UNIT 3

Matrices and Determinants

EXERCISE 3.1

1. Write the number of rows and number of columns in each matrix.

$$A=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad B=\begin{bmatrix}3\\4\\7\end{bmatrix},\quad C=\begin{bmatrix}8&-10&11\end{bmatrix},$$

$$D=\begin{bmatrix}3&2&1\\1&0&3\\4&5&6\end{bmatrix},\quad E=\begin{bmatrix}5&9&-2\\-3&4&5\end{bmatrix}$$

Solution

Number of rows in $A=2$, Number of columns in $A=2$, Number of rows in $B=3$,
Number of columns in $B=1$, Number of rows in $C=1$, Number of columns in $C=3$,
Number of rows in $D=3$, Number of columns in $D=3$, Number of rows in $E=2$,
Number of columns in $E=3$

2. Write the order of each matrix.

$$A=\begin{bmatrix}1\\0\end{bmatrix},\ B=\begin{bmatrix}3&4\end{bmatrix},\ C=\begin{bmatrix}3&5\\5&6\end{bmatrix},\ D=\begin{bmatrix}0&3&4\\5&0&9\end{bmatrix},\ E=\begin{bmatrix}0&1\\3&4\\5&6\end{bmatrix},\ F=\begin{bmatrix}5\end{bmatrix}$$

Solution

2. Order of A = 2-by-1, Order of B = 1-by-2, Order of C = 2-by-2, Order of D = 2-by-3,
Order of E = 3-by-2, Order of F = 1-by-1

3. Which of the following matrices are equal?

$$A=\begin{bmatrix}0&3\\4&1\end{bmatrix},\quad B=\begin{bmatrix}9\\-7\end{bmatrix},\quad C=\begin{bmatrix}2\times3&2-1\\2\times2&4-2\\4+4&3+0\end{bmatrix},$$

$$D=\begin{bmatrix}5+4\\-8+1\end{bmatrix},\quad E=\begin{bmatrix}6&1\\4&2\\8&3\end{bmatrix},\quad F=\begin{bmatrix}3-3&3\\3+1&1\end{bmatrix}$$

Solution

$A=F,\ B=D,\ C=E$

4. If $\begin{bmatrix} a+2 & c-3 \\ b-1 & d+4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 6 & 4 \end{bmatrix}$, then find the values of a, b, c and d.

Solution

$a + 2 = 5 \Rightarrow a = 5 - 2 \Rightarrow a = 3$
$b - 1 = 6 \Rightarrow b = 6 + 1 \Rightarrow b = 7$
$c - 3 = 8 \Rightarrow c = 8 + 3 \Rightarrow c = 11$
$d + 4 = 4 \Rightarrow d = 4 - 4 \Rightarrow d = 0$

5. If $\begin{bmatrix} 2x+1 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 7 & y \end{bmatrix}$, then find the values of x and y.

Solution

$2x + 1 = 9 \Rightarrow 2x = 9 - 1 \Rightarrow 2x = 8 \Rightarrow x = 4$
$y = 5$

6. If $\begin{bmatrix} a+b & 2d-1 \\ 3b+2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 11 & c \end{bmatrix}$, then find the values of a, b, c and d.

Solution

$3b + 2 = 11 \Rightarrow 3b = 11 - 2 \Rightarrow 3b = 9 \Rightarrow b = 3$
$a + b = 10 \Rightarrow a + \mathbf{3} = 10 \Rightarrow a = 10 - 3 \Rightarrow a = 7$
$2d - 1 = 5 \Rightarrow 2d = 5 + 1 \Rightarrow 2d = 6 \Rightarrow d = 3$
$c = 4$

7. If $\begin{bmatrix} p+q & 5 \\ 11 & p-2q \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 11 & 0 \end{bmatrix}$, then find the values of p and q.

Solution

$p + q = 6$ ………. (i) & $p - 2q = 0$ ………. (ii)

$\mathbf{2}p + \mathbf{2}q = \mathbf{12}$ $p - 2q = 0$
$3p = 12$ $p = 4$

Put $p = 4$ in (i)

$\Rightarrow \mathbf{4} + q = 6 \Rightarrow q = 6 - 4$

$\Rightarrow q = 2$

EXERCISE 3.2

1. From the following matrices identify unit matrices, row matrices, column matrices and null matrices.

$A=\begin{bmatrix}5 & 7 & 8\end{bmatrix}$, $B=\begin{bmatrix}0\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$,

$D=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, $E=\begin{bmatrix}6\\0\\8\end{bmatrix}$, $F=\begin{bmatrix}7\\1\\9\end{bmatrix}$

Solution

A = row matrix , B = null matrix , C = unit matrix, D = null matrix,
E = column matrix , F = column matrix

2. Identify type of the given matrices as row, column, square and rectangular matrices.

$A=\begin{bmatrix}5\\3\\4\end{bmatrix}$, $B=\begin{bmatrix}1 & 6\\4 & 1\end{bmatrix}$, $C=\begin{bmatrix}3 & 5 & 8\\0 & 4 & -2\end{bmatrix}$, $D=\begin{bmatrix}5 & -5\\2 & 7\end{bmatrix}$,

$E=\begin{bmatrix}3 & 2\\4 & 1\\5 & 0\end{bmatrix}$, $F=\begin{bmatrix}5 & -3 & 7\end{bmatrix}$, $G=\begin{bmatrix}3 & 0 & 1\\1 & 3 & 4\\5 & 2 & -3\end{bmatrix}$, $H=\begin{bmatrix}3 & 5\\4 & 4\\5 & 2\end{bmatrix}$

Solution

A = column matrix , B = square matrix , C = rectangular matrix,
D = square matrix , E = rectangular matrix , F = row matrix,
G = square matrix , H = rectangular matrix

3. Identify diagonal, scalar and unit matrices.

$A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $B=\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}$, $C=\begin{bmatrix}3 & 0\\0 & 0\end{bmatrix}$,

$D=\begin{bmatrix}7 & 0\\0 & 6\end{bmatrix}$, $E=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$

Solution

A = unit matrix , B = scalar matrix , C = diagonal matrix,
D = diagonal matrix , E = scalar matrix

4. Find transpose of each of the following matrices:

$$A=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix},\quad B=\begin{bmatrix}3\\7\\6\end{bmatrix},\quad C=\begin{bmatrix}5 & -2 & 4\end{bmatrix},\quad D=\begin{bmatrix}2 & 5\\3 & 6\\4 & 7\end{bmatrix}$$

Solution

$$A^t=\begin{bmatrix}2 & 4\\3 & 5\end{bmatrix},\ B^t=\begin{bmatrix}3 & 7 & 6\end{bmatrix},\ C^t=\begin{bmatrix}5\\-2\\4\end{bmatrix},\ D^t=\begin{bmatrix}2 & 3 & 4\\5 & 6 & 7\end{bmatrix}$$

5. Find negative of the following matrices:

$$A=\begin{bmatrix}-3 & 0\\5 & 6\end{bmatrix},\quad B=\begin{bmatrix}-3 & 3\\-2 & 2\end{bmatrix},\quad C=\begin{bmatrix}-9 & 1\\1 & -7\end{bmatrix}$$

Solution

$$-A=\begin{bmatrix}3 & 0\\-5 & -6\end{bmatrix},\ -B=\begin{bmatrix}3 & -3\\2 & -2\end{bmatrix},\ -C=\begin{bmatrix}9 & -1\\-1 & 7\end{bmatrix}$$

6. If $A=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$ and $B=\begin{bmatrix}7 & 6\\5 & 8\end{bmatrix}$, then verify that

(i) $(A^t)^t = A$ (ii) $(B^t)^t = B$

Solution

$A=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$	$B=\begin{bmatrix}7 & 6\\5 & 8\end{bmatrix}$
$A^t=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}^t=\begin{bmatrix}2 & 4\\3 & 5\end{bmatrix}$	$B^t=\begin{bmatrix}7 & 6\\5 & 8\end{bmatrix}^t=\begin{bmatrix}7 & 5\\6 & 8\end{bmatrix}$
$(A^t)^t=\begin{bmatrix}2 & 4\\3 & 5\end{bmatrix}^t=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$	$(B^t)^t=\begin{bmatrix}7 & 5\\6 & 8\end{bmatrix}^t=\begin{bmatrix}7 & 6\\5 & 8\end{bmatrix}$
Hence $(A^t)^t = A$	Hence $(B^t)^t = B$

7. Show that $L=\begin{bmatrix}2 & 3 & 4\\ 3 & -2 & 5\\ 4 & 5 & 0\end{bmatrix}$ is a symmetric matrix.

Solution

$$L=\begin{bmatrix}2 & 3 & 4\\ 3 & -2 & 5\\ 4 & 5 & 0\end{bmatrix}$$

$$L^t=\begin{bmatrix}2 & 3 & 4\\ 3 & -2 & 5\\ 4 & 5 & 0\end{bmatrix}^t=\begin{bmatrix}2 & 3 & 4\\ 3 & -2 & 5\\ 4 & 5 & 0\end{bmatrix}$$

$$L^t=L$$

Hence $L=\begin{bmatrix}2 & 3 & 4\\ 3 & -2 & 5\\ 4 & 5 & 0\end{bmatrix}$ is symmetric matrix

EXERCISE 3.3

1. Which of the following matrices are conformable for addition and subtraction?

$A=\begin{bmatrix}3 & 4\\ -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}3\\ 4\end{bmatrix}$, $C=\begin{bmatrix}5 & 2\end{bmatrix}$, $D=\begin{bmatrix}1 & 7\\ 2 & 5\end{bmatrix}$ $E=\begin{bmatrix}2\end{bmatrix}$,

$F=\begin{bmatrix}7 & 11\end{bmatrix}$, $G=\begin{bmatrix}a\\ b\end{bmatrix}$, $H=\begin{bmatrix}3\end{bmatrix}$, $M=\begin{bmatrix}l\\ m\end{bmatrix}$

Solution

A and D are conformable for addition and subtraction.

B, G and M are conformable for addition and subtraction.

C and F are conformable for addition and subtraction.

E and H are conformable for addition and subtraction.

2. If $X=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}$, $Y=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $Z=\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$, then find the following:

(i) $X+Y$ (ii) $Y+7Z$ (iii) $4X-Z$

(iv) $X+2Y+3Z$ (v) $X-4Y+Z$ (vi) $Z-Z$

Solution

(i) $X+Y$ $X+Y=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ $=\begin{bmatrix}1+1 & -1+2\\ -2+3 & 2+4\end{bmatrix}$ $=\begin{bmatrix}2 & 1\\ 1 & 6\end{bmatrix}.$	(ii) $Y+7Z$ $Y+7Z=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+7\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$ $=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}21 & 0\\ 0 & -14\end{bmatrix}$ $=\begin{bmatrix}1+21 & 2+0\\ 3+0 & 4-14\end{bmatrix}$ $=\begin{bmatrix}22 & 2\\ 3 & -10\end{bmatrix}.$

(iii) $4X - Z$ $4X - Z = 4\begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix} - \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}$ $4X - Z = \begin{bmatrix}4 & -4\\-8 & 8\end{bmatrix} - \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}$ $4X - Z = \begin{bmatrix}4-3 & -4-0\\-8-0 & 8+2\end{bmatrix} = \begin{bmatrix}1 & -4\\-8 & 10\end{bmatrix}$	(iv) $X+2Y+3Z$ $X+2Y+3Z = \begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix} + 2\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + 3\begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}$ $= \begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix} + \begin{bmatrix}2 & 4\\6 & 8\end{bmatrix} + \begin{bmatrix}9 & 0\\0 & -6\end{bmatrix}$ $= \begin{bmatrix}1+2+9 & -1+4+0\\-2+6+0 & 2+8-6\end{bmatrix}$ $= \begin{bmatrix}12 & 3\\4 & 4\end{bmatrix}.$
(v) $X-4Y+Z$ $X-4Y+Z = \begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix} - 4\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}$ $= \begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix} + \begin{bmatrix}-4 & -8\\-12 & -16\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}$ $= \begin{bmatrix}1-4+3 & -1-8+0\\-2-12+0 & 2-16-2\end{bmatrix}$ $= \begin{bmatrix}0 & -9\\-14 & -16\end{bmatrix}.$	(vi) $Z-Z$ $Z - Z = \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix} - \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}$ $= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$

3. Find the additive inverse of the following matrices:

(i) $P = \begin{bmatrix}5\\-7\end{bmatrix}$ (ii) $Q = \begin{bmatrix}9 & -3\end{bmatrix}$ (iii) $R = \begin{bmatrix}2 & -1\\-2 & 3\end{bmatrix}$ (iv) $S = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Solution

(i) $P = \begin{bmatrix}5\\-7\end{bmatrix}.$ $-P = \begin{bmatrix}-5\\7\end{bmatrix}.$	(ii) $Q = \begin{bmatrix}9 & -3\end{bmatrix}.$ $-Q = \begin{bmatrix}-9 & 3\end{bmatrix}.$
(iii) $R = \begin{bmatrix}2 & -1\\-2 & 3\end{bmatrix}.$ $-R = \begin{bmatrix}-2 & 1\\2 & -3\end{bmatrix}.$	(iv) $S = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$ $-S = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}.$

4. If $A=\begin{bmatrix}2 & 3\\ -3 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix}$ and $C=\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}$, then verify the following:

(i) $A+B=B+A$ (ii) $(A+B)+C=A+(B+C)$

(iii) $(2A+B)+C=2A+(B+C)$ (iv) $3(A+B)=3A+3B$

Solution

(i) Verify $A+B=B+A$.

$$A+B=\begin{bmatrix}2+3 & 3+4\\ -3+5 & 2+6\end{bmatrix}=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix},$$

$$B+A=\begin{bmatrix}3+2 & 4+3\\ 5-3 & 6+2\end{bmatrix}=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix}.$$

$$A+B=B+A$$

(ii) Verify $(A+B)+C=A+(B+C)$.

$$(A+B)+C=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix}+\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}=\begin{bmatrix}6 & 5\\ 2 & 13\end{bmatrix},$$

$$B+C=\begin{bmatrix}4 & 2\\ 5 & 11\end{bmatrix},$$

$$A+(B+C)=\begin{bmatrix}2+4 & 3+2\\ -3+5 & 2+11\end{bmatrix}=\begin{bmatrix}6 & 5\\ 2 & 13\end{bmatrix}.$$

$$(A+B)+C=A+(B+C)$$

(iii) Verify $(2A+B)+C=2A+(B+C)$.

$$2A=\begin{bmatrix}4 & 6\\ -6 & 4\end{bmatrix},$$

$$2A+B=\begin{bmatrix}7 & 10\\ -1 & 10\end{bmatrix},$$

$$(2A+B)+C=\begin{bmatrix}8 & 8\\ -1 & 15\end{bmatrix},$$

$$B+C=\begin{bmatrix}4 & 2\\ 5 & 11\end{bmatrix},$$

$$2A+(B+C)=\begin{bmatrix}4+4 & 6+2\\ -6+5 & 4+11\end{bmatrix}=\begin{bmatrix}8 & 8\\ -1 & 15\end{bmatrix}.$$

$$(2A+B)+C=2A+(B+C)$$

(iv) Verify $3(A+B)=3A+3B$.

$$A+B=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix},$$

$$3(A+B)=\begin{bmatrix}15 & 21\\ 6 & 24\end{bmatrix},$$

$$3A=\begin{bmatrix}6 & 9\\ -9 & 6\end{bmatrix},\quad 3B=\begin{bmatrix}9 & 12\\ 15 & 18\end{bmatrix},$$

$$3A+3B=\begin{bmatrix}15 & 21\\ 6 & 24\end{bmatrix}.$$

$$3(A+B)=3A+3B$$

5. If $A=\begin{bmatrix}5 & 6\\ 7 & -2\end{bmatrix}$ and $B=\begin{bmatrix}-5 & -6\\ -7 & 2\end{bmatrix}$, then show that B is additive inverse of A and A is additive inverse of B.

Solution

$$A+B=\begin{bmatrix}5 & 6\\ 7 & -2\end{bmatrix}+\begin{bmatrix}-5 & -6\\ -7 & 2\end{bmatrix}$$
$$=\begin{bmatrix}5-5 & 6-6\\ 7-7 & -2+2\end{bmatrix}$$
$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Thus $B=-A$. Similarly,

$$B+A=\begin{bmatrix}-5 & -6\\ -7 & 2\end{bmatrix}+\begin{bmatrix}5 & 6\\ 7 & -2\end{bmatrix}$$
$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

So $A=-B$.

6. If $A=\begin{bmatrix}6 & -2\\ 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}8 & -1\\ 3 & 0\end{bmatrix}$, then verify that

(i) $(A+B)^t=A^t+B^t$ (ii) $(A-B)^t=A^t-B^t$

Solution

(i) $(A+B)^t=A^t+B^t$

L.H.S. = $(A+B)^t$	**R.H.S.** = A^t+B^t
$A+B=\begin{bmatrix}6 & -2\\ 0 & 3\end{bmatrix}+\begin{bmatrix}8 & -1\\ 3 & 0\end{bmatrix}$ $=\begin{bmatrix}6+8 & -2-1\\ 0+3 & 3+0\end{bmatrix}$ $=\begin{bmatrix}14 & -3\\ 3 & 3\end{bmatrix}.$ $(A+B)^t=\begin{bmatrix}14 & 3\\ -3 & 3\end{bmatrix}$	$A^t+B^t=\begin{bmatrix}6 & 0\\ -2 & 3\end{bmatrix}+\begin{bmatrix}8 & 3\\ -1 & 0\end{bmatrix}$ $=\begin{bmatrix}6+8 & 0+3\\ -2-1 & 3+0\end{bmatrix}$ $=\begin{bmatrix}14 & 3\\ -3 & 3\end{bmatrix}.$ Hence $(A+B)^t=A^t+B^t$

(ii) $(A-B)^t = A^t - B^t$

Solution

L.H.S. = $(\boldsymbol{A}-\boldsymbol{B})^{t}$

$$A - B = \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 8 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6-8 & -2-(-1) \\ 0-3 & 3-0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ -3 & 3 \end{bmatrix}.$$

$$(A-B)^t = \begin{bmatrix} -2 & -3 \\ -1 & 3 \end{bmatrix}.$$

R.H.S. = $\boldsymbol{A}^{t}-\boldsymbol{B}^{t}$

$$A^t - B^t = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6-8 & 0-3 \\ -2-(-1) & 3-0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ -1 & 3 \end{bmatrix}.$$

Hence $(\boldsymbol{A}-\boldsymbol{B})^{t} = \boldsymbol{A}^{t}-\boldsymbol{B}^{t}$

EXERCISE 3.4

1. Find AB and BA, if possible.

(i) $A=\begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix},\ B=\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}$ (ii) $A=[1\ \ -2],\ B=\begin{bmatrix}3\\-4\end{bmatrix}$

(iii) $A=\begin{bmatrix}4\\4\end{bmatrix},\ B=[2\ \ 5]$ (iv) $A=\begin{bmatrix}1 & 2\\-1 & 1\\3 & 0\end{bmatrix},\ B=\begin{bmatrix}-1 & 4 & 1\\2 & 3 & 1\end{bmatrix}$

(v) $A=\begin{bmatrix}1 & 2 & 2\\3 & 1 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & 5\\2 & 4\\1 & 6\end{bmatrix}$

Solution

(i) $A=\begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix},\ B=\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}.$

$$AB=\begin{bmatrix}(1)(3)+(2)(1) & (1)(2)+(2)(-1)\\(-1)(3)+(0)(1) & (-1)(2)+(0)(-1)\end{bmatrix}$$
$$=\begin{bmatrix}3+2 & 2-2\\-3+0 & -2+0\end{bmatrix}=\begin{bmatrix}5 & 0\\-3 & -2\end{bmatrix},$$
$$BA=\begin{bmatrix}(3)(1)+(2)(-1) & (3)(2)+(2)(0)\\(1)(1)+(-1)(-1) & (1)(2)+(-1)(0)\end{bmatrix}$$
$$=\begin{bmatrix}3-2 & 6+0\\1+1 & 2+0\end{bmatrix}=\begin{bmatrix}1 & 6\\2 & 2\end{bmatrix}.$$

(ii) $A=[1\ \ -2],\ B=\begin{bmatrix}3\\-4\end{bmatrix}.$

$$AB=[(1)(3)+(-2)(-4)]=[3+8]=[11],$$
$$BA=\begin{bmatrix}(3)(1) & (3)(-2)\\(-4)(1) & (-4)(-2)\end{bmatrix}=\begin{bmatrix}3 & -6\\-4 & 8\end{bmatrix}.$$

(iii) $A=\begin{bmatrix}4\\4\end{bmatrix},\ B=[2\ \ 5].$

$$AB=\begin{bmatrix}(4)(2) & (4)(5)\\(4)(2) & (4)(5)\end{bmatrix}=\begin{bmatrix}8 & 20\\8 & 20\end{bmatrix},$$
$$BA=[(4)(2)+(4)(5)]=[8+20]=[28].$$

(iv) $A=\begin{bmatrix}1 & 2\\-1 & 1\\3 & 0\end{bmatrix},\ B=\begin{bmatrix}-1 & 4 & 1\\2 & 3 & 1\end{bmatrix}$

$$AB=\begin{bmatrix}(1)(-1)+(2)(2) & (1)(4)+(2)(3) & (1)(1)+(2)(1)\\(-1)(-1)+(1)(2) & (-1)(4)+(1)(3) & (-1)(1)+(1)(1)\\(3)(-1)+(0)(2) & (3)(4)+(0)(3) & (3)(1)+(0)(1)\end{bmatrix}$$
$$AB=\begin{bmatrix}-1+4 & 4+6 & 1+2\\1+2 & -4+3 & -1+1\\-3+0 & 12+0 & 3+0\end{bmatrix}=\begin{bmatrix}3 & 10 & 3\\3 & -1 & 0\\-3 & 12 & 3\end{bmatrix}$$
$$BA=\begin{bmatrix}(-1)(1)+(4)(-1)+(1)(3) & (-1)(2)+(4)(1)+(1)(0)\\(2)(1)+(3)(-1)+(1)(3) & (2)(2)+(3)(1)+(1)(0)\end{bmatrix}$$
$$BA=\begin{bmatrix}-1-4+3 & -2+4+0\\2-3+3 & 4+3+0\end{bmatrix}=\begin{bmatrix}-2 & 2\\2 & 7\end{bmatrix}$$

(v) $A=\begin{bmatrix}1 & 2 & 2\\ 3 & 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 2 & 4\\ 1 & 6\end{bmatrix}$

$$BA=\begin{bmatrix}(1)(1)+(2)(2)+(2)(1) & (1)(5)+(2)(4)+(2)(6)\\ (3)(1)+(1)(2)+(1)(1) & (3)(5)+(1)(4)+(1)(6)\end{bmatrix}$$

$$BA=\begin{bmatrix}1+4+2 & 5+8+12\\ 3+2+1 & 15+4+6\end{bmatrix}=\begin{bmatrix}7 & 25\\ 6 & 25\end{bmatrix}$$

$$BA=\begin{bmatrix}(1)(1)+(5)(3) & (1)(2)+(5)(1) & (1)(2)+(5)(1)\\ (2)(1)+(4)(3) & (2)(2)+(4)(1) & (2)(2)+(4)(1)\\ (1)(1)+(6)(3) & (1)(2)+(6)(1) & (1)(2)+(6)(1)\end{bmatrix}$$

$$BA=\begin{bmatrix}1+15 & 2+5 & 2+5\\ 2+12 & 4+4 & 4+4\\ 1+18 & 2+6 & 2+6\end{bmatrix}=\begin{bmatrix}16 & 7 & 7\\ 14 & 8 & 8\\ 19 & 8 & 8\end{bmatrix}$$

2. Verify each statement, using $A=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}5 & 1\\ 1 & 4\end{bmatrix}$

(i) $AB \neq BA$ (ii) $A(B-C)=AB-AC$

(iii) $A(BC)=(AB)C$ (iv) $(BC)^t=C^tB^t$

(v) $(B+C)A=BA+CA$

Solution

(i) Verify $AB \neq BA$.

$$AB=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$$
$$=\begin{bmatrix}(5)(2)+(1)(0) & (5)(-1)+(1)(3)\\ (-1)(2)+(4)(0) & (-1)(-1)+(4)(3)\end{bmatrix}$$
$$=\begin{bmatrix}10+0 & -5+3\\ -2+0 & 1+12\end{bmatrix}=\begin{bmatrix}10 & -2\\ -2 & 13\end{bmatrix},$$
$$BA=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}$$
$$=\begin{bmatrix}(2)(5)+(-1)(-1) & (2)(1)+(-1)(4)\\ (0)(5)+(3)(-1) & (0)(1)+(3)(4)\end{bmatrix}$$
$$=\begin{bmatrix}10+1 & 2-4\\ 0-3 & 0+12\end{bmatrix}=\begin{bmatrix}11 & -2\\ -3 & 12\end{bmatrix}.$$

$AB \neq BA$ (verified).

(ii) Verify $A(B-C)=AB-AC$.

$$B-C=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}-\begin{bmatrix}5 & 1\\ 1 & 4\end{bmatrix}=\begin{bmatrix}-3 & -2\\ -1 & -1\end{bmatrix},$$

$$A(B-C)=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}\begin{bmatrix}-3 & -2\\ -1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}(5)(-3)+(1)(-1) & (5)(-2)+(1)(-1)\\ (-1)(-3)+(4)(-1) & (-1)(-2)+(4)(-1)\end{bmatrix}$$

$$=\begin{bmatrix}-15-1 & -10-1\\ 3-4 & 2-4\end{bmatrix}=\begin{bmatrix}-16 & -11\\ -1 & -2\end{bmatrix},$$

$$AB=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(5)(2)+(1)(0) & (5)(-1)+(1)(3)\\ (-1)(2)+(4)(0) & (-1)(-1)+(4)(3)\end{bmatrix}$$

$$=\begin{bmatrix}10+0 & -5+3\\ -2+0 & 1+12\end{bmatrix}=\begin{bmatrix}10 & -2\\ -2 & 13\end{bmatrix},$$

$$AC=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}\begin{bmatrix}5 & 1\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}(5)(5)+(1)(1) & (5)(1)+(1)(4)\\ (-1)(5)+(4)(1) & (-1)(1)+(4)(4)\end{bmatrix}$$

$$=\begin{bmatrix}25+1 & 5+4\\ -5+4 & -1+16\end{bmatrix}=\begin{bmatrix}26 & 9\\ -1 & 15\end{bmatrix},$$

$$AB-AC=\begin{bmatrix}10 & -2\\ -2 & 13\end{bmatrix}-\begin{bmatrix}26 & 9\\ -1 & 15\end{bmatrix}=\begin{bmatrix}-16 & -11\\ -1 & -2\end{bmatrix}.$$

$A(B-C)=AB-AC$ **(verified)**

(iii) Verify $A(BC)=(AB)C$.

$$BC = \begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}$$
$$= \begin{bmatrix}(2)(5)+(-1)(1) & (2)(1)+(-1)(4)\\(0)(5)+(3)(1) & (0)(1)+(3)(4)\end{bmatrix}$$
$$= \begin{bmatrix}10-1 & 2-4\\0+3 & 0+12\end{bmatrix} = \begin{bmatrix}9 & -2\\3 & 12\end{bmatrix},$$

$$A(BC) = \begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}9 & -2\\3 & 12\end{bmatrix}$$
$$= \begin{bmatrix}(5)(9)+(1)(3) & (5)(-2)+(1)(12)\\(-1)(9)+(4)(3) & (-1)(-2)+(4)(12)\end{bmatrix}$$
$$= \begin{bmatrix}45+3 & -10+12\\-9+12 & 2+48\end{bmatrix} = \begin{bmatrix}48 & 2\\3 & 50\end{bmatrix},$$

$$AB = \begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}$$
$$= \begin{bmatrix}(5)(2)+(1)(0) & (5)(-1)+(1)(3)\\(-1)(2)+(4)(0) & (-1)(-1)+(4)(3)\end{bmatrix}$$
$$= \begin{bmatrix}10+0 & -5+3\\-2+0 & 1+12\end{bmatrix} = \begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix},$$

$$(AB)C = \begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix}\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}$$
$$= \begin{bmatrix}(10)(5)+(-2)(1) & (10)(1)+(-2)(4)\\(-2)(5)+(13)(1) & (-2)(1)+(13)(4)\end{bmatrix}$$
$$= \begin{bmatrix}50-2 & 10-8\\-10+13 & -2+52\end{bmatrix} = \begin{bmatrix}48 & 2\\3 & 50\end{bmatrix}.$$

$A(BC)=(AB)C$ **(verified)**

(iv) $(BC)^t = C^tB^t$

$$BC = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(5)+(-1)(1) & (2)(1)+(-1)(4) \\ (0)(5)+(3)(1) & (0)(1)+(3)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 10-1 & 2-4 \\ 0+3 & 0+12 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 3 & 12 \end{bmatrix},$$

$$(BC)^T = \begin{bmatrix} 9 & 3 \\ -2 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$C^TB^T = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (5)(2)+(1)(-1) & (5)(0)+(1)(3) \\ (1)(2)+(4)(-1) & (1)(0)+(4)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 10-1 & 0+3 \\ 2-4 & 0+12 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ -2 & 12 \end{bmatrix}.$$

$(BC)^t = C^tB^t$ **(verified)**

(v) Verify $(B+C)A = BA + CA$.

$$B+C=\begin{bmatrix}2&-1\\0&3\end{bmatrix}+\begin{bmatrix}5&1\\1&4\end{bmatrix}=\begin{bmatrix}7&0\\1&7\end{bmatrix},$$

$$(B+C)A=\begin{bmatrix}7&0\\1&7\end{bmatrix}\begin{bmatrix}5&1\\-1&4\end{bmatrix}$$
$$=\begin{bmatrix}(7)(5)+(0)(-1)&(7)(1)+(0)(4)\\(1)(5)+(7)(-1)&(1)(1)+(7)(4)\end{bmatrix}$$
$$=\begin{bmatrix}35+0&7+0\\5-7&1+28\end{bmatrix}=\begin{bmatrix}35&7\\-2&29\end{bmatrix},$$

$$BA=\begin{bmatrix}2&-1\\0&3\end{bmatrix}\begin{bmatrix}5&1\\-1&4\end{bmatrix}$$
$$=\begin{bmatrix}(2)(5)+(-1)(-1)&(2)(1)+(-1)(4)\\(0)(5)+(3)(-1)&(0)(1)+(3)(4)\end{bmatrix}$$
$$=\begin{bmatrix}10+1&2-4\\0-3&0+12\end{bmatrix}=\begin{bmatrix}11&-2\\-3&12\end{bmatrix}.$$

$$CA=\begin{bmatrix}5&1\\1&4\end{bmatrix}\begin{bmatrix}5&1\\-1&4\end{bmatrix}=\begin{bmatrix}23&12\\0&17\end{bmatrix},$$

$$BA+CA=\begin{bmatrix}11&-2\\-3&12\end{bmatrix}+\begin{bmatrix}23&12\\0&17\end{bmatrix}=\begin{bmatrix}34&10\\-3&29\end{bmatrix}.$$

$(B+C)A = BA + CA$ (verified).

3. If $\begin{bmatrix} 4 & a \\ b & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, then find the values of a and b.

Solution

$$\begin{bmatrix} 4 & a \\ b & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 24 + 6a \\ 6b + 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\Rightarrow 24 + 6a = 6 \quad \Rightarrow 6a = -18$$

$$\Rightarrow 6b + 18 = 3 \quad \Rightarrow 6b = -15$$

$$a = -3, \; b = \frac{-5}{2}$$

4. If $\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$, then find the values of x and y.

Solution

$$\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3 & 0 - 1 \\ y + 6 & 0 - 2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3 & -1 \\ y + 6 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$$

$$\Rightarrow x + 3 = 7 \quad \Rightarrow x = 4,$$

$$\Rightarrow y + 6 = 4 \quad \Rightarrow y = -2$$

EXERCISE 3.5

1. Find the values of each of the determinant.

(i) $\begin{vmatrix} 10 & 5 \\ 4 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} -5 & 8 \\ -3 & -7 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & 8 \\ 0 & 2 \end{vmatrix}$

Solution

(i) $\begin{vmatrix} 10 & 5 \\ 4 & 6 \end{vmatrix}$

$$= (10)(6) - (5)(4) = 60 - 20 = 40.$$

(ii) $\begin{vmatrix} -5 & 8 \\ -3 & -7 \end{vmatrix}$

$$= (-5)(-7) - (8)(-3) = 35 + 24 = 59.$$

(iii) $\begin{vmatrix} 3 & 8 \\ 0 & 2 \end{vmatrix}$

$$= (3)(2) - (8)(0) = 6.$$

2. Find whether the following matrices are singular or non-singular.

$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 13 & 5 \\ 7 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $|A| = 10 - 9 = 1 \neq 0$ Non-singular

$B = \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}$, $|B| = 42 - 42 = 0$ (singular).

$C = \begin{bmatrix} 13 & 5 \\ 7 & 3 \end{bmatrix}$, $|C| = 39 - 35 = 4 \neq 0$ Non-singular

$D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $|D| = 0$ (singular).

3. Find the value of x when $A=\begin{bmatrix} x & 6 \\ 5 & 15 \end{bmatrix}$ is a singular matrix.

Solution

$$A=\begin{bmatrix} x & 6 \\ 5 & 15 \end{bmatrix}.$$

$$|A| = 15x - 30 = 0.$$

$$\Longrightarrow \; x = 2.$$

4. Find the adjoint of the following matrices:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\quad B=\begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix},\quad C=\begin{bmatrix} -3 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution

$$\text{Adj}A=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{Adj}B=\begin{bmatrix} 7 & 2 \\ -3 & 5 \end{bmatrix}, \text{Adj}C=\begin{bmatrix} 2 & -5 \\ 3 & -3 \end{bmatrix}$$

5. Find multiplicative inverse of the following matrices:

(i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 40 & 8 \\ 5 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$ (v) $\begin{bmatrix} 10 & 8 \\ 3 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$

Solution

$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$. (i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $A^{-1} = \frac{1}{25}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}.$	$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$. (ii) $\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$, $\det = -8-56 = -64$. $A^{-1} = \frac{1}{-64}\begin{bmatrix} 2 & -8 \\ -7 & -4 \end{bmatrix}.$

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$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$.

(iii) $\begin{bmatrix} 40 & 8 \\ 5 & 2 \end{bmatrix}$, $\det = 80 - 40 = 40$.

$A^{-1} = \frac{1}{40}\begin{bmatrix} 2 & -8 \\ -5 & 40 \end{bmatrix}$.

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$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$.

(iv) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$, $\det = -9 - 25 = -34$.

$A^{-1} = \frac{1}{-34}\begin{bmatrix} -3 & -5 \\ -5 & 3 \end{bmatrix}$.

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$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$.

(v) $\begin{bmatrix} 10 & 8 \\ 3 & 3 \end{bmatrix}$, $\det = 30 - 24 = 6$.

$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -8 \\ -3 & 10 \end{bmatrix}$.

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$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$.

(vi) $\begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$, $\det = -10 + 12 = 2$.

$A^{-1} = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ -4 & -2 \end{bmatrix}$.

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6. If $A = \begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}$, then find A^{-1} and prove that $AA^{-1} = A^{-1}A = I$.

Solution

$$\det(A) = -5 + 6 = 1.$$

$$A^{-1} = \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}.$$

$$AA^{-1} = \begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$A^{-1}A = \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

7. Show that the following matrices are multiplicative inverse of each other.

(i) $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix}, \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix}$

Solution: two matrix are multiplicative inverse of each if their product is identity

(i)	(ii)
$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$	$A = \begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix}$
$AB = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$AB = \begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Prove that $(AB)^{-1} = B^{-1}A^{-1}$, if

(i) $A = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$

Solution

(i) $A = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$

$$|A| = (-3)(6) - (-2)(5) = -18 + 10 = -8$$

$$|B| = (2)(2) - (-1)(-3) = 4 - 3 = 1$$

$$AB = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -8 & 7 \end{bmatrix}$$

$$|AB| = (0)(7) - (-1)(-8) = -8$$

$$A^{-1} = \frac{1}{-8} \begin{bmatrix} 6 & 2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{-8} \begin{bmatrix} 7 & 1 \\ 8 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \frac{1}{-8} \begin{bmatrix} 6 & 2 \\ -5 & -3 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 7 & 1 \\ 8 & 0 \end{bmatrix}$$

Prove that $(AB)^{-1} = B^{-1}A^{-1}$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$.

$$|A| = (1)(0) - (2)(8) = -16$$

$$|B| = (0)(2) - (-1)(5) = 5$$

$$AB = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 0 & -8 \end{bmatrix}$$

$$|AB| = (10)(-8) - (3)(0) = -80$$

$$A^{-1} = \frac{1}{-16} \begin{bmatrix} 0 & -2 \\ -8 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -5 & 0 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{-80} \begin{bmatrix} -8 & -3 \\ 0 & 10 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -5 & 0 \end{bmatrix} \cdot \frac{1}{-16} \begin{bmatrix} 0 & -2 \\ -8 & 1 \end{bmatrix}$$

Prove that $(AB)^{-1} = B^{-1}A^{-1}$

EXERCISE 3.6

1. Solve by matrix inversion method, if possible.

(i) $2x + 5y = 19$
$4x - 3y = -1$

(ii) $3x + 2y = 7$
$5x - y = 16$

(iii) $x - 2y = 9$
$2x + 7y = -4$

(iv) $3x + 2y = 2$
$x - 2y = -2$

Solution

(i) $2x + 5y = 19$, $4x - 3y = -1$.

Matrix form

$$\begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ -1 \end{bmatrix}.$$

Determinant

$$|A| = (2)(-3) - (5)(4) = -26.$$

Inverse

$$A^{-1} = \frac{1}{-26} \begin{bmatrix} -3 & -5 \\ -4 & 2 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{-26} \begin{bmatrix} -3 & -5 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 19 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$x = 2$, $y = 3$.

(ii) $3x + 2y = 7$, $5x - y = 16$.

Matrix form

$$\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}.$$

Determinant

$$|A| = (3)(-1) - (2)(5) = -13.$$

Inverse

$$A^{-1} = \frac{1}{-13} \begin{bmatrix} -1 & -2 \\ -5 & 3 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{-13} \begin{bmatrix} -1 & -2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 16 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$x = 3$, $y = -1$.

(iii) $x-2y=9,\ 2x+7y=-4.$

Matrix form

$$\begin{bmatrix}1 & -2\\ 2 & 7\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}9\\ -4\end{bmatrix}.$$

Determinant

$$|A|=(1)(7)-(-2)(2)=11.$$

Inverse

$$A^{-1}=\frac{1}{11}\begin{bmatrix}7 & 2\\ -2 & 1\end{bmatrix}.$$

Solution

$$\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}B=\frac{1}{11}\begin{bmatrix}7 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}9\\ -4\end{bmatrix}=\begin{bmatrix}5\\ -2\end{bmatrix}.$$

$x=5,\ y=-2.$

(iv) $3x+2y=2,\ x-2y=-2.$

Matrix form

$$\begin{bmatrix}3 & 2\\ 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ -2\end{bmatrix}.$$

Determinant

$$|A|=(3)(-2)-(2)(1)=-8.$$

Inverse

$$A^{-1}=\frac{1}{-8}\begin{bmatrix}-2 & -2\\ -1 & 3\end{bmatrix}.$$

Solution

$$\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}B=\frac{1}{-8}\begin{bmatrix}-2 & -2\\ -1 & 3\end{bmatrix}\begin{bmatrix}2\\ -2\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

$x=0,\ y=1.$

2. Use Cramer's rule to solve the following pair of linear equations, if possible.

(i) $x+4y=4$
$2x-y=5$

(ii) $x+2y=7$
$3x-2y=-3$

(iii) $2x-5y=-6$
$4x-3y=-12$

(iv) $3x+2y=-1$
$5x+6y=5$

Solution

(i) $x+4y=4,\ 2x-y=5.$

$$\mathbf{A}=\begin{bmatrix}1 & 4\\ 2 & -1\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}4\\ 5\end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\mathbf{A}_x=\begin{bmatrix}4 & 4\\ 5 & -1\end{bmatrix}$$

$$\mathbf{A}_y=\begin{bmatrix}1 & 4\\ 2 & 5\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 4\\ 2 & -1\end{vmatrix}=-9$$

$$|A_x|=\begin{vmatrix}4 & 4\\ 5 & -1\end{vmatrix}=-24$$

$$|A_y|=\begin{vmatrix}1 & 4\\ 2 & 5\end{vmatrix}=-3$$

$$x=\frac{|A_x|}{|A|}=\frac{-24}{-9}=\frac{8}{3}$$

$$y=\frac{|A_y|}{|A|}=\frac{-3}{-9}=\frac{1}{3}$$

(ii) $x+2y=7$, $3x-2y=-3$.

$$\mathbf{A}=\begin{bmatrix}1 & 2\\3 & -2\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 2\\3 & -2\end{vmatrix}=-8$$

$$\mathbf{B}=\begin{bmatrix}7\\-3\end{bmatrix}$$

$$|A_x|=\begin{vmatrix}7 & 2\\-3 & -2\end{vmatrix}=-8$$

$$\mathbf{X}=\begin{bmatrix}x\\y\end{bmatrix}$$

$$|A_y|=\begin{vmatrix}1 & 7\\3 & -3\end{vmatrix}=-24$$

$$\mathbf{A}_x=\begin{bmatrix}7 & 2\\-3 & -2\end{bmatrix}$$

$$x=\frac{|A_x|}{|A|}=\frac{-8}{-8}=1$$

$$\mathbf{A}_y=\begin{bmatrix}1 & 7\\3 & -3\end{bmatrix}$$

$$y=\frac{|A_y|}{|A|}=\frac{-24}{-8}=3$$

(iii) $2x-5y=-6$, $4x-3y=-12$.

$$\mathbf{A}=\begin{bmatrix}2 & -5\\4 & -3\end{bmatrix}$$

$$|A|=\begin{vmatrix}2 & -5\\4 & -3\end{vmatrix}=14$$

$$\mathbf{B}=\begin{bmatrix}-6\\-12\end{bmatrix}$$

$$|A_x|=\begin{vmatrix}-6 & -5\\-12 & -3\end{vmatrix}=-42$$

$$\mathbf{X}=\begin{bmatrix}x\\y\end{bmatrix}$$

$$|A_y|=\begin{vmatrix}2 & -6\\4 & -12\end{vmatrix}=0$$

$$\mathbf{A}_x=\begin{bmatrix}-6 & -5\\-12 & -3\end{bmatrix}$$

$$x=\frac{|A_x|}{|A|}=\frac{-42}{14}=-3$$

$$\mathbf{A}_y=\begin{bmatrix}2 & -6\\4 & -12\end{bmatrix}$$

$$y=\frac{|A_y|}{|A|}=\frac{0}{14}=0$$

(iv) $3x+2y=-1$, $5x+6y=5$.

$$\mathbf{A}=\begin{bmatrix}3 & 2\\5 & 6\end{bmatrix}$$

$$|A|=\begin{vmatrix}3 & 2\\5 & 6\end{vmatrix}=8$$

$$\mathbf{B}=\begin{bmatrix}-1\\5\end{bmatrix}$$

$$|A_x|=\begin{vmatrix}-1 & 2\\5 & 6\end{vmatrix}=-16$$

$$\mathbf{X}=\begin{bmatrix}x\\y\end{bmatrix}$$

x$$

$$|A_y|=\begin{vmatrix}3 & -1\\5 & 5\end{vmatrix}=20$$

$$\mathbf{A}_x=\begin{bmatrix}-1 & 2\\5 & 6\end{bmatrix}$$

$$x=\frac{|A_x|}{|A|}=\frac{-16}{8}=-2$$

$$\mathbf{A}_y=\begin{bmatrix}3 & -1\\5 & 5\end{bmatrix}$$

$$y=\frac{|A_y|}{|A|}=\frac{20}{8}=\frac{5}{2}$$

3. An electrical engineer wants to determine the current in two branches A and B of a simple electrical circuit. The system of the equations is:

$$x + y = 7$$
$$2x - y = 2$$

where x is the current in branch A and y is the current in branch B. Find x and y by using matrices.

Solution

$x + y = 7$, $2x - y = 2$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$\mathbf{A}_x = \begin{bmatrix} 7 & 1 \\ 2 & -1 \end{bmatrix}$$

$$|A_x| = \begin{vmatrix} 7 & 1 \\ 2 & -1 \end{vmatrix} = -9$$

$$\mathbf{A}_y = \begin{bmatrix} 1 & 7 \\ 2 & 2 \end{bmatrix}$$

$$|A_y| = \begin{vmatrix} 1 & 7 \\ 2 & 2 \end{vmatrix} = -12$$

$$x = \frac{|A_x|}{|A|} = \frac{-9}{-3} = 3$$

$$y = \frac{|A_y|}{|A|} = \frac{-12}{-3} = 4$$

4. Three forces act on a particle and must be in equilibrium i.e. $F_1 + F_2 + F_3 = 0$, where $F_1 = \begin{bmatrix} 8 \\ x \end{bmatrix}, F_2 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}, F_3 = \begin{bmatrix} y \\ -1 \end{bmatrix}$. Find the value of x and y.

Solution

$$F_1 + F_2 + F_3 = 0$$

$$\begin{bmatrix} 8 \\ x \end{bmatrix} + \begin{bmatrix} -2 \\ -7 \end{bmatrix} + \begin{bmatrix} y \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives two equations:

$$8 - 2 + y = 0$$
$$x - 7 - 1 = 0$$

$$y = -6$$
$$x = 8$$

5. Two support beams, A and B are holding up a combined load of 100 kN. Twice the load on beam A and three times the load on beam B equals 240 kN. Find the load of beam A and beam B by using matrices.

Solution

Let A and B be loads on beams.

Equations:

$$A + B = 100$$
$$2A + 3B = 240$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$\mathbf{A}_A = \begin{bmatrix} 100 & 1 \\ 240 & 3 \end{bmatrix}$$

$$D_A = \begin{vmatrix} 100 & 1 \\ 240 & 3 \end{vmatrix} = 60$$

$$\mathbf{A}_B = \begin{bmatrix} 1 & 100 \\ 2 & 240 \end{bmatrix}$$

$$D_B = \begin{vmatrix} 1 & 100 \\ 2 & 240 \end{vmatrix} = 40$$

$$A = \frac{D_A}{D} = 60$$

$$B = \frac{D_B}{D} = 40$$

6. In a 2D game world, two characters are moving along straight paths. One character moves along a line where the total of twice their horizontal position and vertical position is 5, while the other moves along a line where their horizontal position is one more than their vertical position. Find their point of intersection by using matrices.

Solution

$$2x + y = 5$$
$$x - y = 1$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$\mathbf{A}_x = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|A_x| = \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = -6$$

$$\mathbf{A}_y = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$$

$$|A_y| = \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = -3$$

$$x = \frac{|A_x|}{|A|}. = 2$$

$$y = \frac{|A_y|}{|A|}. = 1$$

7. Two years ago a man was 5 times as old as his son was. After 6 years he will be 3 times as old as his son. Find their present ages by using matrices.

Solution

$$\begin{bmatrix}1 & -5\\1 & -3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-8\\12\end{bmatrix}.$$

$$\Delta=\begin{vmatrix}1 & -5\\1 & -3\end{vmatrix}=2,$$
$$\Delta_x=\begin{vmatrix}-8 & -5\\12 & -3\end{vmatrix}=84,$$
$$\Delta_y=\begin{vmatrix}1 & -8\\1 & 12\end{vmatrix}=20.$$

$$x=\frac{84}{2}=42,$$
$$y=\frac{20}{2}=10.$$

8. Two cyclists are 44 km apart and start out at the same time. If they go towards one another they meet in 2 hours, but if they go in the same direction the faster overtakes the slower in $7\frac{1}{2}$ hours. Find their speeds by using matrices.

Solution

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}22\\\frac{88}{15}\end{bmatrix}.$$

$$\Delta=\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix}=-2,$$
$$\Delta_{v_1}=\begin{vmatrix}22 & 1\\\frac{88}{15} & -1\end{vmatrix}=-26.867,$$
$$\Delta_{v_2}=\begin{vmatrix}1 & 22\\1 & \frac{88}{15}\end{vmatrix}=-15.467.$$

$$v_1=\frac{-26.867}{-2}=13.433,$$
$$v_2=\frac{-15.467}{-2}=7.733.$$

REVIEW EXERCISE 3

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) If $\begin{bmatrix} a+2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, then $a =$

(a) 3 (b) 5 ✓ (c) 6 (d) 7

(ii) $A = [3 \quad 5 \quad 0]$ is a ______ matrix.

(a) row ✓ (b) square (c) column (d) null

(iii) $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a ______ matrix.

(a) identity (b) square (c) row (d) column ✓

(iv) $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a/an ______ matrix.

(a) rectangular (b) identity ✓ (c) column (d) row

(v) If $A' = -A$, then A is ______ matrix.

(a) symmetric (b) row
(c) rectangular (d) skew-symmetric ✓

(vi) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $A + B =$

(a) $[21 \quad 32]$ (b) $[24 \quad 28]$
(c) $[10 \quad 12]$ ✓ (d) $[11 \quad 11]$

(vii) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $B - A =$

(a) $[10 \quad 12]$ (b) $[11 \quad 11]$
(c) $[4 \quad 4]$ ✓ (d) $[-4 \quad -4]$

(viii) What is the additive inverse of $\begin{bmatrix}3\\4\end{bmatrix}$?

(a) $\begin{bmatrix}-3\\4\end{bmatrix}$ ✓(b) $\begin{bmatrix}-3\\-4\end{bmatrix}$ (c) $\begin{bmatrix}3\\4\end{bmatrix}$ (d) $\begin{bmatrix}-1\\1\end{bmatrix}$

(ix) If $A=\begin{bmatrix}3 & 1 & 3\\2 & 0 & 2\end{bmatrix}$, then order of A^t is:

✓(a) 3-by-2 (b) 2-by-3 (c) 3-by-3 (d) 2-by-2

(x) $\begin{vmatrix}3 & 1\\0 & 4\end{vmatrix}=$

(a) 11 ✓(b) 12 (c) 13 (d) −11

2. If $A=\begin{bmatrix}4 & 2\\7 & 6\end{bmatrix}$, $B=\begin{bmatrix}5 & 5\\8 & 8\end{bmatrix}$, then find

(i) $(A-B)^t$ (ii) B^t-A^t (iii) $2A+3B$

Solution

(i) $(A-B)^t$

$$A-B=\begin{bmatrix}4-5 & 2-5\\7-8 & 6-8\end{bmatrix}=\begin{bmatrix}-1 & -3\\-1 & -2\end{bmatrix}.$$

$$(A-B)^t=\begin{bmatrix}-1 & -1\\-3 & -2\end{bmatrix}.$$

(ii) B^t-A^t

$$B^t=\begin{bmatrix}5 & 8\\5 & 8\end{bmatrix}, \quad A^t=\begin{bmatrix}4 & 7\\2 & 6\end{bmatrix}.$$

$$B^t-A^t=\begin{bmatrix}5-4 & 8-7\\5-2 & 8-6\end{bmatrix}=\begin{bmatrix}1 & 1\\3 & 2\end{bmatrix}.$$

(iii) $2A+3B$

$$2A=\begin{bmatrix}8 & 4\\14 & 12\end{bmatrix}.$$

$$3B=\begin{bmatrix}15 & 15\\24 & 24\end{bmatrix}.$$

$$2A+3B=\begin{bmatrix}8+15 & 4+15\\14+24 & 12+24\end{bmatrix}=\begin{bmatrix}23 & 19\\38 & 36\end{bmatrix}.$$

3. If $A=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$, $B=\begin{bmatrix}5 & 6\\2 & 1\end{bmatrix}$, $C=\begin{bmatrix}3 & 2\\1 & 7\end{bmatrix}$, then verify that

(i) $2(A+B)=2A+2B$ (ii) $(A+B)+C=A+(B+C)$

(iii) $(A+B)C=AC+BC$ (iv) $C(A-B)=CA-CB$

(v) $(AB)^{-1}=B^{-1}A^{-1}$ (vi) $AA^{-1}=A^{-1}A=I$

(vii) $(AB)^{t}=B^{t}A^{t}$ (viii) $(AB)C=A(BC)$

Solution

(i) $2(A+B)=2A+2B$

$$A+B=\begin{bmatrix}2+5 & 3+6\\4+2 & 5+1\end{bmatrix}=\begin{bmatrix}7 & 9\\6 & 6\end{bmatrix}$$

Now multiply by 2:

$$2(A+B)=\begin{bmatrix}14 & 18\\12 & 12\end{bmatrix}$$

Now find $2A$ and $2B$:

$$2A=\begin{bmatrix}4 & 6\\8 & 10\end{bmatrix}, \qquad 2B=\begin{bmatrix}10 & 12\\4 & 2\end{bmatrix}$$

Add them:

$$2A+2B=\begin{bmatrix}14 & 18\\12 & 12\end{bmatrix}$$

Hence,

$$2(A+B)=2A+2B$$

(ii) $(A+B)+C=A+(B+C)$

$$A+B=\begin{bmatrix}2+5 & 3+6\\4+2 & 5+1\end{bmatrix}=\begin{bmatrix}7 & 9\\6 & 6\end{bmatrix}$$

Then

$$(A+B)+C=\begin{bmatrix}7+3 & 9+2\\6+1 & 6+7\end{bmatrix}=\begin{bmatrix}10 & 11\\7 & 13\end{bmatrix}$$

Now find $B+C$:

$$B+C=\begin{bmatrix}5+3 & 6+2\\2+1 & 1+7\end{bmatrix}=\begin{bmatrix}8 & 8\\3 & 8\end{bmatrix}$$

Then

$$A+(B+C)=\begin{bmatrix}2+8 & 3+8\\4+3 & 5+8\end{bmatrix}=\begin{bmatrix}10 & 11\\7 & 13\end{bmatrix}$$

Hence,

$$(A+B)+C=A+(B+C)$$

(iii) $(A+B)C = AC + BC$

$$A + B = \begin{bmatrix} 2+5 & 3+6 \\ 4+2 & 5+1 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 6 \end{bmatrix}$$

$$(A+B)C = \begin{bmatrix} 7 & 9 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$$

$$(A+B)C = \begin{bmatrix} 30 & 77 \\ 24 & 54 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 25 \\ 17 & 43 \end{bmatrix}$$

$$BC = \begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 21 & 52 \\ 7 & 11 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 9 & 25 \\ 17 & 43 \end{bmatrix} + \begin{bmatrix} 21 & 52 \\ 7 & 11 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 30 & 77 \\ 24 & 54 \end{bmatrix}$$

Both sides are equal

(iv) $C(A-B) = CA - CB$

$$A - B = \begin{bmatrix} -3 & -3 \\ 2 & 4 \end{bmatrix}.$$

$$C(A-B) = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} -3 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 11 & 25 \end{bmatrix}.$$

$$CA = \begin{bmatrix} 14 & 19 \\ 29 & 38 \end{bmatrix},$$

$$CB = \begin{bmatrix} 19 & 20 \\ 18 & 13 \end{bmatrix}.$$

$$CA - CB = \begin{bmatrix} -5 & -1 \\ 11 & 25 \end{bmatrix}.$$

Both sides are equal.

(v) $(AB)^{-1} = B^{-1}A^{-1}$

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 15 \\ 30 & 29 \end{bmatrix}.$$

$$\det(AB) = (16)(29) - (15)(30) = 14.$$

$$(AB)^{-1} = \frac{1}{14}\begin{bmatrix} 29 & -15 \\ -30 & 16 \end{bmatrix} = \begin{bmatrix} \frac{29}{14} & \frac{-15}{14} \\ \frac{-30}{14} & \frac{16}{14} \end{bmatrix}.$$

$$\det(A) = (2)(5) - (3)(4) = -2,$$
$$\det(B) = (5)(1) - (6)(2) = -7.$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix},$$

$$B^{-1} = \frac{1}{-7}\begin{bmatrix} 1 & -6 \\ -2 & 5 \end{bmatrix}.$$

$$B^{-1}A^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{6}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{bmatrix}\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{29}{14} & \frac{-15}{14} \\ \frac{-30}{14} & \frac{16}{14} \end{bmatrix}.$$

Both sides are equal

(vi) $AA^{-1} = A^{-1}A = I$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$|A| = (2)(5) - (3)(4) = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Both sides are equal

(vii) $(AB)^t = B^t A^t$	(viii) $(AB)C = A(BC)$
$AB = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}\begin{bmatrix}5 & 6\\2 & 1\end{bmatrix} = \begin{bmatrix}10+6 & 12+3\\20+10 & 24+5\end{bmatrix}$ $(AB)^t = \begin{bmatrix}16 & 30\\15 & 29\end{bmatrix}$, $B^t = \begin{bmatrix}5 & 2\\6 & 1\end{bmatrix}$, $A^t = \begin{bmatrix}2 & 4\\3 & 5\end{bmatrix}$, $B^tA^t = \begin{bmatrix}5 & 2\\6 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\3 & 5\end{bmatrix} = \begin{bmatrix}10+6 & 20+10\\12+3 & 24+5\end{bmatrix}$ Both sides are equal	$AB = \begin{bmatrix}16 & 15\\30 & 29\end{bmatrix}$, $BC = \begin{bmatrix}21 & 52\\7 & 11\end{bmatrix}$ $(AB)C = \begin{bmatrix}16 & 15\\30 & 29\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 7\end{bmatrix} = \begin{bmatrix}63 & 137\\119 & 263\end{bmatrix}$ $A(BC) = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}\begin{bmatrix}21 & 52\\7 & 11\end{bmatrix} = \begin{bmatrix}63 & 137\\119 & 263\end{bmatrix}$ Both sides are equal

4. If $A = \begin{bmatrix}7 & 3\\2 & 1\end{bmatrix}$, $B = \begin{bmatrix}5 & 3\\2 & 4\end{bmatrix}$, then find

(i) $|B|$ (ii) $\text{Adj}\,B$ (iii) A^{-1}

(iv) $A^{-1}A$ (v) $(AB)^t$ (vi) $(B^t)^t$

Solution

(i) $\|B\|$	(ii) Adj B
$\|B\| = (5)(4) - (3)(2) = 20 - 6 = 14.$	$\text{Adj } B = \begin{bmatrix}4 & -3\\-2 & 5\end{bmatrix}.$
(iii) A^{-1} $\|A\| = (7)(1) - (3)(2) = 7 - 6 = 1,$ $A^{-1} = \frac{1}{1}\begin{bmatrix}1 & -3\\-2 & 7\end{bmatrix} = \begin{bmatrix}1 & -3\\-2 & 7\end{bmatrix}.$	(iv) $A^{-1}A$ $\|A\| = (7)(1) - (3)(2) = 7 - 6 = 1,$ $A^{-1} = \frac{1}{1}\begin{bmatrix}1 & -3\\-2 & 7\end{bmatrix} = \begin{bmatrix}1 & -3\\-2 & 7\end{bmatrix}.$ $A^{-1}A = \begin{bmatrix}1 & -3\\-2 & 7\end{bmatrix}\begin{bmatrix}7 & 3\\2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
(v) $(AB)^t$ $AB = \begin{bmatrix}7 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}5 & 3\\2 & 4\end{bmatrix} = \begin{bmatrix}41 & 33\\12 & 10\end{bmatrix}$ $(AB)^t = \begin{bmatrix}41 & 33\\12 & 10\end{bmatrix}^t = \begin{bmatrix}41 & 12\\33 & 10\end{bmatrix}$	(vi) $(B^t)^t$ $B = \begin{bmatrix}5 & 3\\2 & 4\end{bmatrix}$ $B^t = \begin{bmatrix}5 & 3\\2 & 4\end{bmatrix}^t = \begin{bmatrix}5 & 2\\3 & 4\end{bmatrix}$ $(B^t)^t = \begin{bmatrix}5 & 2\\3 & 4\end{bmatrix}^t = \begin{bmatrix}5 & 3\\2 & 4\end{bmatrix} = B$

5. Use matrix inversion method and Cramer's rule to solve the following pair of linear equations, if possible:

(i) $3x + 4y = 7$, $5x - y = 2$ (ii) $x - 6y = -15$, $2x + 6y = -3$ (iii) $2x + y = 5$, $x + 3y = 3$

Solution

Matrix inversion method:	Cramer's Rule:
i) $\begin{cases} 3x + 4y = 7 \\ 5x - y = 2 \end{cases}$	**i)** $\begin{cases} 3x + 4y = 7 \\ 5x - y = 2 \end{cases}$
$A = \begin{bmatrix} 3 & 4 \\ 5 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\|A\| = \begin{vmatrix} 3 & 4 \\ 5 & -1 \end{vmatrix}$
$\|A\| = -23$	$= 3(-1) - 4(5) = -23$
$A^{-1} = \frac{1}{-23} \begin{bmatrix} -1 & -4 \\ -5 & 3 \end{bmatrix}$	$\|A_x\| = \begin{vmatrix} 7 & 4 \\ 2 & -1 \end{vmatrix}$
$X = A^{-1}B$	$= 7(-1) - 4(2) = -15$
$= \frac{1}{-23} \begin{bmatrix} -1 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\|A_y\| = \begin{vmatrix} 3 & 7 \\ 5 & 2 \end{vmatrix}$
$= \frac{1}{-23} \begin{bmatrix} -7 - 8 \\ -35 + 6 \end{bmatrix}$	$= 3(2) - 7(5) = -29$
$= \frac{1}{-23} \begin{bmatrix} -15 \\ -29 \end{bmatrix}$	$x = \frac{-15}{-23} = \frac{15}{23}$
$= \begin{bmatrix} 15/23 \\ 29/23 \end{bmatrix}$	$y = \frac{-29}{-23} = \frac{29}{23}$

Matrix inversion method:	**Cramer's Rule:**
ii) $\begin{cases} x - 6y = -15 \\ 2x + 6y = -3 \end{cases}$ $A = \begin{bmatrix} 1 & -6 \\ 2 & 6 \end{bmatrix}, \ B = \begin{bmatrix} -15 \\ -3 \end{bmatrix}$ $\lvert A \rvert = 18$ $A^{-1} = \frac{1}{18}\begin{bmatrix} 6 & 6 \\ -2 & 1 \end{bmatrix}$ $X = A^{-1}B$ $= \frac{1}{18}\begin{bmatrix} 6 & 6 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -15 \\ -3 \end{bmatrix}$ $= \frac{1}{18}\begin{bmatrix} -90 - 18 \\ 30 - 3 \end{bmatrix}$ $= \frac{1}{18}\begin{bmatrix} -108 \\ 27 \end{bmatrix}$ $= \begin{bmatrix} -6 \\ 3/2 \end{bmatrix}$	**ii)** $\begin{cases} x - 6y = -15 \\ 2x + 6y = -3 \end{cases}$ $\lvert A \rvert = \begin{vmatrix} 1 & -6 \\ 2 & 6 \end{vmatrix}$ $= 1(6) - (-6)(2) = 18$ $\lvert A_x \rvert = \begin{vmatrix} -15 & -6 \\ -3 & 6 \end{vmatrix}$ $= (-15)(6) - (-6)(-3) = -108$ $\lvert A_y \rvert = \begin{vmatrix} 1 & -15 \\ 2 & -3 \end{vmatrix}$ $= 1(-3) - (-15)(2) = 27$ $x = \frac{-108}{18} = -6$ $y = \frac{27}{18} = \frac{3}{2}$
Matrix inversion method:	**Cramer's Rule:**
iii) $\begin{cases} 2x + y = 5 \\ x + 3y = 3 \end{cases}$ $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ $\lvert A \rvert = 5$ $A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ $X = A^{-1}B$ $= \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ $= \frac{1}{5}\begin{bmatrix} 15 - 3 \\ -5 + 6 \end{bmatrix}$ $= \frac{1}{5}\begin{bmatrix} 12 \\ 1 \end{bmatrix}$ $= \begin{bmatrix} 12/5 \\ 1/5 \end{bmatrix}$	**iii)** $\begin{cases} 2x + y = 5 \\ x + 3y = 3 \end{cases}$ $\lvert A \rvert = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$ $= 2(3) - 1(1) = 5$ $\lvert A_x \rvert = \begin{vmatrix} 5 & 1 \\ 3 & 3 \end{vmatrix}$ $= 5(3) - 1(3) = 12$ $\lvert A_y \rvert = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$ $= 2(3) - 5(1) = 1$ $x = \frac{12}{5}$ $y = \frac{1}{5}$

6. Find two numbers by using matrices such that twice the first added to the second makes 21 and twice the second added to the first makes 27.

Solution

$$2x + y = 21,$$
$$x + 2y = 27.$$

$$3x + 3y = 48 \implies x + y = 16.$$

Solving,

$$x = 5,$$
$$y = 11.$$

Numbers are **5** and **11**.

7. 4 knives and 6 forks cost Rs. 136, whereas 6 knives and 5 forks cost Rs. 164. Find the cost of a knife and a fork by using matrices.

Solution

Let x = knife cost, y = fork cost.

$$4x + 6y = 136,$$
$$6x + 5y = 164.$$

Matrix form $AX = B$,

$$A = \begin{bmatrix} 4 & 6 \\ 6 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 136 \\ 164 \end{bmatrix}.$$

Using Cramer's rule,

$$\Delta = \begin{vmatrix} 4 & 6 \\ 6 & 5 \end{vmatrix} = 20 - 36 = -16,$$
$$\Delta_x = \begin{vmatrix} 136 & 6 \\ 164 & 5 \end{vmatrix} = -304,$$
$$\Delta_y = \begin{vmatrix} 4 & 136 \\ 6 & 164 \end{vmatrix} = -160.$$

$$x = \frac{-304}{-16} = 19,$$
$$y = \frac{-160}{-16} = 10.$$

Knife = **Rs.19**, fork = **Rs.10**.

8. A shop employs, 5 men and 3 women, pays total daily wages Rs. 3500. If the number of men is reduced to 2 and 3 extra women are taken on, the daily wages amount to Rs. 5000. Find daily wages of a man and a woman by using matrices.

Solution

Let m = man's wage, w = woman's wage.

$$5m + 3w = 3500,$$
$$2m + 6w = 5000.$$

Matrix form $AX = B$,

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3500 \\ 5000 \end{bmatrix}.$$

Using Cramer's rule,

$$\Delta = \begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} = 24,$$
$$\Delta_m = \begin{vmatrix} 3500 & 3 \\ 5000 & 6 \end{vmatrix} = 6000,$$
$$\Delta_w = \begin{vmatrix} 5 & 3500 \\ 2 & 5000 \end{vmatrix} = 18000.$$

$$m = \frac{6000}{24} = 250,$$
$$w = \frac{18000}{24} = 750.$$

Man's wage = **Rs.250**, woman's wage = **Rs.750**.