Differential Geometry: Handwritten notes

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Keywords

Curves with torsion:

Curve, space curve, equation of tangent, normal plane, principal normal curvature, derivation of curvature, plane of the curvature or osculating plane, principal normal or binormal, rectifying plane, equation of binormal, torsion, Serret Frenet formulae, radius of torsion, the circular helix, skew curvature, centre of circle of curvature, spherical curvature, locus of centre of spherical curvature, helices, spherical indicatrix, evolute, involute.

Differential geometry of surfaces:

Surface, tangent plane and normal, equation of tangent plane, equaiton of normal, one parameter family of surfaces, characteristic of surface, envelopes, edge of regression, equation of edge of regression, developable surfaces, osculating developable, polar developable, rectifying developable.

Reference: C. E. Weatherburn, *Differential geometry of three dimensions*, Cambridge at the university press, 1955. (http://archive.org/details/differentialgeom003681mbp)

بسم الله الرحمن الرحيم

DIFFERENTIAL GEOMETRY

CHAPTER 1

"CURVES WITH TORSION."

CURVE : The locus of a point $P(x, y, z)$ whose position vector $\underline{r}$ relative to a fixed origin may be expressed as a function of a single variable parameter is called a curve, The cartesian coordinates can be expressed as $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$. t is called Parameter. f_1, f_2 and f_3 are fns of t.

SPACE CURVE. When the curve is not a plane curve, it is said to be skew or Twisted or Tortuous curve.

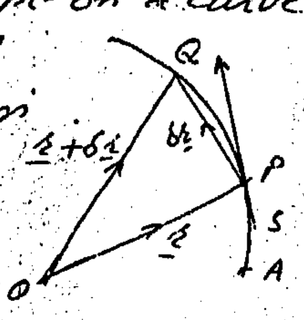
Examples (i) The set of equations
$$\left\{\begin{array}{l} \underline{r} = xi + yj + zk \\ \underline{r} = a\cos t\, i + a\sin t\, j + 0k \end{array}\right.$$
as : $x = a\cos t$ $\quad y = a\sin t$ $\quad z = 0$, $0 \le t \le 2\pi$

represent a circle, with centre at O and radius a.

$$\left.\begin{array}{l} x/a = \cos t \\ y/a = \sin t \end{array}\right\} \text{Squaring \& adding}$$
$$x^2 + y^2 = a^2$$

(ii)
$$\left.\begin{array}{l} \underline{r}(t) = a\cos t\, i + b\sin t\, j + 0k \\ \underline{r}(t) = xi + yj + zk \end{array}\right\} \text{comparing}$$
$$x = a\cos t$$
$$y = b\sin t, \quad z = 0$$
$$(x/a)^2 + (y/b)^2 = \cos^2 t + \sin^2 t = 1$$
$$x^2/a^2 + y^2/b^2 = 1 \qquad \text{Eq of Ellipse}$$

EQUATION OF A TANGENT at a pt on a curve.



Suppose that position vector $\underline{r}$ of a point P on a curve is function of S, (length of arc) from a fixed point A on it

Let P, Q be the neighbouring pts on the curve with p. vectors $\underline{r}$ + $r + \delta r$ resp corresponding to the values S and $S + \delta S$ of

By C.E. Weatherburn (Cambridge Uni Press) the parameter, then δr is the vector $\overrightarrow{PQ}$. The quotient $\frac{\delta r}{\delta s}$ is a vector along δr and in the limit as $\delta s \to 0$, this direction becomes that of the Tangent at P.

Also, in the limit $\frac{\delta r}{\delta s}$ when $Q \to P$ and $\delta s \to 0$ tends to unity & it is a unit vector ∥ to Tangent at P in the positive direction. It is denoted by $\underline{t}$ & called as unit Tangent at P

Thus $\underline{t} = \lim_{\delta s \to 0} \frac{\delta r}{\delta s} = \frac{dr}{ds} = r' \longrightarrow ①$

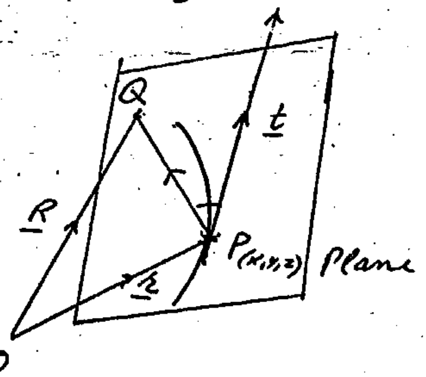
For the point on the Tangent, if $\underline{R}$ is its p.v. then $\underline{R} = \underline{r} + u\underline{t}$ where u is any ~~variable~~ real no. +ve or −ve. This is the equation of Tangent.

NORMAL PLANE:—

The normal plane at P is the plane through P perpendicular to the Tangent to a curve.

Equation of Normal plane:

Let $P(x,y,z)$ be a pt on the curve w.r.t a rectangular coordinate system.



then $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

Suppose $\underline{R}$ be the p.v. of a pt Q in the plane. Then $\underline{R} - \underline{r}$ is the p.v. of the line $\overrightarrow{PQ}$.

then $\frac{dr}{ds} = \frac{dx}{ds}i + \frac{dy}{ds}j + \frac{dz}{ds}k = x'i + y'j + z'k = \underline{t}$

where $\frac{dx}{ds}, \frac{dy}{ds}$ & $\frac{dz}{ds}$ are direction cosines of $\underline{t}$

The equation of normal plane is

$$(\underline{R} - \underline{r}) \cdot \underline{t} = 0 \qquad (\because \underline{R} - \underline{r} \text{ \& } \underline{t} \text{ are } \perp \text{ to each other})$$

Thus every line through P in this plane is a normal to the curve".

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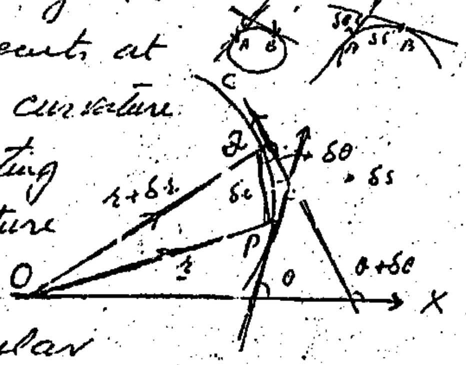
of unit vectors

PRINCIPAL NORMAL, CURVATURE.

Def The curvature of the curve at any pt. is the arc-rate of rotation of the Tangent.

If $\delta\theta$ is the angle between the Tangents at P and Q, $\delta\theta/\delta s$ is the average curvature of the arc PQ and its limiting value as $\delta s \to 0$ is the curvature at the pt. P. This is called the first curvature or the circular curvature and is denoted by K. (Kappa)



Thus

$$K = \underset{\delta s \to 0}{\mathrm{Lt}} \left(\frac{\delta\theta}{\delta\phi}\right) = \frac{d\theta}{ds} = \theta' \quad \text{when } \widehat{PQ} = \delta s$$

Derivation of Curvature.

Let C be a curve and OX as a fixed direction. Suppose P and Q be two pts on the curve C with p.vs $\underline{r}$ and $\underline{r}+\delta\underline{r}$ w.r.t O and ~~θ be the~~ and $\theta + \delta\theta$ be the angles of Tangents at P and Q resp.

The angle $\delta\theta$ is the angle between two Tangents at P and Q, ~~then~~ arc $\delta s = \widehat{PQ}$

Then $\dfrac{\delta\theta}{\delta s}$ is the average curvature of arc PQ. The limiting value of $\dfrac{\delta\theta}{\delta s}$ when $\delta s \to 0$ is called curvature at P

Thus $$K = \underset{\delta s \to 0}{\mathrm{Lt}} \left(\frac{\delta\theta}{\delta s}\right) = \frac{d\theta}{ds} = \theta'.$$

Def The reciprocal of K is defined as $\rho = 1/K$, as radius of curvature which is taken to be +ve.

$1.1\cos 0 = 1.1 = 1$

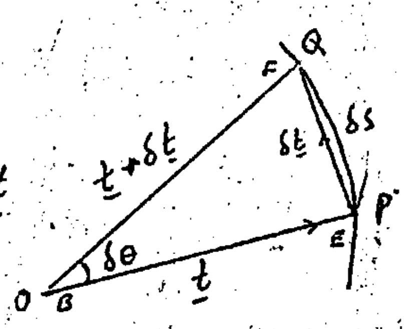
~~Theorem~~ Plane of Curvature or Osculating plane.

Since $\underline{t}$ is unit Tangent & $\underline{t}\cdot\underline{t} = 1$

Diff wrt. s $\quad 2\,\underline{t}\cdot\dfrac{d\underline{t}}{ds} = 0$

$\Rightarrow \underline{t}\cdot\dfrac{d\underline{t}}{ds} = 0$ since direction of $\underline{t}$ changes from pt to pt on the C

$\therefore \frac{d\underline{t}}{ds} \neq 0$ since $\underline{t} \neq 0, \frac{dt}{ds} \neq 0$



The quotient $\delta t/\delta s$ is a vector $\parallel$ to δt and therefore, in the limit as $\delta s \to 0$ its direction is $\perp$ to the Tangent at P since $|\underline{t}| = 1$ $|\underline{t}+\delta\underline{t}| = 1$. The limiting value of $\frac{\delta t}{\delta s}$ is the limiting value of $\delta\theta/\delta s$ which is K. Hence

$$\frac{d\underline{t}}{ds} = \lim_{\delta s \to 0} \frac{\delta \underline{t}}{\delta s} = K\,\hat{\underline{n}}$$

where $\hat{n}$ is a unit vector $\perp$ to $\underline{t}$ and in the plane of the Tangent at P and a consecutive pt. Q.

Def The plane containing two consecutive ~~pts~~ Tangents and therefore containing 3 cons. pts at P, is called the plane of curvature or the osculating plane.

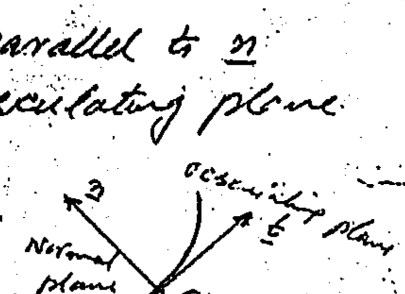
If $\underline{R}$ is any pt in this plane, the vectors $R-\underline{r}$, $\underline{t}$ and $\underline{n}$ are coplanar. Hence the relation $[\underline{R}-\underline{r}\ \underline{t}\ \underline{n}] = 0$ which is the equation of osculating plane. It is also expressed as

$$[\underline{R}-\underline{r},\ \underline{r}',\ \underline{r}''] = 0$$

The unit vectors $\underline{t}$ and $\underline{n}$ are $\perp$ to each other and their plane is plane of curvature.

PRINCIPAL NORMAL & BINORMAL

The straight line through P parallel to $\underline{n}$ ($\perp$ to $\underline{t}$) and lying in the osculating plane is called Principal normal at P and denoted as $\underline{n}$.



Its equation is clearly

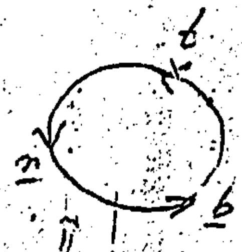
$$\underline{R} = \underline{r} + u\,\underline{n}$$

Where $\underline{R}$ being a current pt. on the line.

Def The normal at P which is $\perp$ to the Osculating plane is called Binormal, at P and it is denoted as $\underline{b}$ (unit Binormal). The vectors $\underline{t}$, $\underline{b}$ and $\underline{n}$ are $\perp$ to one another. Note that $\underline{b}$ is along $\underline{t}\times\underline{n}$ (form R.H. System of unit vectors)

Note that $\underline{t}\cdot\underline{t} = \underline{n}\cdot\underline{n} = \underline{b}\cdot\underline{b} = 1$

& $\underline{t}\cdot\underline{n} = \underline{n}\cdot\underline{b} = \underline{b}\cdot\underline{t} = 0$

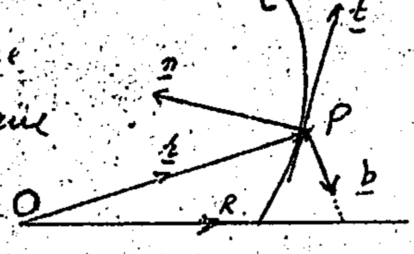


also
$$\left.\begin{aligned} \underline{t}\times\underline{n} &= \underline{b} \\ \underline{n}\times\underline{b} &= \underline{t} \\ \underline{b}\times\underline{t} &= \underline{n} \end{aligned}\right\} \quad \& \quad \underline{t}\times\underline{t} = \underline{n}\times\underline{n} = \underline{b}\times\underline{b} = 0$$

Retifying plane

The plane through the point P and $\perp$ to the normal is called rectifying plane.

Equation of Binormal



Let $P(\underline{r})$ be a point on C. Since a unit vector $\underline{b}$ $\perp$ to osculating plane is called binormal where $\underline{n}$ & $\underline{t}$ are unit normal & unit Tangent at P.

The equation of Principal normal is $\underline{R} = \underline{r} + u\underline{n}$ (i)

where $\underline{R}$ is a current pt on the line. Similarly equation of Binormal can also be written as

$$\underline{R} = \underline{r} + u\underline{b} \quad \text{——— (ii)}$$

or $\quad \underline{R} = \underline{r} + u(\underline{t}\times\underline{n}) \quad \because \underline{b}$ is $\parallel$ to $\underline{t}\times\underline{n}$

$$= \underline{r} + v\left(r' + r''/k\right)$$

Let $v = u/k$ then

$$\underline{R} = \underline{r} + v(r'\times r'') \quad \text{(iii)}$$

(ii) & (iii) are the eqs of binormal at P.

$r' = \underline{t}$

$\frac{dt}{ds} = r''$

$\frac{dt}{ds} = \kappa\underline{n}$

$\underline{n} = \frac{r''}{\kappa}$

K (kappa)

TORSION

The measure of arc-rate of turning of the unit binormal vector $\underline{b}$ is called Tortion of the curve at the point P. It is of course, the rate of rotation of the osculating plane. and it is denoted as τ

Since K is +ve but Tortion may be +ve or -ve

as $\quad \frac{d\underline{b}}{ds} = -\tau\underline{n}$, -ve sign shows →

That the Torsion is regarded as Positive when the rotation of the binormal as s increases is in the same sense that of a right handed screw travelling in the direction of t. ~~From figure, it is clear that in this case b is the opposite direction to ...~~

02 SERRET FRENET FORMULAE.

Three 1852 formulae for the derivatives of dcs for Tangent, principal normal & binormal

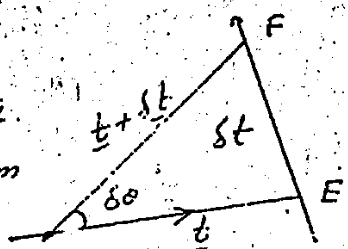
1: For Tangent $\frac{d\underline{t}}{ds} = K\vec{\underline{n}}$

2: For Binormal $\frac{d\underline{b}}{ds} = -T\,\vec{\underline{n}}$

3: For Principal normal $\frac{d\underline{n}}{ds} = T\underline{b} - K\underline{t}$

Proofs.

1 To Prove $\frac{d\underline{t}}{ds} = K\underline{n}$



Since the unit Tangent is not a constant vector as its direction changes from point to point of the curve

Let $\underline{t}$ and $\underline{t}+\delta\underline{t}$ be values of unit Tangent at E and F resp. The vectors $\vec{BE}$ and $\vec{BF}$ are resp. equal to these. Then $\vec{EF} = \delta t$ & let angle $E\hat{B}F = \delta\theta$

The quotient $\frac{\delta t}{\delta s}$ is a vector parallel to $\delta\underline{t}$.

As $\delta s \to 0$ its direction is $\perp$ to Tangent ~~at~~ $\underline{t}$.

Moreover $\vec{BE}$ and $\vec{BF}$ are of unit lengths

The modulus of limiting value of $\delta t/\delta s$ is equal to limiting value of $\delta\theta/\delta s$, which is K (Kappa)

Hence the relation

$$\frac{dt}{ds} = \lim_{\delta s \to 0}\left(\frac{\delta t}{\delta s}\right) = K\underline{n}$$

Where $\underline{n}$ is unit vector $\perp$ to $\underline{t}$ and in the plane of Tangents at two consecutive pts.

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2: To prove $\frac{d\underline{b}}{ds} = -T\,\underline{n}$

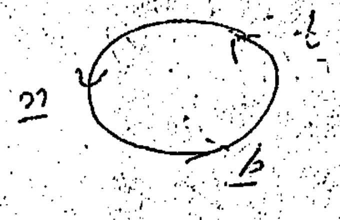
Proof Consider $\underline{b}\cdot\underline{b} = b^2 = 1$; Diff w.r.t s

$$2\underline{b}\cdot\frac{d\underline{b}}{ds} = 0$$

or $\underline{b}\cdot\underline{b}' = 0 \rightarrow (1)$ $\quad \because \frac{d\underline{b}}{ds} = \underline{b}'$

$\Rightarrow \underline{b}' \perp \underline{b}$

Consider the relation:



$$\underline{t}\cdot\underline{b} = 0$$

Diff w r t s

$$\underline{t}\cdot\frac{d\underline{b}}{ds} + \frac{d\underline{t}}{ds}\cdot\underline{b} = 0$$

$$\underline{t}\cdot\underline{b}' + K\underline{n}\cdot\underline{b} = 0 \quad \Big|\because \frac{d\underline{t}}{ds} = K\underline{n}$$

$\Rightarrow \underline{t}\cdot\underline{b}' = 0$ as $\underline{n}\cdot\underline{b} = 0$

$\Rightarrow \underline{b}' \perp \underline{t}$ —— (ii)

But from (i) it is proved that $\underline{b}' \perp \underline{b}$ so b' is parallel to $\underline{n}$, we may write it

$\underline{b}' = -\tau\,\underline{n}$ where τ measures the are rate of turning of unit vector $\underline{b}$ and $-$ve sign ~~shows~~ has been chosen to keep τ ~~post~~ positive.

3: To prove $\frac{d\underline{n}}{ds} = \tau\underline{b} - K\underline{t}$

Consider $\underline{n} = \underline{b}\times\underline{t}$ —— (1)

Diff (1) wrt s

$$\frac{d\underline{n}}{ds} = \underline{b}\times\frac{d\underline{t}}{ds} + \frac{d\underline{b}}{ds}\times\underline{t}$$

$$= \underline{b}\times K\underline{n} + (-\tau\underline{n})\times\underline{t}$$

$$= K(\underline{b}\times\underline{n}) - \tau(\underline{n}\times\underline{t}) \qquad (\underline{b}\times\underline{n} = -\underline{t}$$

$$= -K\underline{t} + \tau\underline{b}$$

$$\frac{d\underline{n}}{ds} = \underline{n}' = \tau\underline{b} - K\underline{t}.$$

Example: To prove that $\tau = \frac{1}{K^2}[\underset{\circ}{r'}\ r''\ r''']$

Sol

Since $\frac{dr}{ds} = \underline{t} \Rightarrow r' = \underline{t}$

$\frac{dt}{ds} = r'' = K\underline{n}$

and $\frac{dr''}{ds} = K'\underline{n} + K\frac{dn}{ds}$

$= K'\underline{n} + K(\tau\underline{b} - K\underline{t})$

In the notation of Scalar Triple product

$r' \cdot r'' \times r''' = [r'\ r''\ r'''] = [\underline{t}\ \ K\underline{n}\ \ K'n + K(\tau\underline{b} - K\underline{t})]$

$= \{\underline{t} \cdot (K\underline{n}) \times (\underline{n}K' + K\tau\underline{b} - K^2\underline{t})\}$

$= [t \cdot K\underline{n} \times \underline{n}K'] + (\underline{t} \cdot K\underline{n} \times K\tau\underline{b})$

$- (\underline{t} \cdot K\underline{n} \times K^2\underline{t}) = 0 + K^2\tau(\underline{t} \cdot \underline{n} \times \underline{b}) + 0$

$= K^2\tau \cdot (1)$

Hence the value of Torsion is given by | $\underline{n} \times \underline{b} = t$

$\tau = \frac{1}{K^2}[r'\ r''\ r''']$ Q.E.D

Def <u>Radius of Torsion</u>

The reciprocal of the Torsion is defined as the radius of Torsion and is denoted by σ.

Thus $\sigma = 1/\tau$. But there is No circle of Torsion or centre of Torsion associated with the curve in the same way as the circle or centre of curvature.

2000, II/03, 02, 99 Example <u>The circular Helix</u>

This is a curve drawn on the Surface of circular cylinder cutting the generators at the constant angle β. The position vector $\underline{r}$ of a point on the curve may be expressed as

$\underline{r} = a\cos\theta\, i + a\sin\theta\, j + a\theta\cot\beta\, K$

Diff wrt S, we have

$r' = \frac{dr}{ds} = \underline{t} = a(-\sin\theta, \cos\theta, \cot\beta)\,\theta'$

but this is a unit vector so that its square is unity and therefore, $|\underline{t}|^2 = 1 \Rightarrow a^2\theta'^2 = \sin^2\beta$

Thus θ' is constant. To find the curvature we have, on diff $\underline{t}$ w.r.t. s

$$K\underline{n} = r'' = -a(\cos\theta, \sin\theta, 0)\,\theta'^2$$

thus the principal normal is ~~a vector~~ the unit vector $\underline{n} = -(\cos\theta, \sin\theta, 0)$

$$\therefore\ K = a\theta'^2 = \tfrac{1}{a}\sin^2\beta.$$

To find the Torsion, we have

$$r''' = a(\sin\theta, -\cos\theta, 0).\,\theta'^3$$

& therefore, $r'' \times r''' = a^2(0,0,1)\,\theta'^5$

Hence $K^2\tau = [r'\ r''\ r'''] = a^3 \cot\beta\ \theta'^6$

On putting the value of K and θ', we have

$$\tau = \frac{1}{a}\sin\beta\cos\beta$$

Thus the curvature and Torsion are both constant & therefore, their ratio is constant.

II/03 Ex. on Page (18)

Q(1) Prove that $r''' = K'\underline{n} - K^2\underline{t} + K\tau\underline{b}$ & hence find $r'''' = (K'' - K^3 - K\tau^2)\underline{n} - 3KK'\underline{t} + (2K'\tau + \tau K')\underline{b}$

<u>Solution</u>:

Let $\underline{r} = \underline{r}(s)$

$$r' = \frac{dr}{ds} = \underline{t}$$

$$r'' = \frac{d\underline{t}}{ds} = \frac{d^2r}{ds^2}$$

also $r'' = K\underline{n}$

$$r''' = \frac{d^3r}{ds^3} = \frac{d}{ds}(K\underline{n}) = K\underline{n}' + \underline{n}K'$$

$$= K[\tau\underline{b} - K\underline{t}] + \underline{n}K'$$

$$= K\tau\underline{b} - K^2\underline{t} + \underline{n}K' = K'\underline{n} - K^2\underline{t} + K\tau\underline{b}$$

For $r'''' = \frac{d^4 r}{ds^4} = \frac{d}{ds}\left(\frac{d^3 r}{ds^3}\right)$

$$\begin{array}{|l} n' = Tb - Kt \\ t' = Kn \\ b' = -nT \end{array}$$

$$= \frac{d}{ds}\left(K'\underline{n} - K^2\underline{t} + KT\underline{b}\right)$$

$$= K''\underline{n} + K'\underline{n}' - K^2\underline{t}' - 2KK'\underline{t} + K'T\underline{b} + KT'\underline{b} + KT\underline{b}'$$

$$= K''\underline{n} + K'(T\underline{b} - K\underline{t}) - K^2(K\underline{n}) - 2KK'\underline{t} + K'T\underline{b} + KT'\underline{b} + KT\underline{b}'$$

$$= K''\underline{n} + K'(T\underline{b} - K\underline{t}) - K^3(\underline{n}) - 2KK'\underline{t} + K'T\underline{b} + KT'\underline{b} - KT(-nT)$$

$$= K''\underline{n} - K\underline{n} - KT^2\underline{n} - 3KK'\underline{t} + 2K'T\underline{b} + KT'\underline{b}$$

$$= (K'' - K^3 - KT^2)\underline{n} - 3KK'\underline{t} + (2K'T + T'K)\underline{b}$$

Q(2) as Q ① Prove that $r' \cdot r'' = 0$

~~$r'' \cdot r''' = KK'$ $\quad r' \cdot r''' = -K^2$~~

$$r'' \cdot r'''' = K(K'' - K^3 - KT^2) \qquad r' \cdot r'''' = -3KK'$$

$$r''' \cdot r'''' = K'K'' + 2K^3K' + K^2TT' + KK'T^2$$

Q(3) If the nth Derivative of r w.r.t. S is given by $r^n = a_n\underline{t} + b_n\underline{n} + c_n\underline{b}$ $\longrightarrow$ ①

Prove the Reduction formulae.

$$a_{n+1} = a_n' - K b_n$$
$$b_{n+1} = b_n' + K a_n - T c_n$$
$$c_{n+1} = c_n' + T b_n$$

Solution

From Given formula, we can write

$$\underline{r}^{n+1} = a_{n+1}\underline{t} + b_{n+1}\underline{n} + c_{n+1}\underline{b} \quad \text{—— ②}$$

Diff ① w.r.t. S

$$\underline{r}^{n+1} = a_n\frac{d\underline{t}}{ds} + \underline{t}\,a_n' + b_n\frac{d\underline{n}}{ds} + \underline{n}(b_n') + c_n\frac{d\underline{b}}{ds} + \underline{b}\,c_n'$$

$$= a_n(K\underline{n}) + \underline{t}\,a_n' + b_n(T\underline{b} - K\underline{t}) + \underline{n}\,b_n' + c_n(-nT) + \underline{b}\,c_n'$$

$= \underline{n}\,(a_n K + b_n' + \tau c_n) + \underline{t}\,(a_n' + K b_n) + \underline{b}\,(c_n' + \tau b_n)$ — (

Comparing coeffs in (A) & (B)

$$\left.\begin{aligned} a_{n+1} &= a_n' - K b_n \\ b_{n+1} &= b_n' + K a_n - \tau c_n \\ c_{n+1} &= c_n' + \tau b_n \end{aligned}\right]$$ as required

Q(4) i. If K is zero at all pts, the curve is a st-line
1999 ii. & if τ is zero at all pts, the curve is plane
The necessary and sufficient condition that the curve is plane is $[r' \; r'' \; r'''] = 0$

Solution Using Serret & Frenet formulas

$$\underline{t}' = K\underline{n}$$

If $K = 0$ then $\underline{t}' = 0$

$\Rightarrow \underline{t} = \text{Constant}$

Tangent is fixed, It is possible only when curve is st-line

(ii) Also $\underline{b}' = -\tau \underline{n}$

if $\tau = 0$ then $\underline{b}' = 0$

$\Rightarrow \underline{b}$ is constant

which is possible only when curve is plane.

iii) Necessary condition

Suppose curve be plane then $\tau = 0$ then we show that $[r' \; r'' \; r'''] = 0$

as $\tau = \dfrac{(r' \; r'' \; r''')}{K^2} = 0$

$\Rightarrow$ (A) $\Rightarrow \left[r' \; r'' \; r'''\right] = 0$

~~Curve is plane~~

Cond is sufficient if $(r' \; r'' \; r''') = 0$ then curve is plane

Given $(r' \; r'' \; r''') = 0 \Rightarrow \tau = \dfrac{(r' \; r'' \; r''')}{K^2} = 0$

$\tau = 0$. curve is plane.

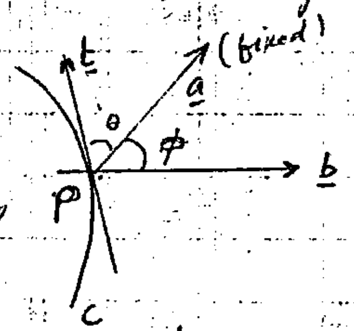
Q(5) Prove that $\underline{t}' \cdot \underline{b}' = -K\tau$

Since $\dfrac{d\underline{t}}{ds} = K\underline{n}$ & $\dfrac{d\underline{b}}{ds} = -\tau\underline{n}$

$$\therefore \quad \underline{t}' \cdot \underline{b}' = K\underline{n} \cdot -\tau\underline{n}$$
$$= -K\tau\, \underline{n}\cdot\underline{n}$$
$$= -K\tau \quad (1)$$

Q(6) If the Tangent and the Binormal at a point of a curve make angle θ and ϕ resp with a fixed direction, then

$$\frac{\sin\theta\, d\theta}{\sin\phi\, d\phi} = -K/\tau$$



Sol Let C be a given curve. Let $\underline{t}$ and $\underline{b}$ be unit tangent and unit binormal at P. Suppose $\hat{\underline{a}}$ be a unit vector along any fixed direction ~~and θ & φ are angles of t and b with it.~~

then $\hat{\underline{a}} \cdot \underline{t} = \cos\theta$ —— ①

$\hat{\underline{a}} \cdot \underline{b} = \cos\phi$ ——→ ②

$|t| = |\hat{a}| = |b| = 1$

Diff ① & ② wrt s

$$\underline{t}' \cdot \hat{a} = -\sin\theta \frac{d\theta}{ds} \Rightarrow K\underline{n}\cdot\hat{a} = -\sin\theta\frac{d\theta}{ds} \quad (3)$$
$$\underline{b}' \cdot \hat{a} = -\sin\phi \frac{d\phi}{ds} \Rightarrow -\tau\underline{n}\cdot\hat{a} = -\sin\phi\frac{d\phi}{ds} \quad (4)$$

Dividing (3) by (4)

$$\frac{K\,\cancel{(\hat{n}\cdot\hat{a})}}{-\tau\,\cancel{(\hat{n}\cdot\hat{a})}} = \frac{-\sin\theta\frac{d\theta}{ds}}{-\sin\phi\frac{d\phi}{ds}} \Rightarrow \frac{\sin\theta\, d\theta}{\sin\phi\, d\phi} = -\frac{K}{\tau}$$

Q(7) Show that the principal normals at consecutive pts do not intersect unless $\tau = 0$

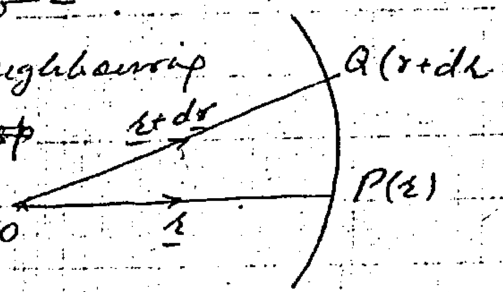
Sol Suppose P & Q be two consecutive pts with p.vs $\underline{r}$ & $\underline{r}+\delta\underline{r}$ and unit principal normals be $\underline{n}$ & $\underline{n}+d\underline{n}$. For intersections of the principal normals, the necessary condition is that the three vectors, $d\underline{r}$, $\underline{n}$ and $n+d\underline{n}$ be coplaner i.e. that r', n, n' be coplanar, This requires

$$\Rightarrow \quad [\underline{t},\ \underline{n}\ \ T\underline{b} - K\underline{t}] = 0 \qquad \Big| \ \begin{array}{l} r' = \underline{t} \\ n' = T\underline{b} - K\underline{t} \end{array}$$

$$T[\underline{t}, n, \underline{b}] = 0 \qquad [t, n, b] \neq 0$$

$$\Rightarrow \quad T = 0$$ (which holds only when $T = 0$)

Q (8) Prove that the shortest distance between the principal normal at consecutive pts, distant s apart is $\dfrac{\rho\, ds}{\sqrt{\rho^2+\sigma^2}}$ and that it divides the radius of curvature in the ratio $\rho^2 : \sigma^2$.



Sol Let P and Q be two neighbouring pts with p.vectors $\underline{r}$ and $\underline{r}+d\underline{r}$ ~~resp~~ and $\underline{n}$ and $\underline{n}+d\underline{n}$ be unit Principal normals at P & Q resp.

The vector $\perp$ to both $\underline{n}$ & $\underline{n}+d\underline{n}$ is $\underline{n}\times(\underline{n}+d\underline{n})$

then $\underline{n}\times(\underline{n}+d\underline{n}) = \underline{n}\times\underline{n} + \underline{n}\times d\underline{n}$

$$= 0 + \underline{n}\times d\underline{n} = \underline{n}\times \frac{d\underline{n}}{ds}\, ds \qquad \Big| \ \because \frac{d\underline{n}}{ds} = T\underline{b} - K\underline{t}$$

$$= \underline{n}\times(T\underline{b} - K\underline{t})\, ds$$

$$= [T(\underline{n}\times\underline{b}) - K(\underline{n}\times\underline{t})]\, ds$$

$$= (T(\underline{t}) - K(-\underline{b}))\, ds$$

$$= (T\underline{t} + K\underline{b})\, ds$$

This is the vector $\perp$ to $\underline{n}$ and $\underline{n}+d\underline{n}$. To find its unit vector. Let $\hat{e}$ be unit vector along it, then

$$\hat{e} = \frac{(T\underline{t} + K\underline{b})\, ds}{|(T\underline{t} + K\underline{b})\, ds|}$$

$$= \frac{(T\underline{t} + K\underline{b})\, ds}{\sqrt{T^2+K^2}\, ds} \Rightarrow \hat{e} = \frac{T\underline{t} + K\underline{b}}{\sqrt{T^2+K^2}}$$

is the unit vector $\perp$ to both $\underline{n}$ & $\underline{n}+d\underline{n}$.

To find shortest distance between two Principal normals at P & Q.

Shortest distance = Projection of $d\underline{r}$ upon $\hat{e}$

$$= \hat{e}\cdot d\underline{r}$$

$$S.D = \left(\hat{e}\cdot\frac{dr}{ds}\right)ds = (\hat{e}\cdot\underline{t})\,ds$$

$$= \left[\frac{(\tau\underline{t}+\kappa\underline{b})}{\sqrt{\tau^2+\kappa^2}}\cdot\underline{t}\right]ds$$

$$= \left[\frac{\tau(\underline{t}\cdot\underline{t})+\kappa(\underline{b}\cdot\underline{t})}{\sqrt{\tau^2+\kappa^2}}\right]ds$$

Put $\tau = 1/\sigma$, $\kappa = 1/\rho$

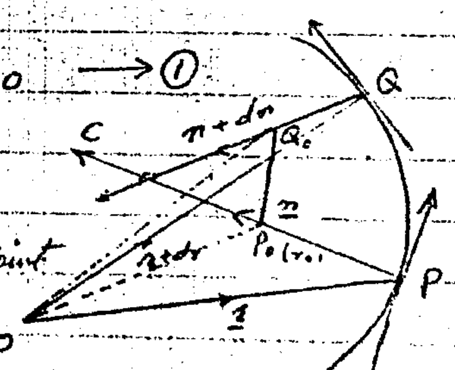
$$= \left(\frac{\tau+\kappa(0)}{\sqrt{\tau^2+\kappa^2}}\right)ds$$

$$S.D = \frac{1/\sigma\, ds}{\sqrt{1/\rho^2+1/\sigma^2}} = \frac{1}{\sigma}\frac{\sigma\rho\, ds}{\sqrt{\rho^2+\sigma^2}}$$

for (ii) Suppose the line of shortest distance meet the unit principal normal $\underline{n}$ at P_0 and $n+dn$ at Q_0, then the vectors $\vec{QP_0}$, $\vec{QQ_0}$ and $\vec{P_0Q_0}$ – ~~Then the vectors QP_0, QQ_0~~ are coplaner

then $[\vec{QP_0}\ \vec{QQ_0}\ \vec{P_0Q_0}] = 0 \longrightarrow ①$

(Scalar Triple product)



If C is the centre of curvature of the circle at point P of the curve, Then

To show that $\dfrac{CP_0}{P_0P} = \dfrac{\tau^2}{\kappa^2} = \dfrac{\rho^2}{\sigma^2}$ $\quad\left[\begin{array}{l}\tau = 1/\sigma\\ \kappa = 1/\rho\end{array}\right.$

Since $\vec{QQ_0}$ is $\parallel$ to $\underline{n}+d\underline{n}$ and the vector $\vec{P_0Q_0}$ is $\parallel$ to the vector $\perp$ to both $\underline{n}$ and $\underline{n}+d\underline{n}$ ie unit vector $\underline{e}$.

then If r_0 is p.v of P_0, Then $\vec{QP_0} = r_0-(\underline{r}+d\underline{r})$ and $\vec{QQ_0}$ is along $\underline{n}+d\underline{n}$ and $\vec{P_0Q_0}$ is along $(\tau\underline{t}+\kappa\underline{b})ds$

By putting values of $\parallel$ vectors in eq ①, we have

$$[r_0-(\underline{r}+d\underline{r})\quad \underline{n}+d\underline{n}\quad (\tau\underline{t}+\kappa\underline{b})ds] = 0 \longrightarrow ②$$

Now eq of Principal normal at P is

$$R = \underline{r} + U\underline{n}$$ & since $P_0(r_0)$ is on this line, therefore

$\underline{r}_0 = \underline{r} + U_0 \underline{n}$ where $U_0 = |P_0 P|$

Hence from eq ② is

$$[\cancel{\underline{r}} + U_0 \underline{n} - \cancel{\underline{r}} - d\underline{r} \quad \underline{n} + d\underline{n} \quad (\tau \underline{b} + \kappa \underline{b}) ds] = 0$$

$$[U_0 \underline{n} - d\underline{r} \quad \underline{n} + d\underline{n} \quad (\tau \underline{t} + \kappa \underline{b}) ds] = 0$$

or $$\left[U_0 \underline{n} - \frac{dr}{ds} ds \quad n + \frac{dn}{ds} ds \quad (\tau \underline{t} + \kappa \underline{b}) ds\right] = 0$$

or $$[U_0 \underline{n} - \underline{t}\, ds \quad \underline{n} + (\tau \underline{b} - \kappa \underline{t}) ds \quad (\tau \underline{t} + \kappa \underline{b}) ds] = 0$$

$$\begin{array}{ccc} \underline{t} & \underline{n} & \underline{b} \end{array}$$
$$\begin{vmatrix} -ds & U_0 & 0 \\ -\kappa ds & 1 & \tau ds \\ \tau ds & 0 & \kappa ds \end{vmatrix} = 0$$

$$(ds)^2 \begin{vmatrix} -1 & U_0 & 0 \\ -\kappa & 1 & \tau \\ \tau & 0 & \kappa \end{vmatrix} = 0 \Rightarrow \text{if } (ds)^2 \neq 0$$

then
$$\begin{vmatrix} -1 & U_0 & 0 \\ -\kappa & 1 & \tau \\ \tau & 0 & \kappa \end{vmatrix} = 0 \qquad -1(\kappa) - U_0(-\kappa^2 - \tau^2) = 0$$

$$\kappa = U_0(\kappa^2 + \tau^2)$$

$$U_0 = \frac{\kappa}{\kappa^2 + \tau^2}$$

So then $U_0 = P_0 P = \dfrac{\kappa}{\kappa^2 + \tau^2}$

Now from figure
$$\overrightarrow{CP_0} = \overrightarrow{CP} - \overrightarrow{P_0 P}$$
$$= \rho - U_0$$
$$= 1/\kappa - \frac{\kappa}{\kappa^2 + \tau^2}$$
$$= \frac{\kappa^2 + \tau^2 - \kappa^2}{\kappa(\kappa^2 + \tau^2)} = \frac{\tau^2}{\kappa(\kappa^2 + \tau^2)}$$

Hence
$$\frac{\overrightarrow{CP_0}}{\overrightarrow{P_0 P}} = \frac{\tau^2}{\kappa(\kappa^2 + \tau^2)} \Big/ U_0$$

$$= \frac{\tau^2}{\kappa(\kappa^2 + \tau^2)} \Big/ \frac{\kappa}{\kappa^2 + \tau^2} = \tau^2/\kappa^2 = \rho^2/\sigma^2$$

Hence the result

Q(9) Prove that $b'' = \tau(\kappa \underline{t} - \tau \underline{b}) - \tau' \underline{n}$

$$n'' = \tau' \underline{b} - (\kappa^2 + \tau^2)\underline{n} - \kappa' \underline{t}$$

and find similar expression for b''' and n'''.

Solution

Since $\underline{b}' = -\tau \underline{n}$

diff wrt s

$$\underline{b}'' = -\tau' \underline{n} - \tau n'$$
$$= -\tau' \underline{n} - \tau(\tau \underline{b} - \kappa \underline{t})$$
$$= \tau(\kappa \underline{t} - \tau \underline{b}) - \tau' \underline{n}$$

again Diff wrt s

$$b''' = \tau'(\kappa \underline{t} - \tau \underline{b}) + \tau(\kappa' \underline{t} + \kappa t' - \tau' \underline{b} - \tau b')$$
$$- \tau'' \underline{n} - \tau' n'$$
$$= \tau'(\kappa \underline{t} - \tau \underline{b}) + \tau[(\kappa' \underline{t} - \tau' \underline{b}) + \kappa(\tau \underline{b} - \kappa \underline{t}) - \tau(-\tau n)]$$
$$- \tau'' \underline{n} - \tau'(\tau \underline{b} - \kappa \underline{t})$$
$$= \tau' \kappa \underline{t} - \tau \tau' \underline{b} + \tau \kappa' \underline{t} - \tau \tau' \underline{b} + \kappa \tau^2 \underline{b} - \kappa^2 \tau \underline{t}$$
$$+ \cancel{\tau^2 \underline{b}} + \tau^2 \underline{n} - \tau'' \underline{n} - \tau \tau' \underline{b} + \kappa \tau' \underline{t}$$
$$= 2\tau' \kappa \underline{t} - 3\tau \tau' \underline{b} + \tau \kappa' \underline{t} + \kappa \tau^2 \underline{b} - \kappa^2 \tau \underline{t}$$
$$+ \tau^2 \underline{n} - \tau'' \underline{m}$$
$$= (2\kappa \tau' + \tau \kappa' - \kappa^2 \tau)\underline{t} + (\kappa \tau^2 - 3\tau \tau')\underline{b} + (\tau^2 - \tau'')\underline{n}$$

Again $n' = \tau \underline{b} - \kappa \underline{t}$

Diff wrt s, we have

$$n'' = \tau \underline{b}' + \tau' \underline{b} - \kappa' \underline{t} - \kappa \underline{t}'$$
$$= \tau' \underline{b} - \kappa' \underline{t} + \tau(-\tau \underline{n}) - \kappa(\kappa \underline{n})$$
$$= \tau' \underline{b} - \kappa' \underline{t} - \tau^2 \underline{n} - \kappa^2 \underline{n}$$
$$= -\kappa' \underline{t} + \tau' \underline{b} - (\tau^2 + \kappa^2)\underline{n}$$

Similarly for n'''

Q(10)

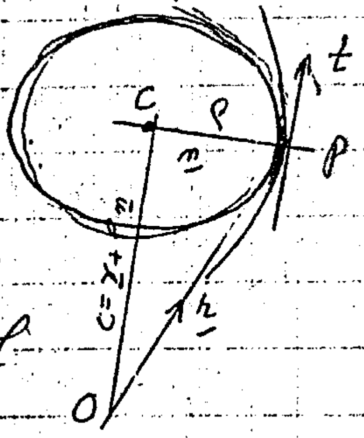
Def. **SKEW CURVATURE**

The arc rate of turning of the Principal normal $\underline{n}$ is called the skew curvature denoted as $\frac{dn}{ds} = n'$ and its magnitude is the modulus of n'.

Since $\frac{dn}{ds} = \tau \underline{b} - K \underline{t}$, the mag. of skew curvature is $\sqrt{\tau^2 + K^2} = |n'|$

Centre of ~~Curv~~ circle of curvature

The centre of curvature at P is the point of Xn of Principal normal at P with the normal at the consecutive pt P′ which lies in the osculating plane at P.



Let $\underline{c}$ be p.v. of centre of curvature and $\underline{r}$ be p.v. of P w.r.t O. Then

$$\underline{C} = \underline{r} + \rho \underline{n}$$ where ρ is the radius of curvature. The tangent to its locus being parallel to $\frac{dc}{ds}$, is therefore parallel to

$$\frac{dc}{ds} = c' = \frac{dr}{ds} + \rho \frac{dn}{ds} + \rho' \underline{n}$$

$$= \underline{t} + \rho(\tau \underline{b} - K \underline{t}) + \rho' \underline{n}$$

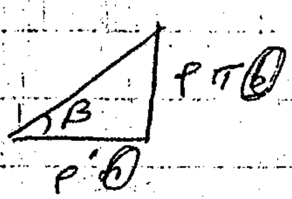
$$= \cancel{\underline{t}} + \rho \tau \underline{b} - \cancel{\underline{t}} + \rho' \underline{n}$$

$$= \rho' \underline{n} + \rho \tau \underline{b}$$

$\frac{K}{1} = 1/\rho$

$\rho K = 1$

It is therefore lies in the normal plane of the original curve. Tangent is inclined to $\underline{n}$ at an angle β s.t. $\tan\beta = \frac{\rho \tau b}{\rho' b}$



$$\tan\beta = \frac{\tau}{\rho' \sigma}$$

Q(a) If the position vector $\underline{r}$ of the current point is a function of any parameter u and dashes denotes Diff w.r.t u then show that

(i) $r' = s't$ (ii) $r'' = s''t + \kappa s'^2 \underline{n}$ and

(iii) $r''' = (s''' - \kappa^2 s'^3)\underline{t} + (3\kappa s'' + \kappa' s')\underline{n} + (\kappa\tau s'^3)\underline{b}$

(iv) $\underline{b} = \dfrac{r' \times r''}{\kappa (s')^3}$ (v) $\kappa^2 = \dfrac{r''^2 - s''^2}{s'^4}$ & $\tau = \dfrac{[r'\, r''\, r''']}{\kappa^2 s'^6}$

Solution (i) $r' = \dfrac{dr}{du} = \dfrac{dr}{ds}\cdot\dfrac{ds}{du} = \underline{t}\, s'$

(ii) $r'' = \dfrac{d}{du}(t s')$

$= \cancel{\dfrac{d}{ds}(t s')\dfrac{ds}{du}}$ $\quad \dfrac{d}{du}(\underline{t}\, s') = \dfrac{d}{ds}\left(t \dfrac{ds}{dt}(s')\right) + \underline{t}\, s''$

$= t'(s')^2 + \underline{t}\, s''$

$= \kappa \underline{n}(s')^2 + \underline{t}\, s''$

and (iii)

$$r''' = \frac{d}{du}\left(\underline{t}\, s'' + \kappa \underline{n} (s')^2\right)$$

$$= s'''\underline{t} + s''\frac{dt}{du} + \kappa'(s')^2\underline{n} + \kappa\, 2(s')s''\underline{n} + \kappa(s')^2\frac{dn}{du}$$

$$= s'''\underline{t} + s''\, t'\frac{ds}{du} + \kappa'(s')^2\underline{n} + 2\kappa s's''\underline{n} + \kappa s''\frac{dn}{ds}\cdot\frac{ds}{du}$$

$$= s'''\underline{t} + s's''t' + \kappa'(s')^2\underline{n} + 2\kappa s's''\underline{n} + \kappa(s')^3 n'$$

$$= s'''\underline{t} + s's''(\kappa n) + \kappa'(s')^2 n + 2\kappa s's''\underline{n} + \kappa(s')^3(\tau b - \kappa \underline{t})$$

$$= (s''' - \kappa^2 s'^3)\,\underline{t} + s'(3\kappa s'' + \kappa' s')\underline{n} + \kappa\tau s'^3\underline{b}$$

To find $r' \times r'' = s'\underline{t} \times (s''\underline{t} + \kappa s'^2 \underline{n})$

$$= 0 + s'\kappa s'^2(\underline{t} \times \underline{n}) = \kappa(s')^3\underline{b}$$

$$\underline{b} = \frac{r' \times r''}{\kappa(s')^3} \quad \text{(proved)}$$

To prove $s'r'' - s''r' = s'(s''t + \kappa s'^2 \underline{n}) - s''s'\underline{t}$

$$= \kappa(s')^3\underline{n}$$

$$\underline{n} = \frac{s'r'' - s''r'}{\kappa s'^3}$$ Similarly for $\kappa^2 = \dfrac{r''^2 - s''^2}{s'^4}$

& others $\quad$ & $\tau = \dfrac{[r'\, r''\, r''']}{\kappa^2 s'^6}$

Exercise 11 For the curve

$$x = 4a\cos^3 u \;,\; y = 4a\sin^3 u \;,\; Z = 3c\cos 2u$$

Prove that $\underline{n} = (\sin u \;\cos u \; 0)$

& $\quad \underline{K} = \dfrac{a}{6(a^2+c^2)\sin 2u}$

Sol

Let $\underline{r} = (4a\cos^3 u \,,\, 4a\sin^3 u \,,\, 3c\cos 2u)$

$$\frac{dr}{ds} = (-12a\cos^2 u\sin u \,,\, 12a\sin^2 u\cos u \;\; -6c\sin u)\frac{du}{ds}$$

$$\underline{t} = -6\sin 2u\,(-a\cos u,\; a\sin u,\; -c)\frac{du}{ds} \quad ①$$

$$\underline{t}\cdot\underline{t} = 36\sin^2 2u\,(a^2\cos^2 u + a^2\sin^2 u + c^2)\left(\frac{du}{ds}\right)^2$$

$$1 = 36\sin^2 2u\,(a^2+c^2)\left(\frac{du}{ds}\right)^2$$

$$\left(\frac{du}{ds}\right)^2 = \frac{1}{36\sin^2 2u\,(a^2+c^2)} \Rightarrow \frac{du}{ds} = \frac{1}{6\sqrt{a^2+c^2}\sin 2u} \rightarrow ②$$

Putting in ①

$$\underline{t} = \frac{-6\sin 2u\,(-a\cos u,\; a\sin u,\; -c)}{6\sqrt{a^2+c^2}\sin 2u}$$

$$\underline{t} = \frac{1}{\sqrt{a^2+c^2}}(-a\cos u,\; a\sin u,\; -c)$$

again diff w.r.t S

$$\frac{dt}{ds} = \frac{1}{\sqrt{a^2+c^2}}(a\sin u \;\; a\cos u \;\; 0)\frac{du}{ds}$$

$$K\underline{n} = \frac{1}{\sqrt{a^2+c^2}}\;\frac{1}{6\sqrt{a^2+c^2}\sin 2u}(a\sin u,\, a\cos u,\, 0) \longrightarrow ③$$

$$K\underline{n} = \frac{(a\sin u,\, a\cos u,\, 0)}{6(a^2+c^2)\sin 2u}$$

$$\Rightarrow K\underline{n}\cdot K\underline{n} = \frac{a^2\sin^2 u + a^2\cos^2 u}{36(a^2+c^2)^2\sin^2 2u}$$

$$K^2 = \frac{a^2}{36(a^2+c^2)^2\sin^2 2u} \qquad K = \frac{a}{6\sin 2u\,(a^2+c^2)}$$

Now from eq (3), we have

$$\underline{n} = \frac{1}{K} \frac{1}{6 \sin 2u (a^2+c^2)} (a \sin u, a \cos u, 0)$$

$$= \frac{6 \sin 2u (a^2+c^2)}{a\, 6 \sin 2u (a^2+c^2)} (a \sin u, a \cos u, 0)$$

$$\underline{n} = (\sin u, \cos u, 0) \quad (\cancel{\text{good}})$$

03/98 Q(12) Find K and T $\quad x = a(u - \sin u)$

$y = a(1-\cos u)$

Sol. $\quad z = bu$

Let $\underline{r} = (a(u-\sin u), a(1-\cos u), bu)$

Diff w.r.t u

$$\dot{r} = (a(1-\cos u), a \sin u, b) \quad \text{—— (i)}$$

$$|\dot{r}| = \sqrt{a^2(1-\cos u)^2 + a^2 \sin^2 u + b^2}$$

$$= \sqrt{b^2 + 2a^2(1-\cos u)}$$

Again Diff (i) w.r.t u

$$\ddot{r} = (a \sin u, a \cos u, 0) \quad \text{—— (ii)}$$

Then $\dot{r} \times \ddot{r} = \begin{vmatrix} i & j & k \\ a(1-\cos u) & a \sin u & b \\ a \sin u & a \cos u & 0 \end{vmatrix} = i(0 - ab\cos u) - j(ab \sin u) + k(a^2(\cos u - \cos^2 u) - a^2 \sin^2 u)$

$$= (-ab \cos u, ab \sin u, a^2(\cos u - 1))$$

$$|\dot{r} \times \ddot{r}| = \sqrt{a^2b^2\cos^2 u + a^2b^2 \sin^2 u + a^4(\cos u - 1)^2}$$

$$= a\sqrt{b^2 + a^2(\cos u - 1)^2}$$

Diff eq (ii) w.r.t u

$$\dddot{r} = (a \cos u, -a \sin u, 0)$$

$$K = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{a\sqrt{b^2+a^2(\cos u-1)^2}}{[b^2+2a^2(1-\cos u)]^{3/2}}$$

Also $[\dot{r}\ \ddot{r}\ \dddot{r}] = -a^2 b$

$$T = \frac{-a^2 b}{a^2[b^2+a^2(\cos u-1)^2]}$$

when $T = \dfrac{[\dot{r} \cdot \ddot{r} \times \dddot{r}]}{(\dot{r} \times \ddot{r})^2}$

Q(13) If the plane of curvature at every point of a curve passes through a fixed pt, show that curve is plane.

Solution: The equation of plane of curvature (Osculating plane) at a point P with p.v. $\underline{r}$ is given as $(\underline{R}-\underline{r})\cdot\underline{b}=0$

Let R_0 be a fixed pt, then R_0 satisfies the above eq

$$(R_0-\underline{r})\cdot\underline{b}=0 \quad \text{---} (1)$$

Diff w.r.t s

$$\left(0-\frac{dr}{ds}\right)\cdot\underline{b}+(R_0-r)\cdot\frac{db}{ds}=0$$

$$-\underline{r}'\cdot\underline{b}+(\underline{R_0}-\underline{r})\cdot b'=0 \qquad \underline{t}\cdot\underline{b}=0$$

$$-\underline{t}\cdot\underline{b}+(\underline{R_0}-\underline{r})\cdot(-\tau\,\underline{n})=0$$

$$0\ \cancel{+\{R_0\cdot(-\tau\,\underline{n})\}+\underline{r}\cdot\tau\,\underline{n}}=0$$

$$-\tau(R_0-\underline{r})\cdot\underline{n}=0$$

If $\tau\neq 0$ then $(\underline{R_0}-\underline{r})\cdot\underline{n}=0$

$\Rightarrow$ $\underline{R_0}-\underline{r}$ is $\perp$ to $\underline{n}$ also from equation (1) $(\underline{R_0}-\underline{r})$ is $\perp$ to $\underline{b}$ — (11)

From these results, we conclude that $(R_0-\underline{r})$ is $\parallel$ to $\underline{t}$

Therefore, $(\underline{R_0}-\underline{r})=\lambda\,\underline{t}$ $\quad \lambda$ is real no.

then $\boxed{\underline{R_0}=\underline{r}+\lambda\underline{t}}$ This is eq of Tangent.

Hence R_0 satisfies eq of Tangent at every pt.

$\Rightarrow$ $\underline{R_0}$ is the point of int of all Tangents to the curve $\Rightarrow$ curve is a straight line

This is contradiction to our assumption that the curve is not plane

Hence the curve is a plane.

Q(14) If m_1, m_2, m_3 are the moments about the origin of unit vectors $\underline{t}, \underline{n}, \underline{b}$ localized in the Tangent, normal & binormal and dashes denotes diff w.r.t s we have $m_1' = K m_2$, $m_2' = b - K m_1 + T m_3$, $m_3' = -n - T m_2$

Solution If $\underline{r}$ is a current point, then by definition of moment of forces about a pt

$$m_1 = \underline{r} \times \underline{t}, \quad m_2 = \underline{r} \times \underline{n} \quad \& \quad m_3 = \underline{r} \times \underline{b}$$

Diff $m_1 = \underline{r} \times \underline{t}$ w.r.t s

$$m_1' = \underline{r}' \times \underline{t} + \underline{r} \times \underline{t}'$$

$$= \underline{t} \times \underline{t} + \underline{r} \times (K \underline{n})$$

$$m_1' = 0 + K(\underline{r} \times \underline{n}) = K m_2$$

Diff $m_2 = \underline{r} \times \underline{n}$ w.r.t s

$$m_2' = \underline{r}' \times \underline{n} + \underline{r} \times \underline{n}'$$

$$= \underline{t} \times \underline{n} + \underline{r} \times (T b - K \underline{t})$$

$$= \underline{b} + T(\underline{r} \times \underline{b}) - K(\underline{r} \times \underline{t})$$

$$= \underline{b} + T m_3 - K m_1$$

Diff $m_3 = \underline{r} \times \underline{b}$ w.r.t s

$$m_3' = \underline{r}' \times \underline{b} + \underline{r} \times \underline{b}'$$

$$= \underline{t} \times \underline{b} + \underline{r} \times (-T \underline{n})$$

$$= -\underline{n} - \underline{r} \times T \underline{n}$$

$$m_3' = -[\underline{n} + T m_2]$$

Q(15) Prove that the position vector of the current point on a curve satisfies the Diff Eq

$$\frac{d}{ds}\left\{\sigma \frac{d}{ds}\left(\rho \frac{d^2 r}{ds^2}\right)\right\} + \frac{d}{ds}\left(\frac{\sigma}{\rho} \frac{dr}{ds}\right) + \frac{\rho}{\sigma} \frac{d^2 r}{ds^2} = 0$$

Hint (Use Seret Frenet Formulae)

Sol Since $\sigma = 1/T$ & $\rho = 1/K$

LHS. $$\frac{d}{ds}\left\{1/T \frac{d}{ds}\left(1/K \frac{dt}{ds}\right)\right\} + \frac{d}{ds}\left[\frac{K}{T} \underline{t}\right] + \frac{T}{K} t' = 0$$

$$\Rightarrow \frac{d}{ds}\left\{ \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{K}K\underline{n}\right)\right\} + \frac{d}{ds}\left(\frac{K}{\tau}\underline{t}\right) + \frac{\tau}{K}(K\underline{n})$$

$$\Rightarrow \frac{d}{ds}\left\{ \frac{1}{\tau}\left(\frac{dn}{ds}\right)\right\} + \frac{d}{ds}\left(\frac{K}{\tau}\underline{t}\right) + \tau\underline{n}$$

$$= \frac{d}{ds}\left(\frac{1}{\tau}(\tau\underline{b} - K\underline{t})\right) + \frac{d}{ds}\left(\frac{K}{\tau}\underline{t}\right) + \tau\underline{n}$$

$$\Rightarrow \frac{db}{ds} - \frac{K}{\tau}\frac{dt}{ds} + \frac{K}{\tau}\frac{dt}{ds} + \tau\underline{n}$$

$$\Rightarrow -\tau\underline{n} - \cancel{\frac{K^2\underline{n}}{\tau}} + \cancel{\frac{K}{\tau}Kn} + \tau\underline{n}$$

$$= -\tau\underline{n} + \tau\underline{n} = 0 \quad \text{R.H.S.}$$

Q(16)) If s_1 is the arc length of the locus of centre of curvature, show that $\frac{ds_1}{ds} = \frac{1}{K^2}\sqrt{K^2\tau^2 + K'^2}$
$= \sqrt{\left(\left(\frac{\rho}{\sigma}\right)^2 + \rho'^2\right)}$

Solution

Since $\underline{t} = \frac{dr}{ds}$, $\underline{b}$, n are Tangent binormal & normal for the curve C, Similarly $t_1 = \frac{dc}{ds_1}$, b_1, n_1 are Tangent, binormal and normal for the curve formed by the locus of centre of curvature.

The Eq of locus of centre of curvature is

$$\underline{C} = \underline{r} + \rho\underline{n} \longrightarrow ①$$

Diff wrt s

$$\frac{dc}{ds} = \frac{dr}{ds} + \rho\frac{dn}{ds} + \frac{d\rho}{ds}\underline{n}$$

$$\frac{dc}{ds_1}\cdot\frac{ds_1}{ds} = \underline{t} + \rho(\tau\underline{b} - K\underline{t}) + \rho'\underline{n} \qquad \rho = 1/K$$

$$t_1\frac{ds_1}{ds} = \underline{t} + \rho\tau\underline{b} - \frac{K}{K}\underline{t} + \rho'\underline{n}$$

$$t_1\frac{ds_1}{ds} = \rho\tau\underline{b} + \rho'\underline{n} \longrightarrow ②$$

Taking dot product of ② with itself

$$\left(\frac{ds_1}{ds}\right)^2(t_1\cdot t_1) = \rho^2\tau^2(\underline{b}\cdot\underline{b}) + (\rho')^2\,\underline{n}\cdot\underline{n}$$

$$\left(\frac{ds_1}{ds}\right)^2 = \rho^2\tau^2 + \rho'^2$$

$$\frac{ds_1}{ds} = \sqrt{\frac{\rho^2}{\sigma^2} + \rho'^2}$$

let $\rho = 1/\kappa$

$$\frac{d\rho}{ds} = -1/\kappa^2 \, \kappa'$$

$$(\rho')^2 = \frac{1}{\kappa^4}(\kappa')^2 \quad \text{Hence } \frac{ds_1}{ds} = \sqrt{\frac{\rho^2}{\sigma^2} + \frac{\kappa'^2}{\kappa^4}}$$

$$\frac{ds_1}{ds} = \frac{1}{\kappa^2}\sqrt{\tau^2\kappa^2 + \kappa'^2}$$

Q(17) In the case of a curve of constant curvature Find the curvature and Torsion of the locus of its centre of curvature C.

<u>Sol</u> The p.r of $\underline{C}$ on the curve of centre of curvature is

$$\underline{C} = \underline{r} + \rho\,\underline{n} \qquad \text{Hence } \rho \text{ is constant.}$$

Diff wrt s

$$d\underline{c} = [\underline{t} + \rho(\tau\underline{b} - \kappa\underline{t})]\,ds$$

$$= (\cancel{\underline{t}} + \rho\tau\underline{b} - \cancel{\tfrac{\kappa}{\kappa}\underline{t}})\,ds$$

$$d\underline{c} = (\rho\tau\underline{b})\,ds = \tau/\kappa\,\underline{b}\,ds$$

or $$\frac{d\underline{c}}{ds} = \tau/\kappa\,\underline{b}$$

$$\frac{d\underline{c}}{ds_1}\cdot\frac{ds_1}{ds} \Rightarrow t_1\,\frac{ds_1}{ds} = \tau/\kappa\,\underline{b} \longrightarrow (1)$$

Taking dot product of (1) with itself.

$$\underline{t}_1\cdot\underline{t}_1\left(\frac{ds_1}{ds}\right)^2 = \frac{\tau^2}{\kappa^2}(\underline{b}\cdot\underline{b})$$

$$\left(\frac{ds_1}{ds}\right)^2 = \frac{\tau^2}{\kappa^2} = \rho^2\tau^2$$

$$\frac{ds_1}{ds} = \rho\tau \longrightarrow (2)$$

From (1) $\quad t_1 = \underline{b}$

Diff last relation w.r.t. s

$$\frac{dt_1}{ds} = \frac{d\underline{b}}{ds} = -\tau\,\underline{n}$$

$$\frac{dt_1}{ds} = \frac{dt_1}{ds_1}\cdot\frac{ds_1}{ds} = -\tau \underline{n} \qquad b' = -\tau \underline{n}$$

$$t_1' = -\kappa_1 n_1$$

$$(\kappa_1 \underline{n}_1)(\rho\,\tau) = -\tau \underline{n} \quad \text{from (ii)}$$

$$\kappa_1 \underline{n}_1 = -\frac{\underline{n}}{\rho}$$

$$\kappa_1 \underline{n}_1 = -\kappa \underline{n} \Rightarrow \boxed{\kappa_1 = \kappa} \quad \text{(iii)}$$

$$\& \quad \underline{n}_1 = -\underline{n} \quad \text{(iv)}$$

For Torsion, $b_1 = t_1 \times n_1$

$$= \underline{b} \times n_1 \qquad \because t_1 = \underline{b}$$

$$= \underline{b} \times (-\underline{n}) = \underline{n} \times \underline{b}$$

$$b_1 = \underline{t}$$

Diff w.r.t. s

$$\frac{db_1}{ds} = \frac{dt}{ds}$$

$$\Rightarrow \frac{db_1}{ds_1}\cdot\frac{ds_1}{ds} = +\kappa \underline{n}$$

$$-\tau_1 n_1 \frac{ds_1}{ds} = +\kappa \underline{n} \qquad \because \frac{ds_1}{ds} = \tau/\kappa$$

$$-\tau_1 n_1 \,\tau/\kappa = +\kappa \underline{n} \qquad -n_1 = n$$

$$\tau_1 \underline{n}\, \tau/\kappa = \kappa \underline{n} \qquad \boxed{\tau_1 = \frac{\kappa^2}{\tau}}$$

6 (a)

Q (18) Prove that for any curve

$$[t'\ t''\ t'''] = [r''\ r'''\ r'''']$$

$$= \kappa^3(\kappa\tau' - \kappa'\tau) = \kappa^5 \frac{d}{ds}(\tau/\kappa)$$

$$\& \ [b'\ b''\ b'''] = \tau^3(\kappa'\tau - \kappa\tau') = \tau^5 \frac{d}{ds}(\kappa/\tau)$$

Solution: Since $\frac{dr}{ds} = \underline{t} \Rightarrow r' = \underline{t}$

Diff w.r.t. s

$$r'' = t'$$

& again diff $r''' = t''$ & again diff $r^{iv} = t'''$

So we have $[t'\ t''\ t'''] = [r''\ r'''\ r'''']$

Q 6(a)

Also, since $t' = K\underline{n}$

$$t'' = K'\underline{n} + K\underline{n}'$$
$$= K'\underline{n} + K(T\underline{b} - K\underline{t})$$
$$= K'\underline{n} + KT\underline{b} - K^2\underline{t}$$

Diff w.r.t s again

$$t''' = K''n + K'n' + K'T\underline{b} + KT'\underline{b} + KT\underline{b}'$$
$$- 2KK't - K^2t'$$
$$= K''\underline{n} + K'(T\underline{b} - K\underline{t}) + K'T\underline{b} + KT'\underline{b} + KT(-T\underline{n})$$
$$- 2KK'\underline{t} - K^2(-K\underline{n})$$
$$= K''\underline{n} + K'T\underline{b} - KK'\underline{t} + K'T\underline{b} + KT'\underline{b} - KT^2\underline{n}$$
$$- 2KK't + K^3\underline{n}$$
$$= (K'' - KT^2 + K^3)\underline{n} - 3KK'\underline{t} + (2K'T + KT')\underline{b}$$

Then, we have

$$[t', t'', t'''] = \begin{vmatrix} 0 & K & 0 \\ -K^2 & K' & KT \\ -3KK' & K'' - KT^2 + K^3 & 2K'T + KT' \end{vmatrix}$$
$$= -K\left(-K^2(2K'T + KT') - KT(-3KK')\right)$$
$$= -K\left(K^2K'T - K^3T'\right)$$
$$= K^4T' - K^3K'T = K^3[KT' - K'T] \quad QED$$

Also

$$[t'\, t''\, t'''] = K^3K^2\left[\frac{KT' - K'T}{K^2}\right] = K^5\frac{d}{ds}\left(\frac{T}{K}\right)$$

(ii) Since $\underline{b}' = -T\underline{n}$

Diff w.r.t s

$$b'' = -T'n - Tn'$$
$$= -T'n - T(T\underline{b} - K\underline{t})$$
$$= -T'n - T^2\underline{b} + KT\underline{t}$$

again Diff w.r.t s

$$b''' = -\tau'' n - \tau' n' - 2\tau \underline{b}\, \tau' - \tau^2 \underline{b}' + k'\tau \underline{t} + k\tau' \underline{t} + k\tau \underline{t}'$$

$$= -\tau'' \underline{n} - \tau'(\tau \underline{b} - k \underline{t}) - 2\tau\tau' \underline{b} - \tau^2(-\tau \underline{n})$$
$$+ k'\tau \underline{t} + k\tau' \underline{t} + k\tau k \underline{n}$$
$$= (2k\tau' + k'\tau)\underline{t} + (k^2\tau - \tau'' + \tau^3)\underline{n} - 3\tau\tau' \underline{b}$$

Therefore, we have

$$[b'\ b''\ b'''] = \begin{vmatrix} 0 & -\tau & 0 \\ k\tau & -\tau' & -\tau^2 \\ 2k\tau' + k'\tau & k^2\tau - \tau'' + \tau^3 & -3\tau\tau' \end{vmatrix}$$

$$= \tau\left[-3k\tau^2\tau' + \tau^2(2k\tau' + k'\tau)\right]$$

$$= \tau\left[-3k\tau^2\tau' + 2k\tau^2\tau' + \tau^3 k'\right]$$

$$= \tau^3(k'\tau - k\tau')$$

QED

This can also be written as

$$[b'\ b''\ b'''] = \tau^3\tau^2\left(\frac{k'\tau - \tau' k}{\tau^2}\right)$$

$$= \tau^5 \frac{d}{ds}\left(k/\tau\right) \qquad \text{(Proved)}$$

§ SPHERICAL CURVATURE. 13/3

The sphere of closest contact with the curve at P is that which passes through four points on the curve ultimately coincident with P. This is called the Osculating sphere or the sphere of curvature at P.

Its centre S and radius R are called the centre and radius of spherical curvature.

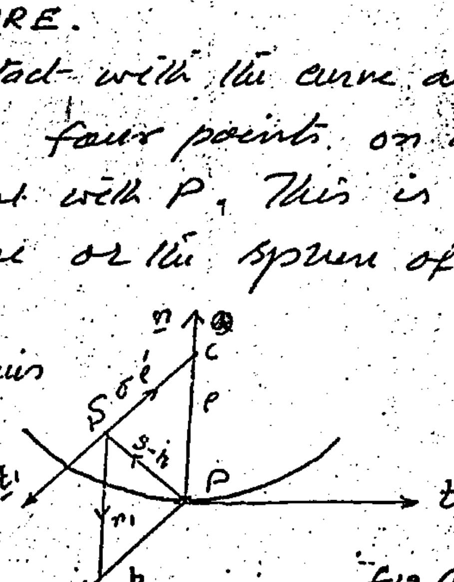


fig ①

1 4 3
~~1~~ 8 6
~~7~~, 8, ~~6~~
9 10 ~~11~~
12 13 ~~14~~
~~15~~ ~~16~~ 17
18 ~~19~~ 20
21 ~~22~~ 23
24 25
26 ~~27~~
28 ~~29~~
36 ~~32~~

<u>Theorem</u>

To derive an expression for the radius of spherical curvature.

Let $\underline{r}$ be the p.v. of P on the curve and $\underline{s}$ is the p.v. of centre of the sperical curvature.

Then the centre of sphere through P and an adjascent point Q on the curve lies on the plane which is the right bisector of PQ and limiting position of this plane is the normal plane at P.

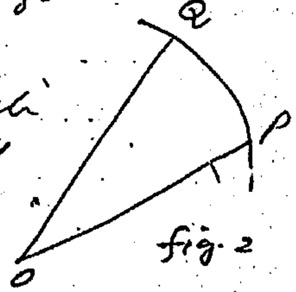


fig. 2

Thus the centre of spherical curvature is the limiting position of the Intersection of three normal planes at adjascent pts. Now eqn of normal plane at point $P(\underline{r})$ is

$$(\underline{S}-\underline{r})\cdot\underline{t} = 0 \longrightarrow ①$$

Where $\underline{S}$ is current pt. on the plane

Diff ① w.r.t s (arc length)

$$(\underline{S}-\underline{r})\cdot\frac{d\underline{t}}{ds} + \frac{d\underline{S}}{ds}\cdot\underline{t} - \frac{d\underline{r}}{ds}\cdot\underline{t} = 0$$

$$(\underline{S}-\underline{r})\kappa\underline{n} + \frac{d\underline{S}}{ds}\cdot\underline{t} - \underline{t}\cdot\underline{t} = 0$$

$$(\underline{S}-\underline{r})\cdot K\vec{n} + \underbrace{\frac{d\underline{s}}{ds}\cdot\underline{t}}_{=0} - 1 = 0$$

As $\underline{s}$ is along $\underline{n}$, $\frac{d\underline{s}}{ds}$ is a unit vector $\perp$ to $\underline{n}$, $\frac{d\underline{s}}{ds}\cdot t = 0$

$$(\underline{S}-\underline{r})\cdot\vec{n} = 1/K$$

$$(\underline{S}-\underline{r})\cdot\underline{n} = \rho \quad \text{---} ②$$

Diff w r t s

$$(\underline{S}-\underline{r})\cdot\frac{dn}{ds} + \left(\frac{d\vec{s}}{ds} - \frac{dr}{ds}\right)\cdot\vec{n} = \rho'$$

$$(\underline{S}-\underline{r})\cdot(\tau\underline{b} - K\underline{t}) + \frac{ds}{ds}\cdot\vec{n} - \frac{dr}{ds}\cdot\vec{n} = \rho'$$

$$(\underline{S}-\underline{r})\cdot(\tau\underline{b} - K\underline{t}) + \frac{ds}{ds}\cdot\vec{n} - \underline{t}\cdot\vec{n} = \rho'$$

$$(\underline{S}-\underline{r})\cdot\tau\underline{b} - (\underline{S}-\underline{r})K\underline{t} + 0 - 0 = \rho'$$

$$\Rightarrow \tau(S-\underline{r})\cdot\underline{b} - K(\underline{S}-\underline{r})\,\underline{t} = \rho'$$

$$\tau(\underline{S}-\underline{r})\cdot\underline{b} - K(0) = \rho' \quad \text{by eq ①}$$

$$(S-\underline{r})\cdot\underline{b} = \rho'/\tau = \sigma\rho' \quad \text{---} ③$$

The vector $\underline{S}-\underline{r}$ satisfies ① ② + ③, then it is clear that

$$\underline{S}-\underline{r} = \rho\,\underline{n} + \sigma\rho'\underline{b} \quad \text{---} ④$$

$$\underline{S} = \underline{r} + \rho\underline{n} + \sigma\rho'\underline{b}$$

This eq determines the p.v $\underline{S}$ of the centre of spherical curvature. Now $\rho\underline{n}$ is the vector $\vec{PC}$ and therefore, $\sigma\rho'\underline{b}$ is the vector $\vec{CS}$. Therefore, the centre of spherical curvature is on the axis of the circle of curvature at a distance $\sigma\rho'$ from the centre of curvature.

To find the radius of spherical curvature, take square of both sides of ④

$$(\underline{S}-\underline{r})^2 = \rho^2 + (\sigma\rho')^2$$

$$\underline{R}^2 = \rho^2 + \sigma^2 \rho'^2 \qquad \text{as } \underline{S} - \underline{r} = \underline{R}$$

$$\text{or } |R| = \sqrt{\rho^2 + \sigma^2 \rho'^2} \longrightarrow (5)$$

<u>Remark</u> For the curve of constant curvature, $\rho' = 0$, therefore, (5) becomes $R = \rho$

Centre of spherical curvature coincides with the centre of circular curvature

§ <u>Locus of Centre of Spherical Curvature:</u>—

The position vector $\underline{S}$ of the centre of sp. curvature has been shown to be

$$\underline{S} = \underline{r} + \rho\,\underline{n} + \sigma\rho'\,\underline{b} \longrightarrow (i)$$

Hence for a small displacement ds of the current point P along the original curve C, the displacement of $\underline{S}$ is $\quad \begin{matrix} \rho K = 1 \\ \sigma\tau = \end{matrix}$

$$\frac{d\underline{S}}{ds} = \underline{t} + \rho'\,\underline{n} + \rho(\tau\underline{b} - K\underline{t}) + \sigma'\rho'\underline{b} + \sigma\rho''\underline{b} + \sigma\rho'(-\tau\underline{n})$$

$$d\underline{S} = (\cancel{\underline{t}} + \cancel{\rho'\underline{n}} + \rho\tau\underline{b} - \cancel{\underline{t}} + \rho'\sigma'\underline{b} + \sigma\rho''\underline{b} - \cancel{\rho'\underline{n}})\,ds$$

$$d\underline{S} = (\rho\tau\underline{b} + \rho'\sigma'\underline{b} + \sigma\rho''\underline{b})\,ds$$

$$= ds\left(\frac{\rho}{\sigma}\underline{b} + \rho'\sigma'\underline{b} + \sigma\rho''\underline{b}\right)$$

$$= ds\left(\frac{\rho}{\sigma} + \rho'\sigma' + \sigma\rho''\right)\underline{b}$$

Thus the Tangent to locus of $\underline{S}$ is $\parallel$ to $\underline{b}$. (fig (11)), we may measure the arc-length s_1 of the locus $\underline{S}$ in that direction which makes its unit Tangent t_1 share the same direction as $\underline{b}$

Thus $t_1 = \underline{b}$, $\quad$ & since $\dfrac{d\underline{S}}{ds_1} = t_1$, $d s_1$

it follows that

$$\frac{ds_1}{ds} = \frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')$$

To find the curvature K_1 of the locus S_1 diff the eq $\underline{t}_1 = \underline{b}$ w r t s_1

$$\frac{d}{ds_1}(\underline{t}_1) = \frac{d}{ds_1}(\underline{b})$$

$$t_1' = K_1 \underline{n}_1 = \frac{db}{ds} \cdot \frac{ds}{ds_1} = -\tau \underline{n} \frac{ds}{ds_1}$$

$\Rightarrow$

Thus the Principal normal to the locus of $\underline{S}$ is parallel to the principal normal of the original curve. ~~(One~~ We may chose the direction of n_1 as opposite to that of $\underline{n}$. Thus

$$\underline{n}_1 = -\underline{n}$$

The unit binormal $\underline{b}_1$ of the locus S is then

$$b_1 = t_1 \times n_1 = \underline{b} \times (-\underline{n}) = t_1$$

and is thus equal to the unit Tangent of the original curve and the curvature

$$K_1 = \tau \frac{ds}{ds_1}.$$

Again $\quad \underline{t}_1 = \underline{b}$ & $\underline{n}_1 = -\underline{n}$.

$$\underline{t}_1 \times \underline{n}_1 = -\underline{b} \times \underline{n}$$

$$\underline{b}_1 = \underline{t}$$

$\Rightarrow$ Binormal of C_1 is $\parallel$ to the Tangent of C.

The curvature K_1 as found above is thus equal to

$$K_1 = \tau \frac{ds}{ds_1}.$$

The Torsion τ_1 is obtained by diff $b_1 = \underline{t}$

$$\frac{d}{ds_1}(b_1) = \frac{dt}{ds_1}$$

$$\frac{d}{ds_1}(b_1) = \frac{dt}{ds} \cdot \frac{ds}{ds_1}$$

$-n_1 = n$

$$-\tau_1 n_1 = \kappa n \frac{ds}{ds_1}$$

$$\tau_1 n = \kappa n \frac{ds}{ds_1} \Rightarrow \tau_1 = \kappa \frac{ds}{ds_1}$$

From the last two Results it follows $KK_1 = \tau\tau_1$

* Example: Prove that for curves drawn on the Surface of a Sphere (or for spherical curve), we have $\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma \rho') = 0$ or $\rho/\sigma + \sigma'\rho' + \sigma\rho'' = 0$

Solution :-

For curves drawn on the surface of a sphere, the osculating sphere at every pt. of the curve is the same sphere on the surface of which it is drawn. ——— already done — as locus of centre of spherical curvature.

✓ Example ②: If the Radius of a spherical curvature is Constant. Prove that the curve either lies on the surface of a sphere or else has a constant curvature.

Solution: Let R be the Radius of spherical curvature then

$$R^2 = \rho^2 + (\sigma\rho')^2 \quad \text{——} ① \quad (R^2 \text{ is const})$$

Diff w.r.t s

$$0 = 2\rho\rho' + 2(\sigma\rho')\left[\frac{d}{ds}(\sigma\rho')\right]$$

$$0 = 2\rho'\left(\rho + \sigma\frac{d}{ds}(\sigma\rho')\right) = 0$$

Then either $\rho' = 0 \Rightarrow \rho$ is constant

The curve has a constant curvature

or if $\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0$

then locus of the centre of sp curvature δ is given by $\delta = \underline{r} + \rho\underline{n} + \sigma\rho'\underline{b}$ (∵ eq ④)

Diff w.r.t s: $\frac{d\delta}{ds} = \underline{r}' + \rho\underline{n}' + \rho'\underline{n} + \sigma'\rho'\underline{b}$

$$+ \sigma\rho''\underline{b} + \sigma\rho'\underline{b}'$$

$$\frac{d\delta}{ds} = \underline{t} + \rho(\tau\underline{b} - \kappa\underline{t}) + \rho'\underline{n} + \sigma'\rho'\underline{b}$$

$$+ \sigma\rho''\underline{b} - \sigma\rho'(\tau\underline{n})$$

$$= \underline{t} + \rho\tau\underline{b} - \underline{t} + \rho'\underline{n} + \sigma'\rho'\underline{b} + \sigma\rho''\underline{b} - \rho'\underline{n}$$

$$\frac{d\delta}{ds} = \left[\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right]\underline{b}$$

Hence, if $\rho + \sigma \frac{d}{ds}(\sigma \rho') = 0$

or $\rho/\sigma + \frac{d}{ds}(\sigma \rho') = 0 \Rightarrow \underline{S}' = 0$

$\Rightarrow \underline{S}$ is constant.

Therefore curve lies on the surface of a sphere.

§ HELICES: :

Def A curve traced on the surface of the cylinder and cutting the generators at a constant angle, is called a HELIX.

If $\underline{t}$ is unit tangent to the helix and $\underline{a}$ is a constant vector ∥ to the generator of cylinder, we have $\underline{t} \cdot \underline{a}$ = Constant

& Diff w.r.t. S, we have $K\underline{n} \cdot \underline{a} = 0$

Thus since the curvature of the Helix does not vanish, the principal normal is every where perpendicular to the generator. Hence fixed direction of the generator is parallel to the plane of $\underline{t}$ and $\underline{b}$, and since it makes a constant angle with $\underline{t}$, it also makes a constant angle with $\underline{b}$.

Theorem:

The necessary and sufficient condition for a curve to be a helix is that the ratio of its curvature and torsion is constant i.e. K/τ = Constant

Proof: If $\underline{t}$ is a unit tangent to the helix and $\underline{a}$ is constant vector ~~along~~ ∥ to generator of cylinder

Then $\underline{t} \cdot \underline{a}$ = constant. —① | $\underline{t} \cdot \underline{a} = 1.1 \cos$

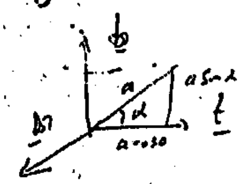
Diff wrt S $\frac{d\underline{t}}{ds} \cdot \underline{a} = 0$ $t' = K\underline{n}$

$K\underline{n} \cdot \underline{a} = 0$

If $K = 0$ then curve is st. line & theorem is proved

& if $K \neq 0$ then $\underline{n} \cdot \underline{a} = 0 \Rightarrow \underline{n} \perp \underline{a}$

Then $\underline{a}$ will lie in the plane determined by $\underline{t}$ and $\underline{b}$, ~~Hence~~ To prove this,



Diff $\underline{n}\cdot\underline{a}=0$, wrt s, we have

$$\underline{n}'\cdot\underline{a}=0$$

Thus

$$(T\underline{b}-K\underline{t})\cdot\underline{a}=0 \Rightarrow \underline{a} \text{ is } \perp \text{ to}$$

the ~~plane of b and t~~ vector $T\underline{b}-K\underline{b}$. But $\underline{a}$ is parallel to the plane of $\underline{t}$ & $\underline{b}$.

Hence $\underline{a} = \cos\alpha\,\underline{t} + \sin\alpha\,\underline{b}$

$\underline{t}$ & $\underline{b}$ are $\perp$ to each other. a makes $\cos\alpha$ with $\underline{t}$ & $\sin\alpha$ with $\underline{b}$

Diff wrt s: $0 = \cos\alpha\, t' + \sin\alpha\, b'$

$$\Rightarrow \cos\alpha\,(K\underline{n}) + \sin\alpha\,(-T\,\underline{n}) = 0$$

$$\vec{n}\,(K\cos\alpha - T\sin\alpha) = 0$$

Since $\vec{n}\neq 0$,

$$\Rightarrow K\cos\alpha - T\sin\alpha = 0 \qquad \tan\alpha = \frac{K}{T}$$

$$\alpha = \tan^{-1}(K/T) \Rightarrow K/T \text{ is constant}$$

(as α is constant)

<u>Condition is Sufficient</u>

Given K/T = constant, To show curve is Helix or $\underline{t}\cdot\underline{a}$ is constant.

Let $T = cK \longrightarrow (1)$, c is constant.

Then since $t' = K\underline{n}$ and $b' = -T\underline{n} = -cK\underline{n}$

It follows that

$$\underline{b}' + c\,\underline{t}' = -T n + cK\underline{n} = 0$$

from (1): $b' + ct' = -Tn + cKn$, $\frac{d}{ds}(b+ct) = 0$

$$\therefore \frac{d}{ds}(\underline{b} + c\,\underline{t}) = 0$$

$$\Rightarrow \underline{b} + c\,\underline{t} = \underline{a} \quad (\underline{a} \text{ is constant vector})$$

Taking scalar product with $\underline{t}$

$$\underline{b}\cdot\underline{t} + c\,\underline{t}\cdot\underline{t} = \underline{a}\cdot\underline{t}$$

$$0 + c(1) = \underline{a}\cdot\underline{t} \qquad \Rightarrow \underline{a}\cdot\underline{t} = c$$

$$\Rightarrow \underline{t}\cdot\underline{a} \text{ is constant}$$

<u>Remarks</u> (i) Thus $\underline{t}$ is inclined at a constant angle to the fixed direction ~~of~~ $\underline{a}$ and curve is therefore Helix

(ii) If ratio $K/T = 0$ then curve is St-line

(iii) If $K/T = \alpha$, then curve is plane ~~p~~ curve (α being angle between $\underline{t}$ and $\underline{a}$)

§ SPHERICAL INDICATRIX.

Def The locus of a point, whose position vector is equal to the unit Tangent $\underline{t}$ of a given curve, is called the Spherical Indicatrix of the Tangent to the curve. Such a locus lies on the surface of a unit sphere. (Hence the name)

Theorem

To show that the curvature of the Spherical Indicatrix of Tangents is the ratio of skew curvature to the circular curvature of the curve: ie $K_1 = \frac{\sqrt{K^2+T^2}}{K}$, Also prove $T_1 = \frac{KT'-TK'}{K(K^2+T^2)}$

Proof :

Let $\underline{r}_1$ be the p.v. of a point of the spherical indicatrix of the Tangent to a curve, then $\underline{r}_1 = \underline{t}$ → ① Diff w r t s

$$\frac{d\underline{r}_1}{ds} = \frac{dt}{ds}$$

$$\Rightarrow \frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = t'$$

$$\underline{t}_1 \frac{ds_1}{ds} = K n \quad \text{, we measure}$$

s_1 so that $\frac{ds_1}{ds} = K$ Then $t_1 = \underline{n}$ —— ⑪

Diff again w r t s

$$\frac{d\underline{t}_1}{ds} = \frac{dn}{ds}$$

$$\Rightarrow \frac{dt_1}{ds_1} \cdot \frac{ds_1}{ds} = n'$$

$\because \frac{dt_1}{ds_1} = K_1 \underline{n}_1$

$$K(K_1 \underline{n}_1) = T\underline{b} - K\underline{t} \qquad \because \frac{ds_1}{ds} = K$$

$$K_1 \underline{n}_1 = \frac{T\underline{b} - K\underline{t}}{K} \longrightarrow \text{III}$$

Squaring both sides

$$K_1^2 (n_1 \cdot n_1) = \left(\frac{T^2+K^2}{K^2}\right)$$

$$K_1 = \frac{\sqrt{T^2+K^2}}{K} = \frac{\text{Skew Curvature}}{\text{Curvature}}$$

Now the equation of osculating sphere is

$$R^2 = \rho^2 + (\sigma \rho')^2 \longrightarrow (1)$$

As the indicatrix lies on the sphere of unit radius $\therefore R = 1$ and eq (1) takes the form

$$1 = \rho_1^2 + (\sigma_1 \rho_1')^2$$

$$1 = \frac{1}{K_1^2} + \frac{1}{T_1^2}\, \rho_1'^2 \quad (A) \qquad \text{as } \rho_1 = \frac{1}{K_1} \text{ \& } \sigma_1 = \frac{1}{T_1}$$

As $\rho_1 = 1/K_1$

$$\rho_1' = (K_1^{-1})' = -K_1^{-2} K_1' = \frac{K_1'}{K_1^2}$$

$$(\rho_1')^2 = \frac{K_1'^2}{K_1^4}$$

Putting in (A) $\quad 1 = \frac{1}{K_1^2} + \frac{1}{T_1^2}\left(\frac{K_1'^2}{K_1^4}\right)$

$$1 - \frac{1}{K_1^2} = \frac{1}{T_1^2}\left(\frac{K_1'^2}{K_1^4}\right)$$

$$K_1^2 - 1 = \frac{K_1'^2}{K_1^2 T_1^2} \Rightarrow T_1^2 = \frac{K_1'^2}{K_1^2(K_1^2 - 1)}$$

$$T_1 = \frac{K_1'}{K_1\sqrt{K_1^2 - 1}} \longrightarrow (ii)$$

Also from $\quad K_1 = \frac{\sqrt{K^2 + T^2}}{K}$

Diff w.r.t S_1

$$K_1' = \frac{d}{ds}\left(\frac{\sqrt{K^2+T^2}}{K}\right)\frac{ds}{ds_1}$$

$$K_1' = \left\{\frac{K\left(\frac{2KK' + 2TT'}{2\sqrt{K^2+T^2}}\right) - \sqrt{K^2+T^2}\cdot K'}{K^2}\right\}\frac{1}{K} \qquad \because \frac{ds}{ds_1} = 1/K$$

$$= \left\{\frac{K(KK' + TT') - (K^2+T^2)K'}{K^2\sqrt{K^2+T^2}}\right\}\frac{1}{K}$$

$$K_1' = \frac{\cancel{K^2K'} + KTT' - \cancel{K^2K'} - T^2K'}{K^3\sqrt{K^2+T^2}} = \frac{KTT' - T^2K'}{K^3\sqrt{K^2+T^2}}$$

Putting values of K_1 and K_1' in (ii), we have

$$T_1 = \frac{KTT' - T^2K'}{K^3\sqrt{K^2+T^2}} \times \frac{1}{\frac{\sqrt{K^2+T^2}}{K}\left(\sqrt{\frac{K^2+T^2}{K^2} - 1}\right)}$$

$$\tau_1 = \frac{\tau(K\tau' - \tau K')}{K^3\sqrt{K^2+\tau^2}} \cdot \frac{K}{\sqrt{K^2+\tau^2}} \frac{1}{\sqrt{\tau^2/K^2}}$$

$$= \frac{\tau(K\tau' - \tau K')}{K^3\sqrt{K^2+\tau^2}} \cdot \frac{K^2}{\tau\sqrt{K^2+\tau^2}}$$

$$\tau_1 = \frac{K\tau' - \tau K'}{K(\sqrt{K^2+\tau^2})^2} = \frac{K\tau' - \tau K'}{K(K^2+\tau^2)}$$

Theorem: Prove that the curvature and Torsion of the spherical indicatrix of the Binormal is given by

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{\tau} \quad \text{and} \quad \tau_1 = \frac{K'\tau - \tau' K}{\tau(K^2+\tau^2)}$$

Proof The eq. of the spherical indicatrix of the binormal is

$$\underline{r}_1 = \underline{b}$$

Diff wrt s_1

$$\frac{d\underline{r}_1}{ds_1} = \frac{d\underline{b}}{ds} \cdot \frac{ds}{ds_1}$$

$$\underline{t}_1 = \underline{b}' \frac{ds}{ds_1}$$

$$\underline{t}_1 = -\tau \underline{n} \frac{ds}{ds_1}$$

Since $\underline{t}_1 = -\underline{n} = -\tau \underline{n} \frac{ds}{ds_1} \Rightarrow \tau \frac{ds}{ds_1} = 1$

$$\Rightarrow \frac{ds_1}{ds} = \tau \quad \text{——— (1)} \quad \Rightarrow \frac{ds}{ds_1} = \frac{1}{\tau}$$

Now $\underline{t}_1 = -\underline{n}$

Diff wrt s_1 $\Rightarrow$

$$\frac{d\underline{t}_1}{d\underline{s}_1} = -\frac{d\underline{n}}{d\underline{s}} \cdot \frac{ds}{d\underline{s}_1}$$

$$K_1 \underline{n}_1 = -[\tau \underline{b} - K\underline{t}] \frac{1}{\tau} \quad \text{by (1)}$$

$$\therefore K_1 \underline{n}_1 = \frac{K\underline{t} - \tau\underline{b}}{\tau}$$

Squaring both sides $\quad K_1^2 = \frac{K^2+\tau^2}{\tau^2}$

$$K_1 = \frac{\sqrt{K^2+\tau^2}}{\tau} \longrightarrow (2)$$

As the indicatrix lies on the unit sphere.

$$R^2 = \rho^2 + (\sigma \rho')^2 \qquad \therefore R = 1$$

$$1 = \rho_1^2 + (\sigma_1 \rho_1')^2$$

$$1 = \frac{1}{K_1^2} + \frac{1}{T_1^2}(\rho_1')^2 \quad \text{——— (B)}$$

As $\rho_1 = 1/K_1 \Rightarrow \rho_1' = -\dfrac{K_1'}{K_1^2}$

$$\therefore (\rho_1')^2 = \frac{K_1'^2}{K_1^4}$$

Putting in B

$$1 = \frac{1}{K_1^2} + \frac{1}{T_1^2}\frac{K_1'^2}{K_1^4}$$

$$1 - \frac{1}{K_1^2} = \frac{1}{T_1^2}\left(\frac{K_1'^2}{K_1^4}\right)$$

$$\frac{K_1^2 - 1}{K_1^2} = \frac{1}{T_1^2}\frac{K_1'^2}{K_1^4}$$

$$T_1^2 = \frac{K_1'^2}{(K_1^2-1)K_1^2} \Rightarrow T_1 = \frac{K_1'}{K_1\sqrt{K_1^2-1}} \quad \text{——— (3)}$$

From (2) $\quad K_1 = \dfrac{\sqrt{K^2+T^2}}{T}$

Diff w.r.t S_1

$$K_1' = \frac{d}{dS_1}\left(\frac{(K^2+T^2)^{1/2}}{T}\right)$$

$$K_1' = \frac{d}{ds}\frac{(K^2+T^2)^{1/2}}{T}\frac{ds}{dS_1}$$

$$= \frac{T\left(\frac{2KK'+2TT'}{2\sqrt{K^2+T^2}} - \sqrt{K^2+T^2}\cdot T'\right)}{T^2}\frac{1}{T}$$

$$= \frac{K(TK'-KT')}{T^3\sqrt{K^2+T^2}}$$

Putting these values of K_1 and K_1' in (3)

$$T_1 = \frac{K(TK'-KT')}{T^3\sqrt{K^2+T^2}} \cdot \frac{1}{\frac{\sqrt{K^2+T^2}}{T}\sqrt{\frac{K^2+T^2}{T_1^2}-1}}$$

$$T_1 = \frac{K(TK' - KT')}{T^3(\sqrt{K^2+T^2})^2} \cdot \frac{T^2}{K} = \frac{K'T - KT'}{T(K^2+T^2)}$$

Example Find all sph. indicatrix (Images) of the circular Helix.

Let $\underline{r} = (a\cos\theta, a\sin\theta, c\theta)$, $c \neq 0$

Diff wrt s

$$\underline{r}' = \underline{t} = (-a\sin\theta, a\cos\theta, c)\frac{d\theta}{ds} \quad \text{--- (1)}$$

Squaring both sides

$$1 = (a^2\sin^2\theta + a^2\cos^2\theta + c^2)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2+c^2)\left(\frac{d\theta}{ds}\right)^2$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2+c^2} \Rightarrow \frac{d\theta}{ds} = \frac{1}{\sqrt{a^2+c^2}}$$

Say, $\frac{ds}{d\theta} = \sqrt{a^2+c^2} = \lambda$ (constant) $\longrightarrow$ (2)

putting in (1)

$$\underline{t} = (-a\sin\theta, a\cos\theta, c)\frac{1}{\lambda} \longrightarrow (3)$$

Diff wrt s

$$\frac{dt}{ds} = \frac{d}{d\theta}(-a\sin\theta, a\cos\theta, c)\frac{1}{\lambda}\frac{d\theta}{ds}$$

$$K\underline{n} = (-a\cos\theta, -a\sin\theta, 0)\,1/\lambda^2 \longleftarrow (4)$$

Squaring both sides

$$K^2 = (a^2\cos^2\theta + a^2\sin^2\theta)\frac{1}{\lambda^4}$$

$$K^2 = a^2/\lambda^4 \quad \boxed{K = \frac{a}{\lambda^2}} \quad \text{--- (A)}$$

Eq (4) can be written as

$$K\underline{n} = (-\cos\theta, -\sin\theta, 0)\,a/\lambda^2$$

$$K\underline{n} = (-\cos\theta, -\sin\theta, 0)\,K$$

$$\Rightarrow \vec{n} = (-\cos\theta, -\sin\theta, 0) \longrightarrow (5)$$

Now $\underline{b} = \underline{t} \times \underline{n}$

$$= \begin{vmatrix} i & j & k \\ \frac{-a\sin\theta}{\lambda} & \frac{a\cos\theta}{\lambda} & c/\lambda \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix} \quad \Big\} \text{ From 3 \& 5}$$

$$= \frac{1}{\lambda}\left\{ i\left(0 + c\sin\theta\right) - j\left(c\cos\theta\right) + k\left(a\sin^2\theta + a\cos^2\theta\right)\right\}$$

$$= \frac{1}{\lambda}\left(c\sin\theta, -c\cos\theta, a\right) \longrightarrow (6)$$

From (4), (5) & (6) we can find spherical Images for Tangent as $x = -\frac{a\sin\theta}{\lambda}$, $y = \frac{a\cos\theta}{\lambda}$, $z = c/\lambda$

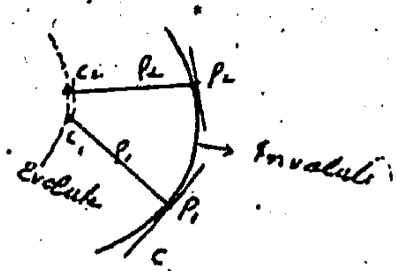
for principal normal,

$$x = -\cos\theta, \quad y = -\sin\theta, \quad z = 0$$

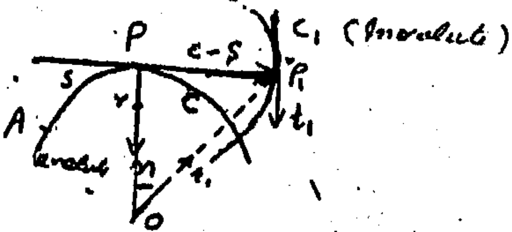
& for Principal Binormal are

$$x = \frac{c\sin\theta}{\lambda}, \quad y = -\frac{c\cos\theta}{\lambda}, \quad z = a/\lambda$$

§. INVOLUTES & EVOLUTES



Def: When the Tangents to a curve C are normal to another curve C_1 Then C_1 is called an Involute of C and C is called an evolute of C_1.



∴ Let $\underline{r}_1$ be the p.v of a pt P_1 on involute ~~and $\underline{r}$ p.v. of P on~~ which lies on the Tangent at the pt $\underline{r}$ of curve C is given by $r_1 = \underline{r} + u\underline{t}$ — (1) where u is to be determined.

Let ds_1 be the arc length of the involute corresponding to the element ds of the curve C. Then Unit Tangent to C_1 is

dot product with $\underline{t}$

$1 = (1+u') + 0$

$+u' = 0$

$$t_1 = \frac{dr_1}{ds}\cdot\frac{ds}{ds_1}$$

$$t_1 = \frac{dr}{ds}\frac{ds}{ds_1} + \left(u'\underline{t} + u\frac{dt}{ds}\right)\frac{ds}{ds_1} \qquad \frac{dr}{ds} = t$$

$$= \left[(1+u')\,\underline{t} + u\kappa\underline{n}\right]\frac{ds}{ds_1} \quad \text{— (2)}$$

To satisfy the condition for an involute, this vector must be $\perp$ to $\underline{t}$, Hence

$$1 + U' = 0$$
$$U' = -1 \quad \text{—— (3)}$$
$$U = -s + c \quad \text{—— (4)}$$
c is constant

Putting the value of U in ①, we have

$$\underline{r}_1 = \underline{r} + (c-s)\,\underline{t}$$

where c is constant, due to c we conclude that there are ∞ nos of involutes for each evolute

and the unit-Tangent (from eq(2)) is

$$\underline{t}_1 = (\underline{t} + U'\underline{t} + UK\underline{n})\frac{ds}{ds_1}$$

from (3) & (4)

$$\underline{t}_1 = (\underline{t} + (-1)\underline{t} + (c-s)K\underline{n})\frac{ds}{ds_1}$$

$$t_1 = (c-s)K\underline{n}\frac{ds}{ds_1}$$

Since $\underline{t}_1 = \vec{n}$ —— ⑤ & $(c-s)K\frac{ds}{ds_1} = 1$

Thus $t_1 \parallel \vec{n}$ & $\frac{ds_1}{ds} = K(c-s)$ ——

From $t_1 \parallel \vec{n}$, we note that Tangent at the point P to C_1 is parallel to the normal of the point P to C.

<u>To find curvature of Involute</u>, we consider

from(5) $\quad \underline{t}_1 = \underline{n}$

Diff wrt s_1

$$\frac{dt_1}{ds_1} = \frac{dn}{ds}\,\frac{ds}{ds_1}$$

$$K_1\underline{n}_1 = \underline{n}'\frac{ds}{ds_1}$$

$$K_1\underline{n}_1 = (T\underline{b} - K\underline{t})\frac{1}{K(c-s)}$$

Squaring both sides

$$K_1^2(1) = \frac{(T^2+K^2)}{K^2(c-s)^2}$$

$$K_1 = \frac{\sqrt{T^2+K^2}}{K(c-s)}$$

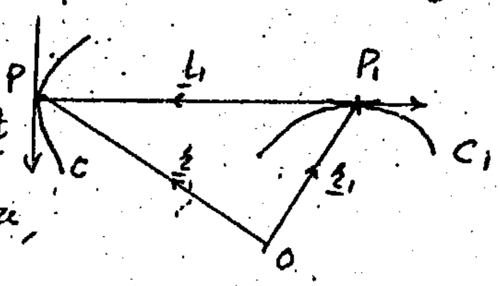
which is the Required expression for curvature of involute C_1

Theorem For Evolutes

Statement: To show there are an infinite family of evolutes for the space curve C.

Proof: Let C be space curve with $\underline{r} = r(s)$ as its eq.

Since the Tangent at P_1 of C_1 is normal at a corresponding pt P of C i.e, the Tangent at P_1 of C_1 lies in the normal plane, therefore, P.v r_1 of P_1 can be expressed as



$$\underline{r}_1 = \underline{r} + U\underline{n} + V\underline{b} \quad \text{—①}$$ where U and V are to be determined

$$\frac{dr_1}{ds} = \underline{r}' + U'\underline{n} + U\underline{n}' + \underline{V}'\underline{b} + V\underline{b}'$$

$$= \underline{t} + U'\underline{n} + U(\tau\underline{b} - K\underline{t}) + V'b + \underline{V}(-\tau\underline{n})$$

$$= \underline{t}(1-UK) + (U' - \tau V)\underline{n} + (U\tau + V')\underline{b} \quad \text{— ②}$$

As $\frac{dr_1}{ds}$ lies in normal plane

we have $$\frac{dr_1}{ds} = U\underline{n} + V\underline{b} \rightarrow ③$$

Comparing ② & ③

$$\Rightarrow 1 - UK = 0 \quad ④$$

$$\& \quad \frac{U' - V\tau}{U} \; ⑤ = \frac{U\tau + V'}{V} \longrightarrow ⑥$$

From ④ $\boxed{U = 1/K = \rho}$. from ⑤ & ⑥

$$U'V - V^2\tau = U^2\tau + UV'$$

$$(U^2 + V^2)\tau = U'V - UV'$$

$$\tau = \frac{U'V - UV'}{U^2 + V^2}$$

Integ. w.r.t s

$$\tau = \frac{1}{1 + V^2/U^2}\left(\frac{U'V - UV'}{V^2}\right)$$

$$\int_0^s \tau\, ds = \tan^{-1}(-(V/U)) \qquad \text{Since } U = \rho$$

we write $\psi = \int_0^s \tau\, ds$

$$= \tan^{-1}(-V/\rho)$$

$$\psi + c = \tan^{-1}(-V/\rho)$$

$$\tan(\psi + c) = -V/\rho$$

$$\nu = -\rho \operatorname{Tan}(\psi + c) \quad \text{where, } \nu = \tfrac{1}{\kappa} = \rho$$

$$\therefore \underline{r}_1 = \underline{r} + \rho \left[\underline{n} - \operatorname{Tan}(\psi + c)\, \underline{b} \right]$$

Which is equation of the evolute C_1 and for different values of arb. constants, we can obtain different evolutes and hence ∞ many different evolutes C_1 for the given curve $\underline{C}$.

Example: Prove that the locus of centre of curvature is an evolute only when the curve is a plane curve.

Solution: The equation of Evolute can be written as

$$\underline{r}_1 = \underline{r} + \rho \underline{n} - \rho \operatorname{Tan}(\psi + a) \cdot \underline{b} \longrightarrow (1)$$

For different values of a we have different evolutes, also the locus of centre of curvature can be written as $\underline{C} = \underline{r} + \rho \underline{n}$ ——— (2)

Equation (1) and (2) are identical, if

* (b is a unit vector so it can't be equal to zero)

$$\therefore \rho \operatorname{Tan}(\psi + a) \cdot \underline{b} = 0$$

$$\Rightarrow \text{if } \underline{b}^{*} \neq 0,\ \rho \neq 0, \text{ Then}$$

$$\operatorname{Tan}(\psi + a) = 0$$

$$\Rightarrow \operatorname{Tan}(\psi + a) = \operatorname{Tan} n\pi \quad n \text{ is any integer,}$$

$$\Rightarrow \psi + a = n\pi$$

$$\psi(s) = n\pi - a$$

$$\psi'(s) = 0 \text{ ——— } (3)$$

سپیس Plane curve کی طرح بن جائیگی Torsion صفر ہو جائے

$$\text{but } \psi = \int \tau\, ds \Rightarrow \tau = \psi'(s)$$

$$\Rightarrow \tau = 0$$

Hence the curve is a plane curve.

Theorem: Prove that the ratio of the Torsion and curvature of an evolute of a space curve (Involute) is

$$\frac{\tau_1}{\kappa_1} = -\operatorname{Tan}(\psi + a) \quad \text{where } \psi = \int \tau\, ds$$

$t_1 = \frac{1}{c}$

Proof The equation of the evolute is

$$\underline{r}_1 = \underline{r} + \rho\,\underline{n} - \rho\,\text{Tan}(\psi+a)\,\underline{b}$$

Diff wrt s_1

$$\frac{d r_1}{d s_1} = \Big\{ \underline{r}' + \rho'\underline{n} + \rho\underline{n}' - \rho'\,\text{Tan}(\psi+a)\,\underline{b} - \rho\,\underline{b}'\,\text{Tan}(\psi+a) - \rho\,\underline{b}\,\text{Sec}^2(\psi+a)\frac{d\psi}{ds}\Big\}\frac{ds}{ds_1}$$

$$= \Big\{ \underline{t} + \rho'\underline{n} + \rho(\tau\underline{b} - \kappa\underline{t}) - \rho'\,\text{Tan}(\psi+a)\,\underline{b} - \rho(\cancel{t} - \tau\underline{n})\,\text{Tan}(\psi+a) - \rho\,\underline{b}\,\text{Sec}^2(\psi+a)\,\tau \Big\}\frac{ds}{ds_1}$$

$$= \Big\{ \underline{t} + \rho'\underline{n} + \rho\tau\underline{b} - \rho\kappa\,\underline{t} - \rho'\,\text{Tan}(\psi+a)\underline{b} + \rho\tau\underline{n}\,\text{Tan}(\psi+a) - \rho\,\underline{b}\,(1+\text{Tan}^2(\psi+a)]\,\tau \Big\}\frac{ds}{ds_1}$$

$$= \Big\{ \cancel{\underline{t}} + \rho'\underline{n} + \cancel{\rho\tau\underline{b}} - \cancel{\underline{t}} - \rho'\,\text{Tan}(\psi+a)\underline{b} + \rho\tau\underline{n}\,\text{Tan}(\psi+a) - \cancel{\rho\tau\underline{b}} - \tau\rho\,\text{Tan}^2(\psi+a)\,\underline{b}\Big\}\frac{ds}{ds_1} \quad \left(\text{By putting } \rho = \frac{1}{\kappa}\right)$$

$$= \rho'\underline{n} + \rho\tau\underline{n}\,\text{Tan}(\psi+a) - \rho'\,\text{Tan}(\psi+a)\,\underline{b} - \rho\tau\,\text{Tan}^2(\psi+a)\frac{ds}{ds_1}$$

$$= \Big\{[\rho' + \rho\tau\,\text{Tan}(\psi+a)]\,\underline{n} - \text{Tan}(\psi+a)\,\{\rho' + \rho\tau\,\text{Tan}(\psi+a)\}\underline{b}\Big\}\frac{ds}{ds_1}$$

$$t_1 = \frac{d r_1}{d s_1} = \Big\{(\rho' + \rho\tau\,\text{Tan}(\psi+a))(\vec{n} - \text{Tan}(\psi+a))\Big\}\frac{ds}{ds_1} \longrightarrow (1)$$

Squaring both sides

$$1 = [\rho' + \rho\tau\,\text{Tan}(\psi+a)]^2\,[1+\text{Tan}^2(\psi+a)]\left(\frac{ds}{ds_1}\right)^2$$

$$1 = [\rho' + \rho\tau\,\text{Tan}(\psi+a)]^2\,\text{Sec}^2(\psi+a)\left(\frac{ds}{ds_1}\right)^2$$

$$\left(\frac{ds_1}{ds}\right)^2 = [\rho' + \rho\tau\,\text{Tan}(\psi+a)]^{-2}\,\text{Sec}^2(\psi+a)$$

$$\frac{ds_1}{ds} = [\rho' + \rho\tau\,\text{Tan}(\psi+a)]\,\text{Sec}(\psi+a)$$

$$\Rightarrow \frac{ds}{ds_1} = \frac{1}{\rho' + \rho\tau\,\text{Tan}(\psi+a)}\cdot\cos(\psi+a)$$

Using result in eq (1), we get

$$t_1 = \{\cancel{\rho' + \rho\tau\,\text{Tan}(\psi+a)}\}\{\underline{n} - [\text{Tan}(\psi+a)]\underline{b}\} \times \frac{\cos(\psi+a)}{\cancel{[\rho' + \rho\tau\,\text{Tan}(\psi+a)]}} = \frac{\underline{n} - \text{Tan}(\psi+a)\,\underline{b}}{\text{Sec}(\psi+a)}$$

$$\underline{t}_1 = \left[n - \frac{\sin(\psi+a)}{\cos(\psi+a)}\, \underline{b} \right] \cos(\psi+a)$$

$$\underline{t}_1 = \left[\underline{n}\cos(\psi+a) - \underline{b}\sin(\psi+a) \right] \quad \text{——} \ (2)$$

Diffwrt s_1

$$\frac{dt_1}{ds_1} = \frac{d}{ds}\left[\underline{n}\cos(\psi+a) - \underline{b}\sin(\psi+a) \right] \frac{ds}{ds_1}$$

$$= \Big\{ \underline{n}'\cos(\psi+a) - \underline{n}\sin(\psi+a)\,\psi' - \underline{b}'\sin(\psi+a) - \underline{b}\cos(\psi+a)\,\psi' \Big\} \frac{ds}{ds_1}$$

$\psi' = \tau$

$b' = -\tau n$

$$K_1 \underline{n}_1 = \Big\{ (\tau\underline{b} - \kappa\underline{t})\cos(\psi+a) - \underline{n}\sin(\psi+a)\,\tau + \tau\underline{n}\sin(\psi+a) - \underline{b}\cos(\psi+a)\,\tau \Big\} \frac{ds}{ds_1}$$

$$K_1 \underline{n}_1 = \Big\{ \cancel{\tau\underline{b}\cos(\psi+a)} - \kappa\underline{t}\cos(\psi+a) - \cancel{\underline{n}\sin(\psi+a)\,\tau} + \cancel{\underline{n}\,\tau\sin(\psi+a)} - \cancel{\underline{b}\,\tau\cos(\psi+a)} \Big\} \frac{ds}{ds_1}$$

$$= \left[-\kappa\underline{t}\cos(\psi+a) \right] \frac{ds}{ds_1}$$

or $\Rightarrow$ $\underline{n}_1 = -\underline{t} \rightarrow (3)$ & $K_1 = \kappa\cos(\psi+a)\dfrac{ds}{ds_1} \longrightarrow (4)$

Now Consider $\underline{b}_1 = t_1 \times n_1$

using (2) & (3)

$$= \left[\underline{n}\cos(\psi+a) - \underline{b}\sin(\psi+a) \right] \times (-\underline{t})$$

$$\underline{b}_1 = -\underline{n}\times\underline{t}\cos(\psi+a) + \underline{b}\times\underline{t}\sin(\psi+a)$$

$$= \underline{b}\cos(\psi+a) + \underline{n}\sin(\psi+a)$$

Now $\dfrac{db_1}{ds_1} = \dfrac{d}{ds}\left[\underline{b}\cos(\psi+a) + \underline{n}\sin(\psi+a) \right] \dfrac{ds}{ds_1}$

$$-\tau_1 \underline{n}_1 = \left[\underline{b}'\cos(\psi+a) - b\sin(\psi+a)\,\psi' + \underline{n}'\sin(\psi+a) + n\cos(\psi+a)\,\psi' \right] \frac{ds}{ds_1}$$

$$-\tau_1 \underline{n}_1 = \left[-\cancel{\tau\underline{n}\cos(\psi+a)} - \cancel{b\sin(\psi+a)\,\tau} + (\cancel{\tau b\sin(\psi+a)} - \kappa\underline{t}\sin(\psi+a)) + \cancel{n\tau\cos(\psi+a)} \right] \frac{ds}{ds_1}$$

$$-\tau_1 n_1 = -\kappa\underline{t}\sin(\psi+a)\frac{ds}{ds_1}$$

$$\tau_1 n_1 = \kappa\underline{t}\sin(\psi+a)\frac{ds}{ds_1} \Big] \Rightarrow n_1 = -t \ \& \ \tau_1 = -\kappa\sin(\psi+a)\frac{ds}{ds_1} \quad \text{——} (5)$$

$\Rightarrow$ Dividing (5) by (4)

$$\frac{\tau_1}{K_1} = \frac{-\kappa\sin(\psi+a)\,{}^{ds}/_{ds_1}}{\kappa\cos(\psi+a)\,{}^{ds}/_{ds_1}} = \frac{-\sin(\psi+a)}{\cos(\psi+a)} = -\tan(\psi+a)$$

where $\psi = \int \tau\, ds$ or $\psi' = \tau$.

CHAPTER - 2

§ Diff. Geometry of Surfaces.

Def: A surface is the locus of a point $P(x,y,z)$ whose coordinates are functions of two indept. parameters u and v, thus

$$x = f_1(u,v), \quad y = f_2(u,v) \quad z = f_3(u,v) \rightarrow (1)$$

are parametric eqs of a surface.

If we eliminate u & v from these eqs we have $F(x,y,z) = 0 \rightarrow (2)$ as eq of surface.

Examples

(I) The parametric eq of a sphere with centre at O and radius a is

$$x = a\cos\theta\cos\varphi$$
$$y = a\cos\theta\sin\varphi$$
$$z = a\sin\theta$$

Eliminating ϕ and θ from these equations

$$x^2+y^2+z^2 = a^2\cos^2\theta\cos^2\varphi + a^2\cos^2\theta\sin^2\varphi + a^2\sin^2\theta$$
$$= a^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + a^2\sin^2\theta$$
$$= a^2(\cos^2\theta + \sin^2\theta) = a^2$$

Eq of a sphere centre at O and rad $= a$

Example (2) The parametric eqs of ellipsoid

$$x = a\cos\theta\cos\varphi$$
$$y = b\cos\theta\sin\varphi$$
$$z = c\sin\theta$$

Eliminating θ and ϕ:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \quad \text{Eq of Ellipsoid}$$

Example (3) The parametric eq of a cone are

$$x = \mu\sin\varphi\cos\psi$$
$$y = \mu\sin\varphi\sin\psi$$
$$z = \mu\cos\varphi$$

$$x^2+y^2 = \mu^2(\sin^2\varphi)(\cos^2\psi + \sin^2\psi)$$
$$= \mu^2\sin^2\varphi$$
$$= \mu^2\frac{\sin^2\varphi}{\cos^2\varphi}\cos^2\varphi$$
$$= \mu^2\cos^2\varphi\tan^2\varphi$$

Required eq. of a cone. $\boxed{x^2+y^2 = z^2\tan^2\varphi}$

§. TANGENT PLANE & NORMAL

Def. The Tangent to any curve drawn of a Surface is called a Tangent line to the surface. ~~Hence all Tangent~~

Def: Tangent plane to a surface, at a point P is the plane containing all Tangent lines to the surface at this point.

§ To find Eq of Tangent plane & eq. of normal at a pt P to the Surface $F(x,y,z)=0$ — (1)

Let $F(x,y,z)=0$ be the equation of Surface. Let C be any curve drawn on it. Suppose s be the arc length measured from a fixed pt. A up to a current pt $P(x,y,z)$. Since $F(x,y,z)=0$ has the same value at all points of the surface, it remains constant along the curve as s varies. Thus

Diff w r t s $\Rightarrow \frac{dF}{ds} = \frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{\partial s} = 0$

$$\Rightarrow F_x x' + F_y y' + F_z \cdot z' = 0$$

or $$(F_x, F_y, F_z)\cdot(x', y', z') = 0 \quad \text{--- (2)}$$

Now the vector (x', y', z') is the unit Tangent to the curve where the vector (x', y', z') is the unit Tangent $\underline{t}$ to the curve at $P(x,y,z)$. Eq (2) shows that it is $\perp$ to the vector (F_x, F_y, F_z). All Tangent lines to the Surface at (x,y,z) are $\perp$ to the vector (F_x, F_y, F_z) and hence lie in the plane through (x,y,z) $\perp$ to this vector. This plane is called Tangent plane & normal to the plane at P, the pt of contact, is called normal to the Surface at that pt. The vector (F_x, F_y, F_z) is called grad $\vec{F}$, denoted as $\nabla \vec{F}$.

Since the line joining any Point $\underline{R}(X,Y,Z)$ on the Tangent plane to the point of contact is $\perp$ to the normal, it follows that $(\underline{R}-\underline{r})\cdot\nabla\vec{F}=0$, $\underline{r}$ is p.v of pt of contact. OR $(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0$

or $(X-x)F_x + (Y-y)F_y + (Z-z)F_z = 0$ — (3) Eq of T.P.

Equation Of Normal.

Since Normal of the surface $F(x,y,z) = 0$ is along the gradient $\vec{\nabla F}$. Hence eq of normal is for any current pt $R(X,Y,Z,)$

$$\underline{R} = \underline{r} + u\vec{\nabla F}$$

$$\underline{R} - \underline{r} = u\vec{\nabla F}$$

or $(X-x,\ Y-y,\ Z-z) = u\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$

or
$$\left.\begin{aligned} X-x &= u\frac{\partial F}{\partial x} \\ Y-y &= u\frac{\partial F}{\partial y} \\ Z-z &= u\frac{\partial F}{\partial z} \end{aligned}\right\}$$
Elliminating u from these eqs, we have
$$\frac{X-x}{F_x} = \frac{Y-y}{F_y} = \frac{Z-z}{F_z}$$
which are eqs of normal at point $P(x,y,z)$.

Ex. on Page 39

Q(1) Prove that the Tangent plane to the Surface $xyz = a^3$ and the coord. planes bound a Tetrahedron of constant volume.

Sol Eq of Surface is $xyz = a^3$

$$F = xyz - a^3$$

$$F_x = yz, \quad F_y = xz, \quad F_z = xy$$

Eq of Tangent plane

$$(\underline{R} - \underline{r}) \cdot \vec{\nabla F} = 0$$

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0$$

$$(X-x)yz + (Y-y)xz + (Z-z)xy = 0$$

$$Xyz - xyz + Yxz - yxz + Zxy - xyz = 0$$

or $Xyz + Yxz + Zxy - 3xyz = 0$

or $Xyz + Yxz + Zxy - 3a^3 = 0$ ——— (1)

pt of Xn with coordinate plane (x axis) $Y = 0 = Z$

$$Xyz = 3a^3 \qquad X = 3a^3/yz$$

Similarly for y & z axis $Y = \frac{3a^3}{xz}$, $Z = \frac{3a^3}{yx}$

(8)

The pts of X_n are $\left(\frac{3a^3}{yz},0,0\right)$, $\left(0,\frac{3a^3}{xz},0\right)$ & $\left(0,0,\frac{3a^3}{xy}\right)$
To find the volume of Tetrahedron through these pts with 4th pt as origin $(0,0,0)$, we have

$$V = \frac{1}{6}\begin{vmatrix} 3a^3/yz & 0 & 0 & 1 \\ 0 & 3a^3/xz & 0 & 1 \\ 0 & 0 & 3a^3/xy & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{6}\left(27\frac{a^9}{x^2y^2z^2}\right)$$

$$= \frac{9}{2}\frac{a^9}{a^6} = \frac{9}{2}a^3$$

which is constant.

Q(2) Show that the Sum of Squares of the Intercepts on the coordinate axes by the Tangent plane to the surface $x^{2/3}+y^{2/3}+z^{2/3} = a^{2/3}$ is constant.

Sol. Eq of Surface $F = x^{2/3}+y^{2/3}+z^{2/3} - a^{2/3} = 0$

$$\frac{\partial F}{\partial x} = \frac{2}{3}x^{-1/3}, \quad \frac{\partial F}{\partial y} = \frac{2}{3}y^{-1/3}$$

$$\partial F/\partial z = \frac{2}{3}z^{-1/3}$$

Eq of Tangent plane $\quad (R-r)\cdot\nabla F = 0$

or $\quad (X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0$

$$(X-x)\,\tfrac{2}{3}x^{-1/3} + (Y-y)\tfrac{2}{3}y^{-1/3} + (Z-z)\tfrac{2}{3}z^{-1/3} = 0$$

$$\Rightarrow \frac{X-x}{x^{1/3}} + \frac{Y-y}{y^{1/3}} + \frac{Z-z}{z^{1/3}} = 0$$

$$x^{-1/3}X + y^{-1/3}Y + z^{-1/3}Z = x^{2/3}+y^{2/3}+z^{2/3} = a^{2/3} \quad \text{— (2)}$$

For x Intercept put $Y = 0 = Z$

$$\therefore \frac{X}{x^{1/3}} = a^{2/3} \Rightarrow X = a^{2/3}x^{1/3}$$

Similarly Y and Z Intercepts are

$$\Rightarrow Y = a^{2/3}y^{1/3} \quad Z = a^{2/3}z^{1/3}$$

Let $A(a^{2/3}x^{1/3},0,0)$
$B(0,a^{2/3}y^{1/3},0)$ and $C = (0,0,a^{2/3}z^{1/3})$

Sum of Squares of Intercepts

$$(OA)^2+(OB)^2+(OC)^2 = a^{4/3}x^{2/3}+a^{4/3}y^{4/3}+a^{4/3}z^{4/3}$$
$$= a^{4/3}(x^{2/3}+y^{2/3}+z^{2/3})$$
$$= a^{4/3}(a^{2/3}) = a^{6/3} = c^2 \text{ (Constant)}$$

Q(3) All pts common to the surface $a(xy+yz+zx)=xyz$ and a sphere whose centre is origin, The Tangent plane to the surface make intercepts on the axis whose sum is constant.

Solution Let Surface be $F(x,y,z) = a(xy+yz+zx)-xyz = 0$ ①
& eq of Sphere is $x^2+y^2+z^2=a^2$ —— ②

For Surface,
$$\frac{\partial F}{\partial x} = ay - yz + az.$$
$$\frac{\partial F}{\partial y} = ax+az-xz$$
$$\frac{\partial F}{\partial z} = ax+ay-xy$$

Eq of Tangent plane.

$$(X-x)(ay+az-yz)+(Y-y)(ax+az-xz)+(Z-z)(ax+ay-xy)=0$$
$$\Rightarrow X(ay+az-yz)+Y(ax+az-xz)+Z(ax+ay-xy)$$
$$-2\left[a(xy+yz+zx)-xyz\right]+xyz=0$$
$$\Rightarrow X(ay+az-yz)+Y(ax+az-xz)+Z(ax+ay-xy)+xyz=0 \quad ③$$

For X Intercept. Put $Y=0=Z$

$$X = \frac{-xyz}{ay+az-yz} = \frac{-x^2yz}{axy+ayz-xyz} = \frac{-x^2yz}{-ayz} = \frac{x^2}{a}$$

Similarly Y and Z Intercepts are

$$Y = \frac{y^2}{a} \quad \text{and} \quad Z = \frac{z^2}{a}$$

Sum of Intercepts
$$\frac{x^2}{a}+\frac{y^2}{a}+\frac{z^2}{a} = \frac{x^2+y^2+z^2}{a} \quad —— ④$$

Pts of the Surface which are common to Sphere will Satisfy the eq of Sphere. $x^2+y^2+z^2=b^2$

Hence Sum of Intercept for such pts is

$$\frac{1}{a}(x^2+y^2+z^2) = \frac{b^2}{a} \quad \text{which is Constant}$$

Q(4) The normal at a point P of Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the coordinate planes in G_1, G_2 & G_3. Prove that the ratios $PG_1 : PG_2 : PG_3$ are constants.

Sol The equation of surface is

$$F(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial F}{\partial y} = \frac{2y}{b^2}, \quad \frac{\partial F}{\partial z} = \frac{2z}{c^2}$$

Equation of normal at $P(x,y,z)$ is

$$\frac{X-x}{2x/a^2} = \frac{Y-y}{2y/b^2} = \frac{Z-z}{2z/c^2}$$

$$\Rightarrow \quad \frac{a^2(X-x)}{x} = \frac{b^2(Y-y)}{y} = \frac{c^2(Z-z)}{z} \quad \text{---}(1)$$

G_1 is pt of X^n of normal —(1) to the yz plane.

Put $X=0$

$$-\frac{a^2x}{x} = \frac{b^2(Y-y)}{y} = \frac{c^2(Z-z)}{z}$$

$$Y = \frac{-a^2y+b^2y}{b^2} = \left(\frac{b^2-a^2}{b^2}\right)y$$

Sly

$$Z = \left(\frac{c^2-a^2}{c^2}\right)z$$

$$\therefore \quad G_1 = \left(0, \frac{(b^2-a^2)y}{b^2}, \left(\frac{c^2-a^2}{c^2}\right)z\right)$$

Similarly $G_2 = \left(\left(\frac{a^2-b^2}{a^2}\right)x, 0, \left(\frac{c^2-b^2}{c^2}\right)z\right)$ X^n with zx plane

$G_3 = \left(\left(\frac{a^2-c^2}{a^2}\right)x, \left(\frac{b^2-c^2}{b^2}\right)y, 0\right)$ X^n with xy plane

$$\therefore |PG_1| = \sqrt{(x-0)^2+\left(y-\left(\frac{b^2-a^2}{b^2}\right)y\right)^2+\left(z-z\left(\frac{c^2-a^2}{c^2}\right)\right)^2}$$

$$= \sqrt{x^2+\frac{a^4}{b^4}y^2+\frac{a^4}{c^4}z^2} = a^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

Similarly

$$|PG_2| = b^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

$$|PG_3| = c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

Hence $|PG_1| : |PG_2| : |PG_3| = a^2 : b^2 : c^2$

which is constant.

§ ONE PARAMETER FAMILY OF SURFACES

An equation of the form $F(x,y,z,a)=0$, where 'a' is constant, represents a surface. Since 'a' is arbitrary constant, therefore, there are infinitely many surfaces. The set of all surfaces corresponding to different values of 'a' is called one parameter family of surfaces with parameter a.

<u>Example</u> Family of spheres of constant radius 'b' and having their centres at the ~~circle~~ fixed circle $x^2+y^2=a^2$ & $z=0$

Coordinates of a point on the given circle are $x=a\cos\theta$, $y=a\sin\theta$ & $z=0$

Therefore, eq of sphere will be

$$(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2=b^2$$

It is family of spheres.

---

§ CHARACTERISTICS OF A SURFACE :-

Consider $F(x,y,z,a)=0$ or $F(a)=0$ (i)

& $F(x,y,z,a+\delta a)=0$ or $F(a+\delta a)=0$ (ii)

be two surfaces of the same family. The curve of intersection of two surfaces of the family ~~the above eqs~~ can be written as

$F(a)=0$

& $\dfrac{F(a+\delta a)-F(a)}{\delta a}=0$

surface [illegible] intersect [illegible] surface [illegible] surface [illegible] Antiderivative [illegible]

If $\delta a \to 0$ then the two eqs represent two consecutive surfaces and eqs become as

$$\left.\begin{aligned} &F(a)=0 \\ \& \ &\tfrac{\partial}{\partial a}F(a)=0 \end{aligned}\right\}$$

The curve of X^n of these two consecutive surfaces is called the characteristics of the surface for the parametric value a.

§ ENVELOPES

The locus of all characteristics is called an Envelope of the family of Surfaces. It is a surface whose equation is obtained by Eliminating 'a' from the eqs $F(a)=0$ & $\frac{\partial}{\partial a}F(a)=0$

Exercise ON Page 41

Q(1) Find the Envelope of the family of Paraboloids $x^2+y^2=4a(z-a)$ is the circular cone $x^2+y^2=z^2$

Sol

Let $F(a) = x^2+y^2-4a(z-a)=0$ (i)

Diff Partially w.r.t 'a'

$$\frac{\partial F(a)}{\partial a} = -4z+8a = 0$$

Eliminating a, put $z/2 = a$ in eq (1)

$$x^2+y^2-4\,z/2\,(z-z/2)=0$$

$$x^2+y^2-z^2=0 \quad \text{or} \quad x^2+y^2=z^2$$

Required Envelope.

Q(2) Spheres of constant Radius b have their Centres on the fixed circle $x^2+y^2=a^2$, $z=0$, prove that their envelope is the Surface

$$(x^2+y^2+z^2+a^2-b^2)^2 = 4a^2(x^2+y^2)$$

Sol The eq of Sphere of radius b with centre on given circle $x=a\cos\theta$, $y=a\sin\theta$, will be

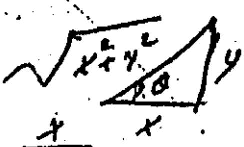
$$(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2=b^2$$

$$\Rightarrow F = x^2+y^2+z^2-2a(x\cos\theta+y\sin\theta)+a^2-b^2=0 \quad (1)$$

$$\frac{\partial F}{\partial \theta} = 2ax\sin\theta - 2ay\cos\theta$$

or $$2a(x\sin\theta-y\cos\theta)=0 \quad (2)$$

$$\tan\theta = y/x$$



$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

Putting these values in eq (1), we get

$$x^2+y^2+z^2-2a\left(\frac{x^2}{\sqrt{x^2+y^2}}+\frac{y^2}{\sqrt{x^2+y^2}}\right)+a^2-b^2=0$$

$$x^2+y^2+z^2+a^2-b^2=2a\sqrt{x^2+y^2}$$

Squaring both sides

$$\left(x^2+y^2+z^2+a^2-b^2\right)^2=4a^2\left(x^2+y^2\right)$$

is the required Envelope

Q(3) Find the Envelope of the family of Surfaces $F(x,y,z,a,b)=0$ in which Parameter a, b are connected by the eq $f(a,b)=0$

Solution: Given $F(x,y,z,a,b)=0$ (i)

$f(a,b)=0$ (ii)

Diff (i) & (ii) w.r.t a

$$\frac{\partial F}{\partial a}+\frac{\partial F}{\partial b}\cdot\frac{\partial b}{\partial a}=0 \quad \text{——— (iii)}$$

and $$\frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial a}=0 \quad \text{——— (iv)}$$

From eq (iv), we have

$$\frac{\partial b}{\partial a}=\frac{-\partial f/\partial a}{\partial f/\partial b}=-\frac{f_a}{f_b}$$

Putting in (3), we have

$$F_a+F_b\left(-\frac{f_a}{f_b}\right)=0$$

$$\Rightarrow \frac{F_a}{f_a}=\frac{F_b}{f_b} \longrightarrow (v)$$

Eqs (i), (ii) & (v) are the required Eqs. of the Envelope

Theorem: To prove that Envelope touches each member of the family of Surfaces at all pts of the characteristic.

Proof: The characteristic corresponding to the parameter value a lies both on the surface with the same parameter value and on the envelope.

Thus all pts of the characteristic are common to the surface and the envelope.

The normal to the surface $F(x,y,z,a)=0$ is parallel to the vector $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \longrightarrow (1)$

The equation of the envelope is obtained by eliminating a from. $F(a)=0$, $\frac{\partial}{\partial a}F(a)=0$

The envelope is therefore, represented by

$F(x,y,z,a)=0$ provided 'a' is regarded as a fn of. x,y,z given by $\frac{\partial F}{\partial a}(x,y,z,a)=0$

The normal to the envelope is then parallel to the vector

$$\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial a}\frac{\partial a}{\partial x},\ \frac{\partial F}{\partial y}+\frac{\partial F}{\partial a}\frac{\partial a}{\partial y},\ \frac{\partial F}{\partial z}+\frac{\partial F}{\partial a}\frac{\partial a}{\partial z}\right) \text{ — (ii)}$$

which in virtue of the preceding eq, is the same as vector (1). Thus at all common pts, the surface and the envelope have the same normal and therefore, the same Tangent plane, so that they touch each other at all pts of the characteristic.

Note: for (ii) $\frac{\partial F}{\partial a}=0$ and it reduces to

eq, $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \longrightarrow (3)$

which is same as (1)

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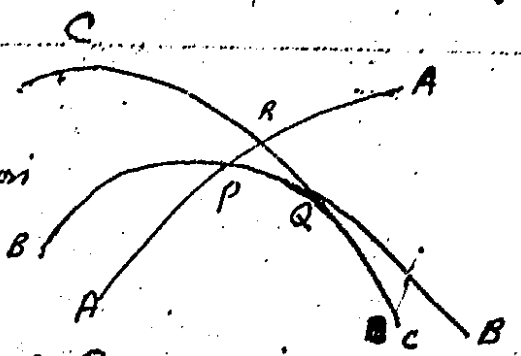


EDGE OF REGRESSION

The locus of ultimate intersection of consecutive characterics of a one-parameter family of surfaces is called the Edge of Regression

Theorem show that each characteristic touche edge of regression (ie to say that two curves have the same tangent at their common point.)

Proof:

Suppose that A, B and C are three consecutive characteristics A and B intersecting at P and B and C intersect at Q. These two pts. are consecutive pts on the characteristic B and also on the edge of regression. Hence ultimately as A and C tends to coincidence with B. The chord PQ becomes a common tangent to the characteristic and edge of regression.

Equation of the Edge of Regression

Let $F(x,y,z,a)=0$ be a family of surfaces then eq of characteristics corresponding to parameter 'a' and $a+\delta a$ are ~~F(a) = 0 & F a+δa~~

$F(x,y,z,a)=0$

& $F_a(x,y,z,a)=0$ and $F(x,y,z,a+\delta a)=0$

& $F_a(x,y,z,a+\delta a)=0$

It follows that

$$F_a(x,y,z,a+\delta a)-F_a(x,y,z,a)=0.$$

ie $F_{aa}(x,y,z,a)=0$

The eqs of edge of regression are obtained by eliminating a from $F(a)=0$ $F_a(a)=0$ and $F_{aa}(a)=0$

* EX ON PAGE 43

Q.1 Find the envelope of the family of planes $3a^2x - 3ay + z = a^3$ and show that its edge of regression is the curve of intersection of the surfaces $xz = y^2$, $xy = z$

Solution Given surface is

$$F(a) = 3a^2x - 3ay + z - a^3 = 0 \longrightarrow ①$$

$$\frac{\partial F}{\partial a} = 6ax - 3y - 3a^2 = 0 \longrightarrow ②$$

$$\frac{\partial^2 F}{\partial a^2} = 6x - 6a = 0 \longrightarrow (3)$$

Multiply eq ① by 3 and eq ② by $-a$

$$9a^2x - 9ay + 3z - 3a^3 = 0$$

adding

$$\frac{-6a^2x + 3ay + 3a^3 = 0}{3a^2x - 6ay + 3z = 0}$$

$$\Rightarrow \quad a^2x - 2ay + z = 0 \longrightarrow ④$$

Eq ② is $3a^2 - 6ax + 3y = 0$

or $a^2 - 2ax + y = 0 \longrightarrow ⑤$

Solving ④ & ⑤ $\dfrac{a^2}{-2y^2+2xz} = \dfrac{-a}{xy - z} = \dfrac{1}{-2x^2+2y}$

$$\Rightarrow a^2 = \frac{-2y^2+2xz}{-2x^2+2y} \quad \& \quad a = \frac{z - xy}{-2x^2+2y}$$

$$a^2 = \frac{-2y^2+2xz}{-2x^2+2y} = \frac{y^2 - xz}{x^2 - y} = \frac{(z - xy)^2}{4(x^2-y)^2}$$

$$(z - xy)^2 = 4(x^2 - y)(y^2 - xz)$$

required eq of Envelope

For edge of regression.

From eq(3)

$$x - a = 0$$

$$x = a$$

putting in ①, $3x^3 - 3xy + z - x^3 = 0$

$$2x^3 - 3xy + z = 0 \quad —— ⑥$$

and in ⑪

$$3x^2 - 3y = 0 \Rightarrow \boxed{x^2 = y}$$

Putting $y = x^2$ in $2x^3 - 3xy + z = 0$

$$2x^3 - 3x^3 + z = 0$$

$$-x^3 + z = 0$$

put $x^2 = y \Rightarrow -xy + z = 0$

$$\boxed{z = xy}$$

Multiply eq $x^2 - y = 0$ by y

$$x^2 y - y^2 = 0$$

$$x(xy) - y^2 = 0$$

$$xz - y^2 = 0 \qquad \boxed{y^2 = xz}$$

The eqs of edge of regression

$$xz = y^2 \text{ and } xy = z$$

Q(2) (b) Find the edge of regression of the envelope of family of planes

$$x \sin\theta - y\cos\theta + z = a\theta, \quad (\theta \text{ parameter})$$

<u>Solution</u>

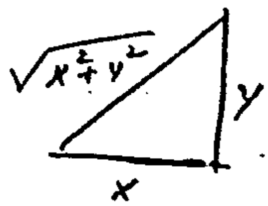
Let $F(\theta) = x\sin\theta - y\cos\theta + z - a\theta = 0$ ①

$F_\theta(\theta) = x\cos\theta + y\sin\theta - a = 0$ ②

$F_{\theta\theta}(\theta) = -x\sin\theta + y\cos\theta = 0$ ③

From (3) $\boxed{+x\sin\theta = y\cos\theta}$

$$\tan\theta = y/x$$



$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$\boxed{y = x\tan\theta}$ ④

Putting in eq ① $\quad z = a\theta \Rightarrow \boxed{\theta = z/a}$

Hence $y = x\tan(z/a)$ —— ⑤

From eq ② Squaring

$$x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta = a^2$$

$$x^2(1 - \sin^2\theta) + y^2\sin^2\theta + 2y(y\cos\theta)\cos\theta = a^2$$

$$x^2 - x^2\sin^2\theta + y^2\sin^2\theta + 2y^2\cos^2\theta = a^2$$

$$x^2 - y^2\cos^2\theta + y^2\sin^2\theta + 2y^2\cos^2\theta = a^2$$

$$x^2 + y^2(\cos^2\theta + \sin^2\theta) = a^2$$

$$\boxed{x^2 + y^2 = a^2} \longrightarrow (6) \quad \& \; y = x \tan(c/a)$$

Eqs (6) & (5) are eqs of edge of regressions

Q(3) Find the Envelope of the family of cones

$$(ax + x + y + z - 1)\cdot(ay + z) = ax(x + y + z - 1) \qquad a \text{ is parameter}$$

Solution:

$$F(a) = (ax + x + y + z - 1)(ay + z) - ax(x + y + z - 1) = 0$$

$$F_a(a) = x(ay + z) + (ax + x + y + z - 1)y - x^2 - xy - xz + x = 0$$

$$\Rightarrow axy + xz + axy + xy + y^2 + zy - y - x^2 - xy - xz + x = 0$$

$$\Rightarrow 2axy = x^2 - y^2 - x + y - zy$$

$$\Rightarrow a = \frac{x^2 - y^2 - x + y - yz}{2xy} \longrightarrow (2)$$

Elliminating (a)

Putting this value of a in (1), we have eq of Envelope.

Q(4) Find the Envelope and the edge of regression of the family of Ellipsoids

Sol

$$c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{z^2}{c^2} = 1 \qquad (c \text{ is Parameter})$$

The given eq

$$F(c) = c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{z^2}{c^2} - 1 = 0 \qquad (1)$$

$$\frac{\partial F}{\partial c} = 2c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{-2z^2}{c^3} = 0$$

$$\Rightarrow c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \frac{z^2}{c^2} = 0 \quad \text{--- (2)}$$

From (2)

$$c^4 = \frac{z^2}{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

Putting in (1) (Squaring eq (1)

$$c^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 + \frac{z^4}{c^4} + 2z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1$$

Putting c^4

$$z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + 2z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1$$

$\Rightarrow 4z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1$ which is eq of Envelope

Now, For the edge of regression

Diff eq ② w r t c

$$\frac{d^2F}{\partial c^2} = 2\left(x^2/a^2 + y^2/b^2\right) + \frac{6z^2}{c^4} = 0$$

$$\text{or} \quad c^2\left(\frac{x^2}{a^2} + y^2/b^2\right) + \frac{3z^2}{c^2} = 0$$

Eliminating c from ①+② and ③

$$c^2\left(x^2/a^2 + y^2/b^2\right) + z^2/c^2 - 1 = 0 \quad (i)$$

$$c^2\left(x^2/a^2 + y^2/b^2\right) - z^2/c^2 = 0 \quad (ii)$$

Subtracting

$$2z^2/c^2 = 1$$

$$\boxed{c^2 = 2z^2}$$

Putting in ①

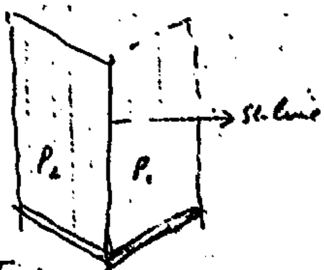
$$z^2\left(x^2/a^2 + y^2/b^2\right) = 1/4$$

and $$z^2\left(x^2/a^2 + y^2/b^2\right) = -3/4$$

which are $\underbrace{\text{the eqs of}}$ edge of regressions

§ DEVELOPABLE SURFACES

We Know, in One-parameter family of planes, the characteristics, being the intersection of ~~planes~~ consecutive planes, are st. lines. These st lines are called the generator of the envelope; and the envelope is called a developable Surface or Simply a develolde.



The reason for the name lies in the fact that the surface may be unrolled or developed into a plane without stretching or Tearing.

Also note that; each plane in One parameter family of planes touches the envelope along its generator. it follows that the Tangent plane to a developable Surface at all pts of a generator corresponding to a plane in the family is the plane itself. Thus a develapable Surface has a Constant Tangent plane along the generator. So That the Tangent planes depends on only one parameter.

The Edge of Regression of the developable is the locus of intersection of consecutive generators and is touched by each of the generators.

The Osculating planes of the edge of regression at any point is the Tangent plane to the developable at that point.

Theorem To find the condition that a surface is a developable. (Let $z = f(x,y)$ be eq of surface)

Proof: The eq. of the Tangent plane at a point (x,y,z) is $Z - z = (X-x)\,\partial F/\partial x - (Y-y)\frac{\partial F}{\partial y}$. Since Tangent plane to a developable depends on only one parameter, therefore, there must be some relation between $\frac{\partial F}{\partial x}$ and $\partial F/\partial y$, ~~~~ which we may write

$$\frac{\partial F}{\partial x} = \phi\left(\partial F/\partial y\right).$$

On differentiation, this gives (w.r.t x & y separately)

$$\frac{\partial^2 F}{\partial x^2} = \phi'\left(\frac{\partial F}{\partial y}\right)\frac{\partial^2 F}{\partial x \partial y} \quad \text{— (i)}$$

and
$$\frac{\partial^2 F}{\partial x \partial y} = \phi'\left(\frac{\partial F}{\partial y}\right)\frac{\partial^2 F}{\partial y^2} \quad \text{— (ii)}$$

From (i) & (ii),

$$\left(\frac{\partial^2 F}{\partial x^2}\right)\cdot\left(\frac{\partial^2 F}{\partial y^2}\right) = \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$$

which is the required condition.

Example Prove that $xy = (z-c)^2$ is a developable surface

$$F(x,y,z) = z^2 - 2cz + c^2 - xy = 0$$

$$\frac{\partial F}{\partial x} = -y \quad \& \quad \frac{\partial^2 F}{\partial x^2} = 0$$

$$\frac{\partial F}{\partial y} = -x \quad \& \quad \frac{\partial^2 F}{\partial y^2} = 0$$

$$\cancel{\frac{\partial^2 F}{\partial x \partial y} = -1}$$

$$z - c = \sqrt{xy}$$

$$z = \sqrt{xy} + c$$

$$\frac{\partial F}{\partial x} = \tfrac{1}{2}(xy)^{-1/2}\, y$$

$$= \frac{y}{2\sqrt{xy}}$$

$$\frac{\partial F}{\partial y} = \frac{1}{2}\frac{x}{\sqrt{xy}}$$

$$\frac{\partial F}{\partial x} = \frac{1}{2}\sqrt{y/x}$$

$$\frac{\partial^2 F}{\partial x \partial y} = 1/4\,(y/x)^{-1/2}\cdot 1/x$$

$$\frac{\partial^2 F}{\partial x^2} = 1/4\,(y/x)^{-1/2}\,(-y/x^2)$$

$$\frac{\partial^2 F}{\partial y^2} = 1/4\,(x/y)^{-1/2}\,(-x/y^2)$$

$$\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = \left(1/4\,(1/x)\,(x/y)^{1/2}\right)^2 = \frac{1}{16}(\ldots$$

$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} \Rightarrow \frac{1}{16}\left(\frac{x}{y}\right)(\ldots)\,\sqrt{x/y}\,\sqrt{y/x} = 1/16\,(1/xy)$

LHS = RHS.

§ OSCULATING DEVELOPABLE

Def. The envelope of the osculating plane is called the osculating developable. Since the intersection of consecutive osculating planes are the tangents to the curve, it follows that the tangents are the generators of the developable. And consecutive tangents intersect at a point on the curve; so that the curve itself is the edge of regression of the osculating developable.

Theorem Prove that

(i) The generator of the osculating developable of a twisted curve are the tangents to the curve.

(ii) The edge of regression is the curve itself.

Proof: At a point $\underline{r}$ on the curve, the eq of the osculating plane is $(\underline{R}-\underline{r})\cdot\underline{b}=0$ ——— (i)

where $\underline{r}$ and $\underline{b}$ are fns of s. Diff (i) wrt s.

$$(0-\underline{r}')\cdot\underline{b}+(\underline{R}-\underline{r})\cdot\underline{b}'=0$$

$$-\underline{t}\cdot\underline{b}+(\underline{R}-\underline{r})\cdot(-\tau\underline{n})=0$$

$$0-\tau(\underline{R}-\underline{r})\cdot\underline{n}=0 \qquad \text{but } \underline{t}\cdot\underline{b}=0$$

$$\Rightarrow (\underline{R}-\underline{r})\cdot\underline{n}=0 \text{ ——— (ii)}$$

which is the equation of rectifying plane. Thus the characteristic which is given by (i) & (ii) is the intersection of the osculating and rectifying planes and is therefore, the tangent to the curve at $\underline{r}$.

To find the edge of regression, differentiating (2) wrt s,

$$(\underline{R}-\underline{r})\cdot\underline{n}'+(0-\underline{r}')\cdot\underline{n}=0$$

$$(\tau\underline{b}-\kappa\underline{t})\cdot(\underline{R}-\underline{r})-\underline{t}\cdot\underline{n}=0$$

$$\tau(\underline{R}-\underline{r})\cdot\underline{b}-\kappa(\underline{R}-\underline{r})\cdot\underline{t}-\underline{t}\cdot\underline{n}=0$$

$$\Rightarrow \tau(\underline{R}-\underline{r})\cdot\underline{b}-\kappa(\underline{R}-\underline{r})\cdot\underline{t}-0=0 \qquad \because \underline{t}\cdot\underline{n}=0$$

From eq ① $(\underline{R}-\underline{r})\cdot\underline{b} = 0$. Hence from the last eq, we have $K(\underline{R}-\underline{r})\cdot\underline{b} = 0$ —③

From ① & ② it is clear that: $\underline{R}-\underline{r}$ is ll to [illegible]

Hence eq ③ implies that

$$(\underline{R}-\underline{r}) = 0 \Rightarrow \underline{R} = \underline{r}$$

The edge of regression is the curve itself.

§ POLAR DEVELOPABLE.

The envelope of the normal plane of a twisted curve is called the polar developable and its generators are called polar lines.

Theorem. Show that a polar line is the axis of the circle of curvature and the edge of regression of the polar developable is the locus of centre of spherical curvature.

Proof Let $P(\underline{r})$ be a point on the curve whose normal plane is $(\underline{R}-\underline{r})\cdot\underline{t} = 0 \longrightarrow ①$

Diff w r t s

$$(\underline{R}-\underline{r})\cdot\underline{t}' + (0-\underline{r}')\cdot\underline{t} = 0$$

$$K(\underline{R}-\underline{r})\cdot\underline{n} - \underline{t}\cdot\underline{t} = 0 \qquad \because \underline{t}\cdot\underline{t}=1$$

$$\Rightarrow (\underline{R}-\underline{r})\cdot\underline{n} = 1/K$$

$$\Rightarrow (\underline{R}-\underline{r})\cdot\underline{n} = \rho\,(\underline{n}\cdot\underline{n}) \qquad \because \underline{n}\cdot\underline{n} =$$

$$\Rightarrow \left[(\underline{R}-\underline{r}) - \rho\,\underline{n}\right]\cdot\underline{n} = 0 \longrightarrow ②$$

which represents a plane through the centre of curvature $\perp$ to the principal normal.

It intersects the normal plane in a st. line through the centre of curvature ll to the binormal. Thus the polar line is the axis of the circle of curvature.

For II part.

From eq : $K(\underline{R}-\underline{r})\cdot\underline{n} = 1$

$$(\underline{R}-\underline{r})\cdot\underline{n} = 1/K = \rho$$

Diff w r t s

$$(\underline{R}-\underline{r})\cdot \underline{n}' + (0-\underline{r}')\cdot \underline{n} = \rho'$$

$$(\underline{R}-\underline{r})\cdot(\tau\underline{b}-\kappa\underline{t}) - \underline{t}\cdot\underline{n} = \rho'$$

$\underline{t}\cdot\underline{n}=0$

$$\tau(\underline{R}-\underline{r})\cdot\underline{b} - \kappa(\underline{R}-\underline{r})\cdot\underline{t} = \rho'$$

Since from ① $\kappa(\underline{R}-\underline{r})\cdot\underline{t} = 0$

$$\therefore \tau(\underline{R}-\underline{r})\cdot\underline{b} = \rho'$$

$$(\underline{R}-\underline{r})\cdot\underline{b} = \sigma\rho' \longrightarrow ③ \qquad \frac{1}{\tau}=\sigma$$

From (1) (2) and (3) it follows that

$$(\underline{R}-\underline{r}) = \rho\underline{n} + \sigma\rho'\underline{b}$$

(Take dot prod with $\underline{b}$ for ③)

$$\text{or} \quad \underline{R} = \underline{r} + \rho\underline{n} + \sigma\rho'\underline{b}$$

This is the equation of spherical curvature. Hence the edge of regression of Polar developable is the locus of the centre of spherical curvature.

RECTIFYING DEVELOPABLE:—

The envelope of the retifying plane of a curve is called the rectifying developable and its generators are the rectifying lines.

Thus the recifying lines at a pt P of the curve is the intersection of consecutive rectifying planes.

Theorem. Prove that (i) The rectifying line is ∥ to the vector $(\tau\underline{t}+\kappa\underline{b})$

(ii) A point on the edge of regression corresponding to a point $\underline{r}$ on the curve is given by $\underline{R} = \underline{r} + \dfrac{\kappa(\tau\underline{t}+\kappa\underline{b})}{\kappa'\tau-\kappa\tau'}$

Proof The equation of rectifying plane at the point $\underline{r}$ is $(\underline{R}-\underline{r})\cdot\underline{n} = 0 \longrightarrow ①$

Diff it w.r.t. s

we have $(\underline{R}-\underline{r})\cdot \bar{n} + (0-\underline{r}')\cdot \underline{n} = 0$

$$(\underline{R}-\underline{r})\cdot(\tau \underline{b} - \kappa \underline{t}) - \underline{t}\cdot\underline{n} = 0$$

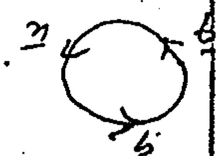
$\underline{t}\cdot\underline{n} = 0$

$$(\underline{R}-\underline{r})\cdot(\tau \underline{b} - \kappa \underline{t}) = 0 \quad \text{--- (2)}$$

Since $(\underline{R}-\underline{r})$ is $\perp$ to $\underline{n}$ and $(\tau\underline{b} - \kappa\underline{t})$

So it is $\parallel$ to the vector product of these two.

From eqs (1) and (2), it follows that rectifying line is $\perp$ to both $\underline{n}$ & $(\tau\underline{b} - \kappa\underline{t})$ & hence $\parallel$ to $\underline{n} \times (\tau\underline{b} - \kappa\underline{t})$



$$= \tau(\underline{n}\times\underline{b}) - \kappa(\underline{n}\times\underline{t})$$

$$= \tau\,\underline{t} + \kappa\,\underline{b}$$

$\underline{n}\times\underline{t} = -$

So the rectifying line is $\parallel$ to

$$\tau\underline{t} + \kappa\underline{b}$$

(ii) Diff (2) $(\underline{R}-\underline{r})\cdot(\tau\underline{b} - \kappa\underline{t}) = 0$ w.r.t. s for the edge of regression

$$\Rightarrow (\underline{R}-\underline{r})\cdot(\tau\underline{b}' + \tau'\underline{b} - \kappa'\underline{t} - \kappa\underline{t}') + (0-\underline{r}')\cdot(\tau\underline{b} - \kappa\underline{t}) = 0$$

$$\Rightarrow -\tau^2(\underline{R}-\underline{r})\cdot\underline{n} + (\underline{R}-\underline{r})\cdot\tau'\underline{b} - \kappa^2(\underline{R}-\underline{r})\cdot\underline{n}$$
$$- \kappa'(\underline{R}-\underline{r})\cdot\underline{t} - \tau\,\underline{t}\cdot\underline{b} + \kappa\,\underline{t}\cdot\underline{t} = 0$$

& From (1) $(\underline{R}-\underline{r})\cdot\underline{n} = 0$

$$\Rightarrow (\underline{R}-\underline{r})\cdot(\tau'\underline{b} - \kappa'\underline{t}) + \kappa = 0 \quad \text{--- (3)}$$

Also since $(\underline{R}-\underline{r})$ is $\parallel$ to $(\tau\underline{t} + \kappa\underline{b})$ we can write

$$(\underline{R}-\underline{r}) = l\,(\tau\underline{t} + \kappa\underline{b}) \quad \text{--- (4)}$$

Putting in (3)

$\underline{t}\cdot\underline{b} = 0$

$$l(\tau\underline{t} + \kappa\underline{b})\cdot(\tau'\underline{b} - \kappa'\underline{t}) + \kappa = 0$$

$$\Rightarrow l(\tau'\kappa - \kappa'\tau) + \kappa = 0 \qquad \Rightarrow l = \frac{\kappa}{\kappa'\tau - \kappa\tau'}$$

Putting in (4)

a(x²y) + xyz = 0 meets it again in a conic which projection on the plane of xy is a rectangular hyperbola.

$$\underline{R} = \underline{r} + \frac{\kappa(\tau \underline{b} - \kappa \underline{b})}{\kappa\tau - \kappa\tau'}$$

Hence the result.

~~XX~~ Ex. on Page (39)

Q No. 5 Any Tangent plane to the surface $a(x^2+y^2)+xyz=0$ meets it again in a conic which projection on the plane of XY is a rectangular hyperbola

Eq of Surface, $F(x,y,z) = a(x^2+y^2) + xyz = 0$

$$\frac{\partial F}{\partial x} = 2ax + yz, \quad \frac{\partial F}{\partial y} = 2ay + xz$$

$$\frac{\partial F}{\partial z} = xy$$

For any pt on the surface $P(\alpha, \beta, \gamma)$

$$\left(\frac{\partial F}{\partial x}\right)_{at\ P} = 2a\alpha + \beta\gamma$$

$$\left.\frac{\partial F}{\partial y}\right)_{at\ P} = 2a\beta + \alpha\gamma \qquad \left.\frac{\partial F}{\partial z}\right)_{at\ P} = \alpha\beta$$

The eq of Tangent plane in cartesian form is

$$(X-\alpha)\frac{\partial F}{\partial x} + (Y-\beta)\frac{\partial F}{\partial y} + (Z-\gamma)\frac{\partial F}{\partial z} = 0$$

$$(X-\alpha)(2a\alpha+\beta\gamma) + (Y-\beta)(2a\beta+\alpha\gamma) + (Z-\gamma)\alpha\beta = 0$$

$$\Rightarrow X(2a\alpha+\beta\gamma) + Y(2a\beta+\alpha\gamma) + Z\alpha\beta - 2a\alpha^2 - \alpha\beta\gamma - 2a\beta^2$$
$$- \alpha\beta\gamma - \alpha\beta\gamma = 0$$

$$\Rightarrow (2a\alpha+\beta\gamma)X + (2a\beta+\alpha\gamma)Y + \alpha\beta Z - 2a(\alpha^2+\beta^2) - 3\alpha\beta\gamma = 0$$

As the pt lies on the surface, therefore we have

$$a(\alpha^2+\beta^2) + \alpha\beta\gamma = 0 \Rightarrow \gamma = -\frac{a(\alpha^2+\beta^2)}{\alpha\beta}$$

For Projection on xy plane $z=0$

$$X\left(2a\alpha - \beta\left(\frac{a\alpha^2+a\beta^2}{\alpha\beta}\right)\right) + \left(2a\beta - \alpha\left(\frac{a\alpha^2+a\beta^2}{\alpha\beta}\right)\right)Y$$
$$- 2a(\alpha^2+\beta^2) + 3a(\alpha^2+\beta^2) = 0$$

$$\Rightarrow X\left[\frac{2a\alpha^2 - a\alpha^2 - a\beta^2}{\alpha}\right] + Y\left[\frac{2a\beta^2 - a\alpha^2 - a\beta^2}{\beta}\right] + a(\alpha^2+\beta^2) = 0$$

$$\Rightarrow \frac{X\,a(\alpha^2-\beta^2)}{\alpha} - Y\left(\frac{a(\alpha^2-\beta^2)}{\beta}\right) = a(\alpha^2+\beta^2)$$

$$\frac{X}{\alpha} - \frac{Y}{\beta} = 1$$

Ex. on Page (50)

Q (1) Find the envelope of the planes through the centre of an Ellipsoid and cutting it in sections of constant area.

Sol

Let Eq of Ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

& let $lx+my+nz=0$ —— (2) be eq of plane passing through the centre of Ellipsoid.

Then the area of the Section of the Ellipsoid cut by plane 2 is given by

$$=\frac{\pi abc}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}=\text{Constant}=K \quad —— (3)$$

$$F(x,y,z;\,l,m,n)=lx+my+nz=0$$

$$F_l=x\ ,\quad F_m=y,\quad F_n=z$$

$$f(l,m,n)=\frac{\pi abc}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}-K=0$$

$$f_l=\frac{-\pi a^3bc\,l}{(a^2l^2+b^2m^2+c^2n^2)^{3/2}}\quad \text{Similarly}\quad f_m=\frac{-\pi ab^3c\,m}{(a^2l^2+b^2m^2+c^2n^2)^{3/2}}$$

$$f_n=\frac{-\pi abc^3\,n}{(a^2l^2+b^2m^2+c^2n^2)^{3/2}}$$

Now $\frac{F_l}{f_l}=\frac{F_m}{f_m}=\frac{F_n}{f_n}$ is $\frac{x}{a^2l}=\frac{y}{b^2m}=\frac{z}{c^2n}=K$ (sa

$$l=\frac{x}{a^2K}\ ,\quad m=\frac{y}{b^2K}\quad n=\frac{z}{c^2K}$$

Putting these values of l,m,n in —— (2), we have

$$\frac{x^2}{a^2K}=\frac{y^2}{b^2K}=\frac{z^2}{c^2K}=0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Which is the required eq of envelope.

Q(2). A Plane makes intercepts a, b, c on the coordinate axes s.t. $1/a^2 + 1/b^2 + 1/c^2 = 1/k^2$, Prove that its Envelope is a conicoid with equi conjugate diameters along the axes.

Solution: Let the eq of plane be $x/a + y/b + z/c = 1$ ⟶ (0)

then $F(x,y,z,a,b,c) = x/a + y/b + z/c - 1 = 0$ ⟶ (1)

$$f(a,b,c) = 1/a^2 + 1/b^2 + 1/c^2 - 1/k^2 = 0 \longrightarrow (2)$$

$$F_a = -x/a^2, \quad F_b = -y/b^2, \quad F_c = -z/c^2$$

$$f_a = -2/a^3, \quad f_b = -2/b^3 \quad f_c = -2/c^3$$

then $\frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}$ is $\frac{-x/a^2}{-2/a^3} = \frac{-y/b^2}{-2/b^3} = \frac{-z/c^2}{-2/c^3}$

$$\Rightarrow \frac{ax}{2} = \frac{by}{2} = \frac{cz}{2} = l$$

$$a = l/x, \quad b = l/y \quad c = l/z$$

putting in (1) $x^2/l + y^2/l + z^2/l = 1$

$$x^2+y^2+z^2 = l$$

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Q(3) Prove that the envelope of a plane, the Sum of Squares of whose intercepts on the axes is Constant & it is a Surface $x^{2/3}+y^{2/3}+z^{2/3} =$ Constant.

Solution

Let Eq of a plane be $lx+my+nz-1=0$

This planes intersects Coordinate axes at pts

$(1/l,0,0)$ $(0,1/m,0)$ & $(0,0,1/n)$

Sum of Squares of intercepts is

$$1/l^2 + 1/m^2 + 1/n^2 = \text{Constant (Given)} \quad (1)$$

so $f(l,m,n) = 1/l^2 + 1/m^2 + 1/n^2 - C = 0$

$$f_l = -2/l^3, \quad f_m = -2/m^3, \quad f_n = -2/n^3$$

Also $F(x,y,z,l,m,n) = lx+my+nz-1 = 0$

$$F_l = x, \quad F_m = y \quad F_n = z$$

Now $\frac{F_l}{f_l} = \frac{F_m}{f_m} = \frac{F_n}{f_n}$ is written as

$$\frac{x}{-2/l^3} = \frac{y}{-2/m^3} = \frac{z}{-2/n^3} = k$$

$$l^3 x = m^3 y + n^3 z = -2k = K$$

$$x = \frac{K}{l^3}, \quad y = \frac{K}{m^3}, \quad z = \frac{K}{n^3}$$

Then $x^{2/3} + y^{2/3} + z^{2/3} = \left(\frac{K}{l^3}\right)^{2/3} + \left(\frac{K}{m^3}\right)^{2/3} + \left(\frac{K}{n^3}\right)^{2/3}$

$$= K^{2/3}\left[\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}\right)$$

$=$ Constant by 1.

Q(3) Prove that the envelope of surface $F(x,y,z,a,b,c) = 0$ where a,b,c are parameters connected by the relation $f(a,b,c) = 0$ is obtained by eliminating a,b,c from the eqs $F=0$ and $f=0$ is $\frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}$.

Solution:

Let $F(x,y,z,a,b,c) = 0 \longrightarrow (1)$

& $f(a,b,c) = 0 \longrightarrow (2)$

Diff (1) & (2) totally,

$$dF = \frac{\partial F}{\partial a}da + \frac{\partial F}{\partial b}db + \frac{\partial F}{\partial c}dc = 0 \longrightarrow (3)$$

$$df = \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db + \frac{\partial f}{\partial c}dc = 0 \quad (4)$$

Multiply eq (3) by $\frac{\partial f}{\partial c}$ and (4) by $\frac{\partial F}{\partial c}$ & subtracting

$$\left(\frac{\partial f}{\partial c}\frac{\partial F}{\partial a} - \frac{\partial F}{\partial c}\frac{\partial f}{\partial a}\right)da + \left(\frac{\partial f}{\partial c}\cdot\frac{\partial F}{\partial b} - \frac{\partial f}{\partial b}\cdot\frac{\partial F}{\partial c}\right)db = 0$$

or, $(f_c F_a - F_c f_a)da + (F_b f_c - F_c f_b)db = 0$

$\longrightarrow (5)$

Let $\frac{da}{db} = K$, when da, db are the changes in parameters a & b. For different values of da & db, K will be difference and non zero. Equation (5) is satisfied only if

$$f_c F_a - F_c f_a = 0 \quad \& \quad f_c F_b - F_c f_b = 0$$

$$f_c F_a = F_c f_a \quad \& \quad f_c F_b = F_c f_b \text{ also.}$$

$$\frac{F_a}{f_a} = \frac{F_c}{f_c} \quad (6) \quad \& \quad \frac{F_b}{f_b} = \frac{F_c}{f_c} \quad \text{—— (7)}$$

From (6) and (7) implies

$$\frac{F_a}{f_a} = \frac{F_b}{f_b} = \frac{F_c}{f_c}$$

Required result.

Q(4) Prove that the envelope of a plane which forms with the coordinate planes a tetrahedron of constant volume is a surface xyz = constant.

Solution: Let the eq of a plane be

$$lx + my + nz = 1$$

This plane meets with the coordinate axes in the point $(1/l, 0, 0)$, $(0, 1/m, 0)$ & $(0, 0, 1/n)$ so the volume of tetrahedron is

$$|V| = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1/l & 0 & 0 & 1 \\ 0 & 1/m & 0 & 1 \\ 0 & 0 & 1/n & 1 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} 1/l & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/n \end{vmatrix} = \frac{1}{6lmn}$$

As volume is constant

$$\frac{1}{6lmn} = c \Rightarrow \frac{1}{lmn} = 6c = C$$

$$\frac{1}{lmn} - C = 0$$

$$f_l = \frac{1}{-l^2mn}, \quad f_m = -\frac{1}{lm^2n}, \quad f_n = \frac{-1}{lmn^2}$$

Also $F(x,y,z,l,m,n) = lx+my+nz-1=0$

$$F_l = x, \quad F_m = y \quad \& \quad F_n = z$$

Then $\dfrac{F_l}{f_l} = \dfrac{F_m}{f_m} = \dfrac{F_n}{f_n}$

$$\Rightarrow \frac{x}{\frac{-1}{l^2mn}} = \frac{y}{\frac{-1}{lm^2n}} = \frac{z}{\frac{-1}{lmn^2}} = k \text{ (say)}$$

$$\Rightarrow \frac{lx}{\frac{-1}{lmn}} = \frac{my}{\frac{-1}{lmn}} = \frac{nz}{\frac{-1}{lmn}} = k$$

$$\Rightarrow lx = my = nz = \frac{-k}{lmn} \qquad \text{by (1)}$$

$$\Rightarrow lx = my = nz = -kc$$

$$\Rightarrow lx = my = nz = k'$$

$$x = k'/l, \quad y = \frac{k'}{m} \quad z = \frac{k'}{n}$$

Then $xyz = \dfrac{(k')^3}{lmn} = k'^3 c$ by 1

$= \text{constant}$