frac{\partial K}{\partial \alpha_1} = \int \frac{u_{r1}}{\sqrt{f_r q_r}} \, dq_r.$$

Define the bilinear Covariant and Show That a transformation is Canonical if and only if

$$\sum \left(\delta p_i\, dq_i - \delta q_i\, dp_i\right) \to (1) \quad \text{is Invariant}$$

Under this transformation

Sol:- The Relation $\sum_{i=1}^{n} (\delta p_i dq_i - \delta q_i dp_i)$ is linear in δp_i and dp_i and is also linear δq_i and dq_i. Therefore it is Called bilinear Covariant

now to Prove (1) is Invariant

Consider $p_i = p_i\,(P_K, Q_K)$

$q_i = q_i\,(P_K, Q_K)$

$$\sum_{i=1}^{n} \left(\delta p_i\, dq_i - \delta q_i\, dp_i\right) = \sum_{i=1}^{n} \left[\left(\frac{\partial p_i}{\partial Q_k} \delta Q_k + \frac{\partial p_i}{\partial P_k} \delta P_k \right) \left(\frac{\partial q_i}{\partial Q_m} dQ_m + \frac{\partial q_i}{\partial P_m} dP_m \right) - \left(\frac{\partial q_i}{\partial Q_k} \delta Q_k + \frac{\partial q_i}{\partial P_k} \delta P_k \right) \left(\frac{\partial p_i}{\partial Q_m} dQ_m + \frac{\partial p_i}{\partial P_m} dP_m \right) \right.$$

$$= \sum_{i=1}^{n} \left[\left\{ \frac{\partial p_i}{\partial Q_k} \frac{\partial q_i}{\partial Q_m} \delta Q_k\, dQ_m + \frac{\partial p_i}{\partial Q_k} \frac{\partial q_i}{\partial P_m} \delta Q_K\, dP_m + \frac{\partial p_i}{\partial P_k} \frac{\partial q_i}{\partial Q_m} \delta P_k\, dQ_m + \frac{\partial p_i}{\partial P_k} \frac{\partial q_i}{\partial P_m} \delta P_k\, dP_m \right\} \right.$$

$$\left. - \left\{ \frac{\partial q_i}{\partial Q_k} \frac{\partial p_i}{\partial Q_m} \delta Q_k\, dQ_m + \frac{\partial q_i}{\partial Q_k} \frac{\partial p_i}{\partial P_m} \delta Q_K\, dP_m + \frac{\partial q_i}{\partial P_k} \frac{\partial p_i}{\partial Q_m} dQ_m\, \delta P_k + \frac{\partial q_i}{\partial P_k} \frac{\partial p_i}{\partial P_m} \delta P_k\, dP_m \right\} \right]$$

$$= \sum_{i=1}^{n} \left[\left(\frac{\partial q_i}{\partial Q_m} \frac{\partial p_i}{\partial Q_k} - \frac{\partial p_i}{\partial Q_m} \frac{\partial q_i}{\partial Q_k} \right) \delta Q_k\, dQ_m + \left(\frac{\partial q_i}{\partial P_m} \frac{\partial p_i}{\partial Q_k} - \frac{\partial p_i}{\partial P_m} \frac{\partial q_i}{\partial Q_k} \right) \delta Q_k\, dP_m + \left(\frac{\partial q_i}{\partial Q_m} \frac{\partial p_i}{\partial P_k} - \frac{\partial p_i}{\partial Q_m} \frac{\partial q_i}{\partial P_k} \right) dQ_m\, \delta P_k \right.$$

$$\left. + \left(\frac{\partial q_i}{\partial P_m} \frac{\partial p_i}{\partial P_k} - \frac{\partial p_i}{\partial P_m} \frac{\partial q_i}{\partial P_k} \right) \delta P_k\, dP_m \right]$$

Using the Def. of Lagrange Bracket we have

$$= \sum_{i=1}^{n} \Big[\{Q_m\, Q_k\} \delta Q_k dQ_m + \{P_m\, Q_k\} \delta Q_k dP_m + \{Q_m\, P_k\} dQ_m \delta P_k + \{P_m P_k\} \delta P_k dP_m \Big]$$

$$= \sum_{i=1}^{n} \Big[(0) + \{P_m\, Q_k\} \delta Q_k dP_m + \{Q_m\, P_k\} dQ_m \delta P_k + 0 \Big\}$$

$$= \sum_{i=1}^{n} \Big[-\{Q_k\, P_m\} \delta Q_k dP_m + \{Q_m\, P_k\} dQ_m \delta P_k \Big]$$

$$= \sum_{i=1}^{n} \Big[\delta_{mk} \delta P_k dQ_m - \delta_{km} \delta Q_k dP_m \Big]$$

for $m = k$

$$= \sum_{i=1}^{n} \Big(\delta P_k dQ_k \delta_{kk} - \delta Q_k dP_k \delta_{kk} \Big)$$

$$= \sum_{i=1}^{n} \Big(\delta P_k dQ_k - \delta Q_k dP_k \Big)$$

Shows that

$$= \sum_{i=1}^{n} \big(\delta P_i dQ_i - \delta Q_i dP_i \big)$$

Conversely to prove the Converse of the Proof. Consider the transformation

$$Q_k = Q_k(q_i, p_i)$$
$$P_k = P_k(q_i, p_i)$$

$$\sum_{i=1}^{n} \big(\delta P_k dQ_k - \delta Q_k dP_k \big) = \sum_{k=1}^{n} \left(\frac{\partial P_k}{\partial q_i} \delta q_i + \frac{\partial P_k}{\partial p_i} \delta p_i \right) \left(\frac{\partial Q_k}{\partial q_j} dq_j + \frac{\partial Q_k}{\partial p_j} dp_j \right) - \left(\frac{\partial Q_k}{\partial q_i} \delta q_i + \frac{\partial Q_k}{\partial p_i} \delta p_i \right) \left(\frac{\partial P_k}{\partial q_j} dq_j + \frac{\partial P_k}{\partial p_j} dp_j \right)$$

$$= \sum_{i=1}^{n} \Bigg[\Big\{ \frac{\partial P_h}{\partial q_i} \frac{\partial Q_h}{\partial q_j} \delta q_i \, dq_j + \frac{\partial P_h}{\partial q_i} \frac{\partial Q_h}{\partial p_j} \delta q_i \, dp_j$$

$$+ \frac{\partial P_h}{\partial p_i} \frac{\partial Q_h}{\partial q_j} \delta p_i \, dq_j + \frac{\partial P_h}{\partial p_i} \frac{\partial Q_h}{\partial p_j} \delta p_i \, dp_j \Big\}$$

$$- \Big\{ \frac{\partial Q_h}{\partial q_i} \frac{\partial P_h}{\partial q_j} \delta q_i \, dq_j + \frac{\partial Q_h}{\partial q_i} \frac{\partial P_h}{\partial p_j} \delta q_i \, dp_j$$

$$+ \frac{\partial Q_h}{\partial p_i} \frac{\partial P_h}{\partial q_j} \delta p_i \, dq_j + \frac{\partial Q_h}{\partial p_i} \frac{\partial P_h}{\partial p_j} \delta p_i \, dp_j \Big\} \Bigg]$$

$$= \sum_{i=1}^{n} \Bigg[\left(\frac{\partial Q_h}{\partial q_j} \frac{\partial P_h}{\partial q_i} - \frac{\partial P_h}{\partial q_j} \frac{\partial Q_h}{\partial q_i} \right) \delta q_i \, dq_j$$

$$+ \left(\frac{\partial Q_h}{\partial p_j} \frac{\partial P_h}{\partial q_i} - \frac{\partial P_h}{\partial p_j} \frac{\partial Q_h}{\partial q_i} \right) \delta q_i \, dp_j + \left(\frac{\partial Q_h}{\partial q_j} \frac{\partial P_h}{\partial p_i} - \frac{\partial P_h}{\partial q_j} \frac{\partial Q_h}{\partial p_i} \right)$$

$$\delta p_i \, dq_j + \left(\frac{\partial Q_h}{\partial p_j} \frac{\partial P_h}{\partial p_i} - \frac{\partial P_h}{\partial p_j} \frac{\partial Q_h}{\partial p_i} \right) \delta p_i \, dp_j \Bigg]$$

$$= \sum_{i=1}^{n} \Big[\{q_j \, q_i\} \delta q_i \, dq_j + \{p_j \, q_i\} \delta q_i \, dp_j + \{q_j, p_i\} \delta p_i \, dq_j + \{p_j \, p_i\} \delta p_i \, dp_j \Big]$$

$$= \sum_{i=1}^{n} \Big[0 \; - \underset{\text{primary}}{\{q_i \, p_j\}} \delta q_i \, dp_j + \{q_j \, p_i\} \delta p_i \, dq_j + 0 \Big]$$

$$= \sum_{i=1}^{n} \Big[-\delta_{ij} \, \delta q_i \, dp_j + \delta_{ji} \, \delta p_i \, dq_j \Big]$$

for $i = j$ $\quad = \sum_{i=1}^{n} \delta p_i \, dq_i - \delta q_i \, dp_i$

Q. Show by method of bi linear Co-variant that the transformation defined by

$$Q = \sqrt{e^{-2q} - p^2} \qquad P = \cos^{-1}(p e^{q})$$

is Canonical transformation or in Contact

$Q = \sqrt{e^{-2q} - p^2} \qquad P = \cos^{-1}(p e^q)$

$$\frac{\partial Q}{\partial q} = \frac{1}{\cancel{2}\sqrt{e^{-2q} - p^2}} e^{-2q} \cancel{(-2)}$$

$$\frac{\partial Q}{\partial q} = \frac{-e^{-2q}}{\sqrt{e^{-2q} - p^2}}$$

$$\frac{\partial Q}{\partial p} = \frac{1}{2\sqrt{e^{-2q} - p^2}}(-2p) = \frac{-p}{\sqrt{e^{-2q} - p^2}}$$

$$\frac{\partial P}{\partial p} = \frac{-1}{\sqrt{1 - p^2 e^{2q}}} e^q$$

$$\frac{\partial P}{\partial q} = \frac{-1}{\sqrt{1 - p^2 e^{2q}}} p \cdot e^q$$

$$\delta P\, dQ - dP\, \delta Q = \left(\frac{-pe^q}{\sqrt{1-p^2e^{2q}}} \delta q + \frac{-e^q}{\sqrt{1-p^2e^{2q}}} \delta p \right) \left(\frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}} dq - \frac{p}{\sqrt{e^{-2q}-p^2}} dp \right)$$

$$- \left(\frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}} \delta q + \frac{-p}{\sqrt{e^{-2q}-p^2}} \delta p \right) \left(\frac{-pe^q}{\sqrt{1-p^2e^{2q}}} dq + \frac{-e^q}{\sqrt{1-p^2e^{2q}}} dp \right)$$

$$= \cancel{\frac{pe^{-q}}{\sqrt{1-p^2e^{2q}}\sqrt{e^{-2q}-p^2}} \delta q\, dq} + \frac{p^2 e^q\, \delta q\, dp}{\sqrt{1-p^2e^{2q}}\sqrt{e^{-2q}-p^2}} + \frac{e^{-q}}{\sqrt{1-p^2e^{2q}}\sqrt{e^{-2q}-p^2}} \delta p\, dq$$

$$+ \cancel{\frac{e^q p}{\sqrt{1-p^2e^{2q}}\sqrt{e^{-2q}-p^2}} \delta p\, dp} - \cancel{\frac{pe^{-q}}{\sqrt{e^{-2q}-p^2}\sqrt{1-p^2e^{2q}}} \delta q\, dq} - \frac{e^{-q}}{\sqrt{e^{-2q}-p^2}\sqrt{1-p^2e^{2q}}} \delta q\, dp$$

$$- \frac{p^2 e^q}{\sqrt{e^{-2q}-p^2}\sqrt{1-p^2e^{2q}}} \delta p\, dq \cancel{+ \frac{pe^q}{\sqrt{e^{-2q}-p^2}\sqrt{1-p^2e^{2q}}} \delta p\, dp}$$

$$\left(\frac{p^2 e^q - e^{-q}}{\sqrt{e^{-2q}-p^2}\sqrt{1-p^2e^{2q}}} \right) \delta q\, dp + \left(\frac{e^{-q} - p^2 e^q}{\sqrt{1-p^2e^{2q}}\sqrt{e^{-2q}-p^2}} \right) \delta p\, dq$$

$$= - \frac{\cancel{e^{-q}}\cancel{(1 - p^2 e^{2q})}}{\cancel{e^{-q}}\cancel{\sqrt{1-p^2e^{2q}}\sqrt{1-p^2e^{2q}}}} \delta q\, dp + \frac{\cancel{e^{-q}}\cancel{(1 - p^2 e^{2q})}}{\cancel{e^{-q}}\cancel{\sqrt{1-p^2e^{2q}}\sqrt{1-p^2e^{2q}}}} \delta p\, dq$$

$$= \delta p\, dq - \delta q\, dp$$

$= R.H.S.$

Show that by the method of bilinear co-variant that the transformation is defined by the equation $Q = \log\left(\frac{\sin p}{q}\right)$ and $P = q\cot p$ is contact transformation

we show that

$$\delta P\, dQ - dP\,\delta Q = \delta p\, dq - dp\,\delta q$$

$$\delta P\, dQ - dP\,\delta Q = \left(\frac{\partial P}{\partial q}\delta q + \frac{\partial P}{\partial p}\delta p\right)\left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\right) - \left(\frac{\partial Q}{\partial q}\delta q + \frac{\partial Q}{\partial p}\delta p\right)\left(\frac{\partial P}{\partial q}dq + \frac{\partial P}{\partial p}dp\right)$$

$$Q = \log\left(\frac{\sin p}{q}\right) = \log\sin p - \log q$$

$$\frac{\partial Q}{\partial q} = 0 - \frac{1}{q} \qquad \frac{\partial Q}{\partial p} = \frac{1}{\sin p}\cos p = \cot p$$

$$P = q\cot p$$

$$\frac{\partial P}{\partial q} = \cot p \qquad \frac{\partial P}{\partial p} = -q\,\mathrm{cosec}^2 p$$

$$\text{L.H.S.} = \left(\cot p\,\delta q - q\,\mathrm{cosec}^2 p\,\delta p\right)\left(-\frac{1}{q}dq + \cot p\,dp\right) - \left(-\frac{1}{q}\delta q + \cot p\,\delta p\right)\left(\cot p\,dq - q\,\mathrm{cosec}^2 p\,dp\right)$$

$$= -\frac{\cot p}{q}\delta q\,dq + \cot^2 p\,\delta q\,dp + \mathrm{cosec}^2 p\,\delta p\,dq - q\,\mathrm{cosec}^2 p\cot p\,\delta p\,dp - \left(-\frac{1}{q}\cot p\,\delta q\,dq + \frac{q}{q}\mathrm{cosec}^2 p\,\delta q\,dp + \cot^2 p\,\delta p\,dq - q\,\mathrm{cosec}^2 p\cot p\,\delta p\,dp\right)$$

$$= -\cancel{\frac{\cot p}{q}\delta q\,dq} + \cot^2 p\,\delta q\,dp + \mathrm{cosec}^2 p\,\delta p\,dq - \cancel{q\,\mathrm{cosec}^2 p\cot p\,\delta p\,dp} + \cancel{\frac{1}{q}\cot p\,\delta q\,dq} - \mathrm{cosec}^2 p\,\delta q\,dp - \cot^2 p\,\delta p\,dq + \cancel{q\,\mathrm{cosec}^2 p\cot p\,\delta p\,dp}$$

$$= -\left(\mathrm{cosec}^2 p - \cot^2 p\right)\delta q\,dp + \left(\mathrm{cosec}^2 p - \cot^2 p\right)\delta p\,dq$$

$$= \delta p\,dq - \delta q\,dp$$

∴ Show that by the method of bilinear Co-variant that the transformation defined by

$$Q = \sqrt{\frac{2q}{K}}\cos p \qquad P = \sqrt{2qK}\,\sin p$$

According to bilinear Co-variant method, to show a given transformation is Contact transformation we show that $\delta P\, dQ - \delta Q\, dP = \delta p\, dq - \delta q\, dp$

$$Q = \frac{\sqrt{2q}}{\sqrt{K}}\cos p \qquad P = \sqrt{2q}\sqrt{K}\,\sin p$$

$$\frac{\partial Q}{\partial q} = \frac{\sqrt{2}}{\sqrt{K}}\cdot\frac{1}{\cancel{2}\sqrt{q}}\cos p = \frac{\cancel{2}}{\sqrt{2Kq}\,\cancel{2}}\cos p \qquad \frac{\partial P}{\partial q} = \sqrt{2}\cdot\frac{1}{2\sqrt{q}}\sqrt{K}\,\sin p$$

$$\frac{\partial Q}{\partial p} = -\frac{\sqrt{2q}}{\sqrt{K}}\sin p \qquad = \frac{\cancel{2}K}{\cancel{2}\sqrt{2qK}}\sin p$$

$$\frac{\partial P}{\partial p} = \sqrt{2qK}\cos p$$

$\delta P\, dQ - \delta Q\, dP =$

$$\left(\sqrt{\frac{K}{2q}}\sin p\,\delta q + \sqrt{2qK}\cos p\,\delta p\right)\left(\frac{1}{\sqrt{2Kq}}\cos p\, dq + (-)\frac{\sqrt{2q}}{\sqrt{K}}\sin p\, dp\right)$$

$$-\left(\frac{\cos p}{\sqrt{2Kq}}\delta q + \left(-\sqrt{\frac{2q}{K}}\sin p\,\delta p\right)\right)\left(\sqrt{\frac{K}{2q}}\sin p\, dq + \sqrt{2qK}\cos p\, dp\right)$$

$$= \cancel{\sqrt{\frac{K}{2q}}\,\frac{\sin p\cos p}{\sqrt{2Kq}}\,\delta q\, dq} - \sqrt{\frac{2q}{K}}\sqrt{\frac{K}{2q}}\sin^2 p\,\delta q\, dp$$

$$+ \frac{\sqrt{2qK}}{\sqrt{2qK}}\cos^2 p\,\delta p\, dq - \cancel{\sqrt{2qK}\sqrt{\frac{2q}{K}}\sin p\cos p\,\delta p\, dp}$$

$$- \cancel{\frac{\cos p\sin p}{\sqrt{2Kq}}\sqrt{\frac{K}{2q}}\,\delta q\, dq} - \frac{\cos^2 p}{\sqrt{2Kq}}\, dp\,\delta q + \sqrt{\frac{2q}{K}}\sqrt{\frac{K}{2q}}\sin^2 p$$

$$dq\,\delta p + \cancel{\sqrt{\frac{2q}{K}}\sqrt{2qK}\sin p\cos p\,\delta p\, dp}$$

$$\delta p\, dq(\cos^2 p + \sin^2 p) - (\cos^2 p + \sin^2 p)\,\delta q\, dp = 0$$

$\Rightarrow \delta p\, dq - \delta q\, dp = 0 \Rightarrow$ bilinear Co-variant is invariant

∴ the transformation is in Contact transformation.

Define Constant of motion and give detailed account of Noether theorem on Constant motion of dynamical System

A function $f(q_1, q_2, \dots q_n, t)$ is a Constant of motion or gives a Conservation Law iff $\frac{df}{dt} = 0$

Let Equation of motion $E(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \rightarrow 1$

$\Rightarrow \quad E = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \rightarrow (1)$

where E is Euler's operator. Let the System is Subjected to transformation
$$\bar{t} = \bar{t}(q_s, \dot{q}_s, t, \epsilon)$$
$$\bar{q}_s = \bar{q}_s(q_s, \dot{q}_s, t, \epsilon) \rightarrow (2)$$
where ϵ is time dependent parameter s.t. the identity transformation Corresponding to ϵ is
$$\left.\begin{aligned}(\bar{t})_{\epsilon=0} &= t\\ (\bar{q}_s)_{\epsilon=0} &= q_s\end{aligned}\right] \rightarrow (3)$$
Now by Taylor's theorem about $\epsilon = 0$ then (2) $\Rightarrow$
$$\left.\begin{aligned}\bar{t} &= \bar{t}_{\epsilon=0} + \epsilon\left(\frac{\partial \bar{t}}{\partial t}\right)_{\epsilon=0} + \dots\\ \bar{q}_s &= (\bar{q}_s)_{\epsilon=0} + \epsilon\left(\frac{\partial \bar{q}_s}{\partial t}\right)_{\epsilon=0} + \dots\end{aligned}\right] \rightarrow (4)$$
Here $\left(\frac{\partial \bar{t}}{\partial t}\right)_{\epsilon=0} = \tau(q_s, \dot{q}_s, t)$

& $\left(\frac{\partial \bar{q}_s}{\partial t}\right)_{\epsilon=0} = \psi(q_s, \dot{q}_s, t)$

$\therefore$ (4) $\Rightarrow$
$$\bar{t} = \bar{t} + \epsilon\,\tau(q_s, \dot{q}_s, t) \dots$$
$$\bar{q}_s = \bar{q}_s + \epsilon\,\psi(q_s, \dot{q}_s, t)$$

<u>Guage Variate</u> :- Let us Suppose under the transformation
$$\bar{t} = t + \epsilon\,\tau(q_s, \dot{q}_s, t) + \dots$$
$$\bar{q}_s = \bar{q}_s + \epsilon\,\psi(q_s, \dot{q}_s, t) + \dots$$
then
$$L = L(q_s, \dot{q}_s, t) \text{ Satisfies}$$
$$L(\bar{t}, \bar{q}_s, \dot{\bar{q}}_s)\,d\bar{t} - L(q_s, \dot{q}_s, t)\,dt = \epsilon\, dF$$
where F is a function of $(q_s, \dot{q}_s, t)$, then L is Called <u>Guage Variate</u>

Theorem:- If the Lagragian L is a gauge under the transformation

$$\bar{t} = t + \varphi(q_s, \dot{q}_s, t) + \dots\dots$$

$$\bar{q}_s = q_s + \psi(q_s, \dot{q}_s, t) + \dots\dots$$

then L satisfies

$$\frac{dF}{dt} = \left(\varphi \frac{\partial L}{\partial t} + \psi_s \frac{\partial L}{\partial q_s}\right) + \frac{\partial L}{\partial \dot{q}_s}\left(\dot{\psi}_s - \dot{\varphi}\dot{q}_s\right) + L(q_s, \dot{q}_s, t)\dot{\varphi}$$

Pf. As L is gauge variant

$$L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t})d\bar{t} - L(q_s, \dot{q}_s, t)dt = \epsilon dF$$

$$L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t})\frac{d\bar{t}}{dt} - L(q_s, \dot{q}_s, t)\frac{dt}{dt} = \epsilon\frac{dF}{dt}$$

Differentiate w.r.t ϵ & put $\epsilon = 0$

$$\left[L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t})\frac{d}{d\epsilon}\left(\frac{d\bar{t}}{dt}\right) + \frac{d\bar{t}}{dt}\frac{d}{d\epsilon}L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t}) - \frac{d}{d\epsilon}L(q_s, \dot{q}_s, t)\right]_{\epsilon=0} = \frac{d\epsilon}{d\epsilon}\frac{dF}{dt}$$

$$L(q_s, \dot{q}_s, t)\frac{d}{d\epsilon}\left(\frac{d\bar{t}}{dt}\right) + \frac{d\bar{t}}{dt}\frac{d}{d\epsilon}L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t}) - 0 = \frac{dF}{dt} \quad (1)$$

$$\because \bar{t} = t + \epsilon\varphi(t, q_s, \dot{q}_s) \longrightarrow (a)$$

$$\frac{d\bar{t}}{dt} = 1 + \epsilon\frac{d\varphi}{dt} \longrightarrow (b)$$

If $\epsilon = 0$ $\quad \frac{d\bar{t}}{dt} = 1$

$$\frac{d}{d\epsilon}\left(\frac{d\bar{t}}{dt}\right) = \frac{d^2\bar{t}}{d\epsilon\, dt} = \frac{d\varphi}{dt} \rightarrow (c)$$

$$\left(\because \frac{d}{d\epsilon}\left(\frac{d\bar{t}}{dt}\right) = 0 + \frac{d\varphi}{dt}\right)$$

Also $\quad \bar{q}_s = q_s + \epsilon\psi_s(t, q_s, \dot{q}_s) + \dots\dots$

$$\frac{d\bar{q}_s}{dt} = \dot{q}_s + \epsilon\dot{\psi}_s$$

Now $\quad \bar{\dot{q}}_s = \frac{d\bar{q}_s}{d\bar{t}} = \frac{d\bar{q}_s}{dt}\cdot\frac{dt}{d\bar{t}}$

$$\bar{\dot{q}}_s = \frac{d\bar{q}_s}{dt}\Big/\frac{d\bar{t}}{dt} = \frac{\dot{q}_s + \epsilon\dot{\psi}_s + \dots}{1 + \epsilon\dot{\varphi} + \dots\dots} \longrightarrow \text{(b)}$$

$$\bar{\dot{q}}_s = (\dot{q}_s + \varepsilon \dot{\psi}_s)(1+\varepsilon\dot{\varphi} + \cdots)^{-1}$$

$$\bar{\dot{q}}_s = (\dot{q}_s + \varepsilon \dot{\psi}_s)(1+(-1)\varepsilon\dot{\varphi} + \cdots)$$

$$\bar{\dot{q}}_s = \dot{q}_s - \varepsilon\dot{\varphi}\dot{q}_s + \varepsilon\dot{\psi}_s - \varepsilon^2\dot{\psi}_s\dot{\varphi}$$

$$= \dot{q}_s - \varepsilon\dot{\varphi}\dot{q}_s + \varepsilon\dot{\psi}_s \qquad (\because \varepsilon^2(0)=0$$

$$\frac{d}{d\varepsilon}\left(L(\bar{t}, \bar{q}_s, \bar{\dot{q}}_s)\right)\Big|_{\varepsilon=0} = \left|\frac{d}{d\varepsilon} L(t+\varepsilon\varphi, q_s+\varepsilon\psi_s, \dot{q}_s+\varepsilon\dot{\psi}_s-\varepsilon\dot{\varphi}\dot{q}_s)\right|_{\varepsilon=0}$$

(using taylors theor) $\left(\underline{L(t, q_s, \dot{q}_s)}\right)$

$$= \frac{d}{d\varepsilon}\left|L(t, q_s, \dot{q}_s) + \varepsilon\varphi\frac{\partial L}{\partial t} + \varepsilon\psi_s\frac{\partial L}{\partial q_s} + \varepsilon(\dot{\psi}_s - \dot{\varphi}\dot{q}_s)\frac{\partial L}{\partial \dot{q}_s} + \cdots\right|_{\varepsilon=0}$$

$$= 0 + \varphi\frac{\partial L}{\partial t} + \psi_s\frac{\partial L}{\partial q_s} + (\dot{\psi}_s - \dot{\varphi}\dot{q}_s)\frac{\partial L}{\partial \dot{q}_s} \rightarrow (d) \qquad \left(\frac{d\varepsilon}{d\varepsilon}=1\right.$$

Putting (b) (c) (d) in (1) we have

$$\left(\Rightarrow L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t})\frac{d\bar{t}}{dt} - L(q_s, \dot{q}_s, t) = 0\right.$$

$$L(\bar{q}_s, \bar{\dot{q}}_s, t)\left(\frac{d\bar{t}}{dt}\right)_{\varepsilon=0} = L(q_s, \dot{q}_s, t)$$

$$L(\bar{q}_s, \bar{\dot{q}}_s, t) = L(q_s, \dot{q}_s, t)\Big)$$

using (d)

$$(1) \Rightarrow L(\bar{q}_s, \bar{\dot{q}}_s, t)\frac{d\varphi}{dt} + (1)\Big\{\Big(\varphi\frac{\partial L}{\partial t} + \psi_s\frac{\partial L}{\partial q_s}\Big)$$

$$+ (\dot{\psi}_s - \dot{\varphi}\dot{q}_s)\frac{\partial L}{\partial \dot{q}_s}\Big) = \frac{dF}{dt} \rightarrow (2)$$

$\varepsilon = 0 \quad \therefore$ by conjugate variable (Def)

$$L(\bar{q}_s, \bar{\dot{q}}_s, \bar{t})\,d\bar{t} - L(q_s\,\dot{q}_s, t)\,dt = 0$$

$$\Rightarrow \quad L(\bar{q}_s, \dot{q}_s, t) = L(q_s, \dot{q}_s, t)$$

$$(2) \Rightarrow L(q_s, \dot{q}_s, t)\frac{d\varphi}{dt} + \Big(\varphi\frac{\partial L}{\partial t} + \psi_s\frac{\partial L}{\partial q_s}\Big)$$

$$+ (\dot{\psi}_s - \dot{\varphi}\dot{q}_s)\frac{\partial L}{\partial \dot{q}_s} = \frac{dF}{dt}$$

$$L(q_s\,\dot{q}_s, t)\dot{\varphi} + \Big(\varphi\frac{\partial L}{\partial t} + \psi_s\frac{\partial L}{\partial q_s}\Big)$$

$$+ (\dot{\psi}_s - \dot{\varphi}\dot{q}_s)\frac{\partial L}{\partial \dot{q}_s} = \frac{dF}{dt}.$$

$\bar{t} = t+\varepsilon\varphi$ $(t, q_s, \dot{q}_s)$

$\frac{d\bar{t}}{dt}$

$\frac{d}{d\varepsilon}$ (b)

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{t}}{dt}\right) = 0 + \frac{d\varphi}{dt}$$

$$\frac{d\bar{t}}{dt} = 1$$

As $\frac{d\bar{t}}{dt} = 1 + \varepsilon\frac{d\varphi}{dt}$

As $\varepsilon = 0$

$$\frac{d\bar{t}}{dt} = 1$$

Nother's Theorem:— If under transformation

$\bar{t} = t + \epsilon \varphi(q_s, \dot{q}_s, t) + \cdots \qquad \bar{q}_s = q_s + \epsilon \psi_s(q_s, \dot{q}_s, t) + \cdots$

then the Lagragian L is the guage variate and the Expression

$$\varphi L + \frac{\partial L}{\partial \dot{q}_s}(\psi_s - \varphi \dot{q}_s) - F \text{ is a constant of motion}$$

Proof:– We shall prove this theorem by the identity that L is a guage variate under the transformation

$$\left(\varphi \frac{\partial L}{\partial t} + \psi_s \frac{\partial L}{\partial q_s}\right) + \frac{\partial L}{\partial \dot{q}_s}(\dot{\psi}_s - \dot{\varphi}\dot{q}_s) + L\dot{\varphi} = \frac{dF}{dt} \rightarrow (A)$$

(Reference taken Theorem before)

where $\mathcal{E} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_s}\right) - \frac{\partial}{\partial q_s}$

As $L = L(q_s, \dot{q}_s, t)$

$$\cancel{\frac{dL}{dt} = \frac{\partial L}{\partial q_s}\frac{\partial q_s}{\partial t} + \frac{\partial L}{\partial \dot{q}_s}\frac{\partial \dot{q}_s}{\partial t} + \frac{\partial L}{\partial t}}$$

$$\cancel{\frac{dL}{dt} = \frac{\partial L}{\partial q_s}\dot{q}_s + \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s + \frac{\partial L}{\partial t}}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_s}\dot{q}_s + \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s + \frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial t} = \frac{dL}{dt} - \frac{\partial L}{\partial q_s}\dot{q}_s - \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s$$

$$\varphi\frac{\partial L}{\partial t} = \varphi\frac{dL}{dt} - \varphi\frac{\partial L}{\partial q_s}\dot{q}_s - \varphi\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s \rightarrow (1)$$

$$\frac{d}{dt}\left(\varphi \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right) = \dot{\varphi}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s} + \varphi \ddot{q}_s \frac{\partial L}{\partial \dot{q}_s} + \varphi \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)$$

$$\dot{\varphi}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s} = \frac{d}{dt}\left(\varphi \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right) - \varphi \ddot{q}_s\frac{\partial L}{\partial \dot{q}_s} - \varphi \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \rightarrow (2)$$

$$\frac{d}{dt}\left(\psi_s \frac{\partial L}{\partial \dot{q}_s}\right) = \dot{\psi}_s\frac{\partial L}{\partial \dot{q}_s} + \psi_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)$$

$$\dot{\psi}_s\frac{\partial L}{\partial \dot{q}_s} = \frac{d}{dt}\left(\psi_s\frac{\partial L}{\partial \dot{q}_s}\right) - \psi_s\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \rightarrow (3)$$

(1) (2) (3) in (A)

using (1)

$$\varphi\frac{dL}{dt} - \varphi\frac{\partial L}{\partial q_s}\dot{q}_s - \varphi\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s + \psi_s\frac{\partial L}{\partial q_s} + \frac{d}{dt}\left(\psi_s\frac{\partial L}{\partial \dot{q}_s}\right) - \psi_s\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{d}{dt}\left(\varphi\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\right) + \varphi\ddot{q}_s\frac{\partial L}{\partial \dot{q}_s} + \varphi\dot{q}_s\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) + L\frac{d\varphi}{dt} = \frac{dF}{dt}$$

$$\left(\varphi \frac{dL}{dt} + L\frac{d\varphi}{dt}\right) + \frac{d}{dt}\left(\psi_s \frac{\partial L}{\partial \dot q_s}\right) - \frac{d}{dt}\left(\varphi \dot q_s \frac{\partial L}{\partial \dot q_s}\right)$$

$$- \frac{dF}{dt} = \varphi \frac{\partial L}{\partial q_s}\dot q_s + \cancel{\varphi \frac{\partial L}{\partial \dot q_s}\ddot q_s} - \psi_s \frac{\partial L}{\partial q_s} + \psi_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_s}\right)$$

$$- \cancel{\varphi \ddot q_s \frac{\partial L}{\partial \dot q_s}} - \varphi \dot q_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_s}\right)$$

$$\frac{d}{dt}\left(\varphi L + \frac{\partial L}{\partial \dot q_s}(\psi_s - \varphi \dot q_s) - F\right) = \frac{\partial L}{\partial q_s}(\varphi \dot q_s - \psi_s)$$

$$+ (\psi_s - \varphi\dot q_s)\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_s}\right)$$

$$\frac{d}{dt}\left(\varphi L + \frac{\partial L}{\partial \dot q_s}(\psi_s - \varphi \dot q_s) - F\right) = (\psi_s - \varphi \dot q_s)\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_s}\right) - \frac{\partial L}{\partial q_s}\right)$$

now if under the transformation (A), the Lagrange L is guage variate. then the expression R.H.S. is zero

So the expression $\varphi L + \frac{\partial L}{\partial \dot q_s}(\psi_s - \varphi \dot q_s) - F$ is constant of motion. This is called Nöther theorem

Example:- If the transformation equations are $\bar{t} = t + \varepsilon$ and $\bar{q}_s = q_s + 0$ and $F = 0$ then Prove that H is a Constant of motion

Sol:- Direct Method

If the transformation equations are defined as $r_i = r_i(q_1, q_2 \dots q_n)$ independent of time. Let T be K.E. then by Euler's theorem

$$\dot{q}_s \frac{\partial T}{\partial \dot{q}_s} = 2T$$

we have $L - \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} = T - V - 2T = -(T+V) = -\text{Const.}$

But $H = \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L$

$$\Rightarrow H = -\left(-\dot{q}_s \frac{\partial L}{\partial \dot{q}_s} + L\right) = \text{Const.} \Rightarrow H = \text{Const.}$$

By Noether theorem

Since $\bar{t} = t + \varepsilon(1) \rightarrow (1)$

But $\bar{t} = t + \varepsilon\phi \rightarrow (2)$

from (1) & (2) $\Rightarrow \phi = 1$ Also given that $\bar{q}_s = q_s + \varepsilon\, 0 \rightarrow (3)$

$\bar{q}_s = q_s + \varepsilon\psi(s) \rightarrow (4)$

(3) & (4) $\Rightarrow \psi(s) = 0$ ~~Subtraction theorem~~ $\Rightarrow F = 0$

By Noether theorem $\phi L + \frac{\partial L}{\partial \dot{q}_s}(\psi_s - \phi\dot{q}_s) - F = \text{Const}$

$$\Rightarrow (1)L + (0 - (1)\dot{q}_s)\frac{\partial L}{\partial \dot{q}_s} - 0 = \text{Const}$$

$$\Rightarrow L - \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} = \text{Const}$$

$$\Rightarrow -H = \text{Const} \Rightarrow H = \text{Const.}$$

Example No. 2:- If q_1 is Ignorable Co-ordinate, then p_1 is a Const. of motion

Sol:- Let us Suppose that transformation

$\bar{t} = t$ & $\bar{q}_1 = q_1 + \varepsilon$

So $\bar{q}_i = 0$ for $i = 2, 3, \dots n$

then $\bar{t} = t + \varepsilon\, 0 \rightarrow (1)$

But $\bar{t} = t + \varepsilon\phi \rightarrow (2)$

$\Rightarrow \phi = 0$ ~~But~~ Also $\bar{q}_1 = q_1 + \varepsilon(1) \rightarrow (3)$

But $\bar{q}_1 = q_1 + \varepsilon\psi_1 \rightarrow (4)$

③ d ④ $\Rightarrow \psi_1 = 1$ Also $\psi_i = 0$ $i = 2, 3, \ldots$

Choose $F = 0$

by Noether theorem $\phi L + (\psi_s - \phi \dot{q}_s)\dfrac{\partial L}{\partial \dot{q}_s} - F = \text{const}$

$$\Rightarrow \phi L + (\psi_1 - \phi \dot{q}_1)\left(\frac{\partial L}{\partial \dot{q}_1}\right) - F = \text{const}$$

$i = 2, 3, \ldots n$

$$0L + (1 - 0)\frac{\partial L}{\partial \dot{q}_1} - 0 = \text{const}$$

$$\frac{\partial L}{\partial \dot{q}_1} = \text{const} \Rightarrow p_1 = \text{const}.$$

: Direct Method

Since q_1 is ignorable coordinate q_1 will be absent from L so $\dfrac{\partial L}{\partial q_1} = 0$

by L.E. $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_1}\right) - \dfrac{\partial L}{\partial q_1} = 0 \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_1}\right) = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_1} = \text{const}.$$

Poission Theorem:- Statement:- If $u(q_i, p_i, t)$ and $v(q_i, p_i, t)$ are two integrals of Canonical Equation then $[u, v]$ gives a 3rd integral of these Equations

Proof:- Hamilton's Equations Canonical Equations are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \cdots n$$

Since u is first integral $\Rightarrow \frac{du}{dt} = 0$

$$\frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial u}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial u}{\partial t} = 0$$

$$\Rightarrow [u\ H]_{q,p} + \frac{\partial u}{\partial t} = 0 \rightarrow (1)$$

Since v is the first integral

$$\therefore [v, H]_{q,p} + \frac{\partial v}{\partial t} = 0 \rightarrow (2)$$

we have to prove that $[u, v]$ is 3rd integral

i.e. $\left[[u, v], H\right] + \frac{\partial}{\partial t}[u, v] = 0$

L.H.S. $\left[[u, v] + H\right] + \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] = 0$

$= \left[(u, v) + H\right] + \left[-[u, H], v\right] + \left[u, -[v, H]\right]$ using (1) and (2)

$= -\left[H, [u, v]\right] - \left[v, [H, u]\right] - \left[u, [v, H]\right]$ · using twice

$= 0$ by Jacobi identity

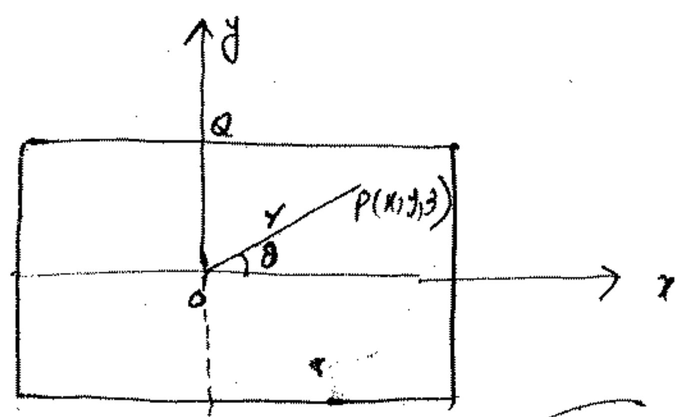
$\Rightarrow [u, v]$ gives the 3rd integral

2. (a) A Particle is a Smooth horizontal table is attached to a String passes through a Small hole in a table and Carries an equal particle hanging vertically. If at any time t, the particle is at a distance r from the hole and the String makes an angle θ with some fixed line on the table, then obtain the Lagrange Equations of motion in terms of $r+\theta$ and their time derivative.

b) The Particle on the table is initially projected at right angle to the String with velocity $\sqrt{2gh}$ when distance a from the ~~fixed~~ hole.

Prove that $\dot{r}^2 = gh\left(1-\frac{a^2}{r^2}\right) + g(a-r)$

Sol.



Let at any time t, the Particle at Point P which is at a distance r from O

We also Suppose that length of String is 'l'

2nd Particle from O is $l-r$

We draw a fixed line ∥ to x-axis passing through O making an angle θ with OP

(Total length is l ∴ remaining $l-r$)

$x = r\cos\theta \quad y = r\sin\theta$

$\dot{x} = -r\sin\theta\,\dot{\theta} + \dot{r}\cos\theta$

$\dot{y} = r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta$

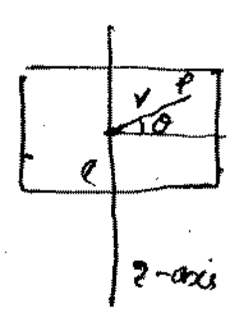
T_P = K.E. of Point P $= \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{m}{2}\left[(-r\sin\theta\,\dot{\theta} + \dot{r}\cos\theta)^2 + (r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta)^2 + 0\right]$

$= \frac{m}{2}\left[r^2\dot{\theta}^2 + \dot{r}^2\right]$

For the Particle Q $\quad x=0 \quad y=0$

$z = l-r$



∴ $\dot{x}=0 \quad \dot{y}=0 \quad \dot{z} = 0-\dot{r}$

$T_Q = \frac{1}{2}m\dot{r}^2$

$T = T_P + T_Q = \frac{m}{2}\left(r^2\dot\theta^2 + \dot r^2\right) + \frac{m}{2}\dot r^2$

$= \frac{m}{2}\left(r^2\dot\theta^2 + 2\dot r^2\right)$

now P.E. $V_P = 0$ $\quad V_Q = mgh = -mg(l-r)$ (vertically down from the ...)

$\therefore V = V_P + V_Q = 0 - mg(l-r)$

$L = T - V = \frac{1}{2} m\left(r^2\dot\theta^2 + 2\dot r^2\right) + mg(l-r) \longrightarrow (A)$

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}\left(0 + \frac{1}{2} m\, 4\dot r\right) - \left(\frac{1}{2} m\left(2r\dot\theta^2 - mg\right)\right) = 0$$

$$\Rightarrow 2m\ddot r - mr\dot\theta^2 + mg = 0$$

$$2\ddot r - r\dot\theta^2 + g = 0$$

$$\Rightarrow r\dot\theta^2 = 2\ddot r + g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{1}{2} m\left(r^2 2\dot\theta\right) + 0\right) - 0 = 0$$

$$\Rightarrow \cancel{m r^2 \ddot\theta} \Rightarrow \frac{d}{dt}\left(m r^2 \dot\theta\right) = 0$$

$$\Rightarrow m r^2 \dot\theta = Const = C \Rightarrow r^2\dot\theta = C \longrightarrow (i)$$

Since by Conservative Law of Energy $T + V = Const.$

Initially at $t = 0$ $\quad r = a$ $\quad \dot r = 0$ and velocity $= r\dot\theta = \sqrt{2gh}$

$\therefore T+V \longrightarrow$

$$\frac{m}{2}\left(\left(\sqrt{2gh}\right)^2 + 2(0)\right) - mg(l-a) = 0$$

$$\frac{m}{2}(2gh) - mgl + mga = 0$$

$$mgh - mgl + mga = 0 \rightarrow I$$

$\therefore$ (A)

$$\frac{m}{2}\left(r^2\dot\theta^2 + 2\dot r^2\right) - mg(l-r) = mgh - mgl + mga \quad \text{(from (I) \& (A))}$$

$$\frac{m}{2}\left(2\dot r^2 + r^2\dot\theta^2\right) - \cancel{mgl} + mgr = mgh - \cancel{mgl} + mga$$

$$\Rightarrow \frac{1}{2}\left(2\dot r^2 + r^2\dot\theta^2\right) = gh + ga - gr \rightarrow II$$

$$(i) \Rightarrow m r^2\dot\theta = Const \Rightarrow r^2\dot\theta = Const.$$

$$\Rightarrow r(r\dot\theta) = Const.$$

At $t = 0$ $\quad r = a$ $\quad r\dot\theta = \sqrt{2gh}$

$$a\left(\sqrt{2gh}\right) = \cos H$$

$$\therefore \quad r^2\dot{\theta} = a\left(\sqrt{2gh}\right)$$

$$\Rightarrow \dot{\theta} = a\left(\frac{\sqrt{2gh}}{r^2}\right) \quad \text{Put in } \text{II}$$

$$\frac{1}{2}\left(2\dot{r}^2 + r^2\left(a^2\left(\frac{2gh}{r^4}\right)\right)\right) = g(h+a-r)$$

$$\frac{1}{2}\left(2\dot{r}^2 + a^2\frac{(2gh)}{r^2}\right) = g(h+a-r)$$

$$\dot{r}^2 + \frac{gha^2}{r^2} = g(h+a-r)$$

$$\dot{r}^2 = gh + ga - gr - \frac{gha^2}{r^2}$$

$$\dot{r}^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a-r)$$