UNIT 2

Quadratic Equations and Inequalities

EXERCISE 2.1

1. Write the following quadratic equations in standard form:

(i) $3x - 1 = 2x^2$ (ii) $2x(x+1) = 4(2x+3)$

(iii) $2x^2 - 4x = 4x + 7$ (iv) $4(3x-2) = 9x^2$

(v) $2x + \frac{1}{x} = 5 - \frac{1}{x}$, $x \neq 0$ (vi) $\frac{6x+6}{20-x} = \frac{1}{x}$, $x \neq 0, 20$

Solution

i. $\boldsymbol{3x - 1 = 2x^2}$ **Solution** $3x - 1 = 2x^2$ $2x^2 - 3x + 1 = 0$	**ii.** $\boldsymbol{2x(x+1) = 4(2x+3)}$ **Solution** $2x(x+1) = 4(2x+3)$ $2x^2 + 2x = 8x + 12$ $2x^2 + 2x - 8x - 12 = 0$ $2x^2 - 6x - 12 = 0$ $x^2 - 3x - 6 = 0$
iii. $\boldsymbol{2x^2 - 4x = 4x + 7}$ **Solution** $2x^2 - 4x = 4x + 7$ $2x^2 - 4x - 4x - 7 = 0$ $2x^2 - 8x - 7 = 0$	**iv.** $\boldsymbol{4(3x-2) = 9x^2}$ **Solution** $4(3x-2) = 9x^2$ $12x - 8 = 9x^2$ $9x^2 - 12x + 8 = 0$
v. $\boldsymbol{2x + \frac{1}{x} = 5 - \frac{1}{x}\ , x \neq 0}$ **Solution** $2x + \frac{1}{x} = 5 - \frac{1}{x}$ $2x^2 + 1 = 5x - 1$ $2x^2 + 1 - 5x + 1 = 0$ $2x^2 - 5x + 2 = 0$	**vi.** $\boldsymbol{\frac{6x+6}{20-x} = \frac{1}{x}\ , x \neq 0, 20}$ **Solution** $\frac{6x+6}{20-x} = \frac{1}{x}$ $x(6x+6) = 1(20-x)$ $6x^2 + 6x = 20 - x$ $6x^2 + 6x - 20 + x = 0$ $6x^2 + 7x - 20 = 0$

2. Solve the following quadratic equations by factorization method:

Solution

i. $\boldsymbol{x^2 - x - 6 = 0}$

Solution

$x^2 - x - 6 = 0$
$x^2 - 3x + 2x - 6 = 0$
$x(x-3) + 2(x-3) = 0$
$(x-3)(x+2) = 0$
$x - 3 = 0\ ;\ x + 2 = 0$
$x = 3\ ;\ x = -2$
$S.S = \{3, -2\}$

ii. $\boldsymbol{x^2 + 3x - 28 = 0}$

Solution

$x^2 + 3x - 28 = 0$
$x^2 + 7x - 4x - 28 = 0$
$x(x+7) - 4(x+7) = 0$
$(x-4)(x+7) = 0$
$x - 4 = 0\ ;\ x + 7 = 0$
$x = 4\ ;\ x = -7$
$S.S = \{4, -7\}$

iii. $\boldsymbol{6x^2 + 13x - 5 = 0}$

Solution

$6x^2 + 13x - 5 = 0$
$6x^2 + 15x - 2x - 5 = 0$
$3x(2x+5) - 1(2x+5) = 0$
$(3x-1)(2x+5) = 0$
$3x - 1 = 0\ ;\ 2x + 5 = 0$
$x = \frac{1}{3}\ ;\ x = -\frac{5}{2}$
$S.S = \left\{-\frac{5}{2}, \frac{1}{3}\right\}$

iv. $\boldsymbol{x^2 - \frac{3}{2}x = \frac{9}{2}}$

Solution

$x^2 - \frac{3}{2}x = \frac{9}{2}$
$2x^2 - 3x = 9$
$2x^2 - 3x - 9 = 0$
$2x^2 - 6x + 3x - 9 = 0$
$2x(x-3) + 3(x-3) = 0$
$(x-3)(2x+3) = 0$
$x - 3 = 0\ ;\ 2x + 3 = 0$
$x = 3\ ;\ x = -\frac{3}{2}$
$S.S = \left\{3, -\frac{3}{2}\right\}$

v. $\boldsymbol{\frac{3x-8}{x-2} = \frac{5x-2}{x+5}\ , x \neq 2, -5}$

Solution

$\frac{3x-8}{x-2} = \frac{5x-2}{x+5}$
$(3x-8)(x+5) = (5x-2)(x-2)$
$3x^2 + 15x - 8x - 40 = 5x^2 - 10x - 2x + 4$
$3x^2 + 7x - 40 = 5x^2 - 12x + 4$
$5x^2 - 12x + 4 - 3x^2 - 7x + 40 = 0$
$2x^2 - 19x + 44 = 0$
$2x^2 - 8x - 11x + 44 = 0$
$2x(x-4) - 11(x-4) = 0$
$(x-4)(2x-11) = 0$
$x - 4 = 0\ ;\ 2x - 11 = 0$
$x = 4\ ;\ x = \frac{11}{2}$
$S.S = \left\{4, \frac{11}{2}\right\}$

vi. $\boldsymbol{\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}\ , x \neq 1, -3}$

Solution

$\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}$
$\frac{x+3-x+1}{(x-1)(x+3)} = \frac{1}{35}$
$\frac{4}{x^2+3x-x-3} = \frac{1}{35}$
$\frac{4}{x^2+2x-3} = \frac{1}{35}$
$x^2 + 2x - 3 = 140$
$x^2 + 2x - 143 = 0$
$x^2 - 11x + 13x - 143 = 0$
$x(x-11) + 13(x-11) = 0$
$(x-11)(x+13) = 0$
$x - 11 = 0\ ;\ x + 13 = 0$
$x = 11\ ;\ x = -13$
$S.S = \{11, -13\}$

3. Solve the following quadratic equations by completing square method:

(i) $2x^2+5x+2=0$ (ii) $x^2+x=42$

(iii) $12x^2+7x=12$ (iv) $\dfrac{x+3}{2x-7}=\dfrac{2x-1}{x-3}, x\neq\dfrac{7}{2},3$

(v) $\dfrac{1}{1+x}-\dfrac{1}{3-x}=\dfrac{6}{35}, x\neq -1,3$ (vi) $\dfrac{3x-1}{4x+7}=1-\dfrac{6}{x+7}, x\neq\dfrac{-7}{4},-7$

Solution

(i) $2x^2+5x+2=0$

$$2x^2+5x=-2$$
$$x^2+\frac{5}{2}x=-1$$
$$x^2+\frac{5}{2}x+(\frac{5}{4})^2=-1+(\frac{5}{4})^2$$
$$(x+\frac{5}{4})^2=-1+\frac{25}{16}$$
$$(x+\frac{5}{4})^2=\frac{9}{16}$$
$$x+\frac{5}{4}=\pm\frac{3}{4}$$
$$x=-\frac{5}{4}\pm\frac{3}{4}.$$

$x=-2$ or $x=-\frac{1}{2}$.

(ii) $x^2+x=42$

$$x^2+x+(\frac{1}{2})^2=42+(\frac{1}{2})^2$$
$$(x+\frac{1}{2})^2=42+\frac{1}{4}$$
$$(x+\frac{1}{2})^2=\frac{169}{4}$$
$$x+\frac{1}{2}=\pm\frac{13}{2}$$
$$x=-\frac{1}{2}\pm\frac{13}{2}.$$

$x=6$ or $x=-7$.

(iii) $12x^2+7x=12$

$$12x^2+7x-12=0$$
$$x^2+\frac{7}{12}x=1$$
$$x^2+\frac{7}{12}x+(\frac{7}{24})^2=1+(\frac{7}{24})^2$$
$$(x+\frac{7}{24})^2=1+\frac{49}{576}$$
$$(x+\frac{7}{24})^2=\frac{625}{576}$$
$$x+\frac{7}{24}=\pm\frac{25}{24}$$
$$x=-\frac{7}{24}\pm\frac{25}{24}.$$

$x=\frac{3}{4}$ or $x=-\frac{4}{3}$.

(iv) $\frac{x+3}{2x-7}=\frac{2x-1}{x-3}$

$$(x+3)(x-3)=(2x-1)(2x-7)$$
$$x^2-9=4x^2-14x-2x+7$$
$$0=3x^2-16x+16.$$

$$x^2-\frac{16}{3}x=-\frac{16}{3}$$
$$x^2-\frac{16}{3}x+(\frac{8}{3})^2=-\frac{16}{3}+(\frac{8}{3})^2$$
$$(x-\frac{8}{3})^2=-\frac{16}{3}+\frac{64}{9}$$
$$(x-\frac{8}{3})^2=\frac{16}{9}$$
$$x-\frac{8}{3}=\pm\frac{4}{3}.$$

$x=4$ or $x=\frac{4}{3}$.

(v) $\frac{1}{1+x}-\frac{1}{3-x}=\frac{6}{35}$

$$\frac{(3-x)-(1+x)}{(1+x)(3-x)}=\frac{6}{35}$$
$$\frac{2-2x}{3+2x-x^2}=\frac{6}{35}$$
$$35(2-2x)=6(3+2x-x^2)$$
$$70-70x=18+12x-6x^2$$
$$6x^2-82x+52=0.$$

$$x^2-\frac{41}{3}x=-\frac{26}{3}$$
$$x^2-\frac{41}{3}x+(\frac{41}{6})^2=-\frac{26}{3}+(\frac{41}{6})^2$$
$$(x-\frac{41}{6})^2=-\frac{52}{6}+\frac{1681}{36}$$
$$(x-\frac{41}{6})^2=\frac{1369}{36}$$
$$x-\frac{41}{6}=\pm\frac{37}{6}.$$

$$x=13 \text{ or } x=\frac{2}{3}.$$

(vi) $\frac{3x-1}{4x+7}=1-\frac{6}{x+7}$

$$\frac{3x-1}{4x+7}=\frac{x+7-6}{x+7}$$
$$\frac{3x-1}{4x+7}=\frac{x+1}{x+7}$$
$$(3x-1)(x+7)=(4x+7)(x+1)$$
$$3x^2+20x-7=4x^2+11x+7$$
$$0=x^2-9x+14.$$

$$x^2-9x=-14$$
$$x^2-9x+(\frac{9}{2})^2=-14+(\frac{9}{2})^2$$
$$(x-\frac{9}{2})^2=-14+\frac{81}{4}$$
$$(x-\frac{9}{2})^2=\frac{25}{4}$$
$$x-\frac{9}{2}=\pm\frac{5}{2}.$$

$$x=7 \text{ or } x=2.$$

4. Use quadratic formula to solve the following equations:

(i) $2x^2-5x+3=0$ (ii) $2x^2-7x-15=0$

(iii) $2x^2+7x=15$ (iv) $x^2+11=7x$

(v) $\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\frac{1}{3}, x\neq 4,3$ (vi) $\frac{3x-3}{x+1}=\frac{2x-1}{x-1}, x\neq -1,1$

Solution

(i) $2x^2-5x+3=0$.

Quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where
$a=2,\ b=-5,\ c=3$.

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(3)}}{2(2)}.$$

$$x=\frac{5\pm\sqrt{25-24}}{4}.$$

$$x=\frac{5\pm1}{4}.$$

Solutions: $x=\frac{3}{2},\ x=1$.

(ii) $2x^2-7x-15=0$.

$a=2,\ b=-7,\ c=-15$.

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(-15)}}{2(2)}.$$

$$x=\frac{7\pm\sqrt{49+120}}{4}.$$

$$x=\frac{7\pm13}{4}.$$

Solutions: $x=5,\ x=-\frac{3}{2}$.

(iii) $2x^2+7x=15$.

$2x^2+7x-15=0$.

$a=2,\ b=7,\ c=-15$.

$x=\frac{-7\pm\sqrt{7^2-4(2)(-15)}}{2(2)}$.

$x=\frac{-7\pm\sqrt{49+120}}{4}$.

$x=\frac{-7\pm 13}{4}$.

Solutions: $x=\frac{3}{2},\ x=-5$.

(iv) $x^2+11=7x$.

$x^2-7x+11=0$.

$a=1,\ b=-7,\ c=11$.

$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(11)}}{2(1)}$.

$x=\frac{7\pm\sqrt{49-44}}{2}$.

$x=\frac{7\pm\sqrt{5}}{2}$.

(v) $\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\frac{1}{3},\ x\neq 4,3$.

$\frac{x+4}{x-4}+\frac{x-2}{x-3}=\frac{19}{3}$.

Combine LHS,

$\frac{(x+4)(x-3)+(x-2)(x-4)}{(x-4)(x-3)}=\frac{19}{3}$.

$\frac{2x^2-5x-4}{x^2-7x+12}=\frac{19}{3}$.

Cross-multiply,

$3(2x^2-5x-4)=19(x^2-7x+12)$.

$6x^2-15x-12=19x^2-133x+228$.

$13x^2-118x+240=0$.

$a=13,\ b=-118,\ c=240$.

$x=\frac{-(-118)\pm\sqrt{(-118)^2-4(13)(240)}}{2(13)}$.

$x=\frac{118\pm\sqrt{13924-12480}}{26}$.

$x=\frac{118\pm 38}{26}$.

Solutions: $x=6,\ x=\frac{40}{13}$.

(vi) $\frac{3x-3}{x+1}=\frac{2x-1}{x-1},\ x\neq -1,1$.

Cross-multiply,

$(3x-3)(x-1)=(2x-1)(x+1)$.

$3(x-1)^2=(2x-1)(x+1)$.

$3x^2-6x+3=2x^2+x-1$.

$x^2-7x+4=0$.

$a=1,\ b=-7,\ c=4$.

$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(4)}}{2(1)}$.

$x=\frac{7\pm\sqrt{49-16}}{2}$.

$x=\frac{7\pm\sqrt{33}}{2}$.

Solutions: $x=\frac{7+\sqrt{33}}{2},\ x=\frac{7-\sqrt{33}}{2}$.

5. Solve the following quadratic equations graphically:

(i) $x^2 - 3x - 18 = 0$ (ii) $x^2 - 5x - 14 = 0$

(iii) $2x^2 + 13x + 6 = 0$ (iv) $4x^2 + 12x - 27 = 0$

Solution

(i) $x^2 - 3x - 18 = 0$.

$x^2 - 3x - 18 = 0$.

$x^2 - 6x + 3x - 18 = 0$.

$x(x-6) + 3(x-6) = 0$.

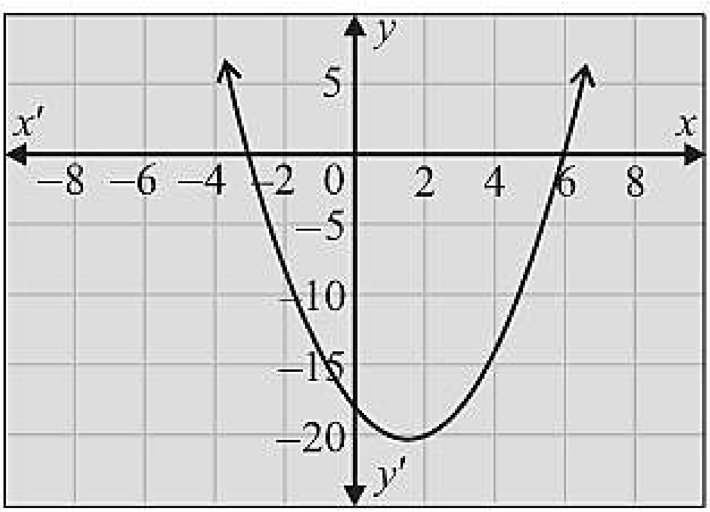
$(x-6)(x+3) = 0$.

$x - 6 = 0 \rightarrow x = 6$.

$x + 3 = 0 \rightarrow x = -3$.

Solutions: $x = 6$, $x = -3$.

Graph cuts x-axis at 6 and -3



(ii) $x^2 - 5x - 14 = 0$.

$x^2 - 5x - 14 = 0$.

$x^2 - 7x + 2x - 14 = 0$.

$x(x-7) + 2(x-7) = 0$.

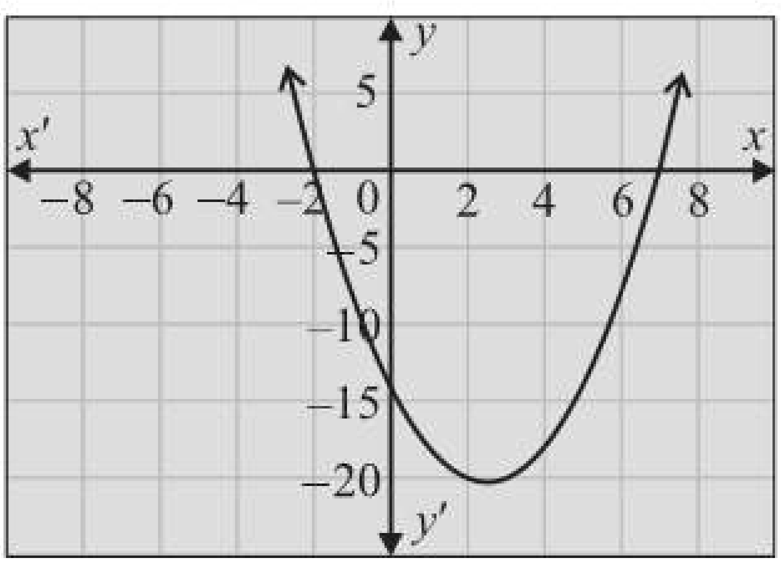
$(x-7)(x+2) = 0$.

$x - 7 = 0 \rightarrow x = 7$.

$x + 2 = 0 \rightarrow x = -2$.

Solutions: $x = 7$, $x = -2$.

Graph cuts x-axis at 7 and -2.



(iii) $2x^2+13x+6=0$.

$2x^2+13x+6=0$.

$x=\frac{-13\pm\sqrt{169-48}}{4}$.

$169-48=121$.

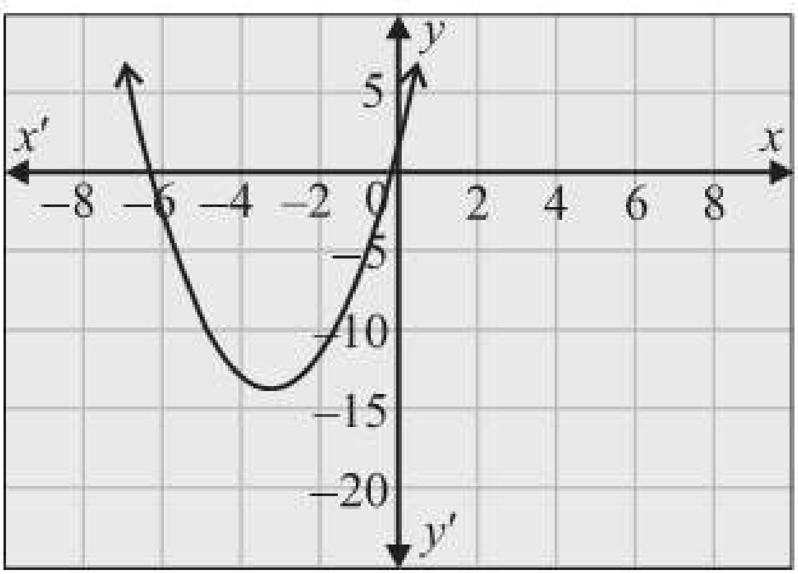
$x=\frac{-13\pm 11}{4}$.

$x=\frac{-13+11}{4}=-\frac{1}{2}$.

$x=\frac{-13-11}{4}=-6$.

Solutions: $x=-\frac{1}{2}$, $x=-6$.

Graph intersects x-axis at $-\frac{1}{2}$ and -6.



(iv) $4x^2+12x-27=0$.

$4x^2+12x-27=0$.

$x=\frac{-12\pm\sqrt{144+432}}{8}$.

$144+432=576$.

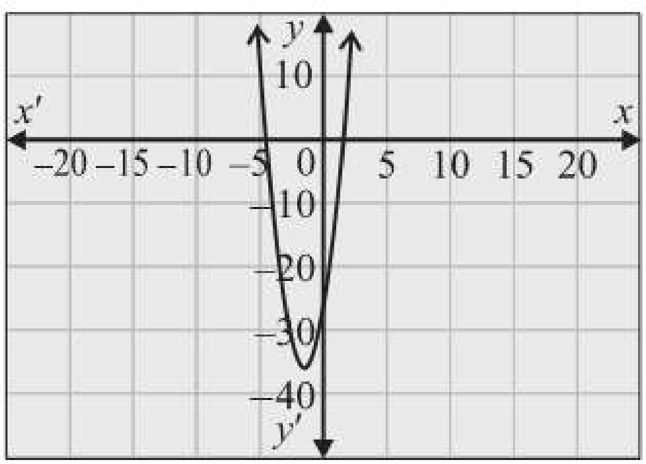
$x=\frac{-12\pm 24}{8}$.

$x=\frac{-12+24}{8}=\frac{3}{2}$.

$x=\frac{-12-24}{8}=-\frac{9}{2}$.

Solutions: $x=\frac{3}{2}$, $x=-\frac{9}{2}$.

Graph cuts x-axis at $\frac{3}{2}$ and $-\frac{9}{2}$.



EXERCISE 2.2

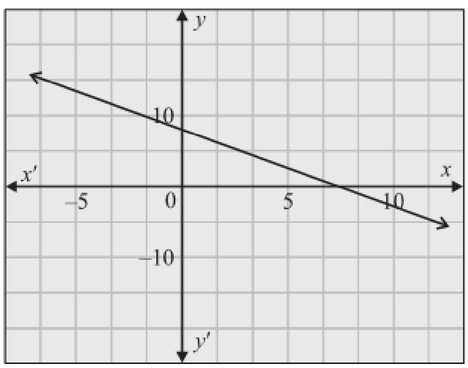
1. Find the points of intersection of the following linear equations with coordinate axes graphically:

(i) $x+y=8$ (ii) $x-y=1$ (iii) $x-2y=1$
(iv) $x-2y+2=0$ (v) $5x-5y=1$

Solution

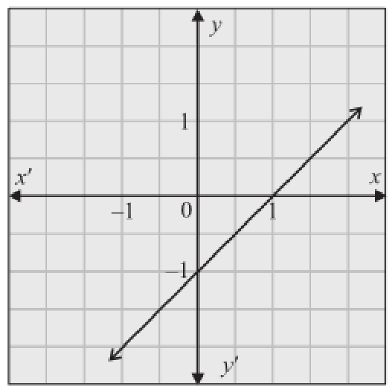
(i) $x+y=8$.

- x-intercept: set $y=0 \implies x=8$. Point $(8,0)$.
- y-intercept: set $x=0 \implies y=8$. Point $(0,8)$.



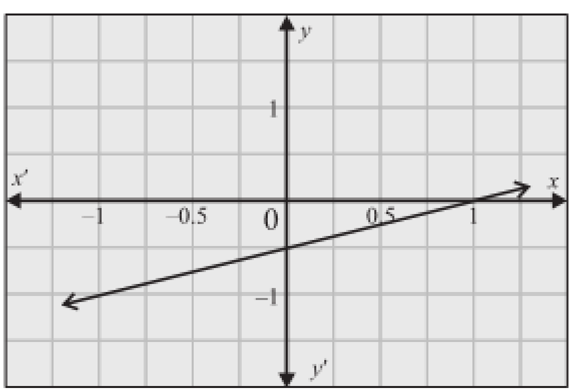
(ii) $x-y=1$.

- x-intercept: $y=0 \implies x=1$. Point $(1,0)$.
- y-intercept: $x=0 \implies y=-1$. Point $(0,\,-1)$.



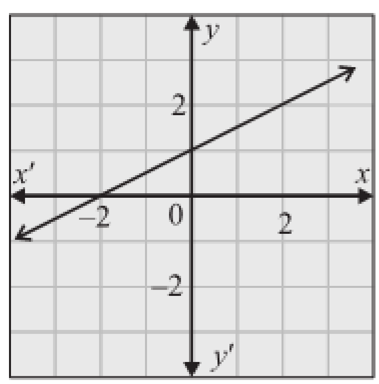
(iii) $x-2y=1$.

- x-intercept: $y=0 \implies x=1$. Point $(1,0)$.
- y-intercept: $x=0 \implies y=-\frac{1}{2}$. Point $(0,\,-\frac{1}{2})$.



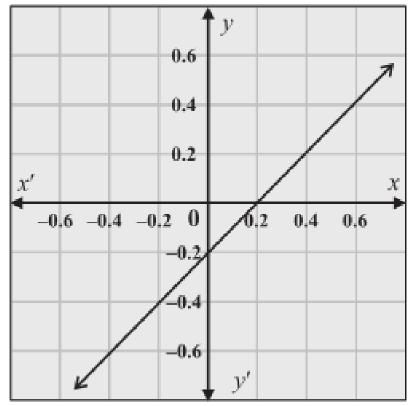
(iv) $x-2y+2=0$.

- x-intercept: $y=0 \implies x=-2$. Point $(-2,0)$.
- y-intercept: $x=0 \implies y=1$. Point $(0,1)$.



(v) $5x - 5y = 1$.

- x-intercept: $y = 0 \implies x = \frac{1}{5}$. Point $(\frac{1}{5}, 0)$.
- y-intercept: $x = 0 \implies y = -\frac{1}{5}$. Point $(0, -\frac{1}{5})$.



2. Solve the following system of linear equations graphically:

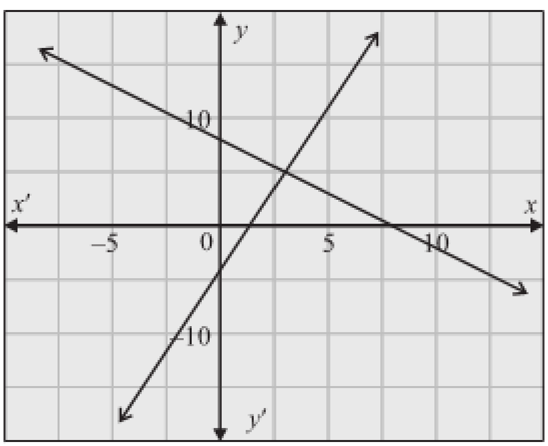
(i) $x + y = 8$, $3x - y = 4$ (ii) $x - y = 1$, $x + 2y = 7$ (iii) $x - 2y = 1$, $2x + y = 2$

(iv) $y = 2x + 2$, $3x + 2y = 4$ (v) $3y = 2x + 8$, $x + y = 1$

Solution

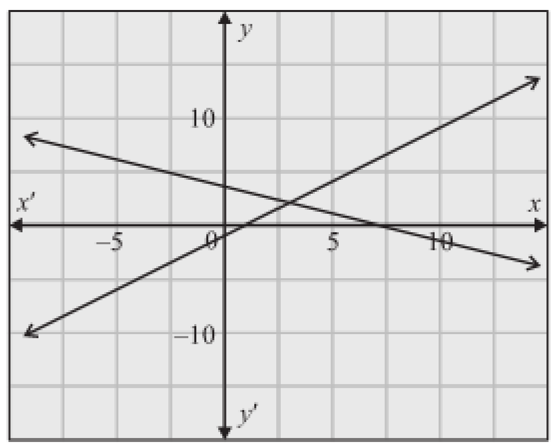
(i) $x + y = 8$ and $3x - y = 4$.

- Solve algebraically: add equations $\implies 4x = 12 \implies x = 3$.
- $y = 8 - x = 8 - 3 = 5$. Solution $(3, 5)$.



(ii) $x - y = 1$ and $x + 2y = 7$.

- Subtract equations $\implies -3y = -6 \implies y = 2$.
- $x = y + 1 = 3$. Solution $(3, 2)$.

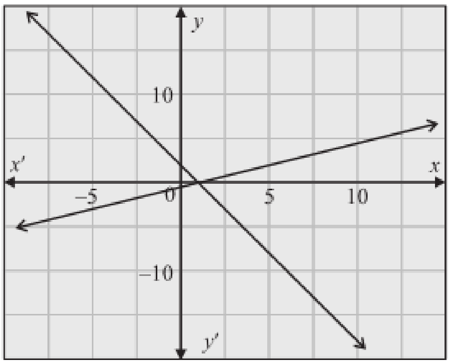


(iii) $x-2y=1$ and $2x+y=2$.

- From first: $x=2y+1$. Substitute into second:

$2(2y+1)+y=2 \implies 5y=0 \implies y=0$.

- $x=1$. Solution $(1,0)$.

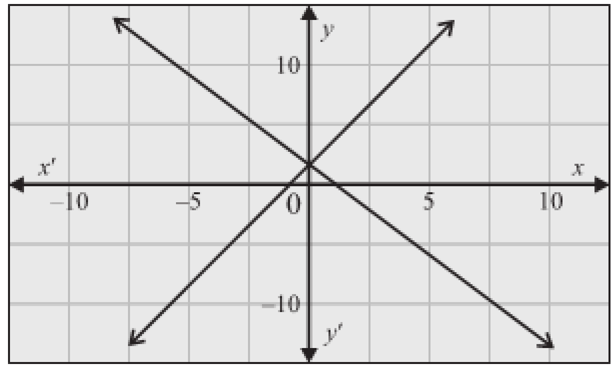


(iv) $y=2x+2$ and $3x+2y=4$.

- Substitute y:

$3x+2(2x+2)=4 \implies 7x=0 \implies x=0$.

- $y=2$. Solution $(0,2)$.

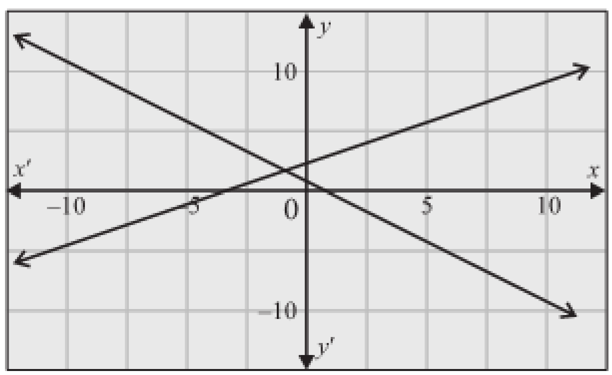


(v) $3y=2x+8$ and $x+y=1$.

- $y=\frac{2}{3}x+\frac{8}{3}$. Substitute into second:

$x+\frac{2}{3}x+\frac{8}{3}=1 \implies \frac{5}{3}x=-\frac{5}{3} \implies x=-1$.

- $y=2$. Solution $(-1,2)$.

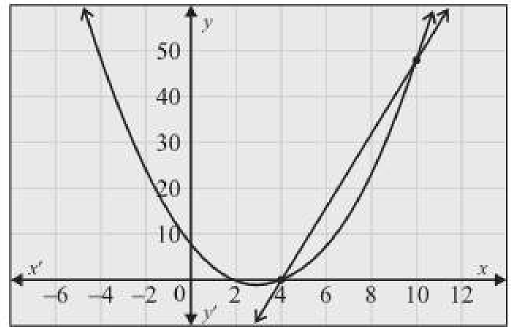


3. Solve the following equations graphically:

(i) $y = 8x - 32$, $y = x^2 - 6x + 8$ (ii) $y + x = 2$, $y = 2x^2 + x - 10$

Solution

(i) $y = 8x - 32$ and $y = x^2 - 6x + 8$.

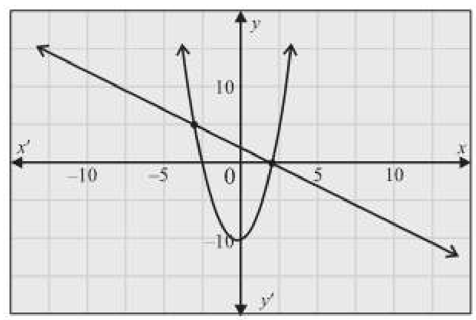


- Set equal: $8x - 32 = x^2 - 6x + 8$.
- $x^2 - 14x + 40 = 0$.
- $(x - 10)(x - 4) = 0$. $x = 10$ or $x = 4$.
- $y = 8(10) - 32 = 48$. $y = 8(4) - 32 = 0$.
- Solutions $(10, 48)$, $(4, 0)$.

(ii) $y + x = 2$ and $y = 2x^2 + x - 10$.

- Substitute $y = 2 - x$ into second:

$2 - x = 2x^2 + x - 10$.



- $2x^2 + 2x - 12 = 0$.
- $x^2 + x - 6 = 0$.
- $(x + 3)(x - 2) = 0$. $x = -3$ or $x = 2$.
- $y = 5$ when $x = -3$. $y = 0$ when $x = 2$.
- Solutions $(-3, 5)$, $(2, 0)$.

EXERCISE 2.3

1. Form a quadratic equation whose roots are given below:

(i) $-4, 9$ (ii) $5, -7$ (iii) $\frac{-7}{5}, \frac{-6}{5}$

(iv) $\frac{-3}{2}, \frac{7}{2}$ (v) $3+\sqrt{5}, 3-\sqrt{5}$ (vi) $-2+\sqrt{3}, -2-\sqrt{3}$

Solution

The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (i) Roots **−4, 9**: $x^2-(-4+9)x+(-4)(9)=0$ $x^2-5x-36=0.$	The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (ii) Roots **5, −7**: $x^2-(5-7)x+(5)(-7)=0$ $x^2+2x-35=0.$
The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (iii) Roots $\frac{-7}{5}$, $\frac{-6}{5}$: $x^2-(\frac{-7}{5}+\frac{-6}{5})x+(\frac{-7}{5})(\frac{-6}{5})=0$ $x^2+\frac{13}{5}x+\frac{42}{25}=0$ $25x^2+65x+42=0.$	The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (iv) Roots $\frac{-3}{2}$, $\frac{7}{2}$: $x^2-(\frac{-3}{2}+\frac{7}{2})x+(\frac{-3}{2})(\frac{7}{2})=0$ $x^2-2x-\frac{21}{4}=0$ $4x^2-8x-21=0.$
The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (v) Roots $3+\sqrt{5}$, $3-\sqrt{5}$: $x^2-(6)x+(3+\sqrt{5})(3-\sqrt{5})=0$ $x^2-6x+(9-5)=0$ $x^2-6x+4=0.$	The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (vi) Roots $-2+\sqrt{3}$, $-2-\sqrt{3}$: $x^2-(-4)x+(-2+\sqrt{3})(-2-\sqrt{3})=0$ $x^2+4x+(4-3)=0$ $x^2+4x+1=0.$

2. Find the quadratic equation with roots exceeding by 2 than those of roots of $x^2+9x+20=0$.

Solution

Roots of $x^2+9x+20=0$ are $-4, -5$.

Roots increased by 2 are $-2, -3$.

$$x^2-(-2-3)x+(-2)(-3)=0$$
$$x^2+5x+6=0.$$

3. Find the equation whose roots are double the roots of $x^2 - px + q = 0$.

Solution

If roots are α, β, then new roots are $2\alpha, 2\beta$.

$$x^2 - (2\alpha + 2\beta)x + (2\alpha)(2\beta) = 0. \Rightarrow x^2 - 2(\alpha+\beta) + 2(\alpha\beta) = 0$$

$$x^2 - 2px + 4q = 0.$$

4. If α, β are the roots of the equation $x^2 + 2x + 4 = 0$, then find the equation whose roots are:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (iii) $2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha}$

(iv) α^2, β^2 (v) $2\alpha - 1, 2\beta - 1$

Solution

$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (i) $\frac{1}{\alpha}, \frac{1}{\beta}$: $x^2 - (\frac{1}{\alpha} + \frac{1}{\beta})x + \frac{1}{\alpha\beta} = 0$ $x^2 - (\frac{\alpha+\beta}{\alpha\beta})x + \frac{1}{4} = 0$ $x^2 + \frac{1}{2}x + \frac{1}{4} = 0$ $4x^2 + 2x + 1 = 0.$	$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (ii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$: $\text{Sum} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4-8}{4} = -1$ $\text{Product} = 1.$ $x^2 + x + 1 = 0.$
$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (iii) $2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha}$: $\text{Sum} = 2(\alpha+\beta) - (\frac{1}{\alpha} + \frac{1}{\beta}) = -4 - (-\frac{1}{2}) = -\frac{7}{2}$ $\text{Product} = 4\alpha\beta - 2 - 2 + \frac{1}{\alpha\beta} = 16 - 4 + \frac{1}{4}.$ $x^2 + \frac{7}{2}x + \frac{49}{4} = 0.$ $4x^2 + 14x + 49 = 0$	$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (iv) α^2, β^2: $\text{Sum} = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 4 - 8 = -4$ $\text{Product} = (\alpha\beta)^2 = 16.$ $x^2 + 4x + 16 = 0.$

$\alpha + \beta = -2, \quad \alpha\beta = 4.$

(v) $2\alpha - 1, 2\beta - 1$:

$$\text{Sum} = 2(\alpha+\beta) - 2 = -4 - 2 = -6.$$

$$\text{Product} = 4\alpha\beta - 2(\alpha+\beta) + 1 = 16 + 4 + 1 = 21$$

$$x^2 + 6x + 21 = 0.$$

5. Find the condition that roots of $ax^2 + bx + c = 0$ should be reciprocals of each other.

Solution

$\alpha = \frac{1}{\beta} \implies \alpha\beta = 1.$

$\frac{c}{a} = 1 \quad \Rightarrow \quad a = c.$

6. Find the value of k, given that one root of $x^2 - (2k + 4)x + (7k + 1) = 0$ is 3.

Solution

$$x^2 - (2k+4)x + (7k+1) = 0$$
$$3^2 - (2k+4)3 + (7k+1) = 0$$
$$9 - 6k - 12 + 7k + 1 = 0$$
$$k - 2 = 0$$
$$k = 2.$$

7. Find the value of m in the equation $2x^2 + 3x + m = 0$ when sum of its roots is equal to double the product of its roots.

Solution

$$\text{Sum} = -\frac{3}{2}, \quad \text{Product} = \frac{m}{2}.$$
$$-\frac{3}{2} = 2 \cdot \frac{m}{2}$$
$$m = -\frac{3}{2}.$$

8. If α, β are the roots of $x^2 + ax + b = 0$ and α^2, β^2 are the roots of $x^2 + Ax + B = 0$, then prove that $A = 2b - a^2$, $B = b^2$.

Solution

$$\alpha + \beta = -a, \quad \alpha\beta = b.$$
$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = a^2 - 2b.$$
$$\alpha^2\beta^2 = b^2.$$

$$A = -(\alpha^2 + \beta^2) = 2b - a^2, \quad B = b^2.$$

9. If α, β are the roots of $x^2 + px + q = 0$, then find the condition that

(i) $\alpha = \beta$ (ii) $\alpha = \frac{1}{\beta}$

Solution

If $\alpha = \beta$ then $\alpha + \beta = 2\alpha = -p \Rightarrow \alpha = -\frac{p}{2}$ $\alpha\beta = \alpha^2 = q \Rightarrow \frac{p^2}{4} = q \Rightarrow \mathbf{p^2 = 4q}$	(ii) $\alpha = \frac{1}{\beta}$: $\alpha\beta = 1 \implies q = 1.$

EXERCISE 2.4

1. Examine the nature of roots of the following quadratic equations:

(i) $3x^2 - 9x - 2 = 0$ (ii) $x^2 + 6x + 9 = 0$

(iii) $2x^2 + 4x + 5 = 0$ (iv) $7x^2 - 6x - 1 = 0$

(v) $5x^2 - 2x + 10 = 0$ (vi) $x^2 - 8x + 16 = 0$

Solution

$D = b^2 - 4ac$ (i) $3x^2 - 9x - 2 = 0$: $D = (-9)^2 - 4(3)(-2) = 81 + 24 = 105 > 0$. Irrational and unequal	$D = b^2 - 4ac$ (ii) $x^2 + 6x + 9 = 0$: $D = 6^2 - 4(1)(9) = 36 - 36 = 0$ Roots are real and equal.
$D = b^2 - 4ac$ (iii) $2x^2 + 4x + 5 = 0$: $D = 4^2 - 4(2)(5) = 16 - 40 = -24 < 0$ Roots are imaginary.	$D = b^2 - 4ac$ (iv) $7x^2 - 6x - 1 = 0$: $D = (-6)^2 - 4(7)(-1) = 36 + 28 = 64 > 0$. Rational and unequal
$D = b^2 - 4ac$ (v) $5x^2 - 2x + 10 = 0$: $D = (-2)^2 - 4(5)(10) = 4 - 200 = -196 < 0$. Roots are imaginary.	$D = b^2 - 4ac$ (vi) $x^2 - 8x + 16 = 0$: $D = (-8)^2 - 4(1)(16) = 64 - 64 = 0$. Roots are real and equal.

2. For what values of t, the roots of $3x^2 + x + 9t = 0$ are real and unequal?

Solution

$D = b^2 - 4ac > 0$

$D = 1^2 - 4(3)(9t) > 0$.

$1 - 108t > 0 \Rightarrow t < \frac{1}{108}$.

3. If the quadratic equation $16x^2 + 7px + 49 = 0$ has equal roots, then find the values of p.

Solution

$D = b^2 - 4ac = 0$

$$D = (7p)^2 - 4(16)(49) = 0.$$

$49p^2 = 4 \cdot 16 \cdot 49 \Rightarrow p^2 = 64 \Rightarrow p = \pm 8.$

4. If the quadratic equation $4u^2 + 8u + q = 0$ has unequal and real roots, find the possible values for q.

Solution

$D = b^2 - 4ac > 0$

$D = 8^2 - 4(4)(q) > 0.$

$64 - 16q > 0 \Rightarrow q < 4.$

5. Find the value for m, if the quadratic equation $mx^2 - 8x + 1 = 0$ has real and equal roots.

Solution

$D = b^2 - 4ac = 0$

$$D = (-8)^2 - 4(m)(1) = 0.$$

$$64 - 4m = 0 \Rightarrow m = 16.$$

EXERCISE 2.5

1. Solve the following inequalities:

(i) $x^2 + 3x - 4 > 0$
(ii) $2x^2 - 8x + 6 > 0$
(iii) $x^2 + x - 6 < 0$
(iv) $x^2 - 6x + 9 < 0$
(v) $4x^2 - 16x + 15 \leq 0$
(vi) $-x^2 + 3x - 2 \geq 0$

Solution

(i) $x^2 + 3x - 4 > 0$

Solution

$x^2 + 3x - 4 > 0$

Solve Associated Equation

$x^2 + 3x - 4 = 0$

$x^2 + 4x - x - 4 = 0$

$x(x + 4) - 1(x + 4) = 0$

$(x + 4)(x - 1) = 0$

$x + 4 = 0 \quad ; \quad x - 1 = 0$

$x = -4 \quad ; \quad x = 1$

Critical Points: $(-4,0), (1,0)$

Find Interval: $(-\infty, -4), (-4,1), (1, \infty)$

Test Points

Test $x = -5$: $(-5+4)(-5-1) = 6 > 0$. Satisfied

Test $x = 0$: $(0+4)(0-1) = -4 < 0$. Satisfied

Test $x = 2$: $(2+4)(2-1) = 6 > 0$. Satisfied

Conclusion

$x \in (-\infty, -4) \cup (1, \infty)$

(ii) $2x^2 - 8x + 6 > 0$

Solution

$2x^2 - 8x + 6 > 0$

Solve Associated Equation

$2x^2 - 8x + 6 = 0$

$2x^2 - 3x - x + 6 = 0$

$x(x - 3) - 1(x - 3) = 0$

$(x - 1)(x - 3) = 0$

$x - 1 = 0 \quad ; \quad x - 3 = 0$

$x = 1 \quad ; \quad x = 3$

Critical Points: $(1,0), (3,0)$

Find Interval: $(-\infty, 1), (1,3), (3, \infty)$

Test Points

Test $x = 0$: $(0-1)(0-3) = 3 > 0$. Satisfied

Test $x = 2$: $(2-1)(2-3) = -1 < 0$. Satisfied

Test $x = 4$: $(4-1)(4-3) = 3 > 0$. Satisfied

Conclusion

$x \in (-\infty, 1) \cup (3, \infty)$.

(iii) $x^2 + x - 6 < 0$

Solution

$x^2 + x - 6 < 0$

Solve Associated Equation

$x^2 + x - 6 = 0$

$x^2 + 3x - 2x - 6 = 0$

$x(x+3) - 2(x+3) = 0$

$(x+3)(x-2) = 0$

$x + 3 = 0 \quad ; \quad x - 2 = 0$

$x = -3 \quad ; \quad x = 2$

Critical Points: $(-3,0), (2,0)$

Find Interval: $(-\infty, -3), (-3,2), (2, \infty)$

Test Points

Test $x = -4: (-4+3)(-4-2) = 6 > 0$. Satisfied

Test $x = 0: (0+3)(0-2) = -6 < 0$. Satisfied

Test $x = 3: (3+3)(3-2) = 6 > 0$. Satisfied

Conclusion

$x \in (-3, 2)$

(iv) $x^2 - 6x + 9 < 0$

Solution

$x^2 - 6x + 9 < 0$

Solve Associated Equation

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x - 3 = 0$

$x = 3$

Critical Points: $(3,0)$

Find Interval: $(3, \infty)$

Test Points

Test $x = 3: (3-3)^2 = 0$

Conclusion

No Solution

(v) $4x^2 - 16x + 15 \leq 0$

Solution

$4x^2 - 16x + 15 \leq 0$

Solve Associated Equation

$4x^2 - 16x + 15 = 0$

$4x^2 - 10x - 6x + 15 = 0$

$2x(x-5) - 3(x-5) = 0$

$(2x-3)(2x-5) = 0$

$2x - 3 = 0 \quad ; \quad 2x - 5 = 0$

$x = \frac{3}{2} \quad ; \quad x = \frac{5}{2}$

Critical Points: $\left(\frac{3}{2}, 0\right), \left(\frac{5}{2}, 0\right)$

Find Interval: $\left(-\infty, \frac{3}{2}\right], \left[\frac{3}{2}, \frac{5}{2}\right], \left[\frac{5}{2}, \infty\right)$

Test Points

Test $x = 1: (2-3)(2-5) = 3 > 0$. Satisfied

Test $x = 2: (4-3)(4-5) = -1 < 0$. Satisfied

Test $x = 3: (6-3)(6-5) = 3 > 0$. Satisfied

Conclusion

$x \in \left[\frac{3}{2}, \frac{5}{2}\right]$

(vi) $-x^2 + 3x - 2 \geq 0$

Solution

$-x^2 + 3x - 2 \geq 0$

Solve Associated Equation

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$(x-1)(x-2) = 0$

$x - 1 = 0 \quad ; \quad x - 2 = 0$

$x = 1 \quad ; \quad x = 2$

Critical Points: $(1,0), (2,0)$

Find Interval: $(-\infty, 1], [1,2], [2, \infty)$

Test Points

Test $x = 0: (0-1)(0-2) = 2 > 0$. Satisfied

Test $x = 1.5: (1.5-1)(1.5-2) = -0.25 < 0$. Satisfied

Test $x = 3: (3-1)(3-2) = 2 > 0$. Satisfied

Conclusion

$x \in [1, 2]$

EXERCISE 2.6

1. Make F the subject of the formula, $C^{o} = \frac{5}{9}(F^{o} - 32)$.

Solution

$C^\circ = \frac{5}{9}(F^\circ - 32)$

$\Rightarrow \frac{9}{5}C^\circ = F^\circ - 32$

$\Rightarrow F = \frac{9}{5}C^\circ + 32$

2. The formula for finding simple interest is $I = PRT$.
   (a) Make P the subject of the formula.
   (b) Make T the subject of the formula.

Solution

(a) We know that $I = PRT$

P the subject of formula $= P = \frac{I}{RT}$

(b) We know that $I = PRT$

T the subject of formula $= T = \frac{I}{PR}$

3. Make 'a' the subject of the formula $S = 2a + (n - 1)d$.

Solution

$S = 2a + (n - 1)d$

$\Rightarrow 2a = S - (n - 1)d$

$\Rightarrow a = \frac{S-(n-1)d}{2}$

4. The volume of a cylinder is given by the formula, $V = \pi r^2 h$. Make 'h' the subject of the formula.

Solution

$V = \pi r^2 h$

$\Rightarrow h = \frac{V}{\pi r^2}$

5. The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, make h the subject of the formula.

Solution

$A = \frac{1}{2}h(b_1 + b_2) \Rightarrow 2A = h(b_1 + b_2) \Rightarrow h = \frac{2A}{b_1+b_2}$

6. If $y = mx + c$, then make 'x' the subject of this equation.

Solution

$y = mx + c$

$\Rightarrow y - c = mx$

$\Rightarrow x = \frac{y-c}{m}$

7. Perimeter (P) of a rectangle is $P = 2(\ell + w)$, make ℓ as the subject of this formula.

Solution

$P = 2(l + w)$

$\Rightarrow \frac{P}{2} = l + w$

$\Rightarrow l = \frac{P}{2} - w$

$\Rightarrow l = \frac{P-2w}{2}$

8. The equation of a parabola is $y^2 = 4ax$, make 'x' as a subject of this equation.

Solution

$y^2 = 4ax$

$\Rightarrow x = \frac{y^2}{4a}$

9. If $P = S - C$, where S is selling price and C is cost price. Make S as subject of the equation.

Solution

$P = S - C$

$\Rightarrow S = P + C$

10. Volume of the cone is $V = \frac{1}{3}\pi r^2 h$, make 'h' as subject of this formula.

Solution

$V = \frac{1}{3}\pi r^2 h$

$\Rightarrow h = \frac{3V}{\pi r^2}$

EXERCISE 2.7

1. A town's population is modeled by $P(t) = -2t^2 + 40t + 800$, where t is years since 2020. Find the years when the population will be at least 1000.

Solution

Model: $P(t) = -2t^2 + 40t + 800$, Find t for $P(t) \geq 1000$ (Years since 2020)

- $-2t^2 + 40t + 800 \geq 1000$
- $-2t^2 + 40t - 200 \geq 0$
- Divide by -2: $t^2 - 20t + 100 \leq 0$
- Factor: $(t - 10)^2 \leq 0$
- Solution: $t = 10$
- **Year:** $2020 + 10 = 2030$

2. A company models its profit P in thousands of rupees by the equation: $P(x) = -5x^2 + 150x - 1000$, where x is the price per item in rupees. Find the price that gives maximum profit.

Solution

Model: $P(x) = -5x^2 + 150x - 1000$

- Max at $x = -\frac{b}{2a}$
- $x = -\frac{150}{2(-5)}$
- $x = \frac{-150}{-10}$
- **Price: Rs. 15**

3. A toy car rolls down an incline and covers a distance given by the equation $d = t^2 - 0.5t$ metres, where t is the time in seconds. Find the time when the car has travelled a distance 12.5 metres.

Solution

Model: $d = t^2 - 0.5t$, Find t for $d = 12.5$

- $t^2 - 0.5t = 12.5$
- $t^2 - 0.5t - 12.5 = 0$
- Quadratic Formula: $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $t = \frac{0.5 \pm \sqrt{(-0.5)^2 - 4(1)(-12.5)}}{2(1)}$
- $t = \frac{0.5 \pm \sqrt{0.25 + 50}}{2}$
- $t = \frac{0.5 + 7.088}{2} \approx 3.794$
- **Time: 3.79 seconds**

4. A ball's height (in metres) after t seconds is $h(t) = -4t^2 + 24t$. For what time interval is the ball at least $20m$ above the ground?

Solution

Model: $h(t) = -4t^2 + 24t$, Find t for $h(t) \geq 20$

- $-4t^2 + 24t \geq 20$
- $-4t^2 + 24t - 20 \geq 0$
- Divide by -4: $t^2 - 6t + 5 \leq 0$
- Factor: $(t-1)(t-5) \leq 0$
- Critical points: $t = 1, t = 5$
- **Interval:** $1 \leq t \leq 5$ **seconds**

5. A ball is thrown upward with an initial velocity of $40\,\text{ms}^{-1}$. Calculate the maximum height it reaches above ground level.

Solution

Given: $u = 40\ \text{m/s}$, $g \approx 10\ \text{m/s}^2$, Max height at $v = 0$

- $v^2 = u^2 + 2as$
- $0^2 = 40^2 + 2(-10)s$
- $0 = 1600 - 20s$
- $20s = 1600$
- $s = \frac{1600}{20}$
- **Max Height: 80 metres**

6. A freelancer's earnings follow the model $E(h) = -2h^2 + 40h$, where E is earning in rupees and h is hours worked per week. What is the maximum number of hours he should work to maximize earnings?

Solution

Model: $E(h) = -2h^2 + 40h$

- Max at $h = -\frac{b}{2a}$
- $h = -\frac{40}{2(-2)}$
- $h = \frac{-40}{-4}$
- **Hours: 10 hours per week**

REVIEW EXERCISE 2

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) The type of the equation $2x^2 - x + 1 = 0$ is:
(a) quadratic (b) linear (c) third degree (d) Pure quadratic

(ii) What is the discriminant of $x^2 + 5x - 5 = 0$?
(a) -20 (b) 20 (c) 25 (d) 45

(iii) The solution set of $3x^2 - 9 = 0$ is:
(a) $\{3\}$ (b) $\{\pm 3\}$ (c) $\{\pm\sqrt{3}\}$ (d) $\{\sqrt{3}\}$

(iv) Sum of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) $\frac{5}{3}$ (b) $-\frac{5}{3}$ (c) $\frac{12}{3}$ (d) $\frac{3}{5}$

(v) Product of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) -4 (b) 3 (c) 4 (d) 5

(vi) What are the roots of $(x - 3)(x + 3) = 0$?
(a) $3, -3$ (b) $3, 3$ (c) $-3, -3$ (d) $9, 0$

(vii) 3 and 2 are the roots of:
(a) $x^2 + 5x + 6 = 0$ (b) $x^2 + 6x + 5 = 0$
(c) $x^2 - 5x + 6 = 0$ (d) $x^2 + 6x - 5 = 0$

(viii) If $b^2 - 4ac > 0$ and is a perfect square, then the roots of $ax^2 + bx + c = 0$ are:
(a) equal (b) unequal (c) imaginary (d) irrational

(ix) If $b^2 - 4ac = 0$, then the roots of $ax^2 + bx + c = 0$ are:
(a) unequal (b) irrational (c) imaginary (d) equal

(x) Subject "c" of $x - 2c = b$ is:
(a) $x + b$ (b) $b - x$ (c) $\frac{x-b}{2}$ (d) $\frac{b-x}{2}$

2. Solve the following quadratic equations by factorization method, by completing square method and by quadratic formula:

(i) $8x^2 = x + 7$ (ii) $2x^2 - x - 10 = 0$

Solution

(i) $8x^2 = x + 7$

Solution

- **Quadratic formula**:

$$8x^2 - x - 7 = 0.$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(8)(-7)}}{2(8)}.$$

$$x = \frac{1 \pm \sqrt{1+224}}{16}.$$

$$x = \frac{1 \pm 15}{16}.$$

$$x = 1, \quad x = -\frac{7}{8}.$$

- **Factorization**:

$$8x^2 - x - 7 = (8x+7)(x-1) = 0.$$

$$x = 1, \quad x = -\frac{7}{8}.$$

- **Completing the square**:

$$x^2 - \frac{1}{8}x = \frac{7}{8}.$$

$$x^2 - \frac{1}{8}x + (\frac{1}{16})^2 = \frac{7}{8} + \frac{1}{256}.$$

$$(x - \frac{1}{16})^2 = \frac{225}{256}.$$

$$x - \frac{1}{16} = \pm\frac{15}{16}.$$

$$x = 1, \quad x = -\frac{7}{8}.$$

(ii) $2x^2 - x - 10 = 0$

Solution

- **Quadratic formula**:

$$x = \frac{-(-1) \pm \sqrt{1 - 4(2)(-10)}}{4}.$$

$$x = \frac{1 \pm \sqrt{81}}{4}.$$

$$x = \frac{5}{2}, \quad x = -2.$$

- **Factorization**:

$$2x^2 - x - 10 = (2x-5)(x+2) = 0.$$

$$x = \frac{5}{2}, \quad x = -2.$$

- **Completing the square**:

$$x^2 - \frac{1}{2}x = 5.$$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = 5 + \frac{1}{16}.$$

$$(x - \frac{1}{4})^2 = \frac{81}{16}.$$

$$x - \frac{1}{4} = \pm\frac{9}{4}.$$

$$x = \frac{5}{2}, \quad x = -2.$$

3. Form a quadratic equation whose roots are; 6, $\frac{3}{2}$.

Solution

$\alpha = 6,\ \beta = \frac{3}{2}$

$S = \alpha + \beta = 6 + \frac{3}{2} = \frac{15}{2}$

$P = \alpha\beta = (6)\left(\frac{3}{2}\right) = 9$

Required Equation is $\boldsymbol{x^2 - Sx + P = 0}$

$x^2 - \frac{15}{2}x + 9 = 0$

$2x^2 - 15x + 18 = 0$

4. Examine the nature of the roots of the following equations:
(i) $15x^2 + 11x + 2 = 0$ (ii) $x^2 - x - 1 = 0$

Solution

(i) $\boldsymbol{15x^2 + 11x + 2 = 0}$	**(ii)** $\boldsymbol{x^2 - x - 1 = 0}$
$D = 11^2 - 4(15)(2) = 121 - 120 = 1.$	$D = (-1)^2 - 4(1)(-1) = 1 + 4 = 5.$
Rational and unequal	Irrational and unequal

5. If a ball is thrown upward with a velocity v, the maximum height it reaches can be determined by using a formula $h = \frac{v^2}{2g}$. Rearrange the formula to make v the subject.

Solution

$h = \frac{v^2}{2g}$

$v^2 = 2gh$

$v = \sqrt{2gh}$

6. If the equation $x^2 + 2(1 + k)x + k^2 = 0$ has equal roots, then find the value of k.

Solution

$D = b^2 - 4ac$

$D = [2(1+k)]^2 - 4(1)(k^2) = 0.$

$4(1 + 2k + k^2) - 4k^2 = 0.$

$4 + 8k = 0.$

$k = -\frac{1}{2}.$