Exercises on Page 84

Q(1) Suppose that $Exp\,z = -1$

i.e. $e^z = -1$

$e^{x+iy} = -1$ (Cond. is necessary)

$e^x(\cos y + i\sin y) = -1 + 0i$

$\Rightarrow e^x\cos y = -1$ (i), $e^x\sin y = 0$ (ii)

$e^x \neq 0$, $\sin y = 0$

$y = \sin^{-1}(0) = n\pi$

From (i) $e^x \cos y = -1$

$e^x\cos(n\pi) = -1$

$e^x \neq 0$ & $\boxed{x = 0}$ $(-1)^n = -1$ n is odd

$n = (2k+1)$

$y = n\pi = (2k+1)\pi$ $k = 0, \pm1, \pm2 \dots$

$z = x + iy$

$= 0 + i(2k+1)\pi$

The condition is sufficient

If $z = (2k+1)\pi i$ then $Exp\,z = -1$

$e^z = -1$

$e^{(2k+1)\pi i} = \cos(2k+1)\pi + i\sin(2k+1)\pi$

$= -1 + 0 = -1$

Hence $e^z = -1$ iff $z = (2k+1)\pi i$

Q(2) $Exp(iz) = e^{iz}$ $Exp(-iz) = e^{-iz}$

are regular fns. (analytic)

Let $f(z) = e^{iz}$

$f'(z) = e^{iz}\cdot i$

| $f(z) = e^{-iz}$

| $f'(z) = e^{-iz}(-i)$

$e^{iz} = \cos z + i\sin z$

$u = \cos z$ $v = \sin z$

$\frac{\partial u}{\partial z} = -\sin z$ $\frac{\partial v}{\partial z} = \cos z$

| $e^{i(x+iy)} = e^{ix-y}$

| $= e^{-y}(\cos x + i\sin x)$

Ex. on Page 84

Q (3) $\quad Exp(iZ) = e^{iZ} = \cos z + i \sin z$

LHS $\quad e^{iz} = e^{i(x+iy)} = e^{ix-y}$

$= e^{-y} \cdot e^{ix} = e^{-y}(\cos x + i \sin x)$ — (1)

RHS. $= \cos z + i \sin z$

$= \cos(x+iy) + i \sin(x+iy)$

$= [\cos x \cosh y - i \sin x \sinh y] + i\,[\sin x \cosh y + i \cos x \sinh y]$

$= (\cos x + i \sin x)\cosh y - i(\sin x \sinh y) - \cos x \sinh y$

$= (\cos x + i \sin x)\cosh y - [\cos x + i \sin x]\sinh y$

$= (\cos x + i \sin x)(\cosh y - \sinh y)$

$= (\cos x + i \sin x)\left[\left(\frac{e^{y}+e^{-y}}{2}\right) - \left(\frac{e^{y}-e^{-y}}{2}\right)\right]$

$= (\cos x + i \sin x)\left(2e^{-y}/2\right) = e^{-y}(\cos x + i \sin x)$ — (2)

(1) & (2) are Same

Q No (4) $\quad |\cos z|^2 = \cos^2 x + \sinh^2 y$

Consider $\cos z = \cos(x+iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$|\cos z|^2 = (\cos x \cosh y)^2 + (\sin x \sinh y)^2$

$= \cos^2 x\,(1+\sinh^2 y) + \sin^2 x \sinh^2 y$

$= (\cos^2 x + \sin^2 x)\sinh^2 y + \cos^2 x$

$= \cos^2 x + \sinh^2 y$

Q No (5)

$\sin \bar{z}$ is not Analytic fn

$f(z) = \sin \bar{z} = \sin(x-iy)$

$u + iv = \sin x \cosh y - i \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \quad , \quad \frac{\partial v}{\partial y} = -\cos x \cosh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y \quad \quad \frac{\partial v}{\partial x} = \sin x \sinh y$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$ Hence $\sin \bar{z}$ is not analytic

Q(6) Find the Roots of the equation $\cos z = 2$

Solution

$$\cos z = 2$$

$$\cos(x+iy) = 2$$

$$\cos x \cosh y - i \sin x \sinh y = 2 + 0i$$

$$\Rightarrow \cos x \cosh y = 2 \quad (i)$$

$$\sin x \sinh y = 0 \quad (ii)$$

From (ii) $\sin x = 0$, $\sinh y \neq 0$

$$x = \sin^{-1}(0) \Rightarrow x = n\pi$$

Putting in (i)

$$\cos(n\pi)\cosh y = 2$$

$$(-1)^n \cosh y = 2$$

n must be even
$n = 2K$

$$\Rightarrow \cosh y = 2$$

$\Rightarrow x = 2K\pi$

$K = 0, \pm1, \pm2$

$$\Rightarrow \frac{e^{y}+e^{-y}}{2} = 2$$

$$e^{2y} - 4e^{y} + 1 = 0$$

$$e^{y} = t \Rightarrow t^2 - 4t + 1 = 0$$

$$t = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$e^{y} = 2 \pm \sqrt{3}$$

$$y = \log(2 \pm \sqrt{3})$$

$$\therefore z = x + iy$$

$$= 2K\pi + i\log(2+\sqrt{3})$$

$K = 0, \pm1, \pm2 \ldots$

Q(7) $$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$= \cancel{\frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}}$$

$$= \frac{2}{2}\,\frac{\sin(x+iy)}{\cos(x+iy)} \cdot \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin(x+iy+x-iy) + \sin[(x+iy)-(x-iy)]}{\cos(x+iy+x-iy) + \cos(x+iy)-(x-iy)}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

Q(8) $\tanh z = \tanh(x+iy) = U+iV$

$$LHS = \frac{\sinh(x+iy)}{\cosh(x+iy)}$$

$$= \frac{2\sinh(x+iy)\cdot\cosh(x-iy)}{2\cosh(x+iy)\cdot\cosh(x-iy)}$$

$$= \frac{\sinh[(x+iy+x-iy)+\sinh(x+iy-x+iy)]}{\cosh(x+iy+x-iy)+\cosh(x+iy-x+iy)}$$

$$= \frac{\sinh 2x+\sinh(2iy)}{\cosh 2x+\cosh(2iy)}$$

$$U+iV = \frac{\sinh 2x + i\sin 2y}{\cosh 2x+\cos 2y}$$

Hence the Result.

Q(9)

$\sinh z = -i+0 \Rightarrow \sinh(x+iy) = -i$

$\sinh x\cos y + i\cosh x\sin y = -i$

$\sinh x\cos y = 0$ (i)

$\cosh x\sin y = -1$ (ii)

$\cosh(iy) = \cos y$

$\sinh(iy) = i\sin y$

From (i) $\sinh x\cos y = 0$

$\Rightarrow \cos y = 0$

$y = \cos^{-1}(0)$

$y = (2n+1)\pi/2 \qquad n = 0, \pm1, \pm2 \ldots$

Putting in (ii)

$\cosh x \cdot \sin(2n+1)\pi/2 = -1$

$\cosh x = -1 \qquad$ if $\sin(2n+1)\pi/2 = +1$

$\frac{e^x+e^{-x}}{2} = -1 \qquad e^{2x}+1+2e^{x} = 0$

$(e^x+1)^2 = 0$

$e^x \neq -1$

If $\sin(2n+1)\pi/2 = -1$

$\cosh x = 1$

$\boxed{x = 0} \qquad y = (2n+1)\pi/2$

$z = x+iy = 0+i\pi/2\cdot(2n+1) = (n+1/2)\pi i$

$n = 0, \pm1, \pm2 \ldots$

Q(9) (ii) $\sinh z = -1$

$\sinh(x+iy) = -1$

$\Rightarrow \sinh x \cos y + i \cosh x \sin y = -1 + 0i$

$\sinh x \cos y = -1$ | $\cosh x \sin y = 0$

$\cosh x \neq 0 \quad \sin y = 0$

$y = n\pi$

$\sinh x \cos n\pi = -1$

$\sinh x \, (-1)^n = -1$

$\sinh x = 1 \qquad n = 2k+1$

$\sinh x = -1 \qquad n = 2k$

Two cases arise.

when $n = 2k \Rightarrow \sinh x = -1 \leftarrow$ Case 1

$\frac{e^x - e^{-x}}{2} = -1$

$e^{2x} - 1 + 2e^x = 0$

$e^{2x} + 2e^x - 1 = 0$

$e^x = \frac{-2 \pm \sqrt{4+4}}{2}$

$e^x = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

$x = \log(-1 \pm \sqrt{2})$ +ve sign, log of +ve real nos

Hence $y = 2k\pi$

$x = \log(-1+\sqrt{2})$

$z = x + iy$

$\boxed{z = \log(-1+\sqrt{2}) + 2k\pi i}$

case (ii) $\sinh x = 1$ when $n = 2k+1$

$\frac{e^x - e^{-x}}{2} = 1$

$e^{2x} - 1 = 2e^x$

$e^{2x} - 2e^x - 1 = 0$

$e^x = \frac{2 \pm \sqrt{4+4}}{2}$

$e^x = \frac{2 \pm 2\sqrt{2}}{2}$

$x = \log(1+\sqrt{2})$

$y = (2k+1)\pi$

$z = x + iy$

$= \log(1+\sqrt{2}) + (2k+1)\pi i$

$k = 0, \pm 1, \pm 2$

Q(10) $\text{Log}(1-i) = \log\sqrt{2} + i\tan^{-1}\left(\frac{-1}{1}\right)$

$= \frac{1}{2}\log 2 + i \, \cancel{\tan}(-\pi/4)$

$= \frac{1}{2}\log 2 - i\pi/4$

Q(10) (ii) $(1+i)^{i}$

$$\Rightarrow e^{\log(1+i)^{i}} = e^{i\log(1+i)}$$

$$= e^{i\,[\log\sqrt{2} + i\tan^{-1}1]}$$

$$= e^{i\,[\log\sqrt{2} + i\,\pi/4]}$$

$$= e^{-\pi/4 + i\,\frac{1}{2}\log 2}$$

(iii) $e^{\log(1+i)}$

$$= (1+i)$$ e & log cancel each other

OR

$$e^{\log\sqrt{2} + i\,\pi/4}$$

$$e^{\frac{1}{2}\log 2} \cdot e^{i\,\pi/4}$$

$$= \sqrt{2}\cdot\left(\cos\pi/4 + i\sin\pi/4\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= 1+i$$

(iv) $(-1+i)^{i}$

(v) $(\sqrt{3}+i)^{i/2}$

Q(11) Show that the function $\tan^{-1} z$ is multivalued and show that each branch is single valued in the Z plane cut along the imaginary axis except for the argument from i to $-i$.

Sol Consider $w = \tan^{-1} z$

(on page 83) then $w = k\pi + \frac{1}{2i}\log\left[\frac{i-z}{i+z}\right]$ multivalued fn. If $k=0$

$$w = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right)$$ is single valued of Z

Q(12) $\pi i = \log(-1)$

$$\pi i = \cancel{\log e^{\pi i}} = \log e^{\pi i} = \log[\cos\pi + i\sin\pi] = \log(-1)$$

also $\log(-1) = \log(i^2) = 2\log i$

$$= 2\left[\ln 1 + i\,\pi/2\right] = i\pi$$

In general, $\log(-1) = \ln 1 + i(2n+1)\pi$

$= (2n+1)\pi i$ Principal value when $n=0$ $\log(-1) = \pi i$

Q(13) $\quad \log(z^4) = 4\log z$

(i) let $z = i \Rightarrow z^4 = (i^4) = (i^2)^2 = 1$

LHS $\log z^4 = \log(i^4) = \log 1 = 2n\pi i$ —— ①

RHS $4\log z = 4\log i = 4(n i \pi/2) = 2n\pi i$ —— (ii)

LHS = RHS

(ii) $z = -1$

LHS $\log(z^4) = \log(-1)^4 = 2n\pi i \qquad n = 0, \pm 1, \pm 2 \ldots$

RHS $4\log(-1) = 4(2n+1)\pi i = 4\pi i(2n+1) \qquad n = 0, \pm 1, \pm 2 \ldots$

Q(14)

$$(-i)^i = e^{\log(-i)^i} = e^{i\log(-i)} = e^{i(\ln 1 - i\pi/2)} = e^{i(-i\pi/2)} = e^{\pi/2}$$

$$(-1)^{2i} = e^{\log(-1)^{2i}} = e^{2i\log(-1)} = e^{2i(i\pi)} = e^{-2\pi}$$

$$(1+i)^i = e^{i\log(1+i)} = e^{i(\log\sqrt{2} + i\pi/4)} = e^{-\pi/4 + i/2 \log 2}$$

Q(15)

$$\sum_{n=0}^{\infty} \frac{(1+\pi i)^n}{\lfloor n} = -e$$

$$= 1 + \left(\frac{1+\pi i}{1}\right) + \frac{(1+\pi i)^2}{\lfloor 2} + \frac{(1+\pi i)^3}{\lfloor 3} + \text{——}$$

$$= [\log(-e)]^0 + (\log(-e))^1 + \frac{[\log(-e)]^2}{\lfloor 2} + \frac{[\log(-e)]^3}{\lfloor 3} + \text{——}$$

$z = 1 + i\pi$

$e^{\log(1+i\pi)}$

$= \log e^{1+i\pi}$

$= \log[e(\cos\pi + i\sin\pi)]$

$1+\pi i = \log[-e]$

Since $e^x = 1 + \frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{——}$

$$= e^{[\log(-e)]} = -e \quad \text{Ans}$$