MATHEMATICS – 10
PTB - SYLLABUS

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This Solution is only

For Teachers

Because

Due to Shortage of time

I only work on few steps and

Not explain all steps

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Scheme

Pattern	Chapter	1	2	3	4	5	6	7	8	9	10	11	12	Marks 75
Objectives	Q.No.1 MCQ's	1	1	1	1	1	1	1	1	2	1	2	2	$1 \times 15 = 15$
	Q.No.2 6/9 Short	At least 2 S.Q. from each chapter												$2 \times 6 = 12$
Subjective Part – I	Q.No.3 6/9 Short					At least 2 S.Q. from each chapter								$2 \times 6 = 12$
	Q.No.4 6/9 Short									At least 2 S.Q. from each chapter				$2 \times 6 = 12$
Subjective Part – II	Attempt any 2 out of 3 Questions Q.No.5: (a) unit 1 + (b) unit 2 Q.No.6: (a) unit 3 + (b) unit 4 Q.No.7: (a) unit 5 + (b) unit 7													$2 \times 8 = 16$
Subjective Part – III	Attempt any 1 out of 2 Questions Q.No.8: (a) unit 6 + (b) unit 8 or 11 Q.No.9: (a) unit 10 + (b) unit 9 or 12													$1 \times 8 = 8$

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UNIT 1 Complex Numbers

EXERCISE 1.1

1. Simplify the following:

(i) i^5 (ii) i^{16} (iii) $(-i)^{-19}$ (iv) $27i^{-26}$

(v) $i^{11}+i^5$ (vi) $(i^4+i^3+i^2+i)^2$ (vii) $\left(\dfrac{i^8}{i^5}\right)^{-5}$ (viii) $i^{13}\times i^{29}$

Solution

i. $\boldsymbol{i^5} = i.i^4 = i.(i^2)^2 = i.(-1)^2 = i.(1) = i$

ii. $\boldsymbol{i^{16}} = (i^2)^8 = (-1)^8 = 1$

iii. $(-\boldsymbol{i})^{-19} = -i^{-19} = -\frac{1}{i^{19}} = -\frac{1}{i.i^{18}} = -\frac{1}{i.(i^2)^9} = -\frac{1}{i.(-1)^9} = -\frac{1}{i.(-1)} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i$

iv. $\boldsymbol{27i^{-26}} = \frac{27}{i^{26}} = \frac{27}{(i^2)^{13}} = \frac{27}{(-1)^{13}} = \frac{27}{(-1)} = -27$

v. $\boldsymbol{i^{11}} + \boldsymbol{i^5} = i.i^{10} + i.i^4 = i.(i^2)^5 + i.(i^2)^2 = i.(-1)^5 + i.(-1)^2$
$= i.(-1) + i.(1) = -i + i = 0$

vi. $(\boldsymbol{i^4} + \boldsymbol{i^3} + \boldsymbol{i^2} + \boldsymbol{i})^2 = [(i^2)^2 + i.i^2 + i^2 + i]^2 = [(-1)^2 + i.(-1) + (-1) + i]^2$
$= [1 - i - 1 + i]^2 = (0)^2 = 0$

vii. $\left(\frac{i^8}{i^5}\right)^{-5} = (i^3)^{-5} = (i.i^2)^{-5} = (-i)^{-5} = -i^{-5} = -\frac{1}{i^5} = -\frac{1}{i.i^4} = -\frac{1}{i.(i^2)^2}$
$= -\frac{1}{i.(-1)^2} = -\frac{1}{i.(1)} = -\frac{1}{i} = -\frac{1}{i} \times \frac{i}{i} = -\frac{i}{i^2} = i$

viii. $\boldsymbol{i^{13}} \times \boldsymbol{i^{29}} = i^{42} = (i^2)^{21} = (-1)^{21} = -1$

2. Write in terms of i.

(i) $2+\sqrt{-4}$ (ii) $3-\sqrt{-7}$ (iii) $\frac{2}{5}+\frac{\sqrt{-16}}{5}$ (iv) $\sqrt{2}-\sqrt{-3}$

Solution

i. $\mathbf{2+\sqrt{-4}} = 2+\sqrt{4\times -1} = 2+\sqrt{4}\times\sqrt{-1} = 2+2i$

ii. $\mathbf{3-\sqrt{-7}} = 3-\sqrt{-1\times 7} = 3-\sqrt{-1}\times\sqrt{7} = 3-i\sqrt{7}$

iii. $\mathbf{\frac{2}{5}+\frac{\sqrt{-16}}{5}} = \frac{2}{5}+\frac{\sqrt{16\times-1}}{5} = \frac{2}{5}+\frac{\sqrt{16}\times\sqrt{-1}}{5} = \frac{2}{5}+\frac{4i}{5}$

iv. $\mathbf{\sqrt{2}-\sqrt{-3}} = \sqrt{2}-\sqrt{-1\times 3} = \sqrt{2}-\sqrt{-1}\times\sqrt{3} = \sqrt{2}-i\sqrt{3}$

3. Find the values of x and y.

(i) $(2x+5)+(y-3)i = 1+2i$ (ii) $(3x+2)-(4-y)i = 5+3i$

(iii) $(2+i)x+(1-2i)y = 3+4i$ (iv) $(1-i)x+(2+i)y = 4-i$

(v) $(3x-1)+(2y-3)i = 8+7i$

Solution

i. $\mathbf{(2x+5)+(y-3)i = 1+2i}$ $2x+5=1$; $y-3=2$ $2x=1-5$; $y=2+3$ $2x=-4$; $y=5$ $x=-2$; $y=5$	**ii.** $\mathbf{(3x+2)-(4-y)i = 5+3i}$ $3x+2=5$; $-(4-y)=3$ $3x=5-2$; $-4+y=3$ $3x=3$; $y=3+4$ $x=1$; $y=7$
iii. $\mathbf{(2+i)x+(1-2i)y = 3+4i}$ $2x+ix+y-2yi = 3+4i$ $2x+y=3$ ……….(i) $x-2y=4$ ……….(ii) $\mathbf{4x+2y=6}$ $x-2y=4$ $5x=10$ $x=2$ Put x = 2 in (i) $\Rightarrow 2(2)+y=3 \Rightarrow 4+y=3$ $\Rightarrow y=3-4 \Rightarrow y=-1$	**iv.** $\mathbf{(1-i)x+(2+i)y = 4-i}$ $x-ix+2y+iy = 4-i$ $x+2y=4$ ……….(i) $-x+y=-1$ ……….(ii) $x+2y=4$ $-x+y=-1$ $3y=3$ $y=1$ Put y = 1 in (i) $\Rightarrow x+2(1)=4 \Rightarrow x+2=4$ $\Rightarrow x=4-2 \Rightarrow x=2$
v. $\mathbf{(3x-1)+(2y-3)i = 8+7i}$ $3x-1=8$; $2y-3=7$ $3x=8+1$; $2y=7+3$ $3x=9$; $2y=10$ $x=3$; $y=5$	

EXERCISE 1.2

1. Simplify and write in the form $a + bi$:

(i)	$(2+5i)+(3-zi)$	(ii)	$(16-3i)+(9+2i)$
(iii)	$(9-2i)-(7-3i)$	(iv)	$(11+9i)-(9-7i)$
(v)	$(3+4i)(2-3i)$	(vi)	$(5-2i)(3-4i)$
(vii)	$(3-5i)\div(2-4i)$	(viii)	$(5+2i)\div(6-3i)$

Solution

i. $\boldsymbol{(2+5i)+(3-zi)} = 2+5i+3-zi = 5+(5-z)i$

ii. $\boldsymbol{(16-3i)+(9+2i)} = 16-3i+9+2i = 25-i$

iii. $\boldsymbol{(9-2i)-(7-3i)} = 9-2i-7+3i = 2+i$

iv. $\boldsymbol{(11+9i)-(9-7i)} = 11+9i-9+7i = 2+16i$

v. $\boldsymbol{(3+4i)(2-3i)} = 6-9i+8i-12i^2 = 6-9i+8i+12 = 18-i$

vi. $\boldsymbol{(5-2i)(3-4i)} = 15-20i-6i+8i^2 = 15-20i-6i-8 = 7-26i$

vii. $\boldsymbol{(3-5i)\div(2-4i)} = \frac{3-5i}{2-4i} = \frac{3-5i}{2-4i}\times\frac{2+4i}{2+4i} = \frac{6+12i-10i-20i^2}{(2)^2-(4i)^2}$

$= \frac{6+12i-10i+20}{4+16} = \frac{26+2i}{20} = \frac{13}{10}+\frac{1}{10}i$

viii. $\boldsymbol{(5+2i)\div(6-3i)} = \frac{5+2i}{6-3i} = \frac{5+2i}{6-3i}\times\frac{6+3i}{6+3i} = \frac{30+15i+12i+6i^2}{(6)^2-(3i)^2}$

$= \frac{30+15i+12i-6}{36+9} = \frac{24+27i}{45} = \frac{8}{15}+\frac{3}{5}i$

2. Write additive inverse for each complex number:

(i) $3+2i$ (ii) $4-3i$ (iii) $5-7i$ (iv) $-\frac{2}{3}+\frac{5}{4}i$

Solution

i. $\boldsymbol{z = 3+2i} \Rightarrow -z = -3-2i$

ii. $\boldsymbol{z = 4-3i} \Rightarrow -z = -4+3i$

iii. $\boldsymbol{z = 5-7i} \Rightarrow -z = -5+7i$

iv. $\boldsymbol{z = -\frac{2}{3}+\frac{5}{4}i} \Rightarrow -z = \frac{2}{3}-\frac{5}{4}i$

3. Find multiplicative inverse for each complex number:

(i) $4+5i$ (ii) $6+2i$ (iii) $7-3i$ (iv) $\sqrt{5}-4i$

Solution

i. $z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=\left(\frac{4}{(4)^2+(5)^2},\frac{-5}{(4)^2+(5)^2}\right)=\left(\frac{4}{16+25},\frac{-5}{16+25}\right)=\left(\frac{4}{41},-\frac{5}{41}\right)=\frac{4}{41}-\frac{5}{41}i$

ii. $z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=\left(\frac{6}{(6)^2+(2)^2},\frac{-2}{(6)^2+(2)^2}\right)=\left(\frac{6}{36+4},\frac{-2}{36+4}\right)=\left(\frac{6}{40},\frac{-2}{40}\right)=\frac{3}{20}-\frac{1}{20}i$

iii. $z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=\left(\frac{7}{(7)^2+(-3)^2},\frac{-(-3)}{(7)^2+(-3)^2}\right)=\left(\frac{7}{49+9},\frac{3}{49+9}\right)=\left(\frac{7}{58},\frac{3}{58}\right)=\frac{7}{58}+\frac{3}{58}i$

iv. $z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=\left(\frac{\sqrt{5}}{(\sqrt{5})^2+(-4)^2},\frac{-(-4)}{(\sqrt{5})^2+(-4)^2}\right)=\left(\frac{\sqrt{5}}{5+16},\frac{4}{5+16}\right)=\left(\frac{\sqrt{5}}{21},\frac{4}{21}\right)=\frac{\sqrt{5}}{21}+\frac{4}{21}i$

4. If $z_1=2+5i$, $z_2=1-3i$ and $z_3=2+i$, then verify that

(i) $z_1+z_2=z_2+z_1$ (ii) $z_1z_2=z_2z_1$

(iii) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ (iv) $(z_1z_2)z_3=z_1(z_2z_3)$

(v) $z_1+(-z_1)=(-z_1)+z_1=0$

Solution

i. $L.H.S=z_1+z_2=(2+5i)+(1-3i)=2+5i+1-3i=3+2i$
$R.H.S=z_2+z_1=(1-3i)+(2+5i)=1-3i+2+5i=3+2i$
Hence $\boldsymbol{z_1+z_2=z_2+z_1}$

ii. $L.H.S=z_1z_2=(2+5i)(1-3i)=2-6i+5i-15i^2=2-6i+5i+15=17-i$
$R.H.S=z_2z_1=(1-3i)(2+5i)=2+5i-6i-15i^2=2+5i-6i+15=17-i$
Hence $\boldsymbol{z_1z_2=z_2z_1}$

iii. $L.H.S=(z_1+z_2)+z_3=[(2+5i)+(1-3i)]+(2+i)$
$=[2+5i+1-3i]+(2+i)=3+2i+2+i=5+3i$
$R.H.S=z_1+(z_2+z_3)=(2+5i)+[(1-3i)+(2+i)]$
$=(2+5i)+1-3i+2+i=2+5i+3-2i=5+3i$
Hence $\boldsymbol{(z_1+z_2)+z_3=z_1+(z_2+z_3)}$

iv. $L.H.S=(z_1z_2)z_3=[(2+5i)(1-3i)](2+i)=(2-6i+5i-15i^2)(2+i)$
$=(2-6i+5i+15)(2+i)=(17-i)(2+i)=34+17i-2i-i^2$
$=34+17i-2i+1=35+15i$
$R.H.S=z_1(z_2z_3)=(2+5i)[(1-3i)(2+i)]=(2+5i)[2+i-6i-3i^2]$
$=(2+5i)[2+i-6i+3]=(2+5i)(5-5i)=10-10i+25i-25i^2$
$=10-10i+25i+25=35+15i$
Hence $\boldsymbol{(z_1z_2)z_3=z_1(z_2z_3)}$

v. $L.H.S=z_1+(-z_1)=(2+5i)+(-2-5i)=2+5i-2-5i=0$
$R.H.S=(-z_1)+z_1=(-2-5i)+(2+5i)=-2-5i+2+5i=0$
Hence $\boldsymbol{z_1+(-z_1)=(-z_1)+z_1}$

5. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the values of x and y.

Solution

$x + iy = \frac{(1+i)^2}{2-i}$

$x + iy = \frac{1+i^2+2i}{2-i} = \frac{1-1+2i}{2-i} = \frac{2i}{2-i} = \frac{2i}{2-i} \times \frac{2+i}{2+i} = \frac{4i+2i^2}{(2)^2-i^2} = \frac{4i-2}{4+1} = \frac{4i-2}{5} = \frac{4}{5}i - \frac{2}{5} = -\frac{2}{5} + \frac{4}{5}i$

$x = -\frac{2}{5}$; $y = \frac{4}{5}$ comparing real and imaginary parts

6. If $(2x+iy)(1-i) = 4+2i$, then find the values of x and y.

Solution

$(2x + iy)(1 - i) = 4 + 2i$

$2x - 2xi + iy - i^2y = 4 + 2i$

$2x - 2xi + iy + y = 4 + 2i$

$2x + y = 4$ ……….(i) and $-2x + y = 2$ ……….(ii)

$2x + y = 4$ $-2x + y = 2$
$2y = 6$ $y = 3$

Put y = 3 in (i)

$\Rightarrow 2x + (3) = 4 \Rightarrow 2x = 4 - 3 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

Hence $x = \frac{1}{2}$; $y = 3$

7. Find the values of a and b, if $(a + bi)(1 + 3i) = -8 + 11i$.

Solution

$(a + bi)(1 + 3i) = -8 + 11i$

$a + 3ai + bi + 3bi^2 = -8 + 11i$

$a + 3ai + bi - 3b = -8 + 11i$

$a - 3b = -8$ ……….(i) and $3a + b = 11$ ……….(ii)

$\mathbf{3}a - \mathbf{9}b = \mathbf{-24}$ $\underline{\pm}3a \underline{\pm} b = \underline{\pm}11$
$-10b = -35$ $b = \frac{7}{2}$

Put b = $\frac{7}{2}$ in (i)

$\Rightarrow a - 3\left(\frac{7}{2}\right) = -8 \Rightarrow a - \frac{21}{2} = -8 \Rightarrow a = -8 + \frac{21}{2} \Rightarrow a = \frac{-16+21}{2} \Rightarrow a = \frac{5}{2}$

Hence $a = \frac{5}{2}$; $b = \frac{7}{2}$

EXERCISE 1.3

1. Find the modulus of the following complex numbers:

(i) $4+3i$ (ii) $-5-4i$ (iii) $\frac{3}{5}-\frac{4}{5}i$ (iv) $-\sqrt{2}-\sqrt{3}i$

Solution

i. $|4+3i| = \sqrt{(4)^2+(3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

ii. $|-5-4i| = \sqrt{(-5)^2+(-4)^2} = \sqrt{25+16} = \sqrt{41}$

iii. $\left|\frac{3}{5}-\frac{4}{5}i\right| = \sqrt{\left(\frac{3}{5}\right)^2+\left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}+\frac{16}{25}} = \sqrt{\frac{9+16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$

iv. $|-\sqrt{2}-\sqrt{3}i| = \sqrt{\left(-\sqrt{2}\right)^2+\left(-\sqrt{3}\right)^2} = \sqrt{2+3} = \sqrt{5}$

2. If $z_1 = 2+7i$ and $z_2 = 4-3i$, then verify that

(i) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ (ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Solution

i. $L.H.S = \overline{z_1+z_2} = \overline{(2+7i)+(4-3i)} = \overline{2+7i+4-3i} = \overline{6+4i} = 6-4i$

$R.H.S = \overline{z_1}+\overline{z_2} = \overline{(2+7i)}+\overline{(4-3i)} = 2-7i+4+3i = 6-4i$

Hence $\boldsymbol{\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}}$

ii. $L.H.S = \overline{z_1 z_2} = \overline{(2+7i)(4-3i)} = \overline{8-6i+28i-21i^2}$

$= \overline{8-6i+28i+21} = \overline{29+22i} = 29-22i$

$R.H.S = \overline{z_1}\,\overline{z_2} = \overline{(2+7i)}.\overline{(4-3i)} = (2-7i)(4+3i)$

$= 8+6i-28i-21i^2 = 8+6i-28i+21 = 29-22i$

Hence $\boldsymbol{\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}}$

iii. $L.H.S = \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{2+7i}{4-3i}\right)} = \overline{\left(\frac{2+7i}{4-3i}\times\frac{4+3i}{4+3i}\right)} = \overline{\left(\frac{8+6i+28i+21i^2}{(4)^2-(3i)^2}\right)} = \overline{\left(\frac{8+6i+28i-21}{16+9}\right)}$

$= \overline{\left(\frac{-13+34i}{25}\right)} = \overline{-\frac{13}{25}+\frac{34}{25}i} = -\frac{13}{25}-\frac{34}{25}i$

$R.H.S = \frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{(2+7i)}}{\overline{(4-3i)}} = \frac{2-7i}{4+3i} = \frac{2-7i}{4+3i}\times\frac{4-3i}{4-3i} = \frac{8-6i-28i+21i^2}{(4)^2-(3i)^2} = \frac{8-6i-28i-21}{16+9}$

$= \frac{-13-34i}{25} = -\frac{13}{25}-\frac{34}{25}i$

Hence $\boldsymbol{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}}$

3. If $z = 5 - 2i$, then verify that

(i) $\bar{\bar{z}} = z$ (ii) $|z| = |\bar{z}|$ (iii) $|z| = |-z|$

(iv) $z\bar{z} = |z|^2$ (v) $|z| = |-\bar{z}|$

Solution

i. $z = 5 - 2i \Rightarrow \bar{z} = 5 + 2i \Rightarrow \bar{\bar{z}} = 5 - 2i \Rightarrow \bar{\bar{\boldsymbol{z}}} = \boldsymbol{z}$

ii. $L.H.S = |z| = |5 - 2i| = \sqrt{(5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29}$

$R.H.S = |\bar{z}| = |5 + 2i| = \sqrt{(5)^2 + (2)^2} = \sqrt{25+4} = \sqrt{29}$

$\Rightarrow |\boldsymbol{z}| = |\bar{\boldsymbol{z}}|$

iii. $L.H.S = |z| = |5 - 2i| = \sqrt{(5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29}$

$R.H.S = |-z| = |-(5 - 2i)| = |-5 + 2i| = \sqrt{(-5)^2 + (2)^2} = \sqrt{25+4} = \sqrt{29}$

$\Rightarrow |\boldsymbol{z}| = |-\boldsymbol{z}|$

iv. $L.H.S = z\bar{z} = (5 - 2i)(5 + 2i) = (5)^2 - (2i)^2 = 25 + 4 = 29$

$R.H.S = |z|^2 = |5 - 2i|^2 = \left(\sqrt{(5)^2 + (-2)^2}\right)^2 = 25 + 4 = 29$

$\Rightarrow \boldsymbol{z\bar{z}} = |\boldsymbol{z}|^2$

v. $L.H.S = |z| = |5 - 2i| = \sqrt{(5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29}$

$R.H.S = |-\bar{z}| = |-(5 + 2i)| = |-5 - 2i| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29}$

$\Rightarrow |\boldsymbol{z}| = |-\bar{\boldsymbol{z}}|$

4. If $z = 4 - 3i$, then verify that $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$.

Solution

$|z| = |4 - 3i| = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$|-z| = |-(4 - 3i)| = |-4 + 3i| = \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$\because \bar{\bar{z}} = z \therefore |\bar{\bar{z}}| = |z| = |4 - 3i| = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$|-\bar{z}| = |-(\overline{4 - 3i})| = |-(4 + 3i)| = |-4 - 3i| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

Clearly $|\boldsymbol{z}| = |-\boldsymbol{z}| = |\bar{\bar{\boldsymbol{z}}}| = |-\bar{\boldsymbol{z}}|$

5. If $z_1 = 2 + 3i$, $z_2 = -1 + i$, then evaluate:

(i) $Re(z_1 z_2)$ (ii) $Im(z_1 z_2)$

Solution

$z_1 z_2 = (2 + 3i)(-1 + i) = -2 + 2i - 3i + 3i^2 = -2 + 2i - 3i - 3$

$z_1 z_2 = -5 - i$

i. $\text{Re}(z_1 z_2) = -5$

ii. $\text{Im}(z_1 z_2) = -1$

EXERCISE 1.4

1. Find the real and imaginary parts of the following complex numbers:

(i) $(8-3i)^2$ (ii) $(5+3i)^{-1}$ (iii) $(4-5i)^{-1}$

(iv) $(4-3i)^{-2}$ (v) $\left(\dfrac{3+2i}{4+3i}\right)^{-1}$ (vi) $\left(\dfrac{2-i}{2+i}\right)^{-2}$

(vii) $\left(\dfrac{1-2i}{1+i}\right)^{2}$

Solution

i. $\mathbf{(8-3i)^2} = (8)^2+(3i)^2-2(8)(3i) = 64+9i^2-48i = 64-9-48i$
$= 55-48i \Rightarrow \text{Re}(z)=55, \text{Im}(z)=-48$

ii. $\mathbf{(5+3i)^{-1}} = \frac{1}{5+3i} = \frac{1}{5+3i}\times\frac{5-3i}{5-3i} = \frac{5-3i}{(5)^2-(3i)^2} = \frac{5-3i}{25+9} = \frac{5-3i}{34} = \frac{5}{34}-\frac{3}{34}i \Rightarrow \text{Re}(z)=\frac{5}{34}, \text{Im}(z)=-\frac{3}{34}$

iii. $\mathbf{(4-5i)^{-1}} = \frac{1}{4-5i} = \frac{1}{4-5i}\times\frac{4+5i}{4+5i} = \frac{4+5i}{(4)^2-(5i)^2} = \frac{4+5i}{16+25} = \frac{4+5i}{41} = \frac{4}{41}+\frac{5}{41}i \Rightarrow \text{Re}(z)=\frac{4}{41}, \text{Im}(z)=\frac{5}{41}$

iv. $\mathbf{(4-3i)^{-2}} = \frac{1}{(4-3i)^2} = \frac{1}{16+9i^2-24i} = \frac{1}{16-9-24i} = \frac{1}{7-24i} = \frac{1}{7-24i}\times\frac{7+24i}{7+24i}$
$= \frac{7+24i}{(7)^2-(24i)^2} = \frac{7+24i}{49+576} = \frac{7+24i}{625} = \frac{7}{625}+\frac{24}{625}i \Rightarrow \text{Re}(z)=\frac{7}{625}, \text{Im}(z)=\frac{24}{625}$

v. $\mathbf{\left(\frac{3+2i}{4+3i}\right)^{-1}} = \frac{4+3i}{3+2i} = \frac{4+3i}{3+2i}\times\frac{3-2i}{3-2i} = \frac{12-8i+9i-6i^2}{(3)^2-(2i)^2} = \frac{12-8i+9i+6}{9+4} = \frac{18+i}{13}$
$= \frac{18}{13}+\frac{1}{13}i \Rightarrow \text{Re}(z)=\frac{18}{13}, \text{Im}(z)=\frac{1}{13}$

vi. $\mathbf{\left(\frac{2-i}{2+i}\right)^{-2}} = \left(\frac{2+i}{2-i}\right)^2 = \left(\frac{2+i}{2-i}\times\frac{2+i}{2+i}\right)^2 = \left(\frac{4+2i+2i+i^2}{(2)^2-(i)^2}\right)^2 = \left(\frac{4+2i+2i-1}{4+1}\right)^2 = \left(\frac{3+4i}{5}\right)^2$
$= \frac{(3+4i)^2}{25} = \frac{9+16i^2+24i}{25} = \frac{9-16+24i}{25} = -\frac{7}{25}+\frac{24}{25}i \Rightarrow \text{Re}(z)=-\frac{7}{25}, \text{Im}(z)=\frac{24}{25}$

vii. $\mathbf{\left(\frac{1-2i}{1+i}\right)^{2}} = \left(\frac{1-2i}{1+i}\times\frac{1-i}{1-i}\right)^2 = \left(\frac{1-i-2i+2i^2}{(1)^2-(i)^2}\right)^2 = \left(\frac{1-i-2i-2}{1+1}\right)^2 = \left(\frac{-1-3i}{2}\right)^2 = \frac{(-1-3i)^2}{4}$
$= \frac{(-1)^2+(-3i)^2+2(-1)(-3i)}{4} = \frac{1-9+6i}{4} = -\frac{8}{4}+\frac{6}{4}i \Rightarrow \text{Re}(z)=-2, \text{Im}(z)=\frac{3}{2}$

2. Solve the following simultaneous linear equations with complex coefficients for w and z:

(i) $3z+(2+i)w=11-i$
$(2-i)z-w=-1+i$

(ii) $2z+(3+i)w=9-i$
$-iz-iw=-1+i$

(iii) $z-4w=3i$
$2z+3w=11-5i$

(iv) $z+w=3i$
$2z+3w=2$

(v) $2z+(3+i)w=1$
$-z-(1-i)w=2$

Solution

i. $\boldsymbol{3z+(2+i)w=11-i}$; $\boldsymbol{(2-i)z-w=-1+i}$

Solution

$3z+(2+i)w=11-i$ ……….(i)

$(2-i)z-w=-1+i$ ……….(ii)

Multiplying eq (i) by 1 and eq (ii) by $(2+i)$ and then adding them

$3z+(2+i)w=11-i$

$(2-i)(2+i)z-(2+i)w=(2+i)(-1+i)$

$3z+(2-i)(2+i)z=11-i+(2+i)(-1+i)$

$3z+(4-i^2)z=11-i-2+2i-i+i^2$

$3z+(4+1)z=11-i-2+2i-i-1$

$3z+5z=8 \Rightarrow 8z=8 \Rightarrow \boldsymbol{z}=1$

Put z = 1 in eq (i)

$3(1)+(2+i)w=11-i$

$3+(2+i)w=11-i$

$(2+i)w=8-i$

$w=\frac{8-i}{2+i}$

$w=\frac{8-i}{2+i}\times\frac{2-i}{2-i}=\frac{16-8i-2i+i^2}{(2)^2-(i)^2}=\frac{16-8i-2i-1}{4+1}=\frac{15-10i}{5}=\frac{15}{5}-\frac{10}{5}i$

$\boldsymbol{w=3-2i}$

ii. $\boldsymbol{2z+(3+i)w=9-i}$; $\boldsymbol{-iz-iw=-1+i}$

Solution

$2z+(3+i)w=9-i$ ……….(i)

$-iz-iw=-1+i$ ……….(ii)

Multiplying eq (i) by i and eq (ii) by 2 and then adding them

$2iz+(3+i)iw=9i-i^2$

$-2iz-2iw=-2+2i$

$(3+i)iw-2iw=9i-i^2-2+2i$

$3iw+i^2w-2iw=9i+1-2+2i$

$iw-w=11i-1$

$(i-1)w = 11i - 1$

$w = \frac{11i-1}{i-1}$

$w = \frac{11i-1}{i-1} \times \frac{i+1}{i+1} = \frac{11i^2+11i-i-1}{(i)^2-(1)^2} = \frac{-11+11i-i-1}{-1-1} = \frac{-12+10i}{-2} = \frac{-12}{-2} + \frac{10}{-2}i$

$\boldsymbol{w = 6 - 5i}$

Put $w = 6 - 5i$ in eq (i)

$2z + (3+i)(6-5i) = 9 - i$

$2z + 18 - 15i + 6i - 5i^2 = 9 - i$

$2z + 18 - 15i + 6i + 5 = 9 - i$

$2z + 23 - 9i = 9 - i$

$2z = 9 - i - 23 + 9i$

$2z = -14 + 8i$

$z = -\frac{14}{2} + \frac{8}{2}i$

$\boldsymbol{z = -7 + 4i}$

iii. $\boldsymbol{z - 4w = 3i \quad ; \quad 2z + 3w = 11 - 5i}$

Solution

$z - 4w = 3i$ ……….(i)

$2z + 3w = 11 - 5i$ ……….(ii)

Multiplying eq (i) by 2 and subtracting both

$2z - 8w = 6i$

$\underline{\pm 2z \pm 3w = \pm 11 \mp 5i}$

$-11w = -11 + 11i$

$w = -\frac{11}{-11} + \frac{11}{-11}i$

$\boldsymbol{w = 1 - i}$

Put $w = 1 - i$ in eq (i)

$z - 4(1-i) = 3i$

$z - 4 + 4i = 3i$

$z = 3i + 4 - 4i$

$\boldsymbol{z = 4 - i}$

iv. $\boldsymbol{z + w = 3i \quad ; \quad 2z + 3w = 2}$

Solution

$z + w = 3i$ ……….(i)

$2z + 3w = 2$ ……….(ii)

Multiplying eq (i) by 2 and subtracting both

$2z + 2w = 6i$
$\underline{\pm 2z \pm 3w = \pm 2}$
$-w = -2 + 6i$
$\boldsymbol{w = 2 - 6i}$
Put $w = 2 - 6i$ in eq (i)
$z + (2 - 6i) = 3i$
$z = 3i - 2 + 6i$
$\boldsymbol{z = -2 + 9i}$

v. $\boldsymbol{2z + (3 + i)w = 1}$ **;** $\boldsymbol{-z - (1 - i)w = 2}$

Solution

$2z + (3 + i)w = 1$ ……….(i)
$-z - (1 - i)w = 2$ ……….(ii)
Multiplying eq (ii) by 2 and adding both
$2z + (3 + i)w = 1$
$\underline{-2z - 2(1 - i)w = 4}$
$(3 + i)w - 2(1 - i)w = 5$
$3w + iw - 2w + 2iw = 5$
$w + 3iw = 5$
$(1 + 3i)w = 5$
$w = \frac{5}{1+3i}$
$w = \frac{5}{1+3i} \times \frac{1-3i}{1-3i} = \frac{5-15i}{(1)^2-(3i)^2} = \frac{5-15i}{1+9} = \frac{5-15i}{10} = \frac{5}{10} - \frac{15}{10}i$
$\boldsymbol{w = \frac{1}{2} - \frac{3}{2}i}$
Put $w = \frac{1}{2} - \frac{3}{2}i$ in eq (i)
$2z + (3 + i)\left(\frac{1}{2} - \frac{3}{2}i\right) = 1$
$2z + \frac{3}{2} - \frac{9}{2}i + \frac{i}{2} - \frac{3}{2}i^2 = 1$
$2z + \frac{3}{2} - \frac{9}{2}i + \frac{i}{2} + \frac{3}{2} = 1$
$2z = 1 - \frac{3}{2} + \frac{9}{2}i - \frac{i}{2} - \frac{3}{2}$
$2z = \frac{2-3+9i-i-3}{2}$
$2z = \frac{-4+8i}{2}$
$z = \frac{-4+8i}{4} = -\frac{4}{4} + \frac{8}{4}i$
$\boldsymbol{z = -1 + 2i}$

REVIEW EXERCISE 1

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) $i^2 + i^4 =$

(a) -1 (b) 0 ✓ (c) 1 (d) 2

(ii) Real part of $(2-3i)(2+3i)$ is:

(a) -3 (b) 1 (c) 4 (d) 13 ✓

(iii) Imaginary part of $(2-i)(2+i)$ is:

(a) 0 ✓ (b) 1 (c) 7 (d) 9

(iv) $x + iy$ will be pure imaginary number, when:

(a) $y = 0$ (b) $x = 0$ ✓ (c) $i = 0$ (d) $x = 0, y = 0$

(v) What is additive inverse of $5 - 2i$?

(a) $5 + 2i$ (b) $-5 - 2i$ (c) $5 - 2i$ (d) $-5 + 2i$ ✓

(vi) What is multiplicative inverse of $z = 1 + i$?

(a) $1 - i$ (b) i (c) $\frac{1}{2} - \frac{1}{2}i$ ✓ (d) $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

(vii) If $z = 4 - 3i$, then $z\bar{z} =$

(a) 3 (b) 9 (c) 16 (d) 25 ✓

(viii) Conjugate of $9 - 4i$ is:

(a) $-9 - 4i$ (b) $9 + 4i$ ✓ (c) $9 + 9i$ (d) $4 - 9i$

(ix) If $z = 4 + 4i$, then $z + \bar{z} =$

(a) 8 ✓ (b) $8 + 8i$ (c) $8i$ (d) 0

(x) If $z = 5 + 4i$, then $|z| =$

(a) 9 (b) 25 (c) 41 (d) $\sqrt{41}$ ✓

2. (i) Is "0" a complex number? Explain.
(ii) What is the result of multiplying a complex number by its conjugate?
(iii) State the condition for two complex numbers to be equal.

Solution

i. The real numbers are a subset of the complex numbers, so **zero is by definition a complex number**. i.e. $0 \in R \subseteq C$

ii. When a complex number is multiplied by its conjugate - the result is **real number**. i.e. $z\bar{z} = |z|^2 \in R$

iii. Two complex numbers are equal **if their real parts are the same and their imaginary parts are the same**. i.e. $(a, b) = (c, d) \Leftrightarrow a = c, b = d$

3. Simplify:
(i) i^{37} (ii) $i^{13} \times i^{11}$ (iii) $(-i)^{-9}$
(iv) $(3 - 4i)(5 - 6i)$ (v) $(3 + 4i) \div (5 - 7i)$

Solution

i. $\boldsymbol{i^{37}} = i.i^{36} = i.(i^2)^{18} = i.(-1)^{18} = i.(1) = i$

ii. $\boldsymbol{i^{13} \times i^{11}} = i^{24} = (i^2)^{12} = (-1)^{12} = 1$

iii. $(-\boldsymbol{i})^{-9} = -i^{-9} = -\frac{1}{i^9} = -\frac{1}{i.i^8} = -\frac{1}{i.(i^2)^4} = -\frac{1}{i.(-1)^4} = -\frac{1}{i.(1)} = -\frac{1}{i} = -\frac{1}{i} \times \frac{i}{i} = -\frac{i}{i^2} = i$

iv. $(\boldsymbol{3 - 4i})(\boldsymbol{5 - 6i}) = 15 - 18i - 20i + 24i^2 = 15 - 18i - 20i - 24 = -9 - 38i$

v. $(\boldsymbol{3 + 4i}) \div (\boldsymbol{5 - 7i}) = \frac{3+4i}{5-7i} = \frac{3+4i}{5-7i} \times \frac{5+7i}{5+7i} = \frac{15+21i+20i+28i^2}{(5)^2-(7i)^2}$

$= \frac{15+21i+20i-28}{25+49} = \frac{-13+41i}{74} = -\frac{13}{74} + \frac{41}{74}i$

4. Find additive and multiplicative inverse of $z = 8 + 9i$.

Solution

Let $z = 8 + 9i$

Additive inverse of z $= -z = -8 - 9i$

Multiplicative inverse of z $= z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \left(\frac{8}{(8)^2+(9)^2}, \frac{-9}{(8)^2+(9)^2}\right)$

$$= \left(\frac{8}{64+81}, \frac{-9}{64+81}\right) = \left(\frac{8}{145}, -\frac{9}{145}\right) = \frac{8}{145} - \frac{9}{145}i$$

5. If $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$, then verify that

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(iv) $|z_1| = |-\overline{z_1}|$ (v) $\overline{\overline{z_2}} = z_2$ (vi) $z_1\overline{z_1} = |z_1|^2$

Solution

i. $L.H.S = \overline{z_1 + z_2} = \overline{(3+4i)+(2+3i)} = \overline{3+4i+2+3i} = \overline{5+7i} = 5-7i$

$R.H.S = \overline{z_1} + \overline{z_2} = \overline{(3+4i)} + \overline{(2+3i)} = 3-4i+2-3i = 5-7i$

Hence $\boldsymbol{\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}}$

ii. $L.H.S = \overline{z_1 z_2} = \overline{(3+4i)(2+3i)} = \overline{6+9i+8i+12i^2}$

$= \overline{6+9i+8i-12} = \overline{-6+17i} = -6-17i$

$R.H.S = \overline{z_1}\,\overline{z_2} = \overline{(3+4i)}.\overline{(2+3i)} = (3-4i)(2-3i)$

$= 6-9i-8i+12i^2 = 6-9i-8i-12 = -6-17i$

Hence $\boldsymbol{\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}}$

iii. $L.H.S = \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{3+4i}{2+3i}\right)} = \overline{\left(\frac{3+4i}{2+3i}\times\frac{2-3i}{2-3i}\right)} = \overline{\left(\frac{6-9i+8i-12i^2}{(2)^2-(3i)^2}\right)} = \overline{\left(\frac{6-9i+8i+12}{4+9}\right)}$

$= \overline{\left(\frac{18-i}{13}\right)} = \overline{\frac{18}{13} - \frac{1}{13}i} = \frac{18}{13} + \frac{1}{13}i$

$R.H.S = \frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{(3+4i)}}{\overline{(2+3i)}} = \frac{3-4i}{2-3i} = \frac{3-4i}{2-3i}\times\frac{2+3i}{2+3i} = \frac{6+9i-8i-12i^2}{(2)^2-(3i)^2} = \frac{6+9i-8i+12}{4+9}$

$= \frac{18+i}{13} = \frac{18}{13} + \frac{1}{13}i$

Hence $\boldsymbol{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}}$

iv. $L.H.S = |z_1| = |3+4i| = \sqrt{(3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$R.H.S = |-\overline{z_1}| = |-\overline{(3+4i)}| = |-(3-4i)| = |-3+4i| = \sqrt{(-3)^2+(4)^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

$\Rightarrow \boldsymbol{|z_1| = |-\overline{z_1}|}$

v. $z_2 = 2+3i \Rightarrow \overline{z_2} = 2-3i \Rightarrow \overline{\overline{z_2}} = 2+3i \Rightarrow \boldsymbol{\overline{\overline{z_2}} = z_2}$

vi. $L.H.S = z_1\overline{z_1} = (3+4i)(3-4i) = (3)^2 - (4i)^2 = 9+16 = 25$

$R.H.S = |z_1|^2 = |3+4i|^2 = \left(\sqrt{(3)^2+(4)^2}\right)^2 = 9+16 = 25$

$\Rightarrow \boldsymbol{z_1\overline{z_1} = |z_1|^2}$

6. If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then find

(i) $z_1 z_2$ (ii) $\frac{z_1}{z_2}$ (iii) $\bar{z}_1 \bar{z}_2$ (iv) $|z_1 z_2|$

Solution

i. $\boldsymbol{z_1 z_2} = (5+4i)(3+2i) = 15 + 10i + 12i + 8i^2 = 15 + 10i + 12i - 8 = 7 + 22i$

ii. $\frac{z_1}{z_2} = \frac{5+4i}{3+2i} = \frac{5+4i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{15-10i+12i-8i^2}{(3)^2-(2i)^2} = \frac{15-10i+12i+8}{9+4} = \frac{23+2i}{13} = \frac{23}{13} + \frac{2}{13} i$

iii. $\bar{z}_1 \bar{z}_2 = \overline{(5+4i)}.\overline{(3+2i)} = (5-4i)(3-2i) = 15 - 10i - 12i + 8i^2$
$= 15 - 10i - 12i - 8 = 7 - 22i$

iv. $|\boldsymbol{z_1 z_2}| = ???$

Solution

$\boldsymbol{z_1 z_2} = (5+4i)(3+2i) = 15 + 10i + 12i + 8i^2 = 15 + 10i + 12i - 8 = 7 + 22i$

$|\boldsymbol{z_1 z_2}| = |7 + 22i| = \sqrt{(7)^2 + (22)^2} = \sqrt{49 + 484} = \sqrt{533}$

7. Find real and imaginary parts of $z = (2 + 7i)^{-1}$.

Solution

$(\boldsymbol{2 + 7i})^{-1} = \frac{1}{2+7i} = \frac{1}{2+7i} \times \frac{2-7i}{2-7i} = \frac{2-7i}{(2)^2-(7i)^2} = \frac{2-7i}{4+49} = \frac{2-7i}{53} = \frac{2}{53} - \frac{7}{53} i$

$\Rightarrow \text{Re}(z) = \frac{2}{53}, \text{Im}(z) = -\frac{7}{53}$

8. Solve the given simultaneous linear equations with complex coefficients for z and w:

$$iz + (2 - i)\,w = 4 + i$$
$$iz + (3 + i)\,w = 3 + 3i$$

Solution

$iz + (2 - i)w = 4 + i$ ……….(i)

$iz + (3 + i)w = 3 + 3i$ ……….(ii)

Subtracting both

$iz + (2 - i)w = 4 + i$

$\underline{\pm iz \pm (3 + i)w = \pm 3 \pm 3i}$

$(2 - i)w - (3 + i)w = 1 - 2i$

$2w - iw - 3w - iw = 1 - 2i$

$-w - 2iw = 1 - 2i$

$(-1 - 2i)w = 1 - 2i$

$w = \frac{1-2i}{-1-2i}$

$w = \frac{1-2i}{-1-2i} \times \frac{-1+2i}{-1+2i} = \frac{-1+2i+2i-4i^2}{(-1)^2-(2i)^2} = \frac{-1+2i+2i+4}{1+4} = \frac{3+4i}{5}$

$\boldsymbol{w = \frac{3}{5} + \frac{4}{5} i}$

Put $w = \frac{3}{5} + \frac{4}{5}i$ in eq (i)

$iz + (2 - i)w = 4 + i$

$iz + (2 - i)\left(\frac{3}{5} + \frac{4}{5}i\right) = 4 + i$

$iz + \frac{6}{5} + \frac{8}{5}i - \frac{3}{5}i - \frac{4}{5}i^2 = 4 + i$

$iz + \frac{6}{5} + \frac{8}{5}i - \frac{3}{5}i + \frac{4}{5} = 4 + i$

$iz = 4 + i - \frac{6}{5} - \frac{8}{5}i + \frac{3}{5}i - \frac{4}{5}$

$iz = 4 - \frac{6}{5} - \frac{4}{5} + i - \frac{8}{5}i + \frac{3}{5}i$

$iz = \frac{20-6-4}{5} + \frac{5-8+3}{5}i$

$iz = \frac{10}{5} + \frac{0}{5}i$

$iz = 2 + 0i$

$iz = 2$

$z = \frac{2}{i} = \frac{2}{i} \times \frac{i}{i} = \frac{2i}{i^2}$

$\boldsymbol{z = -2i}$

9. Solve $(3 - 4i)(a + bi) = 1 + 0i$ and find the values of a and b.

Solution

$(3 - 4i)(a + bi) = 1 + 0i$

$3a + 3bi - 4ai - 4bi^2 = 1 + 0i$

$3a + 3bi - 4ai + 4b = 1 + 0i$

$3a + 4b = 1 \quad \dots\dots\dots.(\text{i})$ and $-4a + 3b = 0 \quad \dots\dots\dots.(\text{ii})$

Multiplying eq (i) by 4, eq (ii) by 3 and adding both

$\mathbf{12}a + \mathbf{16}b = \mathbf{4}$ $-\mathbf{12}a + \mathbf{9}b = 0$
$25b = 4$ $b = \frac{4}{25}$

Put b $= \frac{4}{25}$ in (i)

$\Rightarrow 3a + 4\left(\frac{4}{25}\right) = 1 \Rightarrow 3a + \frac{16}{25} = 1 \Rightarrow 3a = 1 - \frac{16}{25} \Rightarrow 3a = \frac{9}{25} \Rightarrow a = \frac{3}{25}$

Hence $a = \frac{3}{25}$; $b = \frac{4}{25}$

10. Solve the equation for x and y:

$$(3-2i)(x+yi)=2(x-2yi)+2i-1$$

Solution

$(3-2i)(x+yi)=2(x-2yi)+2i-1$
$3x+3yi-2xi-2yi^2=2x-4yi+2i-1$
$3x+3yi-2xi+2y=2x-4yi+2i-1$
$3x+3yi-2xi+2y-2x+4yi=2i-1$
$x+7yi-2xi+2y=2i-1$
$x+2y=-1$ ………(i) and $-2x+7y=2$ ………(ii)

Multiplying eq (i) by 2 and adding both

$\mathbf{2}x+\mathbf{4}y=-\mathbf{2}$ $-2x+7y=2$
$11y=0$ $y=0$

Put y = 0 in (i)
$\Rightarrow x+2(0)=-1 \Rightarrow x=-1$
Hence $x=-1$; $y=0$

UNIT 2

Quadratic Equations and Inequalities

EXERCISE 2.1

1. Write the following quadratic equations in standard form:

(i) $3x - 1 = 2x^2$ (ii) $2x(x+1) = 4(2x+3)$

(iii) $2x^2 - 4x = 4x + 7$ (iv) $4(3x-2) = 9x^2$

(v) $2x + \frac{1}{x} = 5 - \frac{1}{x}, x \neq 0$ (vi) $\frac{6x+6}{20-x} = \frac{1}{x}, x \neq 0, 20$

Solution

i.	$\mathbf{3x - 1 = 2x^2}$ **Solution** $3x - 1 = 2x^2$ $2x^2 - 3x + 1 = 0$	**ii.**	$\mathbf{2x(x+1) = 4(2x+3)}$ **Solution** $2x(x+1) = 4(2x+3)$ $2x^2 + 2x = 8x + 12$ $2x^2 + 2x - 8x - 12 = 0$ $2x^2 - 6x - 12 = 0$ $x^2 - 3x - 6 = 0$
iii.	$\mathbf{2x^2 - 4x = 4x + 7}$ **Solution** $2x^2 - 4x = 4x + 7$ $2x^2 - 4x - 4x - 7 = 0$ $2x^2 - 8x - 7 = 0$	**iv.**	$\mathbf{4(3x-2) = 9x^2}$ **Solution** $4(3x-2) = 9x^2$ $12x - 8 = 9x^2$ $9x^2 - 12x + 8 = 0$
v.	$\mathbf{2x + \frac{1}{x} = 5 - \frac{1}{x}\ , x \neq 0}$ **Solution** $2x + \frac{1}{x} = 5 - \frac{1}{x}$ $2x^2 + 1 = 5x - 1$ $2x^2 + 1 - 5x + 1 = 0$ $2x^2 - 5x + 2 = 0$	**vi.**	$\mathbf{\frac{6x+6}{20-x} = \frac{1}{x}\ , x \neq 0, 20}$ **Solution** $\frac{6x+6}{20-x} = \frac{1}{x}$ $x(6x+6) = 1(20-x)$ $6x^2 + 6x = 20 - x$ $6x^2 + 6x - 20 + x = 0$ $6x^2 + 7x - 20 = 0$

2. Solve the following quadratic equations by factorization method:

Solution

i. $\boldsymbol{x^2 - x - 6 = 0}$ **Solution** $x^2 - x - 6 = 0$ $x^2 - 3x + 2x - 6 = 0$ $x(x-3) + 2(x-3) = 0$ $(x-3)(x+2) = 0$ $x - 3 = 0\ ;\ x + 2 = 0$ $x = 3\ ;\ x = -2$ $\text{S.S} = \{3, -2\}$	**ii.** $\boldsymbol{x^2 + 3x - 28 = 0}$ **Solution** $x^2 + 3x - 28 = 0$ $x^2 + 7x - 4x - 28 = 0$ $x(x+7) - 4(x+7) = 0$ $(x-4)(x+7) = 0$ $x - 4 = 0\ ;\ x + 7 = 0$ $x = 4\ ;\ x = -7$ $\text{S.S} = \{4, -7\}$
iii. $\boldsymbol{6x^2 + 13x - 5 = 0}$ **Solution** $6x^2 + 13x - 5 = 0$ $6x^2 + 15x - 2x - 5 = 0$ $3x(2x+5) - 1(2x+5) = 0$ $(3x-1)(2x+5) = 0$ $3x - 1 = 0\ ;\ 2x + 5 = 0$ $x = \frac{1}{3}\ ;\ x = -\frac{5}{2}$ $\text{S.S} = \left\{-\frac{5}{2}, \frac{1}{3}\right\}$	**iv.** $\boldsymbol{x^2 - \frac{3}{2}x = \frac{9}{2}}$ **Solution** $x^2 - \frac{3}{2}x = \frac{9}{2}$ $2x^2 - 3x = 9$ $2x^2 - 3x - 9 = 0$ $2x^2 - 6x + 3x - 9 = 0$ $2x(x-3) + 3(x-3) = 0$ $(x-3)(2x+3) = 0$ $x - 3 = 0\ ;\ 2x + 3 = 0$ $x = 3\ ;\ x = -\frac{3}{2}$ $\text{S.S} = \left\{3, -\frac{3}{2}\right\}$
v. $\boldsymbol{\frac{3x-8}{x-2} = \frac{5x-2}{x+5}\ , x \neq 2, -5}$ **Solution** $\frac{3x-8}{x-2} = \frac{5x-2}{x+5}$ $(3x-8)(x+5) = (5x-2)(x-2)$ $3x^2 + 15x - 8x - 40 = 5x^2 - 10x - 2x + 4$ $3x^2 + 7x - 40 = 5x^2 - 12x + 4$ $5x^2 - 12x + 4 - 3x^2 - 7x + 40 = 0$ $2x^2 - 19x + 44 = 0$ $2x^2 - 8x - 11x + 44 = 0$ $2x(x-4) - 11(x-4) = 0$ $(x-4)(2x-11) = 0$ $x - 4 = 0\ ;\ 2x - 11 = 0$ $x = 4\ ;\ x = \frac{11}{2}$ $\text{S.S} = \left\{4, \frac{11}{2}\right\}$	**vi.** $\boldsymbol{\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}\ , x \neq 1, -3}$ **Solution** $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}$ $\frac{x+3-x+1}{(x-1)(x+3)} = \frac{1}{35}$ $\frac{4}{x^2+3x-x-3} = \frac{1}{35}$ $\frac{4}{x^2+2x-3} = \frac{1}{35}$ $x^2 + 2x - 3 = 140$ $x^2 + 2x - 143 = 0$ $x^2 - 11x + 13x - 143 = 0$ $x(x-11) + 13(x-11) = 0$ $(x-11)(x+13) = 0$ $x - 11 = 0\ ;\ x + 13 = 0$ $x = 11\ ;\ x = -13$ $\text{S.S} = \{11, -13\}$

3. Solve the following quadratic equations by completing square method:

(i) $2x^2+5x+2=0$ (ii) $x^2+x=42$

(iii) $12x^2+7x=12$ (iv) $\dfrac{x+3}{2x-7}=\dfrac{2x-1}{x-3}, x\neq\dfrac{7}{2},3$

(v) $\dfrac{1}{1+x}-\dfrac{1}{3-x}=\dfrac{6}{35}, x\neq-1,3$ (vi) $\dfrac{3x-1}{4x+7}=1-\dfrac{6}{x+7}, x\neq\dfrac{-7}{4},-7$

Solution

(i) $2x^2+5x+2=0$

$$2x^2+5x=-2$$
$$x^2+\frac{5}{2}x=-1$$
$$x^2+\frac{5}{2}x+(\frac{5}{4})^2=-1+(\frac{5}{4})^2$$
$$(x+\frac{5}{4})^2=-1+\frac{25}{16}$$
$$(x+\frac{5}{4})^2=\frac{9}{16}$$
$$x+\frac{5}{4}=\pm\frac{3}{4}$$
$$x=-\frac{5}{4}\pm\frac{3}{4}.$$

$$x=-2 \text{ or } x=-\tfrac{1}{2}.$$

(ii) $x^2+x=42$

$$x^2+x+(\frac{1}{2})^2=42+(\frac{1}{2})^2$$
$$(x+\frac{1}{2})^2=42+\frac{1}{4}$$
$$(x+\frac{1}{2})^2=\frac{169}{4}$$
$$x+\frac{1}{2}=\pm\frac{13}{2}$$
$$x=-\frac{1}{2}\pm\frac{13}{2}.$$

$$x=6 \text{ or } x=-7.$$

(iii) $12x^2+7x=12$

$$12x^2+7x-12=0$$
$$x^2+\frac{7}{12}x=1$$
$$x^2+\frac{7}{12}x+(\frac{7}{24})^2=1+(\frac{7}{24})^2$$
$$(x+\frac{7}{24})^2=1+\frac{49}{576}$$
$$(x+\frac{7}{24})^2=\frac{625}{576}$$
$$x+\frac{7}{24}=\pm\frac{25}{24}$$
$$x=-\frac{7}{24}\pm\frac{25}{24}.$$

$$x=\tfrac{3}{4} \text{ or } x=-\tfrac{4}{3}.$$

(iv) $\frac{x+3}{2x-7}=\frac{2x-1}{x-3}$

$$(x+3)(x-3)=(2x-1)(2x-7)$$
$$x^2-9=4x^2-14x-2x+7$$
$$0=3x^2-16x+16.$$

$$x^2-\frac{16}{3}x=-\frac{16}{3}$$
$$x^2-\frac{16}{3}x+(\frac{8}{3})^2=-\frac{16}{3}+(\frac{8}{3})^2$$
$$(x-\frac{8}{3})^2=-\frac{16}{3}+\frac{64}{9}$$
$$(x-\frac{8}{3})^2=\frac{16}{9}$$
$$x-\frac{8}{3}=\pm\frac{4}{3}.$$

$$x=4 \text{ or } x=\tfrac{4}{3}.$$

(v) $\frac{1}{1+x}-\frac{1}{3-x}=\frac{6}{35}$ $\frac{(3-x)-(1+x)}{(1+x)(3-x)}=\frac{6}{35}$ $\frac{2-2x}{3+2x-x^2}=\frac{6}{35}$ $35(2-2x)=6(3+2x-x^2)$ $70-70x=18+12x-6x^2$ $6x^2-82x+52=0.$ $x^2-\frac{41}{3}x=-\frac{26}{3}$ $x^2-\frac{41}{3}x+(\frac{41}{6})^2=-\frac{26}{3}+(\frac{41}{6})^2$ $(x-\frac{41}{6})^2=-\frac{52}{6}+\frac{1681}{36}$ $(x-\frac{41}{6})^2=\frac{1369}{36}$ $x-\frac{41}{6}=\pm\frac{37}{6}.$ $x=13$ or $x=\frac{2}{3}.$	(vi) $\frac{3x-1}{4x+7}=1-\frac{6}{x+7}$ $\frac{3x-1}{4x+7}=\frac{x+7-6}{x+7}$ $\frac{3x-1}{4x+7}=\frac{x+1}{x+7}$ $(3x-1)(x+7)=(4x+7)(x+1)$ $3x^2+20x-7=4x^2+11x+7$ $0=x^2-9x+14.$ $x^2-9x=-14$ $x^2-9x+(\frac{9}{2})^2=-14+(\frac{9}{2})^2$ $(x-\frac{9}{2})^2=-14+\frac{81}{4}$ $(x-\frac{9}{2})^2=\frac{25}{4}$ $x-\frac{9}{2}=\pm\frac{5}{2}.$ $x=7$ or $x=2.$

4. Use quadratic formula to solve the following equations:

(i) $2x^2-5x+3=0$ (ii) $2x^2-7x-15=0$

(iii) $2x^2+7x=15$ (iv) $x^2+11=7x$

(v) $\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\frac{1}{3}, x\neq 4,3$ (vi) $\frac{3x-3}{x+1}=\frac{2x-1}{x-1}, x\neq -1,1$

Solution

(i) $2x^2-5x+3=0.$ Quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $a=2,\ b=-5,\ c=3.$ $x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(3)}}{2(2)}.$ $x=\frac{5\pm\sqrt{25-24}}{4}.$ $x=\frac{5\pm 1}{4}.$ Solutions: $x=\frac{3}{2},\ x=1.$	(ii) $2x^2-7x-15=0.$ $a=2,\ b=-7,\ c=-15.$ $x=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(-15)}}{2(2)}.$ $x=\frac{7\pm\sqrt{49+120}}{4}.$ $x=\frac{7\pm 13}{4}.$ Solutions: $x=5,\ x=-\frac{3}{2}.$

(iii) $2x^2+7x=15$.

$2x^2+7x-15=0$.

$a=2,\ b=7,\ c=-15$.

$x=\frac{-7\pm\sqrt{7^2-4(2)(-15)}}{2(2)}$.

$x=\frac{-7\pm\sqrt{49+120}}{4}$.

$x=\frac{-7\pm13}{4}$.

Solutions: $x=\frac{3}{2},\ x=-5$.

(iv) $x^2+11=7x$.

$x^2-7x+11=0$.

$a=1,\ b=-7,\ c=11$.

$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(11)}}{2(1)}$.

$x=\frac{7\pm\sqrt{49-44}}{2}$.

$x=\frac{7\pm\sqrt{5}}{2}$.

(v) $\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\frac{1}{3},\ x\neq 4,3$.

$\frac{x+4}{x-4}+\frac{x-2}{x-3}=\frac{19}{3}$.

Combine LHS,

$\frac{(x+4)(x-3)+(x-2)(x-4)}{(x-4)(x-3)}=\frac{19}{3}$.

$\frac{2x^2-5x-4}{x^2-7x+12}=\frac{19}{3}$.

Cross-multiply,

$3(2x^2-5x-4)=19(x^2-7x+12)$.

$6x^2-15x-12=19x^2-133x+228$.

$13x^2-118x+240=0$.

$a=13,\ b=-118,\ c=240$.

$x=\frac{-(-118)\pm\sqrt{(-118)^2-4(13)(240)}}{2(13)}$.

$x=\frac{118\pm\sqrt{13924-12480}}{26}$.

$x=\frac{118\pm38}{26}$.

Solutions: $x=6,\ x=\frac{40}{13}$.

(vi) $\frac{3x-3}{x+1}=\frac{2x-1}{x-1},\ x\neq -1,1$.

Cross-multiply,

$(3x-3)(x-1)=(2x-1)(x+1)$.

$3(x-1)^2=(2x-1)(x+1)$.

$3x^2-6x+3=2x^2+x-1$.

$x^2-7x+4=0$.

$a=1,\ b=-7,\ c=4$.

$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(4)}}{2(1)}$.

$x=\frac{7\pm\sqrt{49-16}}{2}$.

$x=\frac{7\pm\sqrt{33}}{2}$.

Solutions: $x=\frac{7+\sqrt{33}}{2},\ x=\frac{7-\sqrt{33}}{2}$.

5. Solve the following quadratic equations graphically:

(i) $x^2 - 3x - 18 = 0$ (ii) $x^2 - 5x - 14 = 0$

(iii) $2x^2 + 13x + 6 = 0$ (iv) $4x^2 + 12x - 27 = 0$

Solution

(i) $x^2 - 3x - 18 = 0$.

$x^2 - 3x - 18 = 0$.

$x^2 - 6x + 3x - 18 = 0$.

$x(x-6) + 3(x-6) = 0$.

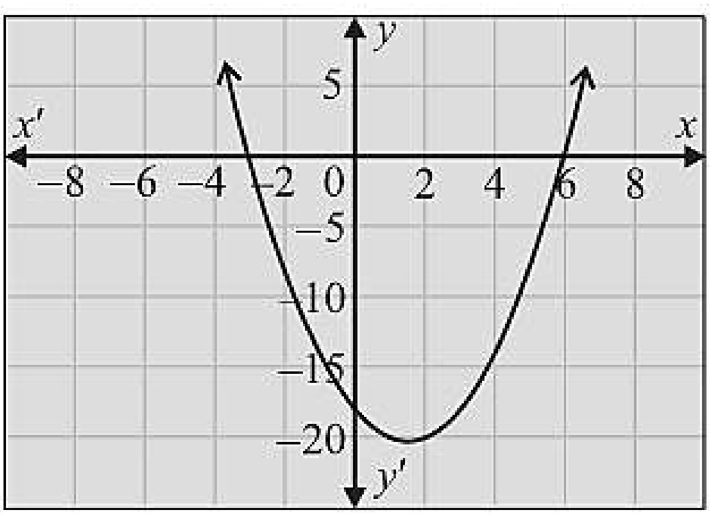
$(x-6)(x+3) = 0$.

$x - 6 = 0 \rightarrow x = 6$.

$x + 3 = 0 \rightarrow x = -3$.

Solutions: $x = 6$, $x = -3$.

Graph cuts x-axis at 6 and -3



(ii) $x^2 - 5x - 14 = 0$.

$x^2 - 5x - 14 = 0$.

$x^2 - 7x + 2x - 14 = 0$.

$x(x-7) + 2(x-7) = 0$.

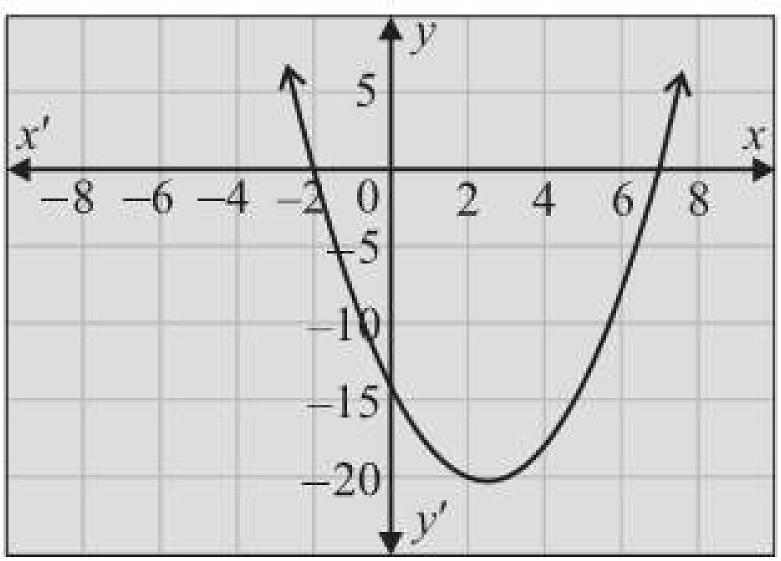
$(x-7)(x+2) = 0$.

$x - 7 = 0 \rightarrow x = 7$.

$x + 2 = 0 \rightarrow x = -2$.

Solutions: $x = 7$, $x = -2$.

Graph cuts x-axis at 7 and -2.



(iii) $2x^2+13x+6=0$.

$2x^2+13x+6=0$.

$x=\frac{-13\pm\sqrt{169-48}}{4}$.

$169-48=121$.

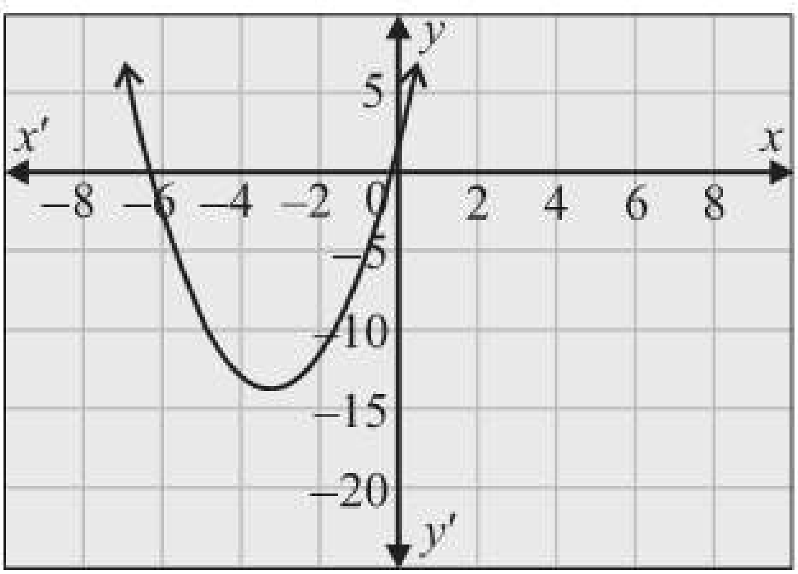
$x=\frac{-13\pm 11}{4}$.

$x=\frac{-13+11}{4}=-\frac{1}{2}$.

$x=\frac{-13-11}{4}=-6$.

Solutions: $x=-\frac{1}{2}$, $x=-6$.

Graph intersects x-axis at $-\frac{1}{2}$ and -6.



(iv) $4x^2+12x-27=0$.

$4x^2+12x-27=0$.

$x=\frac{-12\pm\sqrt{144+432}}{8}$.

$144+432=576$.

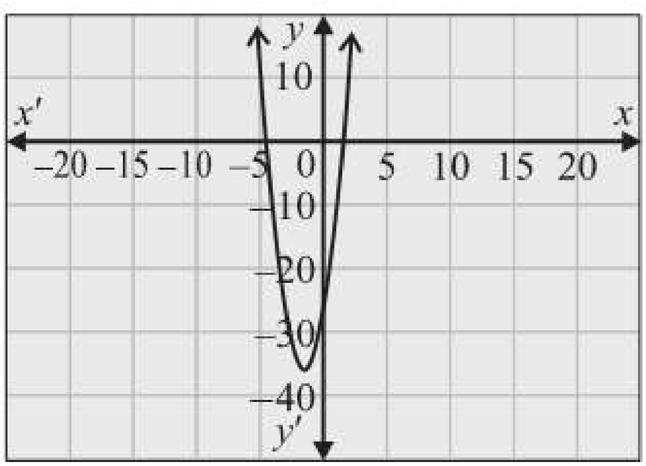
$x=\frac{-12\pm 24}{8}$.

$x=\frac{-12+24}{8}=\frac{3}{2}$.

$x=\frac{-12-24}{8}=-\frac{9}{2}$.

Solutions: $x=\frac{3}{2}$, $x=-\frac{9}{2}$.

Graph cuts x-axis at $\frac{3}{2}$ and $-\frac{9}{2}$.



EXERCISE 2.2

1. Find the points of intersection of the following linear equations with coordinate axes graphically:

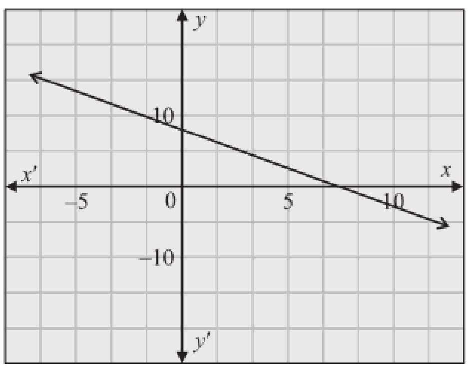
(i) $x + y = 8$ (ii) $x - y = 1$ (iii) $x - 2y = 1$

(iv) $x - 2y + 2 = 0$ (v) $5x - 5y = 1$

Solution

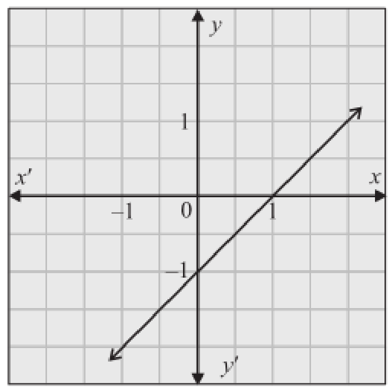
(i) $x + y = 8$.

- x-intercept: set $y = 0 \implies x = 8$. Point $(8, 0)$.
- y-intercept: set $x = 0 \implies y = 8$. Point $(0, 8)$.



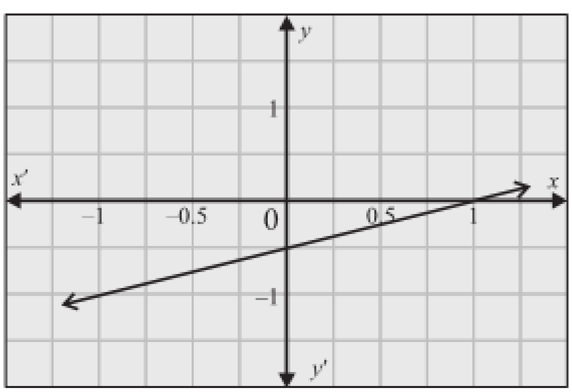
(ii) $x - y = 1$.

- x-intercept: $y = 0 \implies x = 1$. Point $(1, 0)$.
- y-intercept: $x = 0 \implies y = -1$. Point $(0, -1)$.



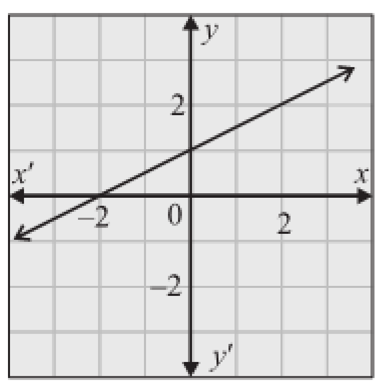
(iii) $x - 2y = 1$.

- x-intercept: $y = 0 \implies x = 1$. Point $(1, 0)$.
- y-intercept: $x = 0 \implies y = -\frac{1}{2}$. Point $(0, -\frac{1}{2})$.



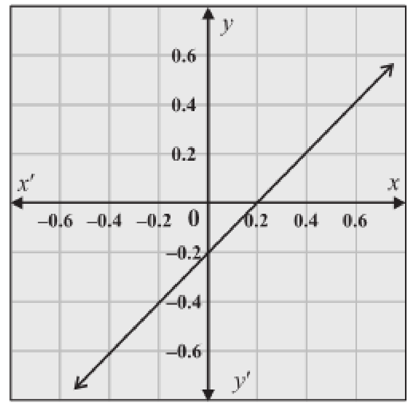
(iv) $x - 2y + 2 = 0$.

- x-intercept: $y = 0 \implies x = -2$. Point $(-2, 0)$.
- y-intercept: $x = 0 \implies y = 1$. Point $(0, 1)$.



(v) $5x - 5y = 1$.

- x-intercept: $y = 0 \implies x = \frac{1}{5}$. Point $\left(\frac{1}{5}, 0\right)$.
- y-intercept: $x = 0 \implies y = -\frac{1}{5}$. Point $\left(0, -\frac{1}{5}\right)$.



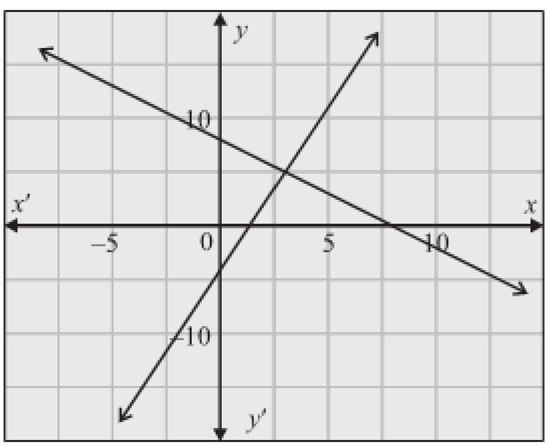
2. Solve the following system of linear equations graphically:

(i) $x + y = 8$, $3x - y = 4$
(ii) $x - y = 1$, $x + 2y = 7$
(iii) $x - 2y = 1$, $2x + y = 2$
(iv) $y = 2x + 2$, $3x + 2y = 4$
(v) $3y = 2x + 8$, $x + y = 1$

Solution

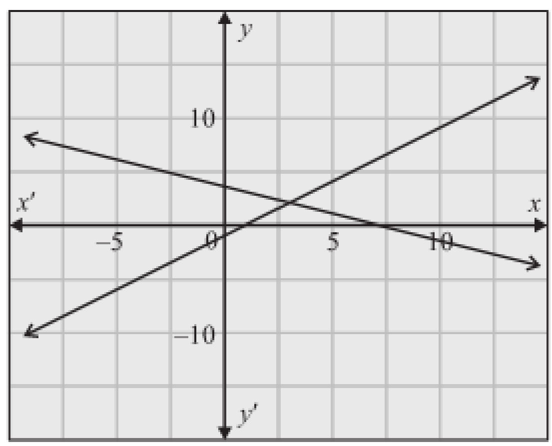
(i) $x + y = 8$ and $3x - y = 4$.

- Solve algebraically: add equations $\implies 4x = 12 \implies x = 3$.
- $y = 8 - x = 8 - 3 = 5$. Solution $(3, 5)$.



(ii) $x - y = 1$ and $x + 2y = 7$.

- Subtract equations $\implies -3y = -6 \implies y = 2$.
- $x = y + 1 = 3$. Solution $(3, 2)$.

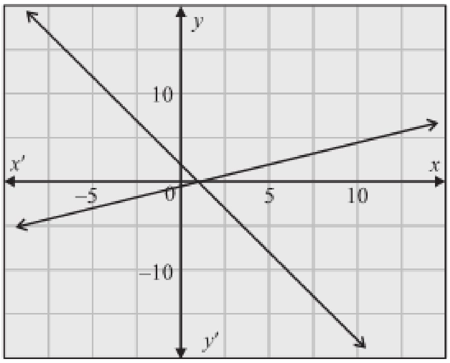


(iii) $x-2y=1$ and $2x+y=2$.

- From first: $x=2y+1$. Substitute into second:

$2(2y+1)+y=2 \implies 5y=0 \implies y=0$.

- $x=1$. Solution $(1,0)$.

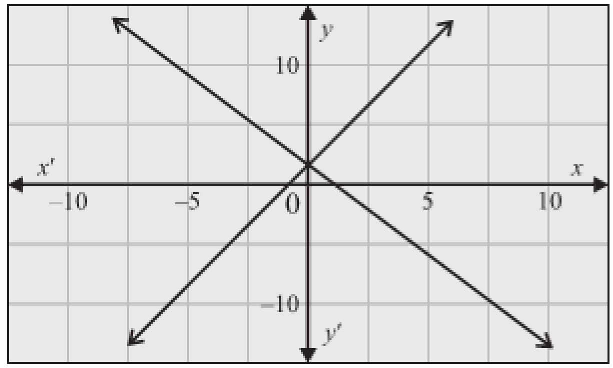


(iv) $y=2x+2$ and $3x+2y=4$.

- Substitute y:

$3x+2(2x+2)=4 \implies 7x=0 \implies x=0$.

- $y=2$. Solution $(0,2)$.

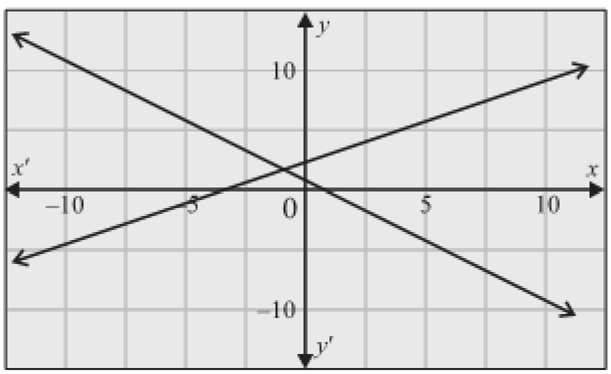


(v) $3y=2x+8$ and $x+y=1$.

- $y=\frac{2}{3}x+\frac{8}{3}$. Substitute into second:

$x+\frac{2}{3}x+\frac{8}{3}=1 \implies \frac{5}{3}x=-\frac{5}{3} \implies x=-1$.

- $y=2$. Solution $(-1,2)$.

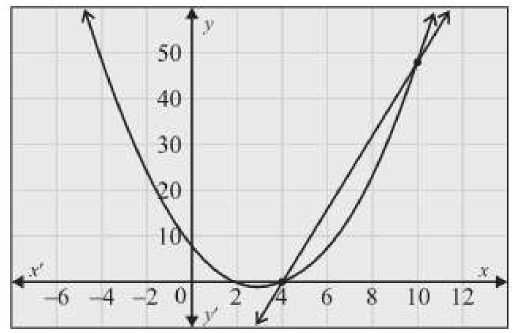


3. Solve the following equations graphically:

(i) $y = 8x - 32$, $y = x^2 - 6x + 8$ (ii) $y + x = 2$, $y = 2x^2 + x - 10$

Solution

(i) $y = 8x - 32$ and $y = x^2 - 6x + 8$.

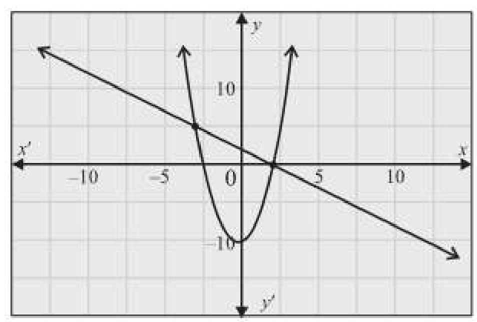


- Set equal: $8x - 32 = x^2 - 6x + 8$.
- $x^2 - 14x + 40 = 0$.
- $(x-10)(x-4) = 0$. $x = 10$ or $x = 4$.
- $y = 8(10) - 32 = 48$. $y = 8(4) - 32 = 0$.
- Solutions $(10, 48)$, $(4, 0)$.

(ii) $y + x = 2$ and $y = 2x^2 + x - 10$.

- Substitute $y = 2 - x$ into second:

$2 - x = 2x^2 + x - 10$.



- $2x^2 + 2x - 12 = 0$.
- $x^2 + x - 6 = 0$.
- $(x+3)(x-2) = 0$. $x = -3$ or $x = 2$.
- $y = 5$ when $x = -3$. $y = 0$ when $x = 2$.
- Solutions $(-3, 5)$, $(2, 0)$.

EXERCISE 2.3

1. Form a quadratic equation whose roots are given below:

(i) $-4, 9$ (ii) $5, -7$ (iii) $\frac{-7}{5}, \frac{-6}{5}$

(iv) $\frac{-3}{2}, \frac{7}{2}$ (v) $3+\sqrt{5}, 3-\sqrt{5}$ (vi) $-2+\sqrt{3}, -2-\sqrt{3}$

Solution

The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (i) Roots **−4, 9**: $x^2-(-4+9)x+(-4)(9)=0$ $x^2-5x-36=0.$	The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (ii) Roots **5, −7**: $x^2-(5-7)x+(5)(-7)=0$ $x^2+2x-35=0.$
The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (iii) Roots $\frac{-7}{5}$, $\frac{-6}{5}$: $x^2-(\frac{-7}{5}+\frac{-6}{5})x+(\frac{-7}{5})(\frac{-6}{5})=0$ $x^2+\frac{13}{5}x+\frac{42}{25}=0$ $25x^2+65x+42=0.$	The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (iv) Roots $\frac{-3}{2}$, $\frac{7}{2}$: $x^2-(\frac{-3}{2}+\frac{7}{2})x+(\frac{-3}{2})(\frac{7}{2})=0$ $x^2-2x-\frac{21}{4}=0$ $4x^2-8x-21=0.$
The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (v) Roots $3+\sqrt{5}$, $3-\sqrt{5}$: $x^2-(6)x+(3+\sqrt{5})(3-\sqrt{5})=0$ $x^2-6x+(9-5)=0$ $x^2-6x+4=0.$	The quadratic equation with roots α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0.$ (vi) Roots $-2+\sqrt{3}$, $-2-\sqrt{3}$: $x^2-(-4)x+(-2+\sqrt{3})(-2-\sqrt{3})=0$ $x^2+4x+(4-3)=0$ $x^2+4x+1=0.$

2. Find the quadratic equation with roots exceeding by 2 than those of roots of $x^2+9x+20=0$.

Solution

Roots of $x^2+9x+20=0$ are $-4, -5$.

Roots increased by 2 are $-2, -3$.

$$x^2-(-2-3)x+(-2)(-3)=0$$
$$x^2+5x+6=0.$$

3. Find the equation whose roots are double the roots of $x^2 - px + q = 0$.

Solution

If roots are α, β, then new roots are $2\alpha, 2\beta$.

$$x^2 - (2\alpha + 2\beta)x + (2\alpha)(2\beta) = 0. \Rightarrow x^2 - 2(\alpha + \beta) + 2(\alpha\beta) = 0$$

$$x^2 - 2px + 4q = 0.$$

4. If α, β are the roots of the equation $x^2 + 2x + 4 = 0$, then find the equation whose roots are:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (iii) $2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha}$

(iv) α^2, β^2 (v) $2\alpha - 1, 2\beta - 1$

Solution

$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (i) $\frac{1}{\alpha}, \frac{1}{\beta}$: $x^2 - (\frac{1}{\alpha} + \frac{1}{\beta})x + \frac{1}{\alpha\beta} = 0$ $x^2 - (\frac{\alpha + \beta}{\alpha\beta})x + \frac{1}{4} = 0$ $x^2 + \frac{1}{2}x + \frac{1}{4} = 0$ $4x^2 + 2x + 1 = 0.$	$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (ii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$: $\text{Sum} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4 - 8}{4} = -1$ $\text{Product} = 1.$ $x^2 + x + 1 = 0.$
$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (iii) $2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha}$: $\text{Sum} = 2(\alpha + \beta) - (\frac{1}{\alpha} + \frac{1}{\beta}) = -4 - (-\frac{1}{2}) = -\frac{7}{2}$ $\text{Product} = 4\alpha\beta - 2 - 2 + \frac{1}{\alpha\beta} = 16 - 4 + \frac{1}{4}.$ $x^2 + \frac{7}{2}x + \frac{49}{4} = 0.$ $4x^2 + 14x + 49 = 0$	$\alpha + \beta = -2, \quad \alpha\beta = 4.$ (iv) α^2, β^2: $\text{Sum} = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 4 - 8 = -4$ $\text{Product} = (\alpha\beta)^2 = 16.$ $x^2 + 4x + 16 = 0.$

$$\alpha + \beta = -2, \quad \alpha\beta = 4.$$

(v) $2\alpha - 1, 2\beta - 1$:

$$\text{Sum} = 2(\alpha + \beta) - 2 = -4 - 2 = -6.$$

$$\text{Product} = 4\alpha\beta - 2(\alpha + \beta) + 1 = 16 + 4 + 1 = 21$$

$$x^2 + 6x + 21 = 0.$$

5. Find the condition that roots of $ax^2 + bx + c = 0$ should be reciprocals of each other.

Solution

$\alpha = \frac{1}{\beta} \implies \alpha\beta = 1.$

$\frac{c}{a} = 1 \quad \Rightarrow \quad a = c.$

6. Find the value of k, given that one root of $x^2 - (2k + 4)x + (7k + 1) = 0$ is 3.

Solution

$$x^2 - (2k+4)x + (7k+1) = 0$$
$$3^2 - (2k+4)3 + (7k+1) = 0$$
$$9 - 6k - 12 + 7k + 1 = 0$$
$$k - 2 = 0$$
$$k = 2.$$

7. Find the value of m in the equation $2x^2 + 3x + m = 0$ when sum of its roots is equal to double the product of its roots.

Solution

$$\text{Sum} = -\frac{3}{2}, \quad \text{Product} = \frac{m}{2}.$$
$$-\frac{3}{2} = 2 \cdot \frac{m}{2}$$
$$m = -\frac{3}{2}.$$

8. If α, β are the roots of $x^2 + ax + b = 0$ and α^2, β^2 are the roots of $x^2 + Ax + B = 0$, then prove that $A = 2b - a^2$, $B = b^2$.

Solution

$$\alpha + \beta = -a, \quad \alpha\beta = b.$$
$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = a^2 - 2b.$$
$$\alpha^2\beta^2 = b^2.$$

$$A = -(\alpha^2 + \beta^2) = 2b - a^2, \quad B = b^2.$$

9. If α, β are the roots of $x^2 + px + q = 0$, then find the condition that

(i) $\alpha = \beta$ (ii) $\alpha = \frac{1}{\beta}$

Solution

If $\alpha = \beta$ then $\alpha + \beta = 2\alpha = -p \Rightarrow \alpha = -\frac{p}{2}$ $\alpha\beta = \alpha^2 = q \Rightarrow \frac{p^2}{4} = q \Rightarrow \mathbf{p^2 = 4q}$	(ii) $\alpha = \frac{1}{\beta}$: $\alpha\beta = 1 \implies q = 1.$

EXERCISE 2.4

1. Examine the nature of roots of the following quadratic equations:

(i) $3x^2 - 9x - 2 = 0$ (ii) $x^2 + 6x + 9 = 0$

(iii) $2x^2 + 4x + 5 = 0$ (iv) $7x^2 - 6x - 1 = 0$

(v) $5x^2 - 2x + 10 = 0$ (vi) $x^2 - 8x + 16 = 0$

Solution

$D = b^2 - 4ac$ (i) $3x^2 - 9x - 2 = 0$: $D = (-9)^2 - 4(3)(-2) = 81 + 24 = 105 > 0.$ Irrational and unequal	$D = b^2 - 4ac$ (ii) $x^2 + 6x + 9 = 0$: $D = 6^2 - 4(1)(9) = 36 - 36 = 0$ Roots are real and equal.
$D = b^2 - 4ac$ (iii) $2x^2 + 4x + 5 = 0$: $D = 4^2 - 4(2)(5) = 16 - 40 = -24 < 0$ Roots are imaginary.	$D = b^2 - 4ac$ (iv) $7x^2 - 6x - 1 = 0$: $D = (-6)^2 - 4(7)(-1) = 36 + 28 = 64 > 0.$ Rational and unequal
$D = b^2 - 4ac$ (v) $5x^2 - 2x + 10 = 0$: $D = (-2)^2 - 4(5)(10) = 4 - 200 = -196 < 0.$ Roots are imaginary.	$D = b^2 - 4ac$ (vi) $x^2 - 8x + 16 = 0$: $D = (-8)^2 - 4(1)(16) = 64 - 64 = 0.$ Roots are real and equal.

2. For what values of t, the roots of $3x^2 + x + 9t = 0$ are real and unequal?

Solution

$D = b^2 - 4ac > 0$

$D = 1^2 - 4(3)(9t) > 0.$

$1 - 108t > 0 \Rightarrow t < \frac{1}{108}.$

3. If the quadratic equation $16x^2 + 7px + 49 = 0$ has equal roots, then find the values of p.

Solution

$D = b^2 - 4ac = 0$

$$D = (7p)^2 - 4(16)(49) = 0.$$

$49p^2 = 4 \cdot 16 \cdot 49 \Rightarrow p^2 = 64 \Rightarrow p = \pm 8.$

4. If the quadratic equation $4u^2 + 8u + q = 0$ has unequal and real roots, find the possible values for q.

Solution

$D = b^2 - 4ac > 0$

$D = 8^2 - 4(4)(q) > 0.$

$64 - 16q > 0 \Rightarrow q < 4.$

5. Find the value for m, if the quadratic equation $mx^2 - 8x + 1 = 0$ has real and equal roots.

Solution

$D = b^2 - 4ac = 0$

$$D = (-8)^2 - 4(m)(1) = 0.$$

$$64 - 4m = 0 \Rightarrow m = 16.$$

EXERCISE 2.5

1. Solve the following inequalities:

(i) $x^2 + 3x - 4 > 0$ (ii) $2x^2 - 8x + 6 > 0$

(iii) $x^2 + x - 6 < 0$ (iv) $x^2 - 6x + 9 < 0$

(v) $4x^2 - 16x + 15 \leq 0$ (vi) $-x^2 + 3x - 2 \geq 0$

Solution

(i) $x^2 + 3x - 4 > 0$

Solution

$x^2 + 3x - 4 > 0$

Solve Associated Equation

$x^2 + 3x - 4 = 0$

$x^2 + 4x - x - 4 = 0$

$x(x + 4) - 1(x + 4) = 0$

$(x + 4)(x - 1) = 0$

$x + 4 = 0 \quad ; \quad x - 1 = 0$

$x = -4 \quad ; \quad x = 1$

Critical Points: $(-4,0), (1,0)$

Find Interval: $(-\infty, -4), (-4,1), (1, \infty)$

Test Points

Test $x = -5$: $(-5+4)(-5-1) = 6 > 0$. Satisfied

Test $x = 0$: $(0+4)(0-1) = -4 < 0$. Satisfied

Test $x = 2$: $(2+4)(2-1) = 6 > 0$. Satisfied

Conclusion

$x \in (-\infty, -4) \cup (1, \infty)$

(ii) $2x^2 - 8x + 6 > 0$

Solution

$2x^2 - 8x + 6 > 0$

Solve Associated Equation

$2x^2 - 8x + 6 = 0$

$2x^2 - 3x - x + 6 = 0$

$x(x - 3) - 1(x - 3) = 0$

$(x - 1)(x - 3) = 0$

$x - 1 = 0 \quad ; \quad x - 3 = 0$

$x = 1 \quad ; \quad x = 3$

Critical Points: $(1,0), (3,0)$

Find Interval: $(-\infty, 1), (1,3), (3, \infty)$

Test Points

Test $x = 0$: $(0-1)(0-3) = 3 > 0$. Satisfied

Test $x = 2$: $(2-1)(2-3) = -1 < 0$. Satisfied

Test $x = 4$: $(4-1)(4-3) = 3 > 0$. Satisfied

Conclusion

$x \in (-\infty, 1) \cup (3, \infty)$.

(iii) $x^2 + x - 6 < 0$

Solution

$x^2 + x - 6 < 0$

Solve Associated Equation

$x^2 + x - 6 = 0$

$x^2 + 3x - 2x - 6 = 0$

$x(x+3) - 2(x+3) = 0$

$(x+3)(x-2) = 0$

$x + 3 = 0 \quad ; \quad x - 2 = 0$

$x = -3 \quad ; \quad x = 2$

Critical Points: $(-3,0), (2,0)$

Find Interval: $(-\infty, -3), (-3,2), (2, \infty)$

Test Points

Test $x = -4: (-4+3)(-4-2) = 6 > 0.$ Satisfied

Test $x = 0: (0+3)(0-2) = -6 < 0.$ Satisfied

Test $x = 3: (3+3)(3-2) = 6 > 0.$ Satisfied

Conclusion

$x \in (-3, 2)$

(iv) $x^2 - 6x + 9 < 0$

Solution

$x^2 - 6x + 9 < 0$

Solve Associated Equation

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x - 3 = 0$

$x = 3$

Critical Points: $(3,0)$

Find Interval: $(3, \infty)$

Test Points

Test $x = 3: (3-3)^2 = 0$

Conclusion

No Solution

(v) $4x^2 - 16x + 15 \le 0$

Solution

$4x^2 - 16x + 15 \le 0$

Solve Associated Equation

$4x^2 - 16x + 15 = 0$

$4x^2 - 10x - 6x + 15 = 0$

$2x(x-5) - 3(x-5) = 0$

$(2x-3)(2x-5) = 0$

$2x - 3 = 0 \quad ; \quad 2x - 5 = 0$

$x = \frac{3}{2} \quad ; \quad x = \frac{5}{2}$

Critical Points: $\left(\frac{3}{2}, 0\right), \left(\frac{5}{2}, 0\right)$

Find Interval: $\left(-\infty, \frac{3}{2}\right], \left[\frac{3}{2}, \frac{5}{2}\right], \left[\frac{5}{2}, \infty\right)$

Test Points

Test $x = 1: (2-3)(2-5) = 3 > 0.$ Satisfied

Test $x = 2: (4-3)(4-5) = -1 < 0.$ Satisfied

Test $x = 3: (6-3)(6-5) = 3 > 0.$ Satisfied

Conclusion

$x \in \left[\frac{3}{2}, \frac{5}{2}\right]$

(vi) $-x^2 + 3x - 2 \ge 0$

Solution

$-x^2 + 3x - 2 \ge 0$

Solve Associated Equation

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$(x-1)(x-2) = 0$

$x - 1 = 0 \quad ; \quad x - 2 = 0$

$x = 1 \quad ; \quad x = 2$

Critical Points: $(1,0), (2,0)$

Find Interval: $(-\infty, 1], [1,2], [2, \infty)$

Test Points

Test $x = 0: (0-1)(0-2) = 2 > 0.$ Satisfied

Test $x = 1.5: (1.5-1)(1.5-2) = -0.25 < 0.$ Satisfied

Test $x = 3: (3-1)(3-2) = 2 > 0.$ Satisfied

Conclusion

$x \in [1, 2]$

EXERCISE 2.6

1. Make F the subject of the formula, $C^o = \frac{5}{9}\left(F^o - 32\right)$.

Solution

$C° = \frac{5}{9}(F° - 32)$

$\Rightarrow \frac{9}{5}C° = F° - 32$

$\Rightarrow F = \frac{9}{5}C° + 32$

2. The formula for finding simple interest is $I = PRT$.

(a) Make P the subject of the formula.

(b) Make T the subject of the formula.

Solution

(a) We know that $I = PRT$

P the subject of formula $= P = \frac{I}{RT}$

(b) We know that $I = PRT$

T the subject of formula $= T = \frac{I}{PR}$

3. Make 'a' the subject of the formula $S = 2a + (n - 1)\,d$.

Solution

$S = 2a + (n-1)d$

$\Rightarrow 2a = S - (n-1)d$

$\Rightarrow a = \frac{S-(n-1)d}{2}$

4. The volume of a cylinder is given by the formula, $V = \pi r^2 h$. Make 'h' the subject of the formula.

Solution

$V = \pi r^2 h$

$\Rightarrow h = \frac{V}{\pi r^2}$

5. The area of a trapezoid is $A = \frac{1}{2}\,h\,(b_1 + b_2)$, make h the subject of the formula.

Solution

$A = \frac{1}{2}h(b_1 + b_2) \Rightarrow 2A = h(b_1 + b_2) \Rightarrow h = \frac{2A}{b_1+b_2}$

6. If $y = mx + c$, then make 'x' the subject of this equation.

Solution

$y = mx + c$

$\Rightarrow y - c = mx$

$\Rightarrow x = \frac{y-c}{m}$

7. Perimeter (P) of a rectangle is $P = 2(\ell + w)$, make ℓ as the subject of this formula.

Solution

$P = 2(l + w)$

$\Rightarrow \frac{P}{2} = l + w$

$\Rightarrow l = \frac{P}{2} - w$

$\Rightarrow l = \frac{P-2w}{2}$

8. The equation of a parabola is $y^2 = 4ax$, make 'x' as a subject of this equation.

Solution

$y^2 = 4ax$

$\Rightarrow x = \frac{y^2}{4a}$

9. If $P = S - C$, where S is selling price and C is cost price. Make S as subject of the equation.

Solution

$P = S - C$

$\Rightarrow S = P + C$

10. Volume of the cone is $V = \frac{1}{3}\pi r^2 h$, make 'h' as subject of this formula.

Solution

$V = \frac{1}{3}\pi r^2 h$

$\Rightarrow h = \frac{3V}{\pi r^2}$

EXERCISE 2.7

1. A town's population is modeled by $P(t) = -2t^2 + 40t + 800$, where t is years since 2020. Find the years when the population will be at least 1000.

Solution

Model: $P(t) = -2t^2 + 40t + 800$, Find t for $P(t) \geq 1000$ (Years since 2020)

- $-2t^2 + 40t + 800 \geq 1000$
- $-2t^2 + 40t - 200 \geq 0$
- Divide by -2: $t^2 - 20t + 100 \leq 0$
- Factor: $(t - 10)^2 \leq 0$
- Solution: $t = 10$
- **Year:** $2020 + 10 = 2030$

2. A company models its profit P in thousands of rupees by the equation: $P(x) = -5x^2 + 150x - 1000$, where x is the price per item in rupees. Find the price that gives maximum profit.

Solution

Model: $P(x) = -5x^2 + 150x - 1000$

- Max at $x = -\frac{b}{2a}$
- $x = -\frac{150}{2(-5)}$
- $x = \frac{-150}{-10}$
- **Price: Rs. 15**

3. A toy car rolls down an incline and covers a distance given by the equation $d = t^2 - 0.5t$ metres, where t is the time in seconds. Find the time when the car has travelled a distance 12.5 metres.

Solution

Model: $d = t^2 - 0.5t$, Find t for $d = 12.5$

- $t^2 - 0.5t = 12.5$
- $t^2 - 0.5t - 12.5 = 0$
- Quadratic Formula: $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $t = \frac{0.5 \pm \sqrt{(-0.5)^2 - 4(1)(-12.5)}}{2(1)}$
- $t = \frac{0.5 \pm \sqrt{0.25 + 50}}{2}$
- $t = \frac{0.5 + 7.088}{2} \approx 3.794$
- **Time: 3.79 seconds**

4. A ball's height (in metres) after t seconds is $h(t) = -4t^2 + 24t$. For what time interval is the ball at least $20m$ above the ground?

Solution

Model: $h(t) = -4t^2 + 24t$, Find t for $h(t) \geq 20$

- $-4t^2 + 24t \geq 20$
- $-4t^2 + 24t - 20 \geq 0$
- Divide by -4: $t^2 - 6t + 5 \leq 0$
- Factor: $(t-1)(t-5) \leq 0$
- Critical points: $t = 1, t = 5$
- **Interval:** $1 \leq t \leq 5$ **seconds**

5. A ball is thrown upward with an initial velocity of $40\,\text{ms}^{-1}$. Calculate the maximum height it reaches above ground level.

Solution

Given: $u = 40\ \text{m/s}$, $g \approx 10\ \text{m/s}^2$, Max height at $v = 0$

- $v^2 = u^2 + 2as$
- $0^2 = 40^2 + 2(-10)s$
- $0 = 1600 - 20s$
- $20s = 1600$
- $s = \frac{1600}{20}$
- **Max Height: 80 metres**

6. A freelancer's earnings follow the model $E(h) = -2h^2 + 40h$, where E is earning in rupees and h is hours worked per week. What is the maximum number of hours he should work to maximize earnings?

Solution

Model: $E(h) = -2h^2 + 40h$

- Max at $h = -\frac{b}{2a}$
- $h = -\frac{40}{2(-2)}$
- $h = \frac{-40}{-4}$
- **Hours: 10 hours per week**

REVIEW EXERCISE 2

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) The type of the equation $2x^2 - x + 1 = 0$ is:
(a) quadratic (b) linear (c) third degree (d) Pure quadratic

(ii) What is the discriminant of $x^2 + 5x - 5 = 0$?
(a) -20 (b) 20 (c) 25 (d) 45

(iii) The solution set of $3x^2 - 9 = 0$ is:
(a) $\{3\}$ (b) $\{\pm 3\}$ (c) $\{\pm\sqrt{3}\}$ (d) $\{\sqrt{3}\}$

(iv) Sum of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) $\frac{5}{3}$ (b) $-\frac{5}{3}$ (c) $\frac{12}{3}$ (d) $\frac{3}{5}$

(v) Product of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) -4 (b) 3 (c) 4 (d) 5

(vi) What are the roots of $(x - 3)(x + 3) = 0$?
(a) $3, -3$ (b) $3, 3$ (c) $-3, -3$ (d) $9, 0$

(vii) 3 and 2 are the roots of:
(a) $x^2 + 5x + 6 = 0$ (b) $x^2 + 6x + 5 = 0$
(c) $x^2 - 5x + 6 = 0$ (d) $x^2 + 6x - 5 = 0$

(viii) If $b^2 - 4ac > 0$ and is a perfect square, then the roots of $ax^2 + bx + c = 0$ are:
(a) equal (b) unequal (c) imaginary (d) irrational

(ix) If $b^2 - 4ac = 0$, then the roots of $ax^2 + bx + c = 0$ are:
(a) unequal (b) irrational (c) imaginary (d) equal

(x) Subject "c" of $x - 2c = b$ is:
(a) $x + b$ (b) $b - x$ (c) $\frac{x-b}{2}$ (d) $\frac{b-x}{2}$

2. Solve the following quadratic equations by factorization method, by completing square method and by quadratic formula:

(i) $8x^2 = x + 7$ (ii) $2x^2 - x - 10 = 0$

Solution

(i) $8x^2 = x + 7$

Solution

- **Quadratic formula**:

$$8x^2 - x - 7 = 0.$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(8)(-7)}}{2(8)}.$$

$$x = \frac{1 \pm \sqrt{1+224}}{16}.$$

$$x = \frac{1 \pm 15}{16}.$$

$$x = 1, \quad x = -\frac{7}{8}.$$

- **Factorization**:

$$8x^2 - x - 7 = (8x+7)(x-1) = 0.$$

$$x = 1, \quad x = -\frac{7}{8}.$$

- **Completing the square**:

$$x^2 - \frac{1}{8}x = \frac{7}{8}.$$

$$x^2 - \frac{1}{8}x + (\frac{1}{16})^2 = \frac{7}{8} + \frac{1}{256}.$$

$$(x - \frac{1}{16})^2 = \frac{225}{256}.$$

$$x - \frac{1}{16} = \pm\frac{15}{16}.$$

$$x = 1, \quad x = -\frac{7}{8}.$$

(ii) $2x^2 - x - 10 = 0$

Solution

- **Quadratic formula**:

$$x = \frac{-(-1) \pm \sqrt{1 - 4(2)(-10)}}{4}.$$

$$x = \frac{1 \pm \sqrt{81}}{4}.$$

$$x = \frac{5}{2}, \quad x = -2.$$

- **Factorization**:

$$2x^2 - x - 10 = (2x-5)(x+2) = 0.$$

$$x = \frac{5}{2}, \quad x = -2.$$

- **Completing the square**:

$$x^2 - \frac{1}{2}x = 5.$$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = 5 + \frac{1}{16}.$$

$$(x - \frac{1}{4})^2 = \frac{81}{16}.$$

$$x - \frac{1}{4} = \pm\frac{9}{4}.$$

$$x = \frac{5}{2}, \quad x = -2.$$

3. Form a quadratic equation whose roots are; 6, $\frac{3}{2}$.

Solution

$\alpha = 6,\ \beta = \frac{3}{2}$

$S = \alpha + \beta = 6 + \frac{3}{2} = \frac{15}{2}$

$P = \alpha\beta = (6)\left(\frac{3}{2}\right) = 9$

Required Equation is $\boldsymbol{x^2 - Sx + P = 0}$

$x^2 - \frac{15}{2}x + 9 = 0$

$2x^2 - 15x + 18 = 0$

4. Examine the nature of the roots of the following equations:

(i) $15x^2 + 11x + 2 = 0$ (ii) $x^2 - x - 1 = 0$

Solution

(i) $\boldsymbol{15x^2 + 11x + 2 = 0}$	(ii) $\boldsymbol{x^2 - x - 1 = 0}$
$D = 11^2 - 4(15)(2) = 121 - 120 = 1.$	$D = (-1)^2 - 4(1)(-1) = 1 + 4 = 5.$
Rational and unequal	Irrational and unequal

5. If a ball is thrown upward with a velocity v, the maximum height it reaches can be determined by using a formula $h = \frac{v^2}{2g}$. Rearrange the formula to make v the subject.

Solution

$h = \frac{v^2}{2g}$

$v^2 = 2gh$

$v = \sqrt{2gh}$

6. If the equation $x^2 + 2(1+k)x + k^2 = 0$ has equal roots, then find the value of k.

Solution

$D = b^2 - 4ac$

$D = [2(1+k)]^2 - 4(1)(k^2) = 0.$

$4(1 + 2k + k^2) - 4k^2 = 0.$

$4 + 8k = 0.$

$k = -\frac{1}{2}.$

UNIT 3

Matrices and Determinants

EXERCISE 3.1

1. Write the number of rows and number of columns in each matrix.

$A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}3\\ 4\\ 7\end{bmatrix}$, $C=\begin{bmatrix}8 & -10 & 11\end{bmatrix}$,

$D=\begin{bmatrix}3 & 2 & 1\\ 1 & 0 & 3\\ 4 & 5 & 6\end{bmatrix}$, $E=\begin{bmatrix}5 & 9 & -2\\ -3 & 4 & 5\end{bmatrix}$

Solution

Number of rows in $A = 2$, Number of columns in $A = 2$, Number of rows in $B = 3$,
Number of columns in $B = 1$, Number of rows in $C = 1$, Number of columns in $C = 3$,
Number of rows in $D = 3$, Number of columns in $D = 3$, Number of rows in $E = 2$,
Number of columns in $E = 3$

2. Write the order of each matrix.

$A=\begin{bmatrix}1\\ 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 4\end{bmatrix}$, $C=\begin{bmatrix}3 & 5\\ 5 & 6\end{bmatrix}$, $D=\begin{bmatrix}0 & 3 & 4\\ 5 & 0 & 9\end{bmatrix}$, $E=\begin{bmatrix}0 & 1\\ 3 & 4\\ 5 & 6\end{bmatrix}$, $F=\begin{bmatrix}5\end{bmatrix}$

Solution

2. Order of A = 2-by-1, Order of B = 1-by-2, Order of C = 2-by-2, Order of D = 2-by-3,
Order of E = 3-by-2, Order of F = 1-by-1

3. Which of the following matrices are equal?

$A=\begin{bmatrix}0 & 3\\ 4 & 1\end{bmatrix}$, $B=\begin{bmatrix}9\\ -7\end{bmatrix}$, $C=\begin{bmatrix}2\times 3 & 2-1\\ 2\times 2 & 4-2\\ 4+4 & 3+0\end{bmatrix}$,

$D=\begin{bmatrix}5+4\\ -8+1\end{bmatrix}$, $E=\begin{bmatrix}6 & 1\\ 4 & 2\\ 8 & 3\end{bmatrix}$, $F=\begin{bmatrix}3-3 & 3\\ 3+1 & 1\end{bmatrix}$

Solution

$A=F,\ B=D,\ C=E$

4. If $\begin{bmatrix} a+2 & c-3 \\ b-1 & d+4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 6 & 4 \end{bmatrix}$, then find the values of a, b, c and d.

Solution

$a+2=5 \Rightarrow a=5-2 \Rightarrow a=3$
$b-1=6 \Rightarrow b=6+1 \Rightarrow b=7$
$c-3=8 \Rightarrow c=8+3 \Rightarrow c=11$
$d+4=4 \Rightarrow d=4-4 \Rightarrow d=0$

5. If $\begin{bmatrix} 2x+1 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 7 & y \end{bmatrix}$, then find the values of x and y.

Solution

$2x+1=9 \Rightarrow 2x=9-1 \Rightarrow 2x=8 \Rightarrow x=4$
$y=5$

6. If $\begin{bmatrix} a+b & 2d-1 \\ 3b+2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 11 & c \end{bmatrix}$, then find the values of a, b, c and d.

Solution

$3b+2=11 \Rightarrow 3b=11-2 \Rightarrow 3b=9 \Rightarrow b=3$
$a+b=10 \Rightarrow a+\mathbf{3}=10 \Rightarrow a=10-3 \Rightarrow a=7$
$2d-1=5 \Rightarrow 2d=5+1 \Rightarrow 2d=6 \Rightarrow d=3$
$c=4$

7. If $\begin{bmatrix} p+q & 5 \\ 11 & p-2q \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 11 & 0 \end{bmatrix}$, then find the values of p and q.

Solution

$p+q=6$ ………. (i) & $p-2q=0$ ………. (ii)

$\mathbf{2}p+\mathbf{2}q=\mathbf{12}$ $p-2q=0$
$3p=12$ $p=4$

Put $p=4$ in (i)

$\Rightarrow \mathbf{4}+q=6 \Rightarrow q=6-4$

$\Rightarrow q=2$

EXERCISE 3.2

1. From the following matrices identify unit matrices, row matrices, column matrices and null matrices.

$A=\begin{bmatrix} 5 & 7 & 8 \end{bmatrix}$, $B=\begin{bmatrix} 0 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $E=\begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$, $F=\begin{bmatrix} 7 \\ 1 \\ 9 \end{bmatrix}$

Solution

A = row matrix , B = null matrix , C = unit matrix, D = null matrix,
E = column matrix , F = column matrix

2. Identify type of the given matrices as row, column, square and rectangular matrices.

$A=\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 6 \\ 4 & 1 \end{bmatrix}$, $C=\begin{bmatrix} 3 & 5 & 8 \\ 0 & 4 & -2 \end{bmatrix}$, $D=\begin{bmatrix} 5 & -5 \\ 2 & 7 \end{bmatrix}$,

$E=\begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 5 & 0 \end{bmatrix}$, $F=\begin{bmatrix} 5 & -3 & 7 \end{bmatrix}$, $G=\begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 4 \\ 5 & 2 & -3 \end{bmatrix}$, $H=\begin{bmatrix} 3 & 5 \\ 4 & 4 \\ 5 & 2 \end{bmatrix}$

Solution

A = column matrix , B = square matrix , C = rectangular matrix,
D = square matrix , E = rectangular matrix , F = row matrix,
G = square matrix , H = rectangular matrix

3. Identify diagonal, scalar and unit matrices.

$A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $C=\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$,

$D=\begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}$, $E=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Solution

A = unit matrix , B = scalar matrix , C = diagonal matrix,
D = diagonal matrix , E = scalar matrix

4. Find transpose of each of the following matrices:

$$A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}, \quad B=\begin{bmatrix}3\\ 7\\ 6\end{bmatrix}, \quad C=\begin{bmatrix}5 & -2 & 4\end{bmatrix}, \quad D=\begin{bmatrix}2 & 5\\ 3 & 6\\ 4 & 7\end{bmatrix}$$

Solution

$$A^t=\begin{bmatrix}2 & 4\\ 3 & 5\end{bmatrix}, \quad B^t=\begin{bmatrix}3 & 7 & 6\end{bmatrix}, \quad C^t=\begin{bmatrix}5\\ -2\\ 4\end{bmatrix}, \quad D^t=\begin{bmatrix}2 & 3 & 4\\ 5 & 6 & 7\end{bmatrix}$$

5. Find negative of the following matrices:

$$A=\begin{bmatrix}-3 & 0\\ 5 & 6\end{bmatrix}, \quad B=\begin{bmatrix}-3 & 3\\ -2 & 2\end{bmatrix}, \quad C=\begin{bmatrix}-9 & 1\\ 1 & -7\end{bmatrix}$$

Solution

$$-A=\begin{bmatrix}3 & 0\\ -5 & -6\end{bmatrix}, \quad -B=\begin{bmatrix}3 & -3\\ 2 & -2\end{bmatrix}, \quad -C=\begin{bmatrix}9 & -1\\ -1 & 7\end{bmatrix}$$

6. If $A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ and $B=\begin{bmatrix}7 & 6\\ 5 & 8\end{bmatrix}$, then verify that

(i) $(A^t)^t = A$ (ii) $(B^t)^t = B$

Solution

$A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ $A^t=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}^t=\begin{bmatrix}2 & 4\\ 3 & 5\end{bmatrix}$ $(A^t)^t=\begin{bmatrix}2 & 4\\ 3 & 5\end{bmatrix}^t=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ Hence $(A^t)^t = A$	$B=\begin{bmatrix}7 & 6\\ 5 & 8\end{bmatrix}$ $B^t=\begin{bmatrix}7 & 6\\ 5 & 8\end{bmatrix}^t=\begin{bmatrix}7 & 5\\ 6 & 8\end{bmatrix}$ $(B^t)^t=\begin{bmatrix}7 & 5\\ 6 & 8\end{bmatrix}^t=\begin{bmatrix}7 & 6\\ 5 & 8\end{bmatrix}$ Hence $(B^t)^t = B$

7. Show that $L = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}$ is a symmetric matrix.

Solution

$$L = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}$$

$$L^t = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}^t = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}$$

$$L^t = L$$

Hence $L = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}$ is symmetric matrix

EXERCISE 3.3

1. Which of the following matrices are conformable for addition and subtraction?

$A=\begin{bmatrix}3 & 4\\ -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}3\\ 4\end{bmatrix}$, $C=\begin{bmatrix}5 & 2\end{bmatrix}$, $D=\begin{bmatrix}1 & 7\\ 2 & 5\end{bmatrix}$ $E=\begin{bmatrix}2\end{bmatrix}$,

$F=\begin{bmatrix}7 & 11\end{bmatrix}$, $G=\begin{bmatrix}a\\ b\end{bmatrix}$, $H=\begin{bmatrix}3\end{bmatrix}$, $M=\begin{bmatrix}l\\ m\end{bmatrix}$

Solution

A and D are conformable for addition and subtraction.

B, G and M are conformable for addition and subtraction.

C and F are conformable for addition and subtraction.

E and H are conformable for addition and subtraction.

2. If $X=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}$, $Y=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $Z=\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$, then find the following:

(i) $X+Y$ (ii) $Y+7Z$ (iii) $4X-Z$

(iv) $X+2Y+3Z$ (v) $X-4Y+Z$ (vi) $Z-Z$

Solution

(i) $X+Y$ $$X+Y=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1+1 & -1+2\\ -2+3 & 2+4\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 6\end{bmatrix}.$$	(ii) $Y+7Z$ $$Y+7Z=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+7\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}21 & 0\\ 0 & -14\end{bmatrix}=\begin{bmatrix}1+21 & 2+0\\ 3+0 & 4-14\end{bmatrix}=\begin{bmatrix}22 & 2\\ 3 & -10\end{bmatrix}.$$

(iii) $4X - Z$ $4X - Z = 4\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $4X - Z = \begin{bmatrix} 4 & -4 \\ -8 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $4X - Z = \begin{bmatrix} 4-3 & -4-0 \\ -8-0 & 8+2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -8 & 10 \end{bmatrix}$	(iv) $X+2Y+3Z$ $X+2Y+3Z = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & -6 \end{bmatrix}$ $= \begin{bmatrix} 1+2+9 & -1+4+0 \\ -2+6+0 & 2+8-6 \end{bmatrix}$ $= \begin{bmatrix} 12 & 3 \\ 4 & 4 \end{bmatrix}.$
(v) $X-4Y+Z$ $X-4Y+Z = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -12 & -16 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $= \begin{bmatrix} 1-4+3 & -1-8+0 \\ -2-12+0 & 2-16-2 \end{bmatrix}$ $= \begin{bmatrix} 0 & -9 \\ -14 & -16 \end{bmatrix}.$	(vi) $Z-Z$ $Z - Z = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

3. Find the additive inverse of the following matrices:

(i) $P = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ (ii) $Q = \begin{bmatrix} 9 & -3 \end{bmatrix}$ (iii) $R = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ (iv) $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution

(i) $P = \begin{bmatrix} 5 \\ -7 \end{bmatrix}.$ $-P = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$	(ii) $Q = \begin{bmatrix} 9 & -3 \end{bmatrix}.$ $-Q = \begin{bmatrix} -9 & 3 \end{bmatrix}.$
(iii) $R = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}.$ $-R = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}.$	(iv) $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$ $-S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

4. If $A=\begin{bmatrix}2 & 3\\ -3 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix}$ and $C=\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}$, then verify the following:

(i) $A+B=B+A$ (ii) $(A+B)+C=A+(B+C)$

(iii) $(2A+B)+C=2A+(B+C)$ (iv) $3(A+B)=3A+3B$

Solution

(i) Verify $A+B=B+A$.

$$A+B=\begin{bmatrix}2+3 & 3+4\\ -3+5 & 2+6\end{bmatrix}=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix},$$

$$B+A=\begin{bmatrix}3+2 & 4+3\\ 5-3 & 6+2\end{bmatrix}=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix}.$$

$$A+B=B+A$$

(ii) Verify $(A+B)+C=A+(B+C)$.

$$(A+B)+C=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix}+\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}=\begin{bmatrix}6 & 5\\ 2 & 13\end{bmatrix},$$

$$B+C=\begin{bmatrix}4 & 2\\ 5 & 11\end{bmatrix},$$

$$A+(B+C)=\begin{bmatrix}2+4 & 3+2\\ -3+5 & 2+11\end{bmatrix}=\begin{bmatrix}6 & 5\\ 2 & 13\end{bmatrix}.$$

$$(A+B)+C=A+(B+C)$$

(iii) Verify $(2A+B)+C=2A+(B+C)$.

$$2A=\begin{bmatrix}4 & 6\\ -6 & 4\end{bmatrix},$$

$$2A+B=\begin{bmatrix}7 & 10\\ -1 & 10\end{bmatrix},$$

$$(2A+B)+C=\begin{bmatrix}8 & 8\\ -1 & 15\end{bmatrix},$$

$$B+C=\begin{bmatrix}4 & 2\\ 5 & 11\end{bmatrix},$$

$$2A+(B+C)=\begin{bmatrix}4+4 & 6+2\\ -6+5 & 4+11\end{bmatrix}=\begin{bmatrix}8 & 8\\ -1 & 15\end{bmatrix}.$$

$$(2A+B)+C=2A+(B+C)$$

(iv) Verify $3(A+B)=3A+3B$.

$$A+B=\begin{bmatrix}5 & 7\\ 2 & 8\end{bmatrix},$$

$$3(A+B)=\begin{bmatrix}15 & 21\\ 6 & 24\end{bmatrix},$$

$$3A=\begin{bmatrix}6 & 9\\ -9 & 6\end{bmatrix},\quad 3B=\begin{bmatrix}9 & 12\\ 15 & 18\end{bmatrix},$$

$$3A+3B=\begin{bmatrix}15 & 21\\ 6 & 24\end{bmatrix}.$$

$$3(A+B)=3A+3B$$

5. If $A=\begin{bmatrix}5 & 6\\7 & -2\end{bmatrix}$ and $B=\begin{bmatrix}-5 & -6\\-7 & 2\end{bmatrix}$, then show that B is additive inverse of A and A is additive inverse of B.

Solution

$$A+B=\begin{bmatrix}5 & 6\\7 & -2\end{bmatrix}+\begin{bmatrix}-5 & -6\\-7 & 2\end{bmatrix}=\begin{bmatrix}5-5 & 6-6\\7-7 & -2+2\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Thus $B=-A$. Similarly,

$$B+A=\begin{bmatrix}-5 & -6\\-7 & 2\end{bmatrix}+\begin{bmatrix}5 & 6\\7 & -2\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

So $A=-B$.

6. If $A=\begin{bmatrix}6 & -2\\0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}8 & -1\\3 & 0\end{bmatrix}$, then verify that

(i) $(A+B)^t=A^t+B^t$ (ii) $(A-B)^t=A^t-B^t$

Solution

(i) $(A+B)^t=A^t+B^t$

L.H.S. = $(A+B)^t$	**R.H.S.** = A^t+B^t
$A+B=\begin{bmatrix}6 & -2\\0 & 3\end{bmatrix}+\begin{bmatrix}8 & -1\\3 & 0\end{bmatrix}=\begin{bmatrix}6+8 & -2-1\\0+3 & 3+0\end{bmatrix}=\begin{bmatrix}14 & -3\\3 & 3\end{bmatrix}.$ $(A+B)^t=\begin{bmatrix}14 & 3\\-3 & 3\end{bmatrix}$	$A^t+B^t=\begin{bmatrix}6 & 0\\-2 & 3\end{bmatrix}+\begin{bmatrix}8 & 3\\-1 & 0\end{bmatrix}=\begin{bmatrix}6+8 & 0+3\\-2-1 & 3+0\end{bmatrix}=\begin{bmatrix}14 & 3\\-3 & 3\end{bmatrix}.$ Hence $(A+B)^t=A^t+B^t$

(ii) $(A-B)^t = A^t - B^t$

Solution

L.H.S. = $(\boldsymbol{A}-\boldsymbol{B})^{\boldsymbol{t}}$

$$A-B=\begin{bmatrix}6 & -2\\ 0 & 3\end{bmatrix}-\begin{bmatrix}8 & -1\\ 3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}6-8 & -2-(-1)\\ 0-3 & 3-0\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}.$$

$$(A-B)^t=\begin{bmatrix}-2 & -3\\ -1 & 3\end{bmatrix}.$$

R.H.S. = $\boldsymbol{A}^{\boldsymbol{t}}-\boldsymbol{B}^{\boldsymbol{t}}$

$$A^t-B^t=\begin{bmatrix}6 & 0\\ -2 & 3\end{bmatrix}-\begin{bmatrix}8 & 3\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}6-8 & 0-3\\ -2-(-1) & 3-0\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -3\\ -1 & 3\end{bmatrix}.$$

Hence $(\boldsymbol{A}-\boldsymbol{B})^{\boldsymbol{t}}=\boldsymbol{A}^{\boldsymbol{t}}-\boldsymbol{B}^{\boldsymbol{t}}$

EXERCISE 3.4

1. Find AB and BA, if possible.

(i) $A=\begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}$ (ii) $A=[1 \quad -2]$, $B=\begin{bmatrix}3\\-4\end{bmatrix}$

(iii) $A=\begin{bmatrix}4\\4\end{bmatrix}$, $B=[2 \quad 5]$ (iv) $A=\begin{bmatrix}1 & 2\\-1 & 1\\3 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 4 & 1\\2 & 3 & 1\end{bmatrix}$

(v) $A=\begin{bmatrix}1 & 2 & 2\\3 & 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\2 & 4\\1 & 6\end{bmatrix}$

Solution

(i) $A=\begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}$.

$$AB=\begin{bmatrix}(1)(3)+(2)(1) & (1)(2)+(2)(-1)\\(-1)(3)+(0)(1) & (-1)(2)+(0)(-1)\end{bmatrix}$$
$$=\begin{bmatrix}3+2 & 2-2\\-3+0 & -2+0\end{bmatrix}=\begin{bmatrix}5 & 0\\-3 & -2\end{bmatrix},$$
$$BA=\begin{bmatrix}(3)(1)+(2)(-1) & (3)(2)+(2)(0)\\(1)(1)+(-1)(-1) & (1)(2)+(-1)(0)\end{bmatrix}$$
$$=\begin{bmatrix}3-2 & 6+0\\1+1 & 2+0\end{bmatrix}=\begin{bmatrix}1 & 6\\2 & 2\end{bmatrix}.$$

(ii) $A=[1 \quad -2]$, $B=\begin{bmatrix}3\\-4\end{bmatrix}$.

$$AB=[(1)(3)+(-2)(-4)]=[3+8]=[11],$$
$$BA=\begin{bmatrix}(3)(1) & (3)(-2)\\(-4)(1) & (-4)(-2)\end{bmatrix}=\begin{bmatrix}3 & -6\\-4 & 8\end{bmatrix}.$$

(iii) $A=\begin{bmatrix}4\\4\end{bmatrix}$, $B=[2 \quad 5]$.

$$AB=\begin{bmatrix}(4)(2) & (4)(5)\\(4)(2) & (4)(5)\end{bmatrix}=\begin{bmatrix}8 & 20\\8 & 20\end{bmatrix},$$
$$BA=[(4)(2)+(4)(5)]=[8+20]=[28].$$

(iv) $A=\begin{bmatrix}1 & 2\\-1 & 1\\3 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 4 & 1\\2 & 3 & 1\end{bmatrix}$

$$AB=\begin{bmatrix}(1)(-1)+(2)(2) & (1)(4)+(2)(3) & (1)(1)+(2)(1)\\(-1)(-1)+(1)(2) & (-1)(4)+(1)(3) & (-1)(1)+(1)(1)\\(3)(-1)+(0)(2) & (3)(4)+(0)(3) & (3)(1)+(0)(1)\end{bmatrix}$$
$$AB=\begin{bmatrix}-1+4 & 4+6 & 1+2\\1+2 & -4+3 & -1+1\\-3+0 & 12+0 & 3+0\end{bmatrix}=\begin{bmatrix}3 & 10 & 3\\3 & -1 & 0\\-3 & 12 & 3\end{bmatrix}$$
$$BA=\begin{bmatrix}(-1)(1)+(4)(-1)+(1)(3) & (-1)(2)+(4)(1)+(1)(0)\\(2)(1)+(3)(-1)+(1)(3) & (2)(2)+(3)(1)+(1)(0)\end{bmatrix}$$
$$BA=\begin{bmatrix}-1-4+3 & -2+4+0\\2-3+3 & 4+3+0\end{bmatrix}=\begin{bmatrix}-2 & 2\\2 & 7\end{bmatrix}$$

(v) $A=\begin{bmatrix}1 & 2 & 2\\3 & 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\2 & 4\\1 & 6\end{bmatrix}$

$$BA=\begin{bmatrix}(1)(1)+(2)(2)+(2)(1) & (1)(5)+(2)(4)+(2)(6)\\(3)(1)+(1)(2)+(1)(1) & (3)(5)+(1)(4)+(1)(6)\end{bmatrix}$$

$$BA=\begin{bmatrix}1+4+2 & 5+8+12\\3+2+1 & 15+4+6\end{bmatrix}=\begin{bmatrix}7 & 25\\6 & 25\end{bmatrix}$$

$$BA=\begin{bmatrix}(1)(1)+(5)(3) & (1)(2)+(5)(1) & (1)(2)+(5)(1)\\(2)(1)+(4)(3) & (2)(2)+(4)(1) & (2)(2)+(4)(1)\\(1)(1)+(6)(3) & (1)(2)+(6)(1) & (1)(2)+(6)(1)\end{bmatrix}$$

$$BA=\begin{bmatrix}1+15 & 2+5 & 2+5\\2+12 & 4+4 & 4+4\\1+18 & 2+6 & 2+6\end{bmatrix}=\begin{bmatrix}16 & 7 & 7\\14 & 8 & 8\\19 & 8 & 8\end{bmatrix}$$

2. Verify each statement, using $A=\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}$

(i) $AB\neq BA$ (ii) $A(B-C)=AB-AC$

(iii) $A(BC)=(AB)C$ (iv) $(BC)^t=C^tB^t$

(v) $(B+C)A=BA+CA$

Solution

(i) Verify $AB\neq BA$.

$$AB=\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}$$
$$=\begin{bmatrix}(5)(2)+(1)(0) & (5)(-1)+(1)(3)\\(-1)(2)+(4)(0) & (-1)(-1)+(4)(3)\end{bmatrix}$$
$$=\begin{bmatrix}10+0 & -5+3\\-2+0 & 1+12\end{bmatrix}=\begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix},$$
$$BA=\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}$$
$$=\begin{bmatrix}(2)(5)+(-1)(-1) & (2)(1)+(-1)(4)\\(0)(5)+(3)(-1) & (0)(1)+(3)(4)\end{bmatrix}$$
$$=\begin{bmatrix}10+1 & 2-4\\0-3 & 0+12\end{bmatrix}=\begin{bmatrix}11 & -2\\-3 & 12\end{bmatrix}.$$

$AB\neq BA$ (verified).

(ii) Verify $A(B-C)=AB-AC$.

$$B-C=\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}-\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}=\begin{bmatrix}-3 & -2\\-1 & -1\end{bmatrix},$$

$$A(B-C)=\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}-3 & -2\\-1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}(5)(-3)+(1)(-1) & (5)(-2)+(1)(-1)\\(-1)(-3)+(4)(-1) & (-1)(-2)+(4)(-1)\end{bmatrix}$$

$$=\begin{bmatrix}-15-1 & -10-1\\3-4 & 2-4\end{bmatrix}=\begin{bmatrix}-16 & -11\\-1 & -2\end{bmatrix},$$

$$AB=\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(5)(2)+(1)(0) & (5)(-1)+(1)(3)\\(-1)(2)+(4)(0) & (-1)(-1)+(4)(3)\end{bmatrix}$$

$$=\begin{bmatrix}10+0 & -5+3\\-2+0 & 1+12\end{bmatrix}=\begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix},$$

$$AC=\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}(5)(5)+(1)(1) & (5)(1)+(1)(4)\\(-1)(5)+(4)(1) & (-1)(1)+(4)(4)\end{bmatrix}$$

$$=\begin{bmatrix}25+1 & 5+4\\-5+4 & -1+16\end{bmatrix}=\begin{bmatrix}26 & 9\\-1 & 15\end{bmatrix},$$

$$AB-AC=\begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix}-\begin{bmatrix}26 & 9\\-1 & 15\end{bmatrix}=\begin{bmatrix}-16 & -11\\-1 & -2\end{bmatrix}.$$

$A(B-C)=AB-AC$ **(verified)**

(iii) Verify $A(BC)=(AB)C$.

$$BC = \begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}$$

$$= \begin{bmatrix}(2)(5)+(-1)(1) & (2)(1)+(-1)(4)\\(0)(5)+(3)(1) & (0)(1)+(3)(4)\end{bmatrix}$$

$$= \begin{bmatrix}10-1 & 2-4\\0+3 & 0+12\end{bmatrix} = \begin{bmatrix}9 & -2\\3 & 12\end{bmatrix},$$

$$A(BC) = \begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}9 & -2\\3 & 12\end{bmatrix}$$

$$= \begin{bmatrix}(5)(9)+(1)(3) & (5)(-2)+(1)(12)\\(-1)(9)+(4)(3) & (-1)(-2)+(4)(12)\end{bmatrix}$$

$$= \begin{bmatrix}45+3 & -10+12\\-9+12 & 2+48\end{bmatrix} = \begin{bmatrix}48 & 2\\3 & 50\end{bmatrix},$$

$$AB = \begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}$$

$$= \begin{bmatrix}(5)(2)+(1)(0) & (5)(-1)+(1)(3)\\(-1)(2)+(4)(0) & (-1)(-1)+(4)(3)\end{bmatrix}$$

$$= \begin{bmatrix}10+0 & -5+3\\-2+0 & 1+12\end{bmatrix} = \begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix},$$

$$(AB)C = \begin{bmatrix}10 & -2\\-2 & 13\end{bmatrix}\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}$$

$$= \begin{bmatrix}(10)(5)+(-2)(1) & (10)(1)+(-2)(4)\\(-2)(5)+(13)(1) & (-2)(1)+(13)(4)\end{bmatrix}$$

$$= \begin{bmatrix}50-2 & 10-8\\-10+13 & -2+52\end{bmatrix} = \begin{bmatrix}48 & 2\\3 & 50\end{bmatrix}.$$

$A(BC)=(AB)C$ **(verified)**

(iv) $(BC)^t = C^tB^t$

$$BC = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} (2)(5)+(-1)(1) & (2)(1)+(-1)(4) \\ (0)(5)+(3)(1) & (0)(1)+(3)(4) \end{bmatrix}$$
$$= \begin{bmatrix} 10-1 & 2-4 \\ 0+3 & 0+12 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 3 & 12 \end{bmatrix},$$

$$(BC)^T = \begin{bmatrix} 9 & 3 \\ -2 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$C^TB^T = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} (5)(2)+(1)(-1) & (5)(0)+(1)(3) \\ (1)(2)+(4)(-1) & (1)(0)+(4)(3) \end{bmatrix}$$
$$= \begin{bmatrix} 10-1 & 0+3 \\ 2-4 & 0+12 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ -2 & 12 \end{bmatrix}.$$

$(BC)^t = C^tB^t$ **(verified)**

(v) Verify $(B+C)A=BA+CA$.

$$B+C=\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}+\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}=\begin{bmatrix}7 & 0\\1 & 7\end{bmatrix},$$

$$(B+C)A=\begin{bmatrix}7 & 0\\1 & 7\end{bmatrix}\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}(7)(5)+(0)(-1) & (7)(1)+(0)(4)\\(1)(5)+(7)(-1) & (1)(1)+(7)(4)\end{bmatrix}$$

$$=\begin{bmatrix}35+0 & 7+0\\5-7 & 1+28\end{bmatrix}=\begin{bmatrix}35 & 7\\-2 & 29\end{bmatrix},$$

$$BA=\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}(2)(5)+(-1)(-1) & (2)(1)+(-1)(4)\\(0)(5)+(3)(-1) & (0)(1)+(3)(4)\end{bmatrix}$$

$$=\begin{bmatrix}10+1 & 2-4\\0-3 & 0+12\end{bmatrix}=\begin{bmatrix}11 & -2\\-3 & 12\end{bmatrix}.$$

$$CA=\begin{bmatrix}5 & 1\\1 & 4\end{bmatrix}\begin{bmatrix}5 & 1\\-1 & 4\end{bmatrix}=\begin{bmatrix}23 & 12\\0 & 17\end{bmatrix},$$

$$BA+CA=\begin{bmatrix}11 & -2\\-3 & 12\end{bmatrix}+\begin{bmatrix}23 & 12\\0 & 17\end{bmatrix}=\begin{bmatrix}34 & 10\\-3 & 29\end{bmatrix}.$$

$(B+C)A=BA+CA$ (verified).

3. If $\begin{bmatrix} 4 & a \\ b & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, then find the values of a and b.

Solution

$$\begin{bmatrix} 4 & a \\ b & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 24 + 6a \\ 6b + 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$\Rightarrow 24 + 6a = 6 \quad \Rightarrow 6a = -18$

$\Rightarrow 6b + 18 = 3 \quad \Rightarrow 6b = -15$

$a = -3,\ b = \dfrac{-5}{2}$

4. If $\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$, then find the values of x and y.

Solution

$$\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3 & 0 - 1 \\ y + 6 & 0 - 2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3 & -1 \\ y + 6 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & -2 \end{bmatrix}$$

$\Rightarrow x + 3 = 7 \quad \Rightarrow x = 4,$

$\Rightarrow y + 6 = 4 \quad \Rightarrow y = -2$

EXERCISE 3.5

1. Find the values of each of the determinant.

(i) $\begin{vmatrix} 10 & 5 \\ 4 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} -5 & 8 \\ -3 & -7 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & 8 \\ 0 & 2 \end{vmatrix}$

Solution

(i) $\begin{vmatrix} 10 & 5 \\ 4 & 6 \end{vmatrix}$

$$= (10)(6) - (5)(4) = 60 - 20 = 40.$$

(ii) $\begin{vmatrix} -5 & 8 \\ -3 & -7 \end{vmatrix}$

$$= (-5)(-7) - (8)(-3) = 35 + 24 = 59.$$

(iii) $\begin{vmatrix} 3 & 8 \\ 0 & 2 \end{vmatrix}$

$$= (3)(2) - (8)(0) = 6.$$

2. Find whether the following matrices are singular or non-singular.

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 13 & 5 \\ 7 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution

$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $|A| = 10 - 9 = 1 \neq 0$ Non-singular

$B = \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}$, $|B| = 42 - 42 = 0$ (singular).

$C = \begin{bmatrix} 13 & 5 \\ 7 & 3 \end{bmatrix}$, $|C| = 39 - 35 = 4 \neq 0$ Non-singular

$D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $|D| = 0$ (singular).

3. Find the value of x when $A=\begin{bmatrix} x & 6 \\ 5 & 15 \end{bmatrix}$ is a singular matrix.

Solution

$$A=\begin{bmatrix} x & 6 \\ 5 & 15 \end{bmatrix}.$$

$$|A| = 15x - 30 = 0.$$

$$\Longrightarrow \; x = 2.$$

4. Find the adjoint of the following matrices:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B=\begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}, \quad C=\begin{bmatrix} -3 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution

$$\text{Adj}A=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{Adj}B=\begin{bmatrix} 7 & 2 \\ -3 & 5 \end{bmatrix}, \text{Adj}C=\begin{bmatrix} 2 & -5 \\ 3 & -3 \end{bmatrix}$$

5. Find multiplicative inverse of the following matrices:

(i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 40 & 8 \\ 5 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$ (v) $\begin{bmatrix} 10 & 8 \\ 3 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$

Solution

$A^{-1}=\frac{1}{\det(A)}\cdot \text{adj}(A)$. (i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $A^{-1}=\frac{1}{25}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}=\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$.	$A^{-1}=\frac{1}{\det(A)}\cdot \text{adj}(A)$. (ii) $\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$, $\det=-8-56=-64$. $A^{-1}=\frac{1}{-64}\begin{bmatrix} 2 & -8 \\ -7 & -4 \end{bmatrix}$.

$A^{-1}=\frac{1}{\det(A)}\cdot\operatorname{adj}(A)$. (iii) $\begin{bmatrix}40 & 8\\ 5 & 2\end{bmatrix}$, $\det=80-40=40$. $A^{-1}=\frac{1}{40}\begin{bmatrix}2 & -8\\ -5 & 40\end{bmatrix}$.	$A^{-1}=\frac{1}{\det(A)}\cdot\operatorname{adj}(A)$. (iv) $\begin{bmatrix}3 & 5\\ 5 & -3\end{bmatrix}$, $\det=-9-25=-34$. $A^{-1}=\frac{1}{-34}\begin{bmatrix}-3 & -5\\ -5 & 3\end{bmatrix}$.
$A^{-1}=\frac{1}{\det(A)}\cdot\operatorname{adj}(A)$. (v) $\begin{bmatrix}10 & 8\\ 3 & 3\end{bmatrix}$, $\det=30-24=6$. $A^{-1}=\frac{1}{6}\begin{bmatrix}3 & -8\\ -3 & 10\end{bmatrix}$.	$A^{-1}=\frac{1}{\det(A)}\cdot\operatorname{adj}(A)$. (vi) $\begin{bmatrix}-2 & -3\\ 4 & 5\end{bmatrix}$, $\det=-10+12=2$. $A^{-1}=\frac{1}{2}\begin{bmatrix}5 & 3\\ -4 & -2\end{bmatrix}$.

6. If $A=\begin{bmatrix}5 & -3\\ 2 & -1\end{bmatrix}$, then find A^{-1} and prove that $AA^{-1}=A^{-1}A=I$.

Solution

$$\det(A)=-5+6=1.$$

$$A^{-1}=\begin{bmatrix}-1 & 3\\ -2 & 5\end{bmatrix}.$$

$$AA^{-1}=\begin{bmatrix}5 & -3\\ 2 & -1\end{bmatrix}\begin{bmatrix}-1 & 3\\ -2 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

$$A^{-1}A=\begin{bmatrix}-1 & 3\\ -2 & 5\end{bmatrix}\begin{bmatrix}5 & -3\\ 2 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

7. Show that the following matrices are multiplicative inverse of each other.

(i) $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix}, \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix}$

Solution: two matrix are multiplicative inverse of each if their product is identity

(i)	(ii)
$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$	$A = \begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix}$
$AB = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$AB = \begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Prove that $(AB)^{-1} = B^{-1}A^{-1}$, if

(i) $A = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$

Solution

(i) $A = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$

$$|A| = (-3)(6) - (-2)(5) = -18 + 10 = -8$$

$$|B| = (2)(2) - (-1)(-3) = 4 - 3 = 1$$

$$AB = \begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -8 & 7 \end{bmatrix}$$

$$|AB| = (0)(7) - (-1)(-8) = -8$$

$$A^{-1} = \frac{1}{-8} \begin{bmatrix} 6 & 2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{-8} \begin{bmatrix} 7 & 1 \\ 8 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \frac{1}{-8} \begin{bmatrix} 6 & 2 \\ -5 & -3 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 7 & 1 \\ 8 & 0 \end{bmatrix}$$

Prove that $(AB)^{-1} = B^{-1}A^{-1}$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$.

$$|A| = (1)(0) - (2)(8) = -16$$

$$|B| = (0)(2) - (-1)(5) = 5$$

$$AB = \begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 0 & -8 \end{bmatrix}$$

$$|AB| = (10)(-8) - (3)(0) = -80$$

$$A^{-1} = \frac{1}{-16} \begin{bmatrix} 0 & -2 \\ -8 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -5 & 0 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{-80} \begin{bmatrix} -8 & -3 \\ 0 & 10 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -5 & 0 \end{bmatrix} \cdot \frac{1}{-16} \begin{bmatrix} 0 & -2 \\ -8 & 1 \end{bmatrix}$$

Prove that $(AB)^{-1} = B^{-1}A^{-1}$

EXERCISE 3.6

1. Solve by matrix inversion method, if possible.

(i) $2x + 5y = 19$
$4x - 3y = -1$

(ii) $3x + 2y = 7$
$5x - y = 16$

(iii) $x - 2y = 9$
$2x + 7y = -4$

(iv) $3x + 2y = 2$
$x - 2y = -2$

Solution

(i) $2x + 5y = 19$, $4x - 3y = -1$.

Matrix form

$$\begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ -1 \end{bmatrix}.$$

Determinant

$$|A| = (2)(-3) - (5)(4) = -26.$$

Inverse

$$A^{-1} = \frac{1}{-26} \begin{bmatrix} -3 & -5 \\ -4 & 2 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{-26} \begin{bmatrix} -3 & -5 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 19 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$x = 2$, $y = 3$.

(ii) $3x + 2y = 7$, $5x - y = 16$.

Matrix form

$$\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}.$$

Determinant

$$|A| = (3)(-1) - (2)(5) = -13.$$

Inverse

$$A^{-1} = \frac{1}{-13} \begin{bmatrix} -1 & -2 \\ -5 & 3 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{-13} \begin{bmatrix} -1 & -2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 16 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$x = 3$, $y = -1$.

(iii) $x-2y=9$, $2x+7y=-4$.

Matrix form

$$\begin{bmatrix}1 & -2\\2 & 7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}9\\-4\end{bmatrix}.$$

Determinant

$$|A|=(1)(7)-(-2)(2)=11.$$

Inverse

$$A^{-1}=\frac{1}{11}\begin{bmatrix}7 & 2\\-2 & 1\end{bmatrix}.$$

Solution

$$\begin{bmatrix}x\\y\end{bmatrix}=A^{-1}B=\frac{1}{11}\begin{bmatrix}7 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}9\\-4\end{bmatrix}=\begin{bmatrix}5\\-2\end{bmatrix}.$$

$x=5$, $y=-2$.

(iv) $3x+2y=2$, $x-2y=-2$.

Matrix form

$$\begin{bmatrix}3 & 2\\1 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\-2\end{bmatrix}.$$

Determinant

$$|A|=(3)(-2)-(2)(1)=-8.$$

Inverse

$$A^{-1}=\frac{1}{-8}\begin{bmatrix}-2 & -2\\-1 & 3\end{bmatrix}.$$

Solution

$$\begin{bmatrix}x\\y\end{bmatrix}=A^{-1}B=\frac{1}{-8}\begin{bmatrix}-2 & -2\\-1 & 3\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}.$$

$x=0$, $y=1$.

2. Use Cramer's rule to solve the following pair of linear equations, if possible.

(i) $x+4y=4$, $2x-y=5$ (ii) $x+2y=7$, $3x-2y=-3$

(iii) $2x-5y=-6$, $4x-3y=-12$ (iv) $3x+2y=-1$, $5x+6y=5$

Solution

(i) $x+4y=4$, $2x-y=5$.

$$\mathbf{A}=\begin{bmatrix}1 & 4\\2 & -1\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}4\\5\end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix}x\\y\end{bmatrix}$$

$$\mathbf{A}_x=\begin{bmatrix}4 & 4\\5 & -1\end{bmatrix}$$

$$\mathbf{A}_y=\begin{bmatrix}1 & 4\\2 & 5\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 4\\2 & -1\end{vmatrix}=-9$$

$$|A_x|=\begin{vmatrix}4 & 4\\5 & -1\end{vmatrix}=-24$$

$$|A_y|=\begin{vmatrix}1 & 4\\2 & 5\end{vmatrix}=-3$$

$$x=\frac{|A_x|}{|A|}=\frac{-24}{-9}=\frac{8}{3}$$

$$y=\frac{|A_y|}{|A|}=\frac{-3}{-9}=\frac{1}{3}$$

(ii) $x+2y=7,\ 3x-2y=-3.$

$$\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & -2\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 2\\ 3 & -2\end{vmatrix}=-8$$

$$\mathbf{B}=\begin{bmatrix}7\\ -3\end{bmatrix}$$

$$|A_x|=\begin{vmatrix}7 & 2\\ -3 & -2\end{vmatrix}=-8$$

$$\mathbf{X}=\begin{bmatrix}x\\ y\end{bmatrix}$$

$$|A_y|=\begin{vmatrix}1 & 7\\ 3 & -3\end{vmatrix}=-24$$

$$\mathbf{A}_x=\begin{bmatrix}7 & 2\\ -3 & -2\end{bmatrix}$$

$$x=\frac{|A_x|}{|A|}=\frac{-8}{-8}=1$$

$$\mathbf{A}_y=\begin{bmatrix}1 & 7\\ 3 & -3\end{bmatrix}$$

$$y=\frac{|A_y|}{|A|}=\frac{-24}{-8}=3$$

(iii) $2x-5y=-6,\ 4x-3y=-12.$

$$\mathbf{A}=\begin{bmatrix}2 & -5\\ 4 & -3\end{bmatrix}$$

$$|A|=\begin{vmatrix}2 & -5\\ 4 & -3\end{vmatrix}=14$$

$$\mathbf{B}=\begin{bmatrix}-6\\ -12\end{bmatrix}$$

$$|A_x|=\begin{vmatrix}-6 & -5\\ -12 & -3\end{vmatrix}=-42$$

$$\mathbf{X}=\begin{bmatrix}x\\ y\end{bmatrix}$$

$$|A_y|=\begin{vmatrix}2 & -6\\ 4 & -12\end{vmatrix}=0$$

$$\mathbf{A}_x=\begin{bmatrix}-6 & -5\\ -12 & -3\end{bmatrix}$$

$$x=\frac{|A_x|}{|A|}=\frac{-42}{14}=-3$$

$$\mathbf{A}_y=\begin{bmatrix}2 & -6\\ 4 & -12\end{bmatrix}$$

$$y=\frac{|A_y|}{|A|}=\frac{0}{14}=0$$

(iv) $3x+2y=-1,\ 5x+6y=5.$

$$\mathbf{A}=\begin{bmatrix}3 & 2\\ 5 & 6\end{bmatrix}$$

$$|A|=\begin{vmatrix}3 & 2\\ 5 & 6\end{vmatrix}=8$$

$$\mathbf{B}=\begin{bmatrix}-1\\ 5\end{bmatrix}$$

$$|A_x|=\begin{vmatrix}-1 & 2\\ 5 & 6\end{vmatrix}=-16$$

$$\mathbf{X}=\begin{bmatrix}x\\ y\end{bmatrix}$$

x$$

$$|A_y|=\begin{vmatrix}3 & -1\\ 5 & 5\end{vmatrix}=20$$

$$\mathbf{A}_x=\begin{bmatrix}-1 & 2\\ 5 & 6\end{bmatrix}$$

$$x=\frac{|A_x|}{|A|}=\frac{-16}{8}=-2$$

$$\mathbf{A}_y=\begin{bmatrix}3 & -1\\ 5 & 5\end{bmatrix}$$

$$y=\frac{|A_y|}{|A|}=\frac{20}{8}=\frac{5}{2}$$

3. An electrical engineer wants to determine the current in two branches A and B of a simple electrical circuit. The system of the equations is:

$$x + y = 7$$
$$2x - y = 2$$

where x is the current in branch A and y is the current in branch B. Find x and y by using matrices.

Solution

$x + y = 7$, $2x - y = 2$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$\mathbf{A}_x = \begin{bmatrix} 7 & 1 \\ 2 & -1 \end{bmatrix}$$

$$|A_x| = \begin{vmatrix} 7 & 1 \\ 2 & -1 \end{vmatrix} = -9$$

$$\mathbf{A}_y = \begin{bmatrix} 1 & 7 \\ 2 & 2 \end{bmatrix}$$

$$|A_y| = \begin{vmatrix} 1 & 7 \\ 2 & 2 \end{vmatrix} = -12$$

$$x = \frac{|A_x|}{|A|} = \frac{-9}{-3} = 3$$

$$y = \frac{|A_y|}{|A|} = \frac{-12}{-3} = 4$$

4. Three forces act on a particle and must be in equilibrium i.e. $F_1 + F_2 + F_3 = 0$, where $F_1 = \begin{bmatrix} 8 \\ x \end{bmatrix}, F_2 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}, F_3 = \begin{bmatrix} y \\ -1 \end{bmatrix}$. Find the value of x and y.

Solution

$$F_1 + F_2 + F_3 = 0$$

$$\begin{bmatrix} 8 \\ x \end{bmatrix} + \begin{bmatrix} -2 \\ -7 \end{bmatrix} + \begin{bmatrix} y \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives two equations:

$$8 - 2 + y = 0$$
$$x - 7 - 1 = 0$$

$$y = -6$$
$$x = 8$$

5. Two support beams, A and B are holding up a combined load of 100 kN. Twice the load on beam A and three times the load on beam B equals 240 kN. Find the load of beam A and beam B by using matrices.

Solution

Let A and B be loads on beams.

Equations:

$$A + B = 100$$
$$2A + 3B = 240$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$\mathbf{A}_A = \begin{bmatrix} 100 & 1 \\ 240 & 3 \end{bmatrix}$$

$$D_A = \begin{vmatrix} 100 & 1 \\ 240 & 3 \end{vmatrix} = 60$$

$$\mathbf{A}_B = \begin{bmatrix} 1 & 100 \\ 2 & 240 \end{bmatrix}$$

$$D_B = \begin{vmatrix} 1 & 100 \\ 2 & 240 \end{vmatrix} = 40$$

$$A = \frac{D_A}{D} = 60$$

$$B = \frac{D_B}{D} = 40$$

6. In a 2D game world, two characters are moving along straight paths. One character moves along a line where the total of twice their horizontal position and vertical position is 5, while the other moves along a line where their horizontal position is one more than their vertical position. Find their point of intersection by using matrices.

Solution

$$2x + y = 5$$
$$x - y = 1$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$\mathbf{A}_x = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|A_x| = \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = -6$$

$$\mathbf{A}_y = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$$

$$|A_y| = \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = -3$$

$$x = \frac{|A_x|}{|A|} = 2$$

$$y = \frac{|A_y|}{|A|} = 1$$

7. Two years ago a man was 5 times as old as his son was. After 6 years he will be 3 times as old as his son. Find their present ages by using matrices.

Solution

$$\begin{bmatrix}1 & -5\\ 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-8\\ 12\end{bmatrix}.$$

$$\Delta=\begin{vmatrix}1 & -5\\ 1 & -3\end{vmatrix}=2,$$
$$\Delta_x=\begin{vmatrix}-8 & -5\\ 12 & -3\end{vmatrix}=84,$$
$$\Delta_y=\begin{vmatrix}1 & -8\\ 1 & 12\end{vmatrix}=20.$$

$$x=\frac{84}{2}=42,$$
$$y=\frac{20}{2}=10.$$

8. Two cyclists are 44 km apart and start out at the same time. If they go towards one another they meet in 2 hours, but if they go in the same direction the faster overtakes the slower in $7\frac{1}{2}$ hours. Find their speeds by using matrices.

Solution

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}22\\ \frac{88}{15}\end{bmatrix}.$$

$$\Delta=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2,$$
$$\Delta_{v_1}=\begin{vmatrix}22 & 1\\ \frac{88}{15} & -1\end{vmatrix}=-26.867,$$
$$\Delta_{v_2}=\begin{vmatrix}1 & 22\\ 1 & \frac{88}{15}\end{vmatrix}=-15.467.$$

$$v_1=\frac{-26.867}{-2}=13.433,$$
$$v_2=\frac{-15.467}{-2}=7.733.$$

REVIEW EXERCISE 3

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) If $\begin{bmatrix} a+2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, then $a =$

(a) 3 ✓(b) 5 (c) 6 (d) 7

(ii) $A = [3 \quad 5 \quad 0]$ is a ______ matrix.

✓(a) row (b) square (c) column (d) null

(iii) $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a ______ matrix.

(a) identity (b) square (c) row ✓(d) column

(iv) $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a/an ______ matrix.

(a) rectangular ✓(b) identity (c) column (d) row

(v) If $A' = -A$, then A is ______ matrix.

(a) symmetric (b) row
(c) rectangular ✓(d) skew-symmetric

(vi) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $A + B =$

(a) $[21 \quad 32]$ (b) $[24 \quad 28]$
✓(c) $[10 \quad 12]$ (d) $[11 \quad 11]$

(vii) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $B - A =$

(a) $[10 \quad 12]$ (b) $[11 \quad 11]$
✓(c) $[4 \quad 4]$ (d) $[-4 \quad -4]$

(viii) What is the additive inverse of $\begin{bmatrix}3\\4\end{bmatrix}$?

(a) $\begin{bmatrix}-3\\4\end{bmatrix}$ ✓(b) $\begin{bmatrix}-3\\-4\end{bmatrix}$ (c) $\begin{bmatrix}3\\4\end{bmatrix}$ (d) $\begin{bmatrix}-1\\1\end{bmatrix}$

(ix) If $A=\begin{bmatrix}3 & 1 & 3\\2 & 0 & 2\end{bmatrix}$, then order of A^t is:

✓(a) 3-by-2 (b) 2-by-3 (c) 3-by-3 (d) 2-by-2

(x) $\begin{vmatrix}3 & 1\\0 & 4\end{vmatrix}=$

(a) 11 ✓(b) 12 (c) 13 (d) −11

2. If $A=\begin{bmatrix}4 & 2\\7 & 6\end{bmatrix}$, $B=\begin{bmatrix}5 & 5\\8 & 8\end{bmatrix}$, then find

(i) $(A-B)^t$ (ii) B^t-A^t (iii) $2A+3B$

Solution

(i) $(A-B)^t$

$$A-B=\begin{bmatrix}4-5 & 2-5\\7-8 & 6-8\end{bmatrix}=\begin{bmatrix}-1 & -3\\-1 & -2\end{bmatrix}.$$

$$(A-B)^t=\begin{bmatrix}-1 & -1\\-3 & -2\end{bmatrix}.$$

(ii) B^t-A^t

$$B^t=\begin{bmatrix}5 & 8\\5 & 8\end{bmatrix},\quad A^t=\begin{bmatrix}4 & 7\\2 & 6\end{bmatrix}.$$

$$B^t-A^t=\begin{bmatrix}5-4 & 8-7\\5-2 & 8-6\end{bmatrix}=\begin{bmatrix}1 & 1\\3 & 2\end{bmatrix}.$$

(iii) $2A+3B$

$$2A=\begin{bmatrix}8 & 4\\14 & 12\end{bmatrix}.$$

$$3B=\begin{bmatrix}15 & 15\\24 & 24\end{bmatrix}.$$

$$2A+3B=\begin{bmatrix}8+15 & 4+15\\14+24 & 12+24\end{bmatrix}=\begin{bmatrix}23 & 19\\38 & 36\end{bmatrix}.$$

3. If $A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$, $B=\begin{bmatrix}5 & 6\\ 2 & 1\end{bmatrix}$, $C=\begin{bmatrix}3 & 2\\ 1 & 7\end{bmatrix}$, then verify that

(i) $2(A+B)=2A+2B$ (ii) $(A+B)+C=A+(B+C)$

(iii) $(A+B)C=AC+BC$ (iv) $C(A-B)=CA-CB$

(v) $(AB)^{-1}=B^{-1}A^{-1}$ (vi) $AA^{-1}=A^{-1}A=I$

(vii) $(AB)^t=B^tA^t$ (viii) $(AB)C=A(BC)$

Solution

(i) $2(A+B)=2A+2B$

$$A+B=\begin{bmatrix}2+5 & 3+6\\ 4+2 & 5+1\end{bmatrix}=\begin{bmatrix}7 & 9\\ 6 & 6\end{bmatrix}$$

Now multiply by 2:

$$2(A+B)=\begin{bmatrix}14 & 18\\ 12 & 12\end{bmatrix}$$

Now find $2A$ and $2B$:

$$2A=\begin{bmatrix}4 & 6\\ 8 & 10\end{bmatrix}, \qquad 2B=\begin{bmatrix}10 & 12\\ 4 & 2\end{bmatrix}$$

Add them:

$$2A+2B=\begin{bmatrix}14 & 18\\ 12 & 12\end{bmatrix}$$

Hence,

$$2(A+B)=2A+2B$$

(ii) $(A+B)+C=A+(B+C)$

$$A+B=\begin{bmatrix}2+5 & 3+6\\ 4+2 & 5+1\end{bmatrix}=\begin{bmatrix}7 & 9\\ 6 & 6\end{bmatrix}$$

Then

$$(A+B)+C=\begin{bmatrix}7+3 & 9+2\\ 6+1 & 6+7\end{bmatrix}=\begin{bmatrix}10 & 11\\ 7 & 13\end{bmatrix}$$

Now find $B+C$:

$$B+C=\begin{bmatrix}5+3 & 6+2\\ 2+1 & 1+7\end{bmatrix}=\begin{bmatrix}8 & 8\\ 3 & 8\end{bmatrix}$$

Then

$$A+(B+C)=\begin{bmatrix}2+8 & 3+8\\ 4+3 & 5+8\end{bmatrix}=\begin{bmatrix}10 & 11\\ 7 & 13\end{bmatrix}$$

Hence,

$$(A+B)+C=A+(B+C)$$

(iii) $(A+B)C = AC + BC$

$$A+B = \begin{bmatrix} 2+5 & 3+6 \\ 4+2 & 5+1 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 6 \end{bmatrix}$$

$$(A+B)C = \begin{bmatrix} 7 & 9 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$$

$$(A+B)C = \begin{bmatrix} 30 & 77 \\ 24 & 54 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 25 \\ 17 & 43 \end{bmatrix}$$

$$BC = \begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 21 & 52 \\ 7 & 11 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 9 & 25 \\ 17 & 43 \end{bmatrix} + \begin{bmatrix} 21 & 52 \\ 7 & 11 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 30 & 77 \\ 24 & 54 \end{bmatrix}$$

Both sides are equal

(iv) $C(A-B) = CA - CB$

$$A - B = \begin{bmatrix} -3 & -3 \\ 2 & 4 \end{bmatrix}.$$

$$C(A-B) = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} -3 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 11 & 25 \end{bmatrix}.$$

$$CA = \begin{bmatrix} 14 & 19 \\ 29 & 38 \end{bmatrix},$$

$$CB = \begin{bmatrix} 19 & 20 \\ 18 & 13 \end{bmatrix}.$$

$$CA - CB = \begin{bmatrix} -5 & -1 \\ 11 & 25 \end{bmatrix}.$$

Both sides are equal.

(v) $(AB)^{-1} = B^{-1}A^{-1}$

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 15 \\ 30 & 29 \end{bmatrix}.$$

$$\det(AB) = (16)(29) - (15)(30) = 14.$$

$$(AB)^{-1} = \frac{1}{14}\begin{bmatrix} 29 & -15 \\ -30 & 16 \end{bmatrix} = \begin{bmatrix} \frac{29}{14} & \frac{-15}{14} \\ \frac{-30}{14} & \frac{16}{14} \end{bmatrix}.$$

$$\det(A) = (2)(5) - (3)(4) = -2,$$
$$\det(B) = (5)(1) - (6)(2) = -7.$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix},$$
$$B^{-1} = \frac{1}{-7}\begin{bmatrix} 1 & -6 \\ -2 & 5 \end{bmatrix}.$$

$$B^{-1}A^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{6}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{bmatrix}\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{29}{14} & \frac{-15}{14} \\ \frac{-30}{14} & \frac{16}{14} \end{bmatrix}.$$

Both sides are equal

(vi) $AA^{-1} = A^{-1}A = I$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$|A| = (2)(5) - (3)(4) = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Both sides are equal

(vii) $(AB)^t = B^t A^t$	(viii) $(AB)C = A(BC)$
$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10+6 & 12+3 \\ 20+10 & 24+5 \end{bmatrix}$ $(AB)^t = \begin{bmatrix} 16 & 30 \\ 15 & 29 \end{bmatrix}$, $B^t = \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$, $A^t = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$, $B^tA^t = \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 10+6 & 20+10 \\ 12+3 & 24+5 \end{bmatrix}$ Both sides are equal	$AB = \begin{bmatrix} 16 & 15 \\ 30 & 29 \end{bmatrix}$, $BC = \begin{bmatrix} 21 & 52 \\ 7 & 11 \end{bmatrix}$ $(AB)C = \begin{bmatrix} 16 & 15 \\ 30 & 29 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 63 & 137 \\ 119 & 263 \end{bmatrix}$ $A(BC) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 21 & 52 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} 63 & 137 \\ 119 & 263 \end{bmatrix}$ Both sides are equal

4. If $A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, then find

(i) $|B|$ (ii) $\text{Adj}\, B$ (iii) A^{-1}
(iv) $A^{-1}A$ (v) $(AB)^t$ (vi) $(B^t)^t$

Solution

(i) $\|B\|$ $\|B\| = (5)(4) - (3)(2) = 20 - 6 = 14.$	(ii) Adj B $\text{Adj } B = \begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}$.
(iii) A^{-1} $\|A\| = (7)(1) - (3)(2) = 7 - 6 = 1,$ $A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$.	(iv) $A^{-1}A$ $\|A\| = (7)(1) - (3)(2) = 7 - 6 = 1,$ $A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$. $A^{-1}A = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(v) $(AB)^t$ $AB = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 41 & 33 \\ 12 & 10 \end{bmatrix}$ $(AB)^t = \begin{bmatrix} 41 & 33 \\ 12 & 10 \end{bmatrix}^t = \begin{bmatrix} 41 & 12 \\ 33 & 10 \end{bmatrix}$	(vi) $(B^t)^t$ $B = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$ $B^t = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}^t = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ $(B^t)^t = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} = B$

5. Use matrix inversion method and Cramer's rule to solve the following pair of linear equations, if possible:

(i) $3x+4y=7$, $5x-y=2$ (ii) $x-6y=-15$, $2x+6y=-3$ (iii) $2x+y=5$, $x+3y=3$

Solution

<table>
<tr><th>Matrix inversion method:</th><th>Cramer's Rule:</th></tr>
<tr><td>

i) $\begin{cases}3x+4y=7\\5x-y=2\end{cases}$

$A=\begin{bmatrix}3&4\\5&-1\end{bmatrix},\ B=\begin{bmatrix}7\\2\end{bmatrix}$

$|A|=-23$

$A^{-1}=\frac{1}{-23}\begin{bmatrix}-1&-4\\-5&3\end{bmatrix}$

$X=A^{-1}B$

$=\frac{1}{-23}\begin{bmatrix}-1&-4\\-5&3\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix}$

$=\frac{1}{-23}\begin{bmatrix}-7-8\\-35+6\end{bmatrix}$

$=\frac{1}{-23}\begin{bmatrix}-15\\-29\end{bmatrix}$

$=\begin{bmatrix}15/23\\29/23\end{bmatrix}$

</td><td>

i) $\begin{cases}3x+4y=7\\5x-y=2\end{cases}$

$|A|=\begin{vmatrix}3&4\\5&-1\end{vmatrix}$

$=3(-1)-4(5)=-23$

$|A_x|=\begin{vmatrix}7&4\\2&-1\end{vmatrix}$

$=7(-1)-4(2)=-15$

$|A_y|=\begin{vmatrix}3&7\\5&2\end{vmatrix}$

$=3(2)-7(5)=-29$

$x=\frac{-15}{-23}=\frac{15}{23}$

$y=\frac{-29}{-23}=\frac{29}{23}$

</td></tr>
</table>

Matrix inversion method:	**Cramer's Rule:**
ii) $\begin{cases} x - 6y = -15 \\ 2x + 6y = -3 \end{cases}$ $A = \begin{bmatrix} 1 & -6 \\ 2 & 6 \end{bmatrix}, \ B = \begin{bmatrix} -15 \\ -3 \end{bmatrix}$ $\lvert A \rvert = 18$ $A^{-1} = \frac{1}{18}\begin{bmatrix} 6 & 6 \\ -2 & 1 \end{bmatrix}$ $X = A^{-1}B$ $= \frac{1}{18}\begin{bmatrix} 6 & 6 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -15 \\ -3 \end{bmatrix}$ $= \frac{1}{18}\begin{bmatrix} -90 - 18 \\ 30 - 3 \end{bmatrix}$ $= \frac{1}{18}\begin{bmatrix} -108 \\ 27 \end{bmatrix}$ $= \begin{bmatrix} -6 \\ 3/2 \end{bmatrix}$	**ii)** $\begin{cases} x - 6y = -15 \\ 2x + 6y = -3 \end{cases}$ $\lvert A \rvert = \begin{vmatrix} 1 & -6 \\ 2 & 6 \end{vmatrix}$ $= 1(6) - (-6)(2) = 18$ $\lvert A_x \rvert = \begin{vmatrix} -15 & -6 \\ -3 & 6 \end{vmatrix}$ $= (-15)(6) - (-6)(-3) = -108$ $\lvert A_y \rvert = \begin{vmatrix} 1 & -15 \\ 2 & -3 \end{vmatrix}$ $= 1(-3) - (-15)(2) = 27$ $x = \frac{-108}{18} = -6$ $y = \frac{27}{18} = \frac{3}{2}$
Matrix inversion method:	**Cramer's Rule:**
iii) $\begin{cases} 2x + y = 5 \\ x + 3y = 3 \end{cases}$ $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ $\lvert A \rvert = 5$ $A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ $X = A^{-1}B$ $= \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ $= \frac{1}{5}\begin{bmatrix} 15 - 3 \\ -5 + 6 \end{bmatrix}$ $= \frac{1}{5}\begin{bmatrix} 12 \\ 1 \end{bmatrix}$ $= \begin{bmatrix} 12/5 \\ 1/5 \end{bmatrix}$	**iii)** $\begin{cases} 2x + y = 5 \\ x + 3y = 3 \end{cases}$ $\lvert A \rvert = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$ $= 2(3) - 1(1) = 5$ $\lvert A_x \rvert = \begin{vmatrix} 5 & 1 \\ 3 & 3 \end{vmatrix}$ $= 5(3) - 1(3) = 12$ $\lvert A_y \rvert = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$ $= 2(3) - 5(1) = 1$ $x = \frac{12}{5}$ $y = \frac{1}{5}$

6. Find two numbers by using matrices such that twice the first added to the second makes 21 and twice the second added to the first makes 27.

Solution

$$2x + y = 21,$$
$$x + 2y = 27.$$

$$3x + 3y = 48 \implies x + y = 16.$$

Solving,

$$x = 5,$$
$$y = 11.$$

Numbers are **5** and **11**.

7. 4 knives and 6 forks cost Rs. 136, whereas 6 knives and 5 forks cost Rs. 164. Find the cost of a knife and a fork by using matrices.

Solution

Let x = knife cost, y = fork cost.

$$4x + 6y = 136,$$
$$6x + 5y = 164.$$

Matrix form $AX = B$,

$$A = \begin{bmatrix} 4 & 6 \\ 6 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 136 \\ 164 \end{bmatrix}.$$

Using Cramer's rule,

$$\Delta = \begin{vmatrix} 4 & 6 \\ 6 & 5 \end{vmatrix} = 20 - 36 = -16,$$
$$\Delta_x = \begin{vmatrix} 136 & 6 \\ 164 & 5 \end{vmatrix} = -304,$$
$$\Delta_y = \begin{vmatrix} 4 & 136 \\ 6 & 164 \end{vmatrix} = -160.$$

$$x = \frac{-304}{-16} = 19,$$
$$y = \frac{-160}{-16} = 10.$$

Knife = **Rs.19**, fork = **Rs.10**.

8. A shop employs, 5 men and 3 women, pays total daily wages Rs. 3500. If the number of men is reduced to 2 and 3 extra women are taken on, the daily wages amount to Rs. 5000. Find daily wages of a man and a woman by using matrices.

Solution

Let m = man's wage, w = woman's wage.

$$5m + 3w = 3500,$$
$$2m + 6w = 5000.$$

Matrix form $AX = B$,

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3500 \\ 5000 \end{bmatrix}.$$

Using Cramer's rule,

$$\Delta = \begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} = 24,$$
$$\Delta_m = \begin{vmatrix} 3500 & 3 \\ 5000 & 6 \end{vmatrix} = 6000,$$
$$\Delta_w = \begin{vmatrix} 5 & 3500 \\ 2 & 5000 \end{vmatrix} = 18000.$$

$$m = \frac{6000}{24} = 250,$$
$$w = \frac{18000}{24} = 750.$$

Man's wage = **Rs.250**, woman's wage = **Rs.750**.

UNIT 4

Functions and Graphs

EXERCISE 4.1

1. If $f(x) = 2x + 5$, $g(x) = 3x - 2$, then find

(i) $f(x) + g(x)$ (ii) $f(x) - g(x)$

Solution

(i) $f(x) + g(x)$ $f(x) + g(x) = (2x+5) + (3x-2)$ $= 2x + 3x + 5 - 2$ $= 5x + 3.$	(ii) $f(x) - g(x)$ $f(x) - g(x) = (2x+5) - (3x-2)$ $= 2x + 5 - 3x + 2$ $= -x + 7.$

2. If $f(x) = x + 2$, $g(x) = 2x + 4$, then find

(i) $f(x) \cdot g(x)$ (ii) $g(x) \cdot f(x)$ (iii) $\dfrac{f(x)}{g(x)}$ (iv) $\dfrac{g(x)}{f(x)}$

Solution

(i) $f(x) \cdot g(x)$ $f(x) \cdot g(x) = (x+2)(2x+4)$ $= 2x^2 + 4x + 4x + 8$ $= 2x^2 + 8x + 8.$	(ii) $g(x) \cdot f(x)$ $g(x) \cdot f(x) = f(x) \cdot g(x) = 2x^2 + 8x + 8.$
(iii) $\frac{f(x)}{g(x)}$ $\frac{f(x)}{g(x)} = \frac{x+2}{2x+4} = \frac{x+2}{2(x+2)} = \frac{1}{2}.$	(iv) $\frac{g(x)}{f(x)}$ $\frac{g(x)}{f(x)} = \frac{2x+4}{x+2} = \frac{2(x+2)}{x+2} = 2.$

3. For the functions f and g, find

(a) $(fog)(x)$ (b) $(gof)(x)$ (c) $(fof)(x)$ (d) $(gog)(x)$

Where,

(i) $f(x) = 2x + 3$; $g(x) = x^3$ (ii) $f(x) = \frac{2}{x}, x \neq 0; g(x) = 2x^2 - 1$

(iii) $f(x) = 2x - 1$; $g(x) = \frac{x+1}{2}$

Solution

(i) $f(x) = 2x + 3$; $g(x) = x^3$	
(a) $(fog)(x)$ $(fog)(x) = f(g(x))$ $= f(x^3)$ $= 2(x^3) + 3$ $= 2x^3 + 3.$	(b) $(gof)(x)$ $(gof)(x) = g(f(x))$ $= g(2x + 3)$ $= (2x + 3)^3.$
(c) $(fof)(x)$ $(fof)(x) = f(f(x))$ $= f(2x + 3)$ $= 2(2x + 3) + 3$ $= 4x + 6 + 3$ $= 4x + 9.$	(d) $(gog)(x)$ $(gog)(x) = g(g(x))$ $= g(x^3)$ $= (x^3)^3$ $= x^9.$
(ii) $f(x) = \frac{2}{x}$, $x \neq 0$; $g(x) = 2x^2 - 1$	
(a) $(fog)(x)$ $(fog)(x) = f(g(x))$ $= f(2x^2 - 1)$ $= \frac{2}{2x^2 - 1}.$	(b) $(gof)(x)$ $(gof)(x) = g(f(x))$ $= g(\frac{2}{x})$ $= 2(\frac{2}{x})^2 - 1$ $= \frac{8}{x^2} - 1.$
(c) $(fof)(x)$ $(fof)(x) = f(f(x))$ $= f(\frac{2}{x})$ $= \frac{2}{\frac{2}{x}}$ $= x.$	(d) $(gog)(x)$ $(gog)(x) = g(g(x))$ $= g(2x^2 - 1)$ $= 2(2x^2 - 1)^2 - 1$ $= 2(4x^4 - 4x^2 + 1) - 1$ $= 8x^4 - 8x^2 + 1.$

(iii) $f(x) = 2x - 1 \quad ; \quad g(x) = \frac{x+1}{2}$	
(a) $(fog)(x)$ $(fog)(x) = f(g(x)) = f(\frac{x+1}{2}) = 2(\frac{x+1}{2}) - 1 = x + 1 - 1 = x.$	(b) $(gof)(x)$ $(gof)(x) = g(f(x)) = g(2x-1) = \frac{(2x-1)+1}{2} = x.$
(c) $(fof)(x)$ $(fof)(x) = f(f(x)) = f(2x-1) = 2(2x-1) - 1 = 4x - 2 - 1 = 4x - 3.$	(d) $(gog)(x)$ $(gog)(x) = g(g(x)) = g(\frac{x+1}{2}) = \frac{\frac{x+1}{2}+1}{2} = \frac{x+1+2}{4} = \frac{x+3}{4}.$

4. Find the value of k, such that $(fog)(x) = (gof)(x)$, where $f(x) = 3x + 2$, $g(x) = 6x - k$.

Solution

$$(fog)(x) = f(g(x)) = 3(6x - k) + 2 = 18x - 3k$$
$$(gof)(x) = g(f(x)) = 6(3x + 2) - k = 18x + 12$$

Equating,

$$\begin{aligned} 18x - 3k + 2 &= 18x + 12 - k \\ -3k + 2 &= 12 - k \\ -2k &= 10 \\ k &= -5. \end{aligned}$$

5. Given that $f(x) = 3x + 2$ and $g(x) = 2x + 3$. Find

(i) $g(f(4))$ (ii) $f(f(3))$ (iii) $f(g(-2))$

Solution

(i) $g(f(4))$

$$f(4) = 3(4) + 2 = 14,$$
$$g(f(4)) = g(14) = 2(14) + 3 = 31.$$

(ii) $f(f(3))$

$$f(3) = 3(3) + 2 = 11,$$
$$f(f(3)) = f(11) = 3(11) + 2 = 35.$$

(iii) $f(g(-2))$

$$g(-2) = 2(-2) + 3 = -1,$$
$$f(g(-2)) = f(-1) = 3(-1) + 2 = -1.$$

6. Find $f^{-1}(x)$ in each of the following:

(i) $f(x) = 2x - 3$ (ii) $f(x) = 4x^3 - 1$

(iii) $f(x) = \sqrt{x-1}, x \geq 1$ (iv) $f(x) = \frac{x+1}{3x-2}, x \neq \frac{2}{3}$

Solution

(i) $f(x) = 2x - 3$ $y = 2x - 3$ $2x = y + 3$ $x = \frac{y+3}{2}$ $f^{-1}(y) = \frac{y+3}{2}$ $f^{-1}(x) = \frac{x+3}{2}$	(ii) $f(x) = 4x^3 - 1$ $y = 4x^3 - 1$ $4x^3 = y + 1$ $x^3 = \frac{y+1}{4}$ $x = \sqrt[3]{\frac{y+1}{4}}$ $f^{-1}(y) = \sqrt[3]{\frac{y+1}{4}}$ $f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}$

(iii) $f(x)=\sqrt{x-1}$, $x\geq 1$	(iv) $f(x)=\frac{x+1}{3x-2}$, $x\neq\frac{2}{3}$
$y=\sqrt{x-1}$ $y^2=x-1$ $x=y^2+1.$ Thus, $f^{-1}(x)=x^2+1, \quad x\geq 0.$	$y=\dfrac{x+1}{3x-2}$ $y(3x-2)=x+1$ $3yx-2y=x+1$ $3yx-x=2y+1$ $x(3y-1)=2y+1$ $x=\dfrac{2y+1}{3y-1}.$ Thus, $f^{-1}(x)=\frac{2x+1}{3x-1}, \quad x\neq\frac{1}{3}.$

7. The functions f and g are defined such that $f(x) = 4x + 2$ and $g(x) = 6x - 18$.
(i) Find $f^{-1}(x)$ and $g^{-1}(x)$. (ii) Find x if $f^{-1}(x) = g^{-1}(x)$.

Solution

(i) Find $f^{-1}(x)$ and $g^{-1}(x)$.	(ii) Find x if $f^{-1}(x)=g^{-1}(x)$.
$f(x)=4x+2$ $y=4x+2$ $4x=y-2$ $x=\dfrac{y-2}{4}.$ $f^{-1}(x)=\frac{x-2}{4}.$ $g(x)=6x-18$ $y=6x-18$ $6x=y+18$ $x=\dfrac{y+18}{6}.$ $g^{-1}(x)=\frac{x+18}{6}.$	$\dfrac{x-2}{4}=\dfrac{x+18}{6}$ $6(x-2)=4(x+18)$ $6x-12=4x+72$ $6x-4x=72+12$ $2x=84$ $x=42.$

8. Verify that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(i) $f(x) = x - 6$ (ii) $f(x) = 7x - 4$

(iii) $f(x) = \dfrac{x-3}{4}$ (iv) $f(x) = \dfrac{x-4}{x+2},\ x \neq -2$

Solution

(i) $f(x) = x - 6$ $f^{-1}(x) = x + 6.$ $f(f^{-1}(x)) = f(x+6) = (x+6) - 6 = x,$ $f^{-1}(f(x)) = f^{-1}(x-6) = (x-6) + 6 = x.$	(ii) $f(x) = 7x - 4$ $f^{-1}(x) = \frac{x+4}{7}.$ $f(f^{-1}(x)) = f(\frac{x+4}{7}) = 7(\frac{x+4}{7}) - 4 = x,$ $f^{-1}(f(x)) = f^{-1}(7x-4) = \frac{7x-4+4}{7} = x.$
(iii) $f(x) = \frac{x-3}{4}$ $f^{-1}(x) = 4x + 3.$ $f(f^{-1}(x)) = f(4x+3) = \frac{4x+3-3}{4} = x,$ $f^{-1}(f(x)) = f^{-1}(\frac{x-3}{4}) = 4(\frac{x-3}{4}) + 3 = x.$	(iv) $f(x) = \frac{x-4}{x+2},\ x \neq -2$ $f^{-1}(x) = \frac{2x+4}{1-x}, \quad x \neq 1.$ $f(f^{-1}(x)) = f(\frac{2x+4}{1-x}) = \frac{\frac{2x+4}{1-x} - 4}{\frac{2x+4}{1-x} + 2} = x,$ $f^{-1}(f(x)) = f^{-1}(\frac{x-4}{x+2}) = x.$

9. Without finding $f^{-1}(x)$, find domain and range of $f^{-1}(x)$:

(i) $f(x) = 12x - 3$ (ii) $f(x) = \dfrac{1}{2}x + 8$

(iii) $f(x) = \dfrac{x}{1+x},\ x \neq -1$ (iv) $f(x) = \sqrt{x-2},\ x \geq 2$

Solution

(i) Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$ **(ii)** Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$

(iii) Domain = $(-\infty, 1) \cup (1, \infty)$; Range = $(-\infty, -1) \cup (-1, \infty)$

(iv) Domain = $[0, \infty)$; Range = $[2, \infty)$

10. Given that $f(x) = x^2 + 9$ and $g(x) = x + 21$. Find the values of a such that: $f(a) = g(a)$

Solution

$$f(a) = g(a)$$
$$a^2 + 9 = a + 21$$
$$a^2 - a - 12 = 0$$
$$(a - 4)(a + 3) = 0.$$

Thus,

$$a = 4 \quad \text{or} \quad a = -3.$$

EXERCISE 4.2

1. Plot graph for the following absolute valued functions:

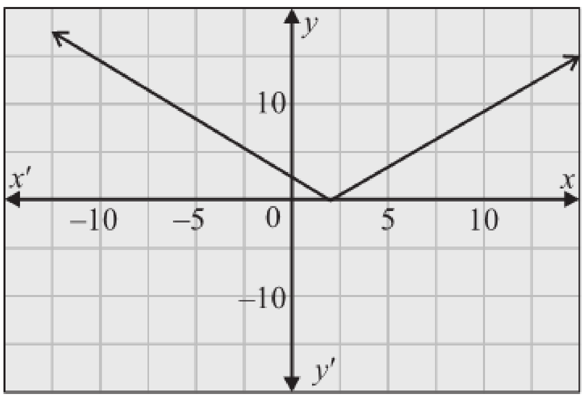
(i) $f(x) = |x - 2|$ (ii) $f(x) = 3|x + 3| - 4$

(iii) $f(x) = 5|x|$ (iv) $f(x) = |x + 2| + 3$

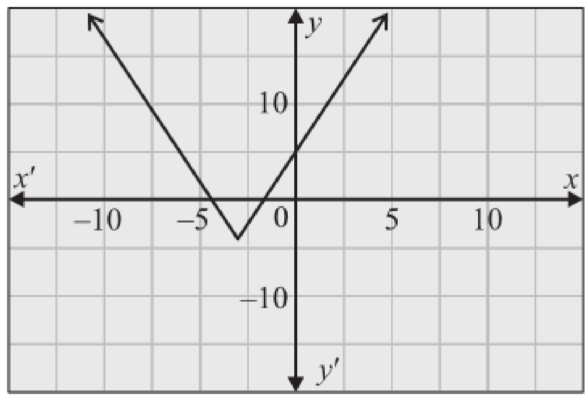
(v) $f(x) = 2|x + 4| - 3$ (vi) $f(x) = 2|x + 1| - 6$

Solution

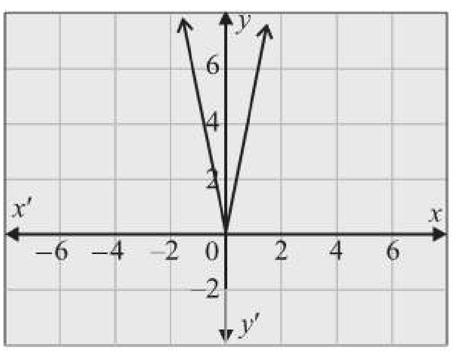
(i)



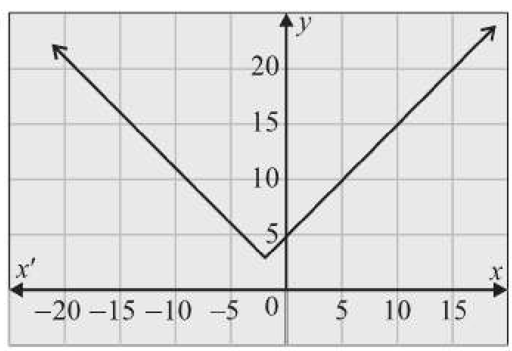
(ii)



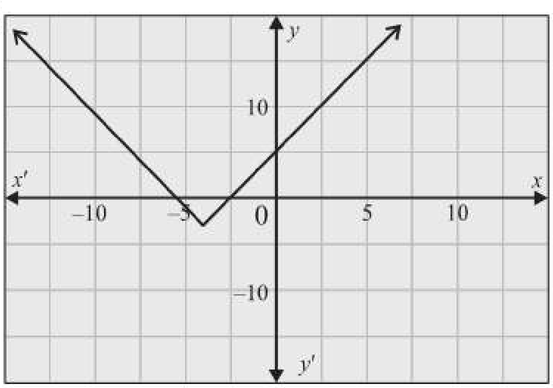
(iii)



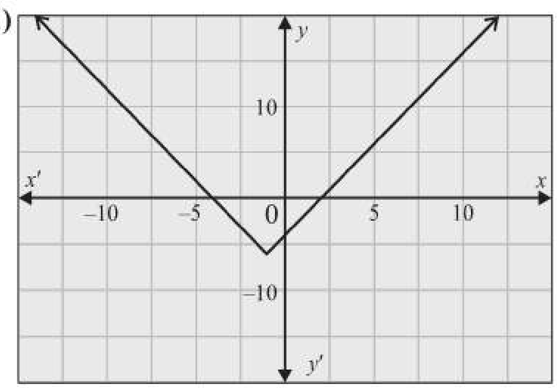
(iv)



(v)



(vi)



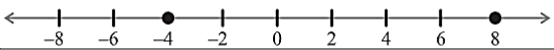
2. Solve and express the solution on number line:

(i) $|x-2|=6$ (ii) $|2x+1|=7$ (iii) $|4x-9|=3$ (iv) $|7-2x|=1$

Solution

(i) $|x-2|=6$

$x-2=\pm 6 \implies x=8$ or $x=-4$.



(ii) $|2x+1|=7$

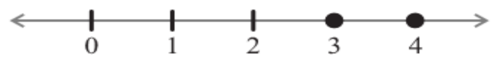
$2x+1=\pm 7 \implies x=3$ or $x=-4$.

(iii) $|4x-9|=3$

$4x-9=\pm 3 \implies x=3$ or $x=\frac{3}{2}$.

(iv) $|7-2x|=1$

$7-2x=\pm 1 \implies x=3$ or $x=4$.



3. Solve and express the solution on number line:

(i) $|5x+8|\le 3$ (ii) $|4x-12|\le 0$

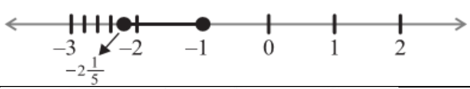
(iii) $|3-4x|>0$ (iv) $1-2\left|\frac{3}{2}x-5\right|>-3$

(v) $|3x-2|\le 12$ (vi) $|1-2x|>5$

Solution

(i) $|5x+8|\le 3$

$$-3\le 5x+8\le 3$$
$$-3-8\le 5x\le 3-8$$
$$-11\le 5x\le -5$$
$$\frac{-11}{5}\le x\le -1.$$
$$\left[-2\frac{1}{5},-1\right]$$

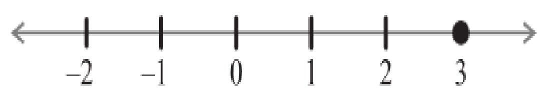


2. (ii) $|4x-12|\le 0$

$4x-12=0$ (absolute value can only be zero)

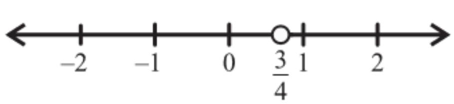
$4x=12$

$x=3.$



(iii) $|3-4x|>0$

$$3-4x\ne 0$$
$$x\ne\frac{3}{4}.$$
$$\left(-\infty,\frac{3}{4}\right)\cup\left(\frac{3}{4},\infty\right)$$

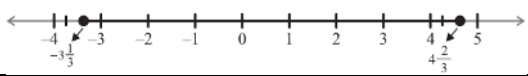


(iv) $1-2|\frac{3}{2}x-5|>-3$

$$1-2|\frac{3}{2}x-5|>-3$$
$$-2|3x-10|>-4$$
$$|3x-10|<2$$
$$-2<3x-10<2$$
$$8<3x<12$$
$$\frac{8}{3}<x<4.$$

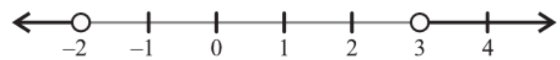
(v) $|3x-2| \le 12$

$$-12 \le 3x-2 \le 12$$
$$-12+2 \le 3x \le 12+2$$
$$-10 \le 3x \le 14$$
$$\frac{-10}{3} \le x \le \frac{14}{3}.$$
$$\left[-3\frac{1}{3}, 4\frac{2}{3}\right]$$



(vi) $|1-2x| > 5$

$$1-2x < -5 \quad \text{or} \quad 1-2x > 5$$
$$-2x < -6 \quad \text{or} \quad -2x > 4$$
$$x > 3 \quad \text{or} \quad x < -2.$$
$$(-\infty, -2) \cup (3, \infty)$$



EXERCISE 4.3

1. A function $B(t) = 5{,}000 + 200t$ represents the total balance (in rupees) after t months. What will be the balance after 6 months?

Solution

$$B(t) = 5{,}000 + 200t$$

$$B(6) = 5\,000 + 200(6) = 6\,200$$ rupees

2. A function $f(k) = 150 + 20k$ represents the total fare (in rupees) for k kilometres. How much will the fare be for a 12 kilometres ride?

Solution

$$f(k) = 150 + 20k$$

$$f(12) = 150 + 20(12) = 390$$ rupees

3. The cost of manufacturing fancy sofa set would be fixed charges Rs. 5500 which is modeled as $f(n) = 5500\,n$, where n is the number of sofa sets. Find the cost of 50 sofa sets.

Solution

$$f(n) = 5500\,n$$

$f(50) = 5500(50) = 275000$ rupees

4. A function $T(d) = \dfrac{d}{60}$ represents the time T in hours to travel a distance d kilometres. How long will it take to travel 180 km?

Solution

$$T(d) = \frac{d}{60}$$

$T(180) = \frac{180}{60} = 3$ hours

5. A company charges Rs. 100 for an encoding work. In addition, the company charges Rs. 5 per page of printed output. The model of function $f(x) = 100 + 5x$, where x represents the number of pages printed out. How much will company charge for 55 page encoding and printing work?

Solution

$$f(x) = 100 + 5x$$

$f(55) = 100 + 5(55) = 375$ rupees

6. A chemical reaction is stable at $37°C$. The process must be stopped if the temperature deviates by more than $2.5°C$. The condition is modeled as: $|T - 37°| > 2.5°$, T be the temperature. For what temperature values must the process be stopped?

Solution

$$|T - 37°| > 2.5°,$$

$$T - 37 < -2.5 \quad \text{or} \quad T - 37 > 2.5$$

$$\Rightarrow \quad T < 34.5 \quad \text{or} \quad T > 39.5.$$

7. A factory produces metal rods that must be 2.5 metres long, with a tolerance of ± 0.04 metres. An absolute value inequality models this: $|x - 2.5| \le 0.04$. What is the range of acceptable lengths?

Solution

$$|x - 2.5| \le 0.04$$

$$2.5 - 0.04 \le x \le 2.5 + 0.04$$

$$\Rightarrow \quad 2.46 \le x \le 2.54$$

8. A machine part must be aligned so that its centre is exactly at 0. If it shifts more than 0.1 mm, the part is rejected. The model is given by $|x| > 0.1$. What positions of the centre cause rejection?

Solution

$$|x| > 0.1$$

$$x < -0.1 \quad \text{or} \quad x > 0.1.$$

Rejection for shifts beyond ± 0.1 mm.

REVIEW EXERCISE 4

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) If $f(x) = \frac{5x-6}{3}$, then what is the value of $f(3)$?

(a) −1 ✓(b) 3 (c) 9 (d) 15

(ii) A function f from X to Y is represented by:

(a) $f: XY$ (b) $f: Y \rightarrow X$ ✓(c) $f: X \rightarrow Y$ (d) $f: \frac{X}{Y}$

(iii) $(fog)(x) =$

(a) $(f+g)(x)$ (b) $(f-g)(x)$ ✓(c) $f(g(x))$ (d) $f(x) \div g(x)$

(iv) If $f(x) = 2x + 3$, $g(x) = x + 1$, then $f(x) + g(x) =$

(a) $3x$ ✓(b) $3x + 4$ (c) 4 (d) $2x^2 + 3$

(v) If $f(x) = 5x + 2$, $h(x) = 2x - 2$, then $f(x) - h(x) =$

(a) $3x$ (b) $5x^2 - 4$ ✓(c) $3x + 4$ (d) $-3x - 4$

(vi) If $f(x) = 3x + 1$, $g(x) = 2x$, then $g(x) \times f(x) =$

(a) $6x + 2x$ (b) $5x^2 + 1$ (c) $x + 1$ ✓(d) $6x^2 + 2x$

(vii) If $f(x) = x^2 - 4$, $g(x) = x + 2, x \neq -2$, then $\frac{f(x)}{g(x)} =$

(a) $\frac{1}{x-2}$ (b) $\frac{1}{x+2}$ (c) $x + 2$ ✓(d) $x - 2$

(viii) What is the shape of the graph of an absolute value function?

(a) U-shaped ✓(b) V-shaped
(c) L-shaped (d) M-shaped

(ix) If a graph represents a function, then every vertical line must intersect it at:

(a) 4 points (b) 3 points (c) 2 points ✓(d) 1 point

(x) If $f(x) = x^3$, then $f(-2) =$

✓(a) −8 (v) 8 (c) 4 (d) −6

2. If $f(x) = 25 - x^2$ and $g(x) = 5 + x$, then find

(i) $f(x) + g(x)$ (ii) $f(x) - g(x)$ (iii) $f(x) \cdot g(x)$

(iv) $\dfrac{f(x)}{g(x)}$ (v) $f(7)$ (vi) $g(-8)$

Solution

(i) $f(x)+g(x)=(25-x^2)+(5+x)=30+x-x^2$.	(ii) $f(x)-g(x)=(25-x^2)-(5+x)=20-x-x^2$.
(iii) $f(x)\cdot g(x)=(25-x^2)(5+x)=125+25x-5x^2$	(iv) $\dfrac{f(x)}{g(x)}=\dfrac{25-x^2}{5+x}=\dfrac{(5-x)(5+x)}{5+x}=5-x$
(v) $f(7)=25-7^2=25-49=-24$	(vi) $g(-8)=5+(-8)=-3$.

3. If $f(x) = x^3$ and $g(x) = 14 + 2x$, then find

(i) $(fog)(x)$ (ii) $(gof)(x)$

(iii) $(fof)(x)$ (iv) $(gog)(x)$

Solution

(i) $(fog)(x)=f(g(x))=(14+2x)^3$.

(ii) $(gof)(x)=g(f(x))=14+2(x^3)=14+2x^3$.

(iii) $(fof)(x)=f(f(x))=(x^3)^3=x^9$.

(iv)

$(gog)(x)=g(g(x))=14+2(14+2x)=42+4x$.

4. Find $f^{-1}(x)$, if

(i) $f(x) = 9x - 1$

(ii) $f(x) = \dfrac{5}{x-1},\ x \neq 1$

(iii) $f(x) = \sqrt{x-5},\ x \geq 5$

(iv) $f(x) = \dfrac{3-x}{2}$

Solution

(i) $f(x) = 9x - 1$ $y = 9x - 1$ $9x = y + 1$ $x = \frac{y+1}{9}$ $f^{-1}(y) = \frac{y+1}{9}$ $f^{-1}(x) = \frac{x+1}{9}$	(ii) $f(x) = \frac{5}{x-1}$ $y = \frac{5}{x-1}$ $yx - y = 5$ $yx = 5 + y$ $x = \frac{5+y}{y}$ $f^{-1}(y) = \frac{5+y}{y}$ $f^{-1}(x) = \frac{5+x}{x}$
(iii) $f(x) = \sqrt{x-5}$ $y = \sqrt{x-5}$ $y^2 = x - 5$ $x = y^2 + 5$ $f^{-1}(y) = y^2 + 5$ $f^{-1}(x) = x^2 + 5$	(iv) $f(x) = \frac{3-x}{2}$ $y = \frac{3-x}{2}$ $2y = 3 - x$ $x = 3 - 2y$ $f^{-1}(y) = 3 - 2y$ $f^{-1}(x) = 3 - 2x$

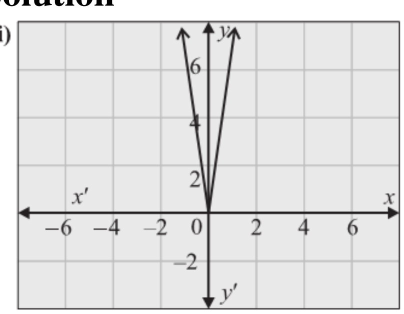
5. Plot graph for the following absolute valued functions:

(i) $f(x) = 7|x|$

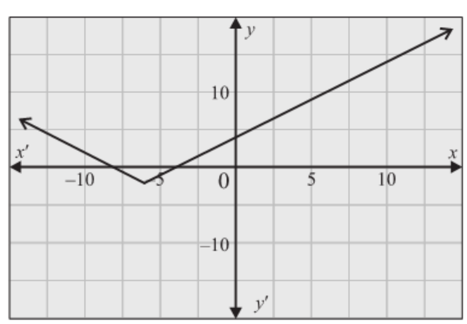
(ii) $f(x) = |x+6| - 2$

Solution

(i)



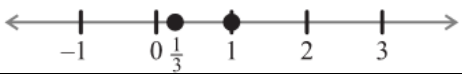
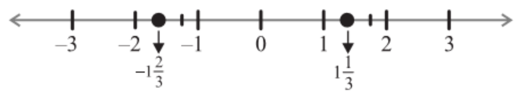
(ii)



6. Solve and express the solution on number line:

(i) $|3x-2|=1$ (ii) $|6x+1|=9$

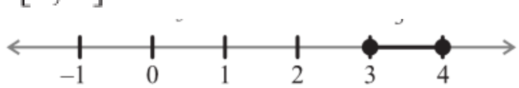
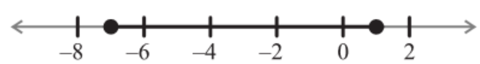
Solution

(i) $\lvert 3x-2\rvert=1$. $3x-2=1$ or $3x-2=-1$ $x=1$ or $x=\frac{1}{3}$. $\left\{\frac{1}{3},1\right\}$ 	(ii) $\lvert 6x+1\rvert=9$. $6x+1=9$ or $6x+1=-9$ $x=\frac{4}{3}$ or $x=-\frac{5}{3}$. $\left\{-1\frac{2}{3},1\frac{1}{3}\right\}$ 

7. Solve and express the solution on number line:

(i) $|7-2x|\le 1$ (ii) $|6x+18|\le 24$

Solution

(i) $\lvert 7-2x\rvert\le 1$. $-1\le 7-2x\le 1$ $-8\le -2x\le -6$ $3\le x\le 4$. [3, 4] 	(ii) $\lvert 6x+18\rvert\le 24$. $-24\le 6x+18\le 24$ $-42\le 6x\le 6$ $-7\le x\le 1$. [–7, 1] 

8. Given that $f(x)=3x+7$, find

(i) $f^{-1}(x)$ (ii) the value of x for which $f(x)=f^{-1}(x)$

Solution

(i) $f(x)=3x+7$ $y=3x+7\Rightarrow 3x=y-7$ $x=\frac{y-7}{3}$ $f^{-1}(y)=\frac{y-7}{3}$ $f^{-1}(x)=\frac{x-7}{3}$	(i) $f(x)=f^{-1}(x)$ $3x+7=\frac{x-7}{3}$ $9x+21=x-7$ $9x-x=-7-21$ $8x=-28$ $x=-\frac{7}{2}$

9. A company earns a profit of Rs. 100 for each item sold after a fixed monthly expense of Rs. 10,000. A function $P(x) = 100x - 10{,}000$ represents the profit after selling x items. How many items must be sold to break even (profit = 0)?

Solution

$P(x) = 100x - 10\,000$.

$$P(x) = 0$$
$$100x - 10\,000 = 0$$
$$x = \frac{10\,000}{100} = 100.$$

100 items must be sold to break even.

10. A store offers a 15% discount on a product. A function $D(p) = 0.85p$ gives the selling price after the discount on an original price p. What is the selling price of an item that originally costs Rs. 2,000?

Solution

$D(p) = 0.85p$.

For an original price $p = 2\,000$,

$$D(p) = 0.85 \times 2\,000 = 1\,700.$$

The selling price after discount is **Rs. 1,700**.

11. A GPS system is considered accurate if the actual position differs from the reported position by no more than 6 metres. If the actual location is at point $x = 100$, the allowed range is defined as: $|x - r| \leq 6$, where r is the reported location. What is the range of acceptable reported locations?

Solution

GPS accuracy range $|r - A| \leq 6$ with

actual location $A = 100$.

$$|r - 100| \leq 6$$
$$-6 \leq r - 100 \leq 6$$
$$94 \leq r \leq 106.$$

The acceptable reported locations are in the interval **[94, 106]**. 📍📊

UNIT 5

Algebraic Fractions

EXERCISE 5.1

1. Reduce the following rational expressions to lowest forms:

(i) $\dfrac{3a^2-6ab}{2a^2b-4ab^2}$ (ii) $\dfrac{abx+bx^2}{acx+cx^2}$ (iii) $\dfrac{ac}{a^2x^2-ax}$

(iv) $\dfrac{15a^2b^2c}{100(a^2-a^2b)}$ (v) $\dfrac{4x^2-9y^2}{4x^2+6xy}$ (vi) $\dfrac{20(x^3-y^3)}{5x^2+5xy+5y^2}$

(vii) $\dfrac{x(2a^2-3ax)}{a(4a^2x-9x^3)}$ (viii) $\dfrac{x^2-5x}{x^2-4x-5}$ (ix) $\dfrac{3x^2+6x}{x^2+4x+4}$

(x) $\dfrac{x^2+xy-2y^2}{x^3-y^3}$ (xi) $\dfrac{2x^2+17x+21}{3x^2+26x+35}$

Solution

(i) $\dfrac{3a^2-6ab}{2a^2b-4ab^2}$ $=\dfrac{3a(a-2b)}{2ab(a-2b)}$ $=\dfrac{3}{2b}$	(ii) $\dfrac{abx+bx^2}{acx+cx^2}$ $=\dfrac{bx(a+x)}{cx(a+x)}$ $=\dfrac{b}{c}$
(iii) $\dfrac{ac}{a^2x^2-ax}$ $=\dfrac{ac}{ax(ax-1)}$ $=\dfrac{c}{x(ax-1)}$	(iv) $\dfrac{15a^2b^2c}{100(a^2-a^2b)}$ $=\dfrac{15a^2b^2c}{100a^2(1-b)}$ $=\dfrac{3b^2c}{20(1-b)}$

(v) $\frac{4x^2-9y^2}{4x^2+6xy}$ $=\frac{(2x)^2-(3y)^2}{2x(2x+3y)}$ $=\frac{(2x-3y)(2x+3y)}{2x(2x+3y)}$ $=\frac{2x-3y}{2x}$	(vi) $\frac{20(x^3-y^3)}{5x^2+5xy+5y^2}$ $=\frac{20(x-y)(x^2+xy+y^2)}{5(x^2+xy+y^2)}$ $= 4(x-y)$
(vii) $\frac{x(2a^2-3ax)}{a(4a^2x-9x^3)}$ $=\frac{ax(2a-3x)}{ax(4a^2-9x^2)}$ $=\frac{(2a-3x)}{(2a)^2-(3x)^2}$ $=\frac{(2a-3x)}{(2a-3x)(2a+3x)}$ $=\frac{1}{(2a+3x)}$	(viii) $\frac{x^2-5x}{x^2-4x-5}$ $=\frac{x(x-5)}{x^2-5x+x-5}$ $=\frac{x(x-5)}{x(x-5)+1(x-5)}$ $=\frac{x(x-5)}{(x-5)(x+1)}$ $=\frac{x}{(x+1)}$
(ix) $\frac{3x^2+6x}{x^2+4x+4}$ $=\frac{3x(x+2)}{x^2+2x+2x+4}$ $=\frac{3x(x+2)}{x(x+2)+2(x+2)}$ $=\frac{3x(x+2)}{(x+2)(x+2)}$ $=\frac{3x}{(x+2)}$	(x) $\frac{x^2+xy-2y^2}{x^3-y^3}$ $=\frac{x^2+2xy-xy-2y^2}{(x-y)(x^2+xy+y^2)}$ $=\frac{x(x+2y)-y(x+2y)}{(x-y)(x^2+xy+y^2)}$ $=\frac{(x+2y)(x-y)}{(x-y)(x^2+xy+y^2)}$ $=\frac{x+2y}{x^2+xy+y^2}$
(xi) $\frac{2x^2+17x+21}{3x^2+26x+35}=\frac{2x^2+14x+3x+21}{3x^2+21x+5x+35}=\frac{2x(x+7)+3(x+7)}{3x(x+7)+5(x+7)}=\frac{(x+7)(2x+3)}{(x+7)(3x+5)}$ $=\frac{2x+3}{3x+5}$	

EXERCISE 5.2

(i) $\dfrac{3}{x-y}+\dfrac{1}{y-x}$

$=\dfrac{3}{(x-y)}+\dfrac{1}{-(x-y)}$

$=\dfrac{3}{(x-y)}-\dfrac{1}{(x-y)}$

$=\dfrac{3-1}{(x-y)}$

$=\dfrac{2}{(x-y)}$

(ii) $\dfrac{4x}{x^2-3x+2}-\dfrac{4}{1-x}-\dfrac{5}{x-2}$

$=\dfrac{4x}{x^2-2x-x+2}-\dfrac{4}{1-x}-\dfrac{5}{x-2}$

$=\dfrac{4x}{x(x-2)-1(x-2)}-\dfrac{4}{1-x}-\dfrac{5}{x-2}$

$=\dfrac{4x}{(x-2)(x-1)}+\dfrac{4}{(x-1)}-\dfrac{5}{(x-2)}$

$=\dfrac{4x+4(x-2)-5(x-1)}{(x-2)(x-1)}$

$=\dfrac{4x+4x-8-5x+5}{(x-2)(x-1)}$

$=\dfrac{4x+4x-5x-8+5}{(x-2)(x-1)}$

$=\dfrac{3x-3}{(x-2)(x-1)}$

$=\dfrac{3\cancel{(x-1)}}{(x-2)\cancel{(x-1)}}$

$=\dfrac{3}{(x-2)}$

(iii) $\dfrac{x+y}{12x-6y}+\dfrac{x-y}{18x-9y}$

$=\dfrac{x+y}{6(2x-y)}+\dfrac{x-y}{9(2x-y)}$

$=\dfrac{9(x+y)+6(x-y)}{54(2x-y)}$

$=\dfrac{9x+9y+6x-6y}{54(2x-y)}$

$=\dfrac{15x+3y}{54(2x-y)}$

$=\dfrac{3(5x+y)}{54(2x-y)}$

$=\dfrac{5x+y}{18(2x-y)}$

(iv) $\dfrac{2}{x+2y}-\dfrac{x-6y}{x^2-4y^2}$

$=\dfrac{2}{x+2y}-\dfrac{x-6y}{(x)^2-(2y)^2}$

$=\dfrac{2}{x+2y}-\dfrac{x-6y}{(x-2y)(x+2y)}$

$=\dfrac{2(x-2y)-(x-6y)}{(x+2y)(x-2y)}$

$=\dfrac{2x-4y-x+6y}{(x+2y)(x-2y)}$

$=\dfrac{\cancel{(x+2y)}}{\cancel{(x+2y)}(x-2y)}$

$=\dfrac{1}{x-2y}$

(v) $\frac{1}{x+1}-\frac{2}{x+2}-\frac{2x+3}{x^2+3x+2}$

$=\frac{1}{x+1}-\frac{2}{x+2}-\frac{2x+3}{x^2+2x+x+2}$

$=\frac{1}{x+1}-\frac{2}{x+2}-\frac{2x+3}{x(x+2)+1(x+2)}$

$=\frac{1}{x+1}-\frac{2}{x+2}-\frac{2x+3}{(x+2)(x+1)}$

$=\frac{(x+2)-2(x+1)-(2x+3)}{(x+2)(x+1)}$

$=\frac{x+2-2x-2-2x-3}{(x+2)(x+1)}$

$=\frac{-3x-3}{(x+2)(x+1)}$

$=\frac{-3(x+1)}{(x+2)(x+1)}$

$=\frac{-3}{x+2}$

(vi) $\frac{5}{x^2+x-6}-\frac{1}{2x^2-7x+6}$

$=\frac{5}{x^2+3x-2x-6}-\frac{1}{2x^2-3x-4x+6}$

$=\frac{5}{x(x+3)-2(x+3)}-\frac{1}{x(2x-3)-2(2x-3)}$

$=\frac{5}{(x+3)(x-2)}-\frac{1}{(2x-3)(x-2)}$

$=\frac{5(2x-3)-(x+3)}{(2x-3)(x+3)(x-2)}$

$=\frac{10x-15-x-3}{(2x-3)(x+3)(x-2)}$s

$=\frac{9x-18}{(2x-3)(x+3)(x-2)}$

$=\frac{9(x-2)}{(2x-3)(x+3)(x-2)}$

$=\frac{9}{(2x-3)(x+3)}$

(vii) $\frac{2x}{x+y}-\frac{y}{x-y}-\frac{2y^2}{x^2-y^2}$

$=\frac{2x}{x+y}-\frac{y}{x-y}-\frac{2y^2}{(x-y)(x+y)}$

$=\frac{2x(x-y)-y(x+y)-2y^2}{(x-y)(x+y)}$

$=\frac{2x^2-2xy-xy-y^2-2y^2}{(x-y)(x+y)}$

$=\frac{2x^2-3xy-3y^2}{(x-y)(x+y)}$

(viii) $\frac{7}{2x^2-x-6}-\frac{8}{3x^2-4x-4}$

$=\frac{7}{2x^2-4x+3x-6}-\frac{8}{3x^2-6x+2x-4}$

$=\frac{7}{2x(x-2)+3(x-2)}-\frac{8}{3x(x-2)+2(x-2)}$

$=\frac{7}{(x-2)(2x+3)}-\frac{8}{(x-2)(3x+2)}$

$=\frac{7(3x+2)-8(2x+3)}{(3x+2)(2x+3)(x-2)}$

$=\frac{21x+14-16x-24}{(3x+2)(2x+3)(x-2)}$

$=\frac{5x-10}{(3x+2)(2x+3)(x-2)}$

$=\frac{5(x-2)}{(3x+2)(2x+3)(x-2)}$

$=\frac{5}{(3x+2)(2x+3)}$

(ix) $\frac{x+2}{x^2-x-12}-\frac{x}{x^2+6x+9}$

$=\frac{x+2}{x^2-4x+3x-12}-\frac{x}{x^2+3x+3x+9}$

$=\frac{x+2}{x(x-4)+3(x-4)}-\frac{x}{x(x+3)+3(x+3)}$

$=\frac{x+2}{(x-4)(x+3)}-\frac{x}{(x+3)(x+3)}$

$=\frac{(x+2)(x+3)-x(x-4)}{(x-4)(x+3)^2}$

$=\frac{x^2+3x+2x+6-x^2+4x}{(x-4)(x+3)^2}$

$=\frac{9x+6}{(x-4)(x+3)^2}$

$=\frac{3(3x+2)}{(x+3)^2(x-4)}$

(x) $\frac{1}{x^2-4x+3}+\frac{1}{x^2-3x+2}-\frac{1}{x^2-5x+6}$

$=\frac{1}{x^2-3x-x+3}+\frac{1}{x^2-2x-x+2}-\frac{1}{x^2-3x-2x+6}$

$=\frac{1}{x(x-3)-1(x-3)}+\frac{1}{x(x-2)-1(x-2)}-\frac{1}{x(x-3)-2(x-3)}$

$=\frac{1}{(x-3)(x-1)}+\frac{1}{(x-2)(x-1)}-\frac{1}{(x-3)(x-2)}$

$=\frac{(x-2)+(x-3)-(x-1)}{(x-1)(x-2)(x-3)}$

$=\frac{x-2+x-3-x+1}{(x-1)(x-2)(x-3)}$

$=\frac{x-4}{(x-1)(x-2)(x-3)}$

(xi) $\frac{x^3}{x-y}+\frac{y^3}{y-x}=\frac{x^3}{x-y}+\frac{y^3}{-(x-y)}=\frac{x^3}{x-y}-\frac{y^3}{x-y}=\frac{x^3-y^3}{(x-y)}$

$=\frac{(x-y)(x^2+xy+y^2)}{(x-y)}=x^2+xy+y^2$

2. Subtract $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

$=\frac{2x^3+x^2+3}{(x^2+2)^2}-\frac{1}{x^2+2}$

$=\frac{(2x^3+x^2+3)-(x^2+2)}{(x^2+2)^2}$

$=\frac{2x^3+x^2+3-x^2-2}{(x^2+2)^2}$

$=\frac{2x^3+1}{(x^2+2)^2}$

EXERCISE 5.3

(i) $\dfrac{24lm}{5l+10m} \times \dfrac{5l^2-20m^2}{16mn}$

$= \dfrac{24lm}{5(l+2m)} \times \dfrac{5(l^2-4m^2)}{16mn}$

$= \dfrac{24lm}{5(l+2m)} \times \dfrac{5[(l)^2-(2m)^2]}{16mn}$

$= \dfrac{3l}{(l+2m)} \times \dfrac{(l-2m)(l+2m)}{2n}$

$= \dfrac{3l(l-2m)}{2n}$

(ii) $\dfrac{x^2-3x+2}{x^2+3x-4} \times \dfrac{2x^2+8x}{3x+6}$

$= \dfrac{x^2-2x-x+2}{x^2+4x-x-4} \times \dfrac{2x(x+4)}{3(x+2)}$

$= \dfrac{x(x-2)-1(x-2)}{x(x+4)-1(x+4)} \times \dfrac{2x(x+4)}{3(x+2)}$

$= \dfrac{(x-2)(x-1)}{(x+4)(x-1)} \times \dfrac{2x(x+4)}{3(x+2)}$

$= \dfrac{2x(x-2)}{3(x+2)}$

(iii) $\dfrac{a^2-4b^2}{a^2+2ba} \times \dfrac{2a^2+10ab}{a^2+3ab-10b^2}$

$= \dfrac{(a)^2-(2b)^2}{a(a+2b)} \times \dfrac{2a(a+5b)}{a^2-2ab+5ab-10b^2}$

$= \dfrac{(a-2b)(a+2b)}{a(a+2b)} \times \dfrac{2a(a+5b)}{a(a-2b)+5b(a-2b)}$

$= \dfrac{(a-2b)(a+2b)}{a(a+2b)} \times \dfrac{2a(a+5b)}{(a-2b)(a+5b)}$

$= 2$

(iv) $\dfrac{(a+b)^2-c^2}{(a+c)^2-b^2} \times \dfrac{a^2-(b-c)^2}{(a+c)^2-c^2}$

$= \dfrac{(a+b-c)(a+b+c)}{(a+c-b)(a+b+c)} \times \dfrac{(a-b+c)(a+b-c)}{(a+c+c)(a+c-c)}$

$= \dfrac{(a+b-c)^2}{a(a+2c)}$

(v) $\dfrac{x^3-8}{x^2-4} \div \dfrac{x^2+2x+4}{x^2+4x+4}$

$= \dfrac{(x)^3-(2)^3}{(x)^2-(2)^2} \div \dfrac{x^2+2x+4}{x^2+2x+2x+4}$

$= \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} \div \dfrac{x^2+2x+4}{x(x+2)+2(x+2)}$

$= \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} \div \dfrac{x^2+2x+4}{(x+2)(x+2)}$

$= \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} \times \dfrac{(x+2)(x+2)}{(x^2+2x+4)}$

$= x+2$

(vi) $\dfrac{(a^2+ab)^2}{(a^2-ab)^2} \div \left(\dfrac{a+b}{a-b}\right)^2$

$= \dfrac{[a(a+b)]^2}{[a(a-b)]^2} \div \dfrac{(a+b)^2}{(a-b)^2}$

$= \dfrac{a^2(a+b)^2}{a^2(a-b)^2} \times \dfrac{(a-b)^2}{(a+b)^2}$

$= 1$

(vii) $\dfrac{x^2-x-6}{x^2-x-20} \div \left\{\dfrac{x^3-3x^2}{x^2+4x} \times \dfrac{x^3-5x^2-14x}{x^2-12x+35}\right\}$

$= \dfrac{x^2-3x+2x-6}{x^2-5x+4x-20} \div \left\{\dfrac{x^2(x-3)}{x(x+4)} \times \dfrac{x(x^2-5x-14)}{x^2-7x-5x+35}\right\}$

$= \dfrac{x(x-3)+2(x-3)}{x(x-5)+4(x-5)} \div \left\{\dfrac{x^2(x-3)}{(x+4)} \times \dfrac{(x^2-7x+2x-14)}{x(x-7)-5(x-7)}\right\}$

$= \dfrac{(x-3)(x+2)}{(x-5)(x+4)} \div \left\{\dfrac{x^2(x-3)}{(x+4)} \times \dfrac{x(x-7)+2(x-7)}{(x-7)(x-5)}\right\}$

$= \dfrac{(x-3)(x+2)}{(x-5)(x+4)} \div \left\{\dfrac{x^2(x-3)}{(x+4)} \times \dfrac{(x-7)(x+2)}{(x-7)(x-5)}\right\}$

$= \dfrac{(x-3)(x+2)}{(x-5)(x+4)} \div \left\{\dfrac{x^2(x-3)(x+2)}{(x+4)(x-5)}\right\} = \dfrac{(x-3)(x+2)}{(x-5)(x+4)} \times \dfrac{(x+4)(x-5)}{x^2(x-3)(x+2)} = \dfrac{1}{x^2}$

(viii) $\dfrac{3x^2-3xy}{10x^2+10xy-20y^2} \times \dfrac{5xy+10y^2}{6x^2+6xy+6y^2} \div \dfrac{2x^2-2y^2}{4x^3-4y^3}$

$= \dfrac{3x(x-y)}{10(x^2+xy-2y^2)} \times \dfrac{5y(x+2y)}{6(x^2+xy+y^2)} \div \dfrac{2[(x)^2-(y)^2]}{4[(x)^3-(y)^3]}$

$= \dfrac{x(x-y)}{2(x^2+2xy-xy-2y^2)} \times \dfrac{y(x+2y)}{2(x^2+xy+y^2)} \div \dfrac{(x-y)(x+y)}{2(x-y)(x^2+xy+y^2)}$

$= \dfrac{x(x-y)}{2[x(x+2y)-y(x+2y)]} \times \dfrac{y(x+2y)}{2(x^2+xy+y^2)} \times \dfrac{2(x^2+xy+y^2)}{(x+y)}$

$= \dfrac{x(x-y)}{2(x+2y)(x-y)} \times \dfrac{y(x+2y)}{2(x^2+xy+y^2)} \times \dfrac{2(x^2+xy+y^2)}{(x+y)}$

$= \dfrac{xy}{2(x+y)}$

(ix) $\dfrac{2x^2-98}{x^3-125}\times\dfrac{x^2-3x-10}{3x-21}\div\dfrac{x^2+5x-14}{x^2+5x+25}$

$=\dfrac{2(x^2-49)}{(x)^3-(5)^3}\times\dfrac{x^2-5x+2x-10}{3(x-7)}\div\dfrac{x^2+7x-2x-14}{x^2+5x+25}$

$=\dfrac{2[(x)^2-(7)^2]}{(x-5)(x^2+5x+25)}\times\dfrac{x(x-5)+2(x-5)}{3(x-7)}\div\dfrac{x(x+7)-2(x+7)}{x^2+5x+25}$

$=\dfrac{2(x-7)(x+7)}{(x-5)(x^2+5x+25)}\times\dfrac{(x-5)(x+2)}{3(x-7)}\div\dfrac{(x+7)(x-2)}{x^2+5x+25}$

$=\dfrac{2(x-7)(x+7)}{(x-5)(x^2+5x+25)}\times\dfrac{(x-5)(x+2)}{3(x-7)}\times\dfrac{(x^2+5x+25)}{(x+7)(x-2)}$

$=\dfrac{2(x+2)}{3(x-2)}$

(x) $\dfrac{x^2-2x}{2x+6}\times\dfrac{x^2+x-6}{x^2-5x}\times\dfrac{6x-30}{x^2-2x}$

$=\dfrac{x(x-2)}{2(x+3)}\times\dfrac{x^2+3x-2x-6}{x(x-5)}\times\dfrac{6(x-5)}{x(x-2)}$

$=\dfrac{x(x-2)}{2(x+3)}\times\dfrac{x(x+3)-2(x+3)}{x(x-5)}\times\dfrac{6(x-5)}{x(x-2)}$

$=\dfrac{x(x-2)}{2(x+3)}\times\dfrac{(x+3)(x-2)}{x(x-5)}\times\dfrac{6(x-5)}{x(x-2)}$

$=\dfrac{3(x-2)}{x}$

EXERCISE 5.4

1. Solve the following equations:

(i) $5x^4 - 19x^2 + 12 = 0$ (ii) $4x^4 - 27x^2 + 18 = 0$

(iii) $5x^4 - 22x^2 + 8 = 0$ (iv) $4\cdot 2^{2x+1} - 9\cdot 2^x + 1 = 0$

(v) $4^{1+x} + 4^{1-x} - 10 = 0$ (vi) $3^x + 3^{3-x} - 12 = 0$

(vii) $5^{1+3x} + 5^{2-3x} - 126 = 0$ (viii) $\left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 33 = 0$

(ix) $2\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) - 17 = 0$ (x) $2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$

(xi) $(x+1)(x+2)(x-4)(x-5) + 8 = 0$ (xii) $(x+3)(x+4)(x+5)(x+6) = -1$

Solutions

(i) $\mathbf{5x^4 - 19x^2 + 12 = 0} \longrightarrow (1)$

Let $y = x^2$

$y^2 = (x^2)^2$

$y^2 = x^4$

puting the value of y and y^2 in equation 1

$5y^2 - 19y + 12 = 0$

$5y^2 - 15y - 4y + 12 = 0$

$5y(y-3) - 4(y-3) = 0$

$(y-3)(5y-4) = 0$

$y - 3 = 0$, $5y - 4 = 0$

$y = 3$, $5y = 4$ $y = \frac{4}{5}$

since $y = x^2$

$x^2 = 3$, $x^2 = \frac{4}{5}$

$x = \pm\sqrt{3}$, $x = \pm\frac{2}{\sqrt{5}}$

$S.S = \left\{\pm\sqrt{3}, \pm\frac{2}{\sqrt{5}}\right\}$

(ii) $\mathbf{4x^4 - 27x^2 + 18 = 0} \longrightarrow (1)$

Let $y = x^2$

$y^2 = (x^2)^2$

$y^2 = x^4$

puting the value of y and y^2 in equation 1

$4y^2 - 27y + 18 = 0$

$4y^2 - 24y - 3y + 18 = 0$

$4y(y-6) - 3(y-6) = 0$

$(y-6)(4y-3) = 0$

$y - 6 = 0$, $4y - 3 = 0$

$y = 6$ $4y = 3$

$y = \frac{3}{4}$

since $y = x^2$

$x^2 = 6$, $x^2 = \frac{3}{4}$

$x = \pm\sqrt{6}$ $x = \pm\frac{\sqrt{3}}{2}$

$S.S = \left\{\pm\sqrt{6}, \pm\frac{\sqrt{3}}{2}\right\}$

(iii) $5x^4 - 22x^2 + 8 = 0 \longrightarrow (1)$

Let $y = x^2$

$y^2 = (x^2)^2$

$y^2 = x^4$

now equation (1) *become*

$5y^2 - 22y + 8 = 0$

$5y^2 - 20y - 2y + 8 = 0$

$5y(y - 40) - 2(y - 4) = 0$

$(y - 4)(5y - 2) = 0$

$y - 4 = 0 \quad , \quad 5y - 2 = 0$

$y = 4 \quad , \quad 5y = 2$

$y = \frac{2}{5}$

since $x^2 = y$

$x^2 = 4 \quad , \quad x^2 = \frac{2}{5}$

$x = \pm 2 \quad x = \pm\sqrt{\frac{2}{5}}$

$S.S = \left\{\pm 2, \pm\sqrt{\frac{2}{5}}\right\}$

(iv) $4.2^{2x+1} - 9.2^x + 1 = 0$

$4.2^{2x}.2 - 9.2^x + 1 = 0$

$8.2^{2x} - 9.x^2 + 1 = 0 \longrightarrow (1)$

let $y = 2^x$

$y^2 = 2^{2x}$

now equation (1) *become*

$8y^2 - 9y + 1 = 0$

$8y^2 - 8y - y + 1 = 0$

$8y(y - 1) - 1(y - 1) = 0$

$(y - 1)(8y - 1) = 0$

$y - 1 = 0 \quad , \quad 8y - 1 = 0$

$y = 1 \quad 8y = 1$

$y = \frac{1}{8}$

since $2^x = y$

$2^x = 1 \quad , \quad 2^x = \frac{1}{8}$

$2^x = 2^0 \quad 2^x = 2^{-3}$

$x = 0 \quad x = -3$

$s.s = \{0, -3\}$

(v) $4^{1+x} + 4^{1-x} - 10 = 0$

$4.4^x + 4.4^{-x} - 10 = 0$

$4.4^x + 4.\frac{1}{4^x} - 10 = 0$

$let \quad y = 4^x$

$4y + 4\frac{1}{y} - 10 = 0$

$\frac{4y^2 + 4 - 10y}{y} = 0$

$4y^2 - 10y + 4 = 0$

$4y^2 - 8y - 2y + 4 = 0$

$4y(y-2) - 2(y-2) = 0$

$(y-2)(4y-2) = 0$

$y - 2 = 0 \quad , \quad 4y - 2 = 0$

$y = 2 \qquad 4y = 2$

$y = \frac{2}{4}$

$y = \frac{1}{2}$

$since \quad y = 4^x$

$4^x = 2 \quad , \quad 4^x = \frac{1}{2}$

$2^{2x} = 2^1 \qquad 2^{2x} = 2^{-1}$

$2x = 1 \qquad 2x = -1$

$x = \frac{1}{2} \qquad x = \frac{-1}{2}$

$s.s = \left\{\frac{-1}{2}, \frac{1}{2}\right\}$

(vi) $3^x + 3^{3-x} - 12 = 0$

$3^x + 3^3.3^{-x} - 12 = 0$

$3^x + 27.\frac{1}{3^x} - 12 = 0$

$let\ y = 3^x$

$y + 27.\frac{1}{y} - 12 = 0$

$\frac{y^2 + 27 - 12y}{y} = 0$

$y^2 - 12y + 27 = 0$

$y^2 - 9y - 3y + 27 = 0$

$y(y-9) - 3(y-9) = 0$

$(y-9)(y-3) = 0$

$y - 9 = 0 \quad , \quad y - 3 = 0$

$y = 9 \qquad y = 3$

$since\ y = 3^x$

$3^x = 9 \qquad 3^x = 3$

$3^x = 3^2 \qquad 3^x = 3^1$

$x = 2 \qquad x = 1$

$S.S = \{1,2\}$

(vii) $5^{1+3x}+5^{2-3x}-126=0$

$5.5^{3x}+5^2.5^{-3x}-126=0$

$5.5^{3x}+25.\frac{1}{5^{3x}}-126=0$

$let\ y=5^{3x}$

$5y+25\frac{1}{y}-126=0$

$\frac{5y^2+25-126y}{y}=0$

$5y^2-126y+25=0$

$5y^2-125y-y+25=0$

$5y(y-25)-1(y-25)=0$

$(y-25)(5y-1)=0$

$y-25=0 \quad , \quad 5y-1=0$

$y=25 \quad\quad 5y=1$

$y=\frac{1}{5}$

$since\ y=5^{3x}$

$5^{3x}=25 \quad , \quad 5^{3x}=\frac{1}{5}$

$5^{3x}=5^2 \quad\quad 5^{3x}=5^{-1}$

$3x=2 \quad\quad 3x=-1$

$x=\frac{2}{3} \quad\quad x=\frac{-1}{3}$

$S.S=\left\{\frac{-1}{3},\frac{2}{3}\right\}$

(viii) $\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-33=0$

$let\ y=x+\frac{1}{x}$

$(y)^2=\left(x+\frac{1}{x}\right)^2$

$y^2=x^2+\frac{1}{x^2}+2$

$y^2-2=x^2+\frac{1}{x^2}$

$(y^2-2)+2y-33=0$

$y^2+2y-35=0$

$y^2+7y-5y-35=0$

$y(y+7)-5(y+7)=0$

$(y+7)(y-5)=0$

$y+7=0 \quad , \quad y-5=0$

$y=-7 \quad\quad y=5$

$since\ y=x+\frac{1}{x}$

$x+\frac{1}{x}=-7 \quad , \quad x+\frac{1}{x}=5$

$\frac{x^2+1}{x}=-7$

$x^2+1=-7x$

$x^2+7x+1=0$

$a=1,\ b=7\ ,\ c=1$

$x=\frac{-(7)\pm\sqrt{(7)^2-4(1)(1)}}{2(1)}$

$x=\frac{-7\pm\sqrt{49-4}}{2}$

$x=\frac{-7\pm\sqrt{45}}{2}$

$x=\frac{-7\pm3\sqrt{5}}{2}$

Now

$x+\frac{1}{x}=5$

$x^2-5x+1=0$

$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(1)}}{2(1)}$

$x=\frac{5\pm\sqrt{25-4}}{2}$

$x=\frac{5\pm\sqrt{21}}{2}$

$S.S=\left\{\frac{-7\pm3\sqrt{5}}{2},\frac{5\pm\sqrt{21}}{2}\right\}$

(ix) $2\left(x^2+\frac{1}{x^2}\right)-25\left(x-\frac{1}{x}\right)-17=0$

Let $y = x-\frac{1}{x}$

$y^2 = \left(x-\frac{1}{x}\right)^2$

$y^2 = x^2+\frac{1}{x^2}-2$

$y^2+2 = x^2+\frac{1}{x^2}$

$2(y^2+2)-25y-17=0$

$2y^2+4-25y-17=0$

$2y^2-25y-13=0$

$2y^2-26y+y-13=0$

$2y(y-13)+1(y-13)=0$

$(y-13)(2y+1)=0$

$y-13=0 \quad , \quad 2y+1=0$

$y=13 \qquad\qquad y=\frac{-1}{2}$

$since\ y = x-\frac{1}{x}$

$x-\frac{1}{x}=13$

$x^2-13x-1=0$

$a=1 \quad b=-13 \quad c=-1$

$x=\frac{-(-13)\pm\sqrt{(-13)^2-4(1)(-1)}}{2(1)}$

$x=\frac{13\pm\sqrt{169+4}}{2}$

$x=\frac{13\pm\sqrt{173}}{2}$

Now

$y=\frac{-1}{2}$

$x-\frac{1}{x}=\frac{-1}{2}$

$\frac{x^2-1}{x}=\frac{-1}{2}$

$2(x^2-1)=-x$

$2x^2-2=-x$

$2x^2+x-2=0$

$a=2 \quad b=1 \quad c=-2$

$x=\frac{-(1)\pm\sqrt{(1)^2-4(2)(-2)}}{2(2)}$

$x=\frac{-1\pm\sqrt{1+16}}{4}$

$x=\frac{-1\pm\sqrt{17}}{4}$

$S.S=\left\{\frac{13\pm\sqrt{173}}{2},\frac{-1\pm\sqrt{17}}{4}\right\}$

(x) $2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$

Dividied By x^2

$$2\frac{x^4}{x^2} - 5\frac{x^3}{x^2} - 14\frac{x^2}{x^2} - 5\frac{x}{x^2} + \frac{2}{x^2} = \frac{0}{x^2}$$

$$2x^2 - 5x - 14 - 5\frac{1}{x} + \frac{2}{x^2} = 0$$

$$2x^2 + \frac{2}{x^2} - 5x - 5\frac{1}{x} - 14 = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) - 14 = 0$$

$Let\ y = x + \frac{1}{x}$

$$(y)^2 = \left(x + \frac{1}{x}\right)^2$$

$$y^2 = x^2 + \frac{1}{x^2} + 2$$

$$y^2 - 2 = x^2 + \frac{1}{x^2}$$

$$2(y^2 - 2) - 5y - 14 = 0$$

$$2y^2 - 4 - 5y - 14 = 0$$

$$2y^2 - 5y - 18 = 0$$

$$2y^2 - 9y + 4y - 18 = 0$$

$$y(2y - 9) + 2(2y - 9) = 0$$

$$(2y - 9)(y + 2) = 0$$

$2y - 9 = 0 \quad , \quad y + 2 = 0$

$2y = 9 \qquad\qquad y = -2$

$$y = \frac{9}{2}$$

$since\ y = x + \frac{1}{x}$

$x + \frac{1}{x} = \frac{9}{2}$	$x + \frac{1}{x} = -2$
$\frac{x^2+1}{x} = \frac{9}{2}$	$x^2 + 1 + 2x = 0$
$2x^2 + 2 = 9x$	$x^2 + 2x + 1 = 0$
$2x^2 - 9x + 2 = 0$	$(x+1)^2 = 0$
$a = 2 \quad b = -9 \quad c = 2$	$x + 1 = 0$
$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(2)}}{2(2)}$	$x = -1$

$$x = \frac{9 \pm \sqrt{81 - 16}}{4}$$

$$x = \frac{9 \pm \sqrt{65}}{4}$$

$$S.S = \left\{-1, \frac{9 \pm \sqrt{65}}{4}\right\}$$

(xi) $(x+1)(x+2)(x-4)(x-5)+8=0$

$[(x+1)(x-4)][(x+2)(x-5)]+8=0$

$[x^2-4x+x-4][x^2-5x+2x-10]+8=0$

$(x^2-3x-4)(x^2-3x-10)+8=0$

$Let\ y=x^2-3x$

$(y-4)(y-10)+8=0$

$y^2-10y-4y+40+8=0$

$y^2-14y+48=0$

$y^2-8y-6y+48=0$

$y(y-8)-6(y-8)=0$

$(y-8)(y-6)=0$

$y-8=0 \quad , \quad y-6=0$

$y=8 \quad , \quad y=6$

$since\ y=x^2-3x$

$x^2-3x=8$

$x^2-3x-8=0$

$a=1 \quad b=-3 \quad c=-8$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-8)}}{2(1)}$$

$$x=\frac{3\pm\sqrt{9+32}}{2}$$

$$x=\frac{3\pm\sqrt{41}}{2}$$

Now

$x^2-3x=6$

$x^2-3x-6=0$

$a=1 \quad b=-3 \quad c=-6$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-6)}}{2(1)}$$

$$x=\frac{3\pm\sqrt{9+24}}{2}$$

$$x=\frac{3\pm\sqrt{33}}{2}$$

$$S.S=\left\{\frac{3\pm\sqrt{41}}{2},\frac{3\pm\sqrt{33}}{2}\right\}$$

(xii) $(\boldsymbol{x}+\mathbf{3})(\boldsymbol{x}+\mathbf{4})(\boldsymbol{x}+\mathbf{5})(\boldsymbol{x}+\mathbf{6})=-\mathbf{1}$

$[(x+3)(x+6)][(x+4)(x+5)]=-1$

$[x^2+6x+3x+18][x^2+5x+4x+20]=-1$

$(x^2+9x+18)(x^2+9x+20)=-1$

$Let\ y=x^2+9x$

$(y+18)(y+20)=-1$

$y^2+20y+18y+360=-1$

$y^2+38y+360+1=0$

$y^2+38y+361=0$

$y^2+2(y)(19)+(19)^2=0$

$(y+19)^2=0$

$y+19=0$

$y=-19$

$Since\ y=x^2+9x$

$x^2+9x=-19$

$x^2+9x+19=0$

$a=1 \quad b=9 \quad c=19$

$$x=\frac{-(9)\pm\sqrt{(9)^2-4(1)(19)}}{2(1)}$$

$$x=\frac{-9\pm\sqrt{81-76}}{2}$$

$$x=\frac{-9\pm\sqrt{5}}{2}$$

$S.S=\left\{\frac{-9\pm\sqrt{5}}{2}\right\}$

EXERCISE 5.5

1. A train travels a distance of 240 km at a uniform rate, if it had finished 4 km an hour slower, it would have taken 2 hours more over the journey. Find its rate of travelling.

Solution

Distance (D) = 240km Initial speed = x Time = $\frac{distance}{speed}$ $T_1 = \frac{240}{x}$ Final speed = x-4 $T_2 = \frac{240}{x-4}$	$T_2 - T_1 = 2$ $\frac{240}{x-4} - \frac{240}{x} = 2$ Divided by 2 $\frac{120}{x-4} - \frac{120}{x} = 1$ $\frac{120x-120(x-4)}{x(x-4)} = 1$ $\frac{120x-120x+480}{x^2-4x} = 1$

$$120x - 120x + 480 = x^2 - 4x$$

$$x^2 - 4x - 480 = 0$$

$$x^2 - 24x + 20x - 480 = 0$$

$$x(x-24) + 20(x-24) = 0$$

$$(x-24)(x+20) = 0$$

$$x - 24 = 0 \quad , x + 20 = 0$$

$$x = 24 \qquad , x = -20$$

Speed cannot be negative, so the answer is 24 km/h

2. Arshia and Ibraheem complete a job together in 4 hours. If Arshia takes 6 hours, then find how much time will Ibraheem take?

Solution

Combined rate = $\frac{1}{4}\left(\frac{job}{hours}\right)$

Arshia rate = $\frac{1}{6}$

Let Ibraheem rate = $\frac{1}{x}$

$$Arshia\ rate + Ibraheem\ rate = Combined\ rate$$

$$\frac{1}{6}+\frac{1}{x}=\frac{1}{4}$$

$$\frac{1}{x}=\frac{1}{4}-\frac{1}{6}$$

$$\frac{1}{x}=\frac{6-4}{24}$$

$$\frac{1}{x}=\frac{2}{24}$$

$$\frac{1}{x}=\frac{1}{12}$$

$x = 12$ hours

3. One pipe fills water in 5 hours, another in 8 hours. Second pipe closed after 2 hours. Find the total time.

Solution

1^{st} pipe rate = $\frac{1}{5}$ $\left(\frac{tank}{hour}\right)$; 2^{nd} pipe rate = $\frac{1}{8}$

Assuming they start together, and 2^{nd} pipe closed ofter 2 hours

$$\left(\frac{1}{5}+\frac{1}{8}\right)\times 2 = \left(\frac{8+5}{40}\right)\times 2$$

$$= \frac{13}{20} \quad \text{(Filling rate)}$$

Remaining work:

$$= 1 - \frac{13}{20}$$

$$= \frac{20-13}{20}$$

$$= \frac{7}{20}$$

Time for 1^{st} pipe to finish rest

Time = $\frac{work}{rate}$

$$= \frac{\frac{7}{20}}{\frac{1}{5}}$$

$$= \frac{7}{20}\times 5$$

$$= \frac{7}{4}$$

$$= 1.75\ hours$$

Total time = time for both pipe + time for 1^{st} pipe

= 2 hours + 1.75 hours

= 3.75 hours

= 3 hours 45 minutes

4. Huria can complete a project in 12 hours by working alone. If Abdul Hadi join her and they finish it together in 5 hours, how long would it take Abdul Hadi to do the project alone?

Solution

Huria works rate = $\frac{1}{12}$ $\left(\frac{job}{hour}\right)$

Combined work rate = $\frac{1}{5}$

let Abdul Hadi time rate x, so

Abdul hadi rate = $\frac{1}{x}$

Huria works rate + Abdul hadi works rate = combined works rate

$$\frac{1}{12}+\frac{1}{x}=\frac{1}{5}$$

$$\frac{1}{x}=\frac{1}{5}-\frac{1}{12}$$

$$\frac{1}{x}=\frac{12-5}{60}$$

$$\frac{1}{x}=\frac{7}{60}$$

$$x=\frac{60}{7}$$

$$x=8.57\ hours$$

$$x=8\ hours\ 34\ minutes$$

5. Two cars start from opposite towns and head towards each other. The distance between them is 240 km. One travels at x km/h and the other at $(x + 10)$ km/h. They meet after 2 hours. Find the speed of each car.

Solution

Total distance = 240 km

Relative speed = x+(x+10)

= 2x+10

Time = 2 hours

Distance = Speed × Time

$240 = (2x + 10) \times 2$

divided by 2

$120 = 2x + 10$

$120 - 10 = 2x$

$2x = 110$

$x = 55$

speed of car A = 55 km/h

speed of car B = 55 + 10

= 65 km/h

6. Rashid can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

Solution

Rashid rate = $\frac{1}{6}$

combined rate = $\frac{1}{4}$

let the helper time x, so

helper time rate = $\frac{1}{x}$

Rashid rate + helper rate = combined rate

$$\frac{1}{6}+\frac{1}{x}=\frac{1}{4}$$

$$\frac{1}{x}=\frac{1}{4}-\frac{1}{6}$$

$$\frac{1}{x}=\frac{6-4}{24}$$

$$\frac{1}{x}=\frac{2}{24}$$

$$\frac{1}{x}=\frac{1}{12}$$

$$x=12\ days$$

7. Ishmal runs 10 km in $\frac{2x}{x+4}$ hours. If her average speed is 8 km/h, find the value of x.

Solution

distance (D) = 10km

time (T) = $\frac{2x}{x+4}$

speed = 8 km/h

distance = speed × time

$10 = 8 \times \frac{2x}{x+4}$

$10(x+4) = 16x$

$10x + 40 = 16x$

$40 = 16x - 10x$

$40 = 6x$

$\frac{40}{6} = x$

$x = 6.67$

8. Fahad runs 600 m at a certain pace, and then doubling his pace, does another 600 m. If he took $2\frac{1}{2}$ to cover the distance 1200 m, find the pace he started at, in metres per seconds.

Solution

first part :-

distance = 600m

speed = x

time (t_1) = $\frac{600}{x}$

second part:-

distance = 600m

speed = 2x

time(t_2) = $\frac{600}{2x} = \frac{300}{x}$

total time = 2.5 minutes= 150 second

t_1+t_2=total time

$\frac{600}{x} + \frac{300}{x} = 150$

$\frac{600+300}{x} = 150$

$900 = 150x$

$x = 6m/s$

9. A cyclist travels 30 km at a certain speed. If the speed had been 5 km/h faster, the journey would have taken 1 hour less. Find the speed.

Solution

distance = 30km	new speed = x+5
original speed = x	time = $\frac{30}{x+5}$
time = $\frac{30}{x}$	

$$\frac{30}{x} - \frac{30}{x+5} = 1$$

$$\frac{30(x+5)-30x}{x(x+5)} = 1$$

$$30x + 150 - 30x = x^2 + 5x$$

$$x^2 + 5x - 150 = 0$$

$$x^2 + 15x - 10x - 150 = 0$$

$$x(x + 15) - 10(x + 15) = 0$$

$$(x + 15)(x - 10) = 0$$

$x + 15 = 0$, $x - 10 = 0$

$x = -15$ (reject) $x = 10km/h$

REVIEW EXERCISE 5

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) The expression $\frac{2x-1}{x^2+4}$ is:

(a) a polynomial (b) an algebraic fraction
(c) a numerical fraction (d) an equation

(ii) Lowest form of $\frac{18a^2bc^2}{12ac}$ is:

(a) $\frac{9abc}{6}$ (b) $\frac{3abc}{2}$ (c) $\frac{18b}{12}$ (d) $\frac{3}{2}$

(iii) Lowest form of $\frac{15xyz}{10}$ is:

(a) $\frac{3xyz}{2}$ (b) $\frac{3}{2}$ (c) xyz (d) $\frac{xyz}{2}$

(iv) Simplified form of $\frac{3x+1}{5}+\frac{2x+3}{5}$ is:

(a) $\frac{5x+4}{10}$ (b) $\frac{5x+4}{5}$ (c) $\frac{5x+4}{25}$ (d) $5x+4$

(v) Simplified form of $\left(\frac{x^2-y^2}{x+y}\right)(x-y)$ is:

(a) $(x+y)^2$ (b) x^2+y^2 (c) $x-y$ (d) $(x-y)^2$

(vi) $(x^3-y^3)\div(x-y)$ in simplified form is:

(a) x^2-xy+y^2 (b) x^2+xy+y^2 (c) x^2-y^2 (d) x^2+y^2

(vii) An equation of the form $\frac{5}{x}+\frac{1-x}{3}=\frac{1}{6x}$ is called:

(a) radical equation (b) reciprocal equation
(c) fractional equation (d) exponential equation

(viii) An equation of the form $5^x+64\cdot5^{-x}-20=0$ is called:

(a) radical equation (b) exponential equation
(c) reciprocal equation (d) fractional equation

(ix) Roots of $y^2-24y+128=0$ are:

(a) 8, –16 (b) 8, 16 (c) –8, 16 (d) –8, –16

(x) Linear factors of $3x^2+10x+3=0$ are:

(a) $(x+3)$, $(3x+1)$ (b) $(x+3)$, $(3x-1)$
(c) $(x-3)$, $(3x+1)$ (d) $(x-3)$, $(3x-1)$

2. Reduce the following to the lowest form:

(i) $\dfrac{x^2-7x+12}{x^2-6x+9}$ (ii) $\dfrac{x^2+3x-18}{7x-21}$

Solution

(i) $\dfrac{x^2-7x+12}{x^2-6x+9}$

$=\dfrac{x^2-4x-3x+12}{x^2-3x-3x+9}$

$=\dfrac{x(x-4)-3(x-4)}{x(x-3)-3(x-3)}$

$=\dfrac{(x-4)(x-3)}{(x-3)(x-3)}$

$=\dfrac{x-4}{x-3}$

(ii) $\dfrac{x^2+3x-18}{7x-21}$

$=\dfrac{x^2+6x-3x-18}{7(x-3)}$

$=\dfrac{x(x+6)-3(x+6)}{7(x-3)}$

$=\dfrac{(x+6)(x-3)}{7(x-3)}$

$=\dfrac{x+6}{7}$

3. Simplify:

(i) $\dfrac{5}{x^2+x-6}-\dfrac{1}{2x^2-7x+6}$ (ii) $\dfrac{x^3-27}{x^2-9}\div\dfrac{x^2+3x+9}{x^2+6x+9}$

(iii) $\dfrac{a^2-(b-c)^2}{(a+b)^2-c^2}\times\dfrac{a^2-(b+c)^2}{(a-b)^2-c^2}$

Solution

(i) $\dfrac{5}{x^2+x-6}-\dfrac{1}{2x^2-7x+6}$

$=\dfrac{5}{x^2+3x-2x-6}-\dfrac{1}{2x^2-4x-3x+6}$

$=\dfrac{5}{x(x+3)-2(x+3)}-\dfrac{1}{2x(x-2)-3(x-2)}$

$$= \frac{5}{(x+3)(x-2)} - \frac{1}{(x-2)(2x-3)}$$

$$= \frac{5(2x-3)-(x+3)}{(x+3)(x-2)(2x-3)}$$

$$= \frac{10x-15-x-3}{(x+3)(x-2)(2x-3)}$$

$$= \frac{9x-18}{(x+3)(x-2)(2x-3)}$$

$$= \frac{9(x-2)}{(x+3)(x-2)(2x-3)}$$

$$= \frac{9}{(x+3)(2x-3)}$$

(ii) $\frac{x^3-27}{x^2-9} \div \frac{x^2+3x+9}{x^2+6x+9}$

$$= \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} \div \frac{x^2+3x+9}{x^2+3x+3x+9}$$

$$= \frac{(x^2+3x+9)}{(x+3)} \div \frac{x^2+3x+9}{x(x+3)+3(x+3)}$$

$$= \frac{(x^2+3x+9)}{(x+3)} \div \frac{x^2+3x+9}{(x+3)(x+3)}$$

$$= \frac{x^2+3x+9}{(x+3)} \times \frac{(x+3)(x+3)}{x^2+3x+9}$$

$$= x + 3$$

(iii) $\frac{a^2-(b-c)^2}{(a+b)^2-c^2} \times \frac{a^2-(b+c)^2}{(a-b)^2-c^2}$

$$= \frac{[a-(b-c)][a+b-c]}{(a+b-c)(a+b+c)} \times \frac{[a-(b+c)][a+b+c]}{(a-b-c)(a-b+c)}$$

$$= \frac{(a-b+c)(a+b-c)}{(a+b-c)(a+b+c)} \times \frac{(a-b-c)(a+b+c)}{(a-b-c)(a-b+c)}$$

$$= 1$$

4. Solve the following equations:

(i) $x^4-16x^2+63=0$ (ii) $4x^4-16x^2+15=0$

(iii) $3^{2x}-12\cdot 3^x+27=0$ (iv) $\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-2=0$

(v) $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=4$ (vi) $(x+9)(x-3)(x-7)(x+5)=385$

(vii) $(x-1)(x-2)(x+5)(x-8)=-360$

Solution

(i) $x^4-16x^2+63=0$	**(ii)** $4x^4-16x^2+15=0$
Let $y=x^2$	let $y=x^2, y^2=x^4$
$y^2=x^4$	$4y^2-16y+15=0$
$y^2-16y+63=0$	$4y^2-10y-6y+15=0$
$y^2-9y-7y+63=0$	$2y(2y-5)-3(2y-5)=0$
$y(y-9)-7(y-9)=0$	$(2y-5)(2y-3)=0$
$(y-9)(y-7)=0$	$2y-5=0\,,\ 2y-3=0$
Y-9=0 , y-7=0	$y=\frac{5}{2}\ ,\ y=\frac{3}{2}$
Y=9 y=7	$x^2=\frac{5}{2}\ ,\ x^2=\frac{3}{2}$
Since $y=x^2$	$x=\pm\sqrt{\frac{5}{2}}\ ,\ x=\pm\sqrt{\frac{3}{2}}$
$x^2=9\ ,\ x^2=7$	$x=\pm\frac{\sqrt{5}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\ ,\ x=\pm\frac{\sqrt{3}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$
$x=\pm 3\quad x=\pm\sqrt{7}$	$x=\pm\frac{\sqrt{10}}{2}\ ,\ x=\pm\frac{\sqrt{3}}{2}$
$S.S=\{\pm 3,\pm\sqrt{7}\}$	$S.S=\left\{\pm\frac{\sqrt{10}}{2},\pm\frac{\sqrt{3}}{2}\right\}$

(iii) $3^{2x} - 12.3^{x} + 27 = 0$

Let $y = 3^x$, $y^2 = 3^{2x}$

$y^2 - 12y + 27 = 0$

$y^2 - 9y - 3y + 27 = 0$

$y(y-9) - 3(y-9) = 0$

$(y-9)(y-3) = 0$

$y - 9 = 0$, $y - 3 = 0$

$y = 9$ $\quad y = 3$

$3^x = 9$ $\quad 3^x = 3$

$x = 2$ $\quad x = 1$

(iv) $\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 2 = 0$

$let\ y = x + \frac{1}{x}$

$y^2 - 2 = x^2 + \frac{1}{x^2}$

$(y^2 - 2) - 3y - 2 = 0$

$y^2 - 3y - 4 = 0$

$y^2 - 4y + y - 4 = 0$

$y(y-4) + 1(y-4) = 0$

$(y-4)(y+1) = 0$

$y = 4$, $y = -1$

$y = 4$

$x + \frac{1}{x} = 4$

$x^2 - 4x + 1 = 0$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$

$x = \frac{4 \pm \sqrt{16-4}}{2}$

$x = \frac{4 \pm \sqrt{12}}{2}$

$x = \frac{2(2 \pm \sqrt{3})}{2}$

$x = 2 \pm \sqrt{3}$

$y = -1$

$x + \frac{1}{x} = -1$

$x^2 + x + 1 = 0$

$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$x = \frac{-1 \pm \sqrt{1-4}}{2}$

$x = \frac{-1 \pm \sqrt{-3}}{2}$

$S.S = \left\{2 \pm \sqrt{3}, \frac{-1 \pm \sqrt{-3}}{2}\right\}$

(v) $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=4$

Let $y = x+\frac{1}{x}$

$y^2-2=x^2+\frac{1}{x^2}$

$y^2-2+y-4=0$

$y^2+y-6=0$

$y^2+3y-2y-6=0$

$y(y+3)-2(y+3)=0$

$(y+3)(y-2)=0$

$y+3=0\ ,\ y-2=0$

$y=-3 \qquad y=2$

$y=-3$

$x+\frac{1}{x}=-3$

$x^2+3x+1=0$

$x=\frac{-(3)\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$

$x=\frac{-3\pm\sqrt{9-4}}{2}$

$x=\frac{-3\pm\sqrt{5}}{2}$

$y=2$

$x+\frac{1}{x}=2$

$x^2-2x+1=0$

$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(1)}}{2(1)}$

$x=\frac{2\pm\sqrt{4-4}}{2}=\frac{2\pm\sqrt{0}}{2}=\frac{2}{2}$

$x=1$

$S.S=\left\{1,\frac{-3\pm\sqrt{5}}{2}\right\}$

(vi) $(x+9)(x-3)(x-7)(x+5)=385$

$[(x+9)(x-7)][(x-3)(x+5)]=385$

$[x^2-7x+9x-63][x^2+5x-3x-15]=385$

$(x^2+2x-63)(x^2+2x-15)=385$

$let\ y=x^2+2x$

$(y-63)(y-15)=385$

$y^2-15y-63y+945=385$

$y^2-78y+945-385=0$

$y^2-78y+560=0$

$y^2-70y-8y+560=0$

$y(y-70)-8(y-70)=0$

$(y-70)(y-8)=0$

$y-70=0\quad,\quad y-8=0$

$y=70\qquad y=8$

$y=70$	$y=8$
$x^2+2x=70$	$x^2+2x=8$
$x^2+2x-70=0$	$x^2+2x-8=0$
$x=\frac{-(2)\pm\sqrt{(2)^2-4(1)(-70)}}{2(1)}$	$x^2+4x-2x-8=0$
$x=\frac{-2\pm\sqrt{4+280}}{2}$	$x(x+4)-2(x+4)=0$
$x=\frac{-2\pm\sqrt{284}}{2}$	$(x+4)(x-2)=0$
$x=\frac{-2\pm\sqrt{4\times 71}}{2}$	$x+4=0\quad,\quad x-2=0$
$x=\frac{-2\pm 2\sqrt{71}}{2}$	$x=-4\quad,\quad x=2$
$x=-1\pm\sqrt{71}$	$S.S=\{-4,2,-1\pm\sqrt{71}\}$

(vii) $(x-1)(x-2)(x+5)(x-8) = -360$

$[(x-1)(x-2)][(x+5)(x-8)] = -360$

$(x^2-2x-x+2)(x^2-8x+5x-40) = -360$

$(x^2-3x+2)(x^2-3x-40) = -360$

$let\ y = x^2-3x$

$(y+2)(y-40) = -360$

$y^2-40y+2y-80 = -360$

$y^2-38y-80+360 = 0$

$y^2-38y+280 = 0$

$y^2-28y-10y+280 = 0$

$y(y-28)-10(y-28) = 0$

$(y-28)(y-10) = 0$

$y-28 = 0 \quad , \quad y-10 = 0$

$y = 28 \qquad y = 10$

$x^2-3x = 28$	$x^2-3x = 10$
$x^2-3x-28 = 0$	$x^2-3x-10 = 0$
$x^2-7x+4x-28 = 0$	$x^2-5x+2x-10 = 0$
$x(x-7)+4(x-7) = 0$	$x(x-5)+2(x-5) = 0$
$(x-7)(x+4) = 0$	$(x-5)(x+2) = 0$
$x-7 = 0 \quad , \quad x+4 = 0$	$x-5 = 0 \quad , \quad x+2 = 0$
$x = 7 \qquad x = -4$	$x = 5 \qquad x = -2$
	$S.S = \{-2,-4,5,7\}$

5. Ahmad takes 2 hours to paint 50 glasses. Faiza takes 2 hours to paint 45 glasses. Working together, how long should it take them to paint 150 glasses?

Solution

Ahamd works 2 hours = 50 glasses

Faiza works in 2 hours = 45 glasses

Total glasses to be paint = 150

Time taken working together =?

Combined rate = ahamd rate + faiza rate

$$= \frac{work}{time} + \frac{work}{time}$$

$$= \frac{50}{2} + \frac{45}{2}$$

$$= 25 + 22.5$$

Combined rate = 47.5 glasses

Time taken together working = $\frac{total\ work}{combined\ rate}$

$$= \frac{150}{47.5}$$

$$= 3.16 \text{ hours}$$

6. A tap can fill a tank in 6 hours. Another tap can fill it in 9 hours. If both taps are opened together, how long will it take to fill the empty tank?

Solution

Time for tap A to fill the tank = 6 hours

Time for tap B to fill the tank = 9 hours

Total time whrn both are open = ?

Work done by Tap A in 1h = $\frac{1}{6}$

Work done by Tap B in 1h = $\frac{1}{9}$

Total work per hour = $\frac{1}{6} + \frac{1}{9}$

$$\frac{1}{T} = \frac{3}{18} + \frac{2}{18}$$

$$\frac{1}{T} = \frac{5}{18}$$

$$T = \frac{18}{5} = 3.6 \text{ hours}$$

7. Maham bought a certain number of toys for Rs. 300. If each toy had cost Rs. 5 less, she could have bought two more for the same amount. How many toys did she buy?

Solution

Let x be the number of toys maham originally bought

The orignal price per toy= $\frac{300}{x}$

If the price were Rs.5 less,

The new price = $\frac{300}{x} - 5$

At this lower price, she could buy x+2 toys for same Rs.300

$$(x+2)\left(\frac{300}{x} - 5\right) = 300$$

$$\frac{300x}{x} - 5x + \frac{600}{x} - 10 = 300$$

$$300 - 5x + \frac{600}{x} - 10 - 300 = 0$$

$$-5x^2 + 600 - 10x = 0$$

$$-5(x^2 - 120 + 2x) = 0$$

$$x^2 + 2x - 120 = 0$$

$$x^2 + 12x - 10x - 120 = 0$$

$$x(x+12) - 10(x+12) = 0$$

$$(x+12)(x-10) = 0$$

$$x + 12 = 0 \quad , \quad x - 10 = 0$$

$$x = -12 \qquad x = 10$$

$$(x = -12\ reject)$$

UNIT 6

Vectors in Plane

EXERCISE 6.1

1. Name the quadrant in which each point lies.

(i) (4, 3) (ii) (5, –4) (iii) (–6,2) (iv) (–4, –4)

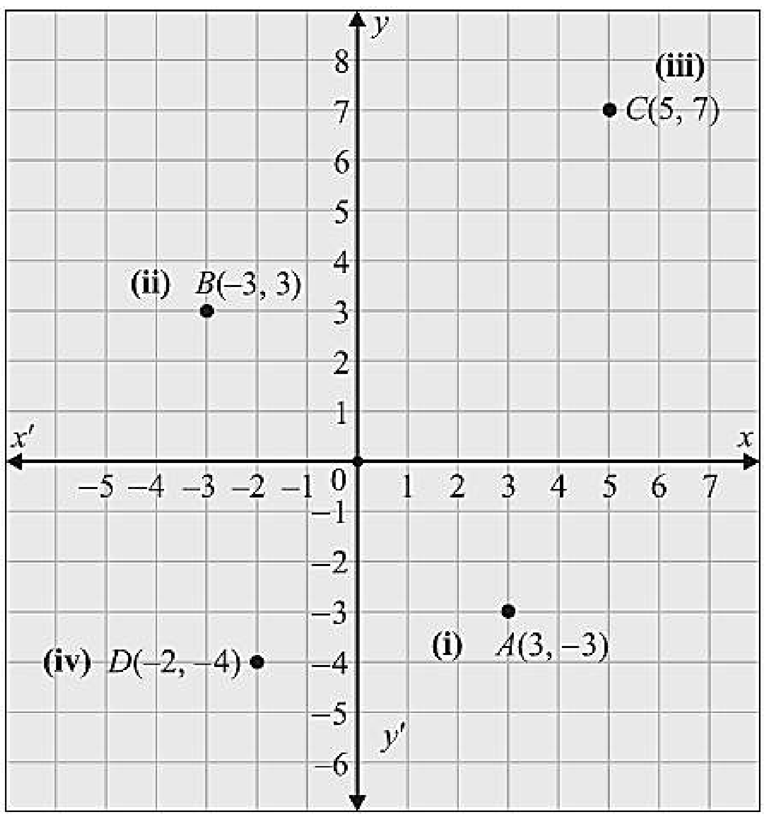
Solution

1. (i) I **(ii)** IV **(iii)** II **(iv)** III

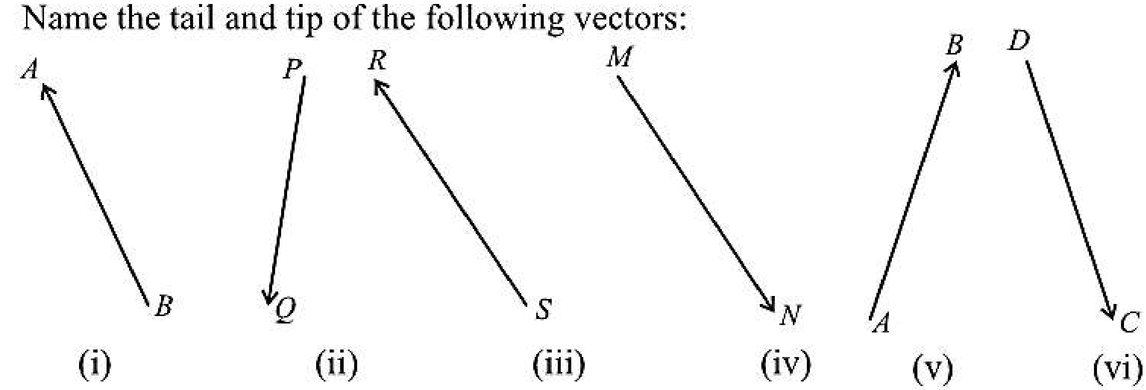
2. Plot the following points on the coordinate plane:

(i) $A(3, -3)$ (ii) $B(-3, 3)$ (iii) $C(5, 7)$ (iv) $D(-2, -4)$

Solution



3. Name the tail and tip of the following vectors:



Solution

(i) Tail $= B$, Tip $= A$
(ii) Tail $= P$, Tip $= Q$
(iii) Tail $= S$, Tip $= R$
(iv) Tail $= M$, Tip $= N$
(v) Tail $= A$, Tip $= B$
(vi) Tail $= D$, Tip $= C$

4. Write the vector AB in the form of $x\underline{i} + y\underline{j}$:

(i) $A(1, -7), B(-2, 4)$ (ii) $A(8, 9), B(12, 3)$

Solution

(i) A(1,-7),B(-2,4)

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$
$$= (-2-1)\underline{i} + (4+7)\underline{j}$$
$$= -3\underline{i} + 11\underline{j}$$

(ii) A(8,9),B(12,3)

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$
$$= (12-8)\underline{i} + (3-9)\underline{j}$$
$$= 4\underline{i} - 6\underline{j}$$

5. Find the magnitude of the $\underline{a}$:

(i) $\underline{a} = -3\underline{i} + 2\underline{j}$ (ii) $\underline{a} = 4\underline{i} - 3\underline{j}$ (iii) $\underline{a} = \frac{1}{2}\underline{i} + \frac{3}{2}\underline{j}$

Solution

(i) $\underline{a} = -3\underline{i}+2\underline{j}$ $\lvert a\rvert = \sqrt{(-3)^2 + (2)^2}$ $= \sqrt{9+4}$ $= \sqrt{13}$	(ii) $\underline{a} = 4\underline{i}-3\underline{j}$ $\lvert a\rvert = \sqrt{(4)^2 + (-3)^2}$ $= \sqrt{16+9}$ $= \sqrt{25} = 5$
(iii) $\underline{a} = \frac{1}{2}\underline{i} + \frac{3}{2}\underline{j}$ $\lvert a\rvert = \sqrt{(\frac{1}{2})^2 + (\frac{3}{2})^2}$ $= \sqrt{\frac{1}{4} + \frac{9}{4}}$ $= \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$	

6. Find a unit vector in the direction of the vector given below:

(i) $\underline{a} = -4\underline{i} + 5\underline{j}$ (ii) $\underline{a} = 6\underline{i} + 8\underline{j}$

(iii) $\underline{a} = \frac{1}{\sqrt{6}}\underline{i} + \frac{1}{\sqrt{6}}\underline{j}$ (iv) $\underline{a} = \frac{1}{2}\underline{i} - \frac{3}{4}\underline{j}$

Solution

(i) $\underline{a} = -4\underline{i} + 5\underline{j}$

$$\hat{a} = \frac{\underline{a}}{|a|} = \frac{-4\underline{i}+5\underline{j}}{\sqrt{41}} = \frac{-4}{\sqrt{41}}\underline{i} + \frac{5}{\sqrt{41}}\underline{j}$$

$$|a| = \sqrt{(-4)^2 + (5)^2} = \sqrt{16+25} = \sqrt{41}$$

(ii) $\underline{a} = 6\underline{i} + 8\underline{j}$

$$\hat{a} = \frac{\underline{a}}{|a|} = \frac{6\underline{i}+8\underline{j}}{\sqrt{(6)^2+(8)^2}} = \frac{6\underline{i}+8\underline{j}}{\sqrt{36+64}} = \frac{6\underline{i}+8\underline{j}}{\sqrt{100}} = \frac{6\underline{i}+8\underline{j}}{10} = \frac{6}{10}\underline{i} + \frac{8}{10}\underline{j} = \frac{3}{5}\underline{i} + \frac{4}{5}\underline{j}$$

(iii) $\underline{a} = \frac{1}{\sqrt{6}}\underline{i} + \frac{1}{\sqrt{6}}\underline{j}$

$$\hat{a} = \frac{\underline{a}}{|a|} = \frac{\frac{1}{\sqrt{6}}\underline{i}+\frac{1}{\sqrt{6}}\underline{j}}{\sqrt{(\frac{1}{\sqrt{6}})^2+(\frac{1}{\sqrt{6}})^2}} = \frac{\frac{1}{\sqrt{6}}\underline{i}+\frac{1}{\sqrt{6}}\underline{j}}{\sqrt{\frac{1}{6}+\frac{1}{6}}} = \frac{\frac{1}{\sqrt{6}}\underline{i}+\frac{1}{\sqrt{6}}\underline{j}}{\sqrt{\frac{1}{3}}} = \frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{3}}}\underline{i} + \frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{3}}}\underline{j} = \frac{\sqrt{3}}{\sqrt{6}}\underline{i} + \frac{\sqrt{3}}{\sqrt{6}}\underline{j} = \frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}$$

(Iv) $\underline{a} = \frac{1}{2}\underline{i} - \frac{3}{4}\underline{j}$

$$\hat{a} = \frac{\underline{a}}{|a|} = \frac{\frac{1}{2}\underline{i}-\frac{3}{4}\underline{j}}{\sqrt{(\frac{1}{2})^2+(\frac{3}{4})^2}} = \frac{\frac{1}{2}\underline{i}-\frac{3}{4}\underline{j}}{\sqrt{\frac{1}{4}+\frac{9}{16}}} = \frac{\frac{1}{2}\underline{i}-\frac{3}{4}\underline{j}}{\sqrt{\frac{13}{16}}} = \frac{\frac{1}{2}\underline{i}-\frac{3}{4}\underline{j}}{\frac{\sqrt{13}}{4}}$$

$$= \frac{1}{2}\times\frac{4}{\sqrt{13}}\underline{i} - \frac{3}{4}\times\frac{4}{\sqrt{13}}\underline{j} = \frac{2}{\sqrt{13}}\underline{i} - \frac{3}{\sqrt{13}}\underline{j}$$

7. If $\underline{a} = 5\underline{i} - 7\underline{j}$, $\underline{b} = -\underline{i} - \underline{j}$ and $\underline{c} = 2\underline{i} + 3\underline{j}$, then find unit vector parallel to $\underline{a} + \underline{b} - 3\underline{c}$.

Solution

$$\underline{a} + \underline{b} - 3\underline{c} = (5\underline{i} - 7\underline{j}) + (-\underline{i} - \underline{j}) - 3(2\underline{i} + 3\underline{j})$$
$$= 5\underline{i} - 7\underline{j} - \underline{i} - \underline{j} - 6\underline{i} - 9\underline{j}$$
$$= -2\underline{i} - 17\underline{j}$$

Let

$$\vec{v} = \underline{a} + \underline{b} - 3\underline{c} = -2\underline{i} - 17\underline{j}$$
$$\vec{v} = -2\underline{i} - 17\underline{j}$$

Now we show that unit vector is parallel to $\underline{a} + \underline{b} - 3\underline{c}$

$$\hat{v} = \frac{v}{|v|}$$
$$= \frac{-2\underline{i} - 17\underline{j}}{\sqrt{(-2)^2 + (-17)^2}}$$
$$= \frac{-2\underline{i} - 17\underline{j}}{\sqrt{4 + 289}}$$
$$= \frac{-2\underline{i} - 17\underline{j}}{\sqrt{293}}$$
$$= \frac{-2}{\sqrt{293}}\underline{i} - \frac{17}{\sqrt{293}}\underline{j}$$

NOTE:-

When a vector is divided by its magnitude its direction remains the same but its magnitude becomes the 1. Therefore the original vector and the unit vector are parallel .

8. If $\underline{a} = 3\underline{i} - \underline{j}$, $\underline{b} = -2\underline{i} + 4\underline{j}$ and $\underline{c} = \underline{i} + 2\underline{j}$, then find unit vector parallel to $3\underline{a} - 2\underline{c} + 4\underline{b}$.

Solution

$$3\underline{a} - 2\underline{c} + 4\underline{b} = 3\left(3\underline{i} - \underline{j}\right) - 2\left(-2\underline{i} + 4\underline{j}\right) + 4\left(\underline{i} + 2\underline{j}\right)$$
$$= 9\underline{i} - 3\underline{j} + 4\underline{i} - 8\underline{j} + 4\underline{i} + 8\underline{j}$$
$$= -\underline{i} + 9\underline{j}$$

Let

$\vec{v} = -\underline{i} + 9\underline{j}$

Now find $\vec{v}$

$$\hat{v} = \frac{\underline{v}}{|v|} = \frac{-\underline{i}+9\underline{j}}{\sqrt{(-1)^2+(9)^2}} = \frac{-\underline{i}+9\underline{j}}{\sqrt{82}} = \frac{-1}{\sqrt{82}}\underline{i} + \frac{9}{\sqrt{82}}\underline{j}$$

NOTE:-

When a vector is divided by its magnitude its direction remains the same but its magnitude becomes the 1. Therefore the original vector and the unit vector are parallel .

9. Which of the following vectors are parallel?

(i) $\underline{a} = 6\underline{i} + \underline{j}$, $\underline{b} = 12\underline{i} + 2\underline{j}$ (ii) $\underline{a} = -2\underline{i} + 3\underline{j}$, $\underline{b} = 6\underline{i} - 9\underline{j}$

(iii) $\underline{a} = 5\underline{i} - 4\underline{j}$, $\underline{b} = 6\underline{i} - 3\underline{j}$ (iv) $\underline{a} = 3\underline{i} - 7\underline{j}$, $\underline{b} = 6\underline{i} - 14\underline{j}$

Solution

(i) $\underline{a} = 6\underline{i} + \underline{j}\ ,\ \underline{b} = 12\underline{i} + 2\underline{j}$

$$\underline{b} = 2(6\underline{i} + \underline{j})$$

$$\underline{b} = 2\underline{a} \quad \text{(parallel)}$$

(ii) $\underline{a} = -2\underline{i} + 3\underline{j}$, $\underline{b} = 6\underline{i} - 9\underline{j}$

$$b = -3(-2\underline{i} + 3\underline{j})$$

$$\underline{b} = -3\underline{a} \text{ (parallel)}$$

(Iii) $\underline{a} = 5\underline{i} - 4\underline{j}$, $\underline{b} = 6\underline{i} - 3\underline{j}$

$$b = 3(2\underline{i} - \underline{j}) \quad \therefore a \neq 2\underline{i} - \underline{j} \text{ so it is not parallel}$$

(Iv) $\underline{a} = 3\underline{i} - 7\underline{j}$, $\underline{b} = 6\underline{i} - 14\underline{j}$

$$\underline{b} = 2(3\underline{i} - 7\underline{j})$$

$$\underline{b} = 2\underline{a} \text{ (parallel)}$$

10. Find a vector thrice in length of $3\underline{i} - 2\underline{j}$, but opposite in direction.

Solution

Given vector = $3\underline{i} - 2\underline{j}$

Three time in length = $3(3\underline{i} - 2\underline{j})$

$$= 9\underline{i} - 6\underline{j}$$

Opposite direction = $-9\underline{i} + 6\underline{j}$

11. Find two vectors that are double in magnitude of $3\underline{i} - 5\underline{j}$, one in the same direction of it and other in its opposite direction.

Solution

Given vector = $3\underline{i} - 5\underline{j}$

Double magnitude = $2(3\underline{i} - 5\underline{j})$

$$= 6\underline{i} - 10\underline{j}$$

Opposite direction = $-2\left(3\underline{i} - 5\underline{j}\right)$

$$= -6\underline{i} + 10\underline{j}$$

EXERCISE 6.2

1. If $\underline{a}=7\underline{i}-3\underline{j}$ and $\underline{b}=\underline{i}+5\underline{j}$, then find the following vectors:

(i) $\underline{a}+\underline{b}$ (ii) $\underline{a}+3\underline{b}$ (iii) $3\underline{a}+\frac{1}{2}\underline{b}$

(iv) $\underline{b}-\underline{a}$ (v) $4\underline{b}-5\underline{a}$ (vi) $\frac{3}{2}\underline{a}-\underline{b}$

Solution

(i) $\underline{a}+\underline{b}$ $=7\underline{i}-3\underline{j}+\underline{i}+5\underline{j}$ $=8\underline{i}+2\underline{j}$	(ii) $\underline{a}+3\underline{b}$ $=7\underline{i}-3\underline{j}+3(\underline{i}+5\underline{j})$ $=10\underline{i}+12\underline{j}$
(iii) $3\underline{a}+\frac{1}{2}\underline{b}$ $=3\left(7\underline{i}-3\underline{j}\right)+\frac{1}{2}(\underline{i}+5\underline{j})$ $=21i-9j+\frac{1}{2}i+\frac{5}{2}j$ $=\frac{43}{2}i-\frac{13}{2}j$	(iv) $\underline{b}-\underline{a}$ $=\left(\underline{i}+5\underline{j}\right)-(7\underline{i}-3\underline{j})$ $=\underline{i}+5\underline{j}-7\underline{i}+3\underline{j}$ $=-6\underline{i}+8\underline{j}$
(v) $4\underline{b}-5\underline{a}$ $=4\left(\underline{i}+5\underline{j}\right)-5(7\underline{i}-3\underline{j})$ $=4\underline{i}+20\underline{j}-35\underline{i}+15\underline{j}$ $=-31\underline{i}+35\underline{j}$	(vi) $\frac{3}{2}\underline{a}-\underline{b}$ $=\frac{3}{2}\left(7\underline{i}-3\underline{j}\right)-(\underline{i}+5\underline{j})$ $=\frac{21}{2}\underline{i}-\frac{9}{2}\underline{j}-\underline{i}-5\underline{j}$ $=\frac{19}{2}\underline{i}-\frac{19}{2}\underline{j}$

2. If $\underline{a} = 6\underline{i} - \underline{j}$, $\underline{b} = \underline{i} + 5\underline{j}$ and $\underline{c} = 3\underline{i} + 5\underline{j}$, then find the magnitudes of the following vectors:

(i) $\underline{b} - \underline{c}$ (ii) $\underline{a} - 2\underline{b} + \underline{c}$ (iii) $\underline{c} - \underline{b} - \underline{a}$

Solution

(i) $\underline{b}$-$\underline{c}$

$$= \underline{i} + 5\underline{j} - (3\underline{i} + 5\underline{j})$$
$$= \underline{i} + 5\underline{j} - 3\underline{i} - 5\underline{j}$$
$$\underline{b}\text{-}\underline{c} = -2\underline{i} + 0\underline{j}$$
$$|\underline{b} - \underline{c}| = \sqrt{(-2)^2 + (0)^2} = 2$$

(ii) $\underline{a} - 2\underline{b} + \underline{c}$

$$\underline{a} - 2\underline{b} + \underline{c} = (6\underline{i} - \underline{j}) - 2(\underline{i} + 5\underline{j}) + (3\underline{i} + 5\underline{j})$$
$$= 6\underline{i} - \underline{j} - 2\underline{i} - 10\underline{j} + 3\underline{i} + 5\underline{j}$$
$$= 7\underline{i} - 6\underline{j}$$

Let $v = \underline{a} - 2\underline{b} + \underline{c} = 7\underline{i} - 6\underline{j}$

$$|\vec{v}| = \sqrt{(7)^2 + (-6)^2}$$
$$= \sqrt{49 + 36}$$
$$= \sqrt{85}$$

(iii) $\underline{c} - \underline{b} - \underline{a}$

$$\underline{c} - \underline{b} - \underline{a} = (3\underline{i} + 5\underline{j}) - (\underline{i} + 5\underline{j}) - (6\underline{i} - \underline{j})$$
$$= 3\underline{i} + 5\underline{j} - \underline{i} - 5\underline{j} - 6\underline{i} + \underline{j}$$
$$\underline{c} - \underline{b} - \underline{a} = -4\underline{i} + \underline{j}$$

Let $v = \underline{c} - \underline{b} - \underline{a} = -4\underline{i} + \underline{j}$

$$|\vec{v}| = \sqrt{(-4)^2 + (1)^2}$$
$$= \sqrt{16 + 1} = \sqrt{17}$$

3. Find the values of x and y in each of the following equations:

(i) $(x\underline{i}+y\underline{j})+(2\underline{i}+3\underline{j})=7\underline{i}+6\underline{j}$ (ii) $(x\underline{i}-5\underline{j})+(3\underline{i}+5\underline{j})=-8\underline{i}+y\underline{j}$

(iii) $(y\underline{i}+3\underline{j})+(-5\underline{i}+2x\underline{j})=9\underline{i}+7\underline{j}$

Solution

(i) $\left(x\underline{i}+y\underline{j}\right)+\left(2\underline{i}+3\underline{j}\right)=(7\underline{i}+6\underline{j})$

$x\underline{i}+y\underline{j}+2\underline{i}+3\underline{j}=7\underline{i}+6\underline{j}$

$(x+2)\underline{i}+(y+3)\underline{j}=7\underline{i}+6\underline{j}$

Comparing the coefficient

$x+2=7$, $y+3=6$

$x=7-2$, $y=6-3$

$x=5$ $y=3$

(ii) $\left(x\underline{i}-5\underline{j}\right)+\left(3\underline{i}+5\underline{j}\right)=-8\underline{i}+y\underline{j}$

$x\underline{i}-5\underline{j}+3\underline{i}+5\underline{j}=-8\underline{i}+y\underline{j}$

$(x+3)\underline{i}+(-5+5)\underline{j}=-8\underline{i}+y\underline{j}$

$(x+3)\underline{i}+0\underline{j}=-8\underline{i}+y\underline{j}$

$x+3=-8$, $0=y$

$x=-11$

(iii) $\left(y\underline{i}+3\underline{j}\right)+\left(-5\underline{i}+2x\underline{j}\right)=9\underline{i}+7\underline{j}$

$y\underline{i}+3\underline{j}-5\underline{i}+2x\underline{j}=9\underline{i}+7\underline{j}$

$(y-5)\underline{i}+(3+2x)\underline{j}=9\underline{i}+7\underline{j}$

Comparing the coefficient

$y-5=9$, $3+2x=7$

$y=9+5$ $2x=7-3$

$y=14$ $2x=4$

$x=2$

4. If $\underline{a}=\underline{i}+3\underline{j}$, $\underline{c}=2\underline{i}+\underline{j}$ and $\underline{a}+2\underline{b}=\underline{c}$, then find $|\underline{b}|$.

Solution

$\underline{a}+2\underline{b}=\underline{c}$

$\underline{i}+3\underline{j}+2\underline{b}=2\underline{i}+\underline{j}$

$2\underline{b}=2\underline{i}+\underline{j}-\underline{i}-3\underline{j}$

$2\underline{b}=\underline{i}-2\underline{j}$

$\underline{b}=\frac{1}{2}\underline{i}-\underline{j}$

$|\underline{b}|=\sqrt{(\frac{1}{2})^2+(-1)^2}=\sqrt{\frac{1}{4}+1}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$

5. If $\underline{a} = -2\underline{i} + 7\underline{j}$ and $\underline{b} = 3\underline{i} - 5\underline{j}$, then find the vector components of $\underline{a} + 5\underline{b}$.

Solution

$$\underline{a} + 5\underline{b} = \left(-2\underline{i} + 7\underline{j}\right) + 5(3\underline{i} - 5\underline{j})$$

$$= -2\underline{i} + 7\underline{j} + 15\underline{i} - 25\underline{j}$$

$$= 13\underline{i} - 18\underline{j}$$

6. If $5\underline{i} - 3\underline{j} = m(\underline{i} - 10\underline{j}) + n(4\underline{i} - 3\underline{j})$, then find the values of m and n.

Solution

$$5\underline{i} - 3\underline{j} = m\underline{i} - 10m\underline{j} + 4n\underline{i} - 7n\underline{j}$$

$$5\underline{i} - 3\underline{j} = (m + 4n)\underline{i} - (10m + 7n)\underline{j}$$

comparing the coefficient

$$m + 4n = 5 \quad , \quad -3 = -(10m + 7n)$$

$$10m + 7n = 3$$

$$10m + 7n = 3$$

$$\pm 10m \pm 40n = \pm 50$$

$$-37n = -47$$

$$n = \frac{47}{37}$$

Now putting the value of n in $m + 4n = 5$

$$m + 4\left(\frac{47}{37}\right) = 5$$

$$m + \frac{188}{37} = 5$$

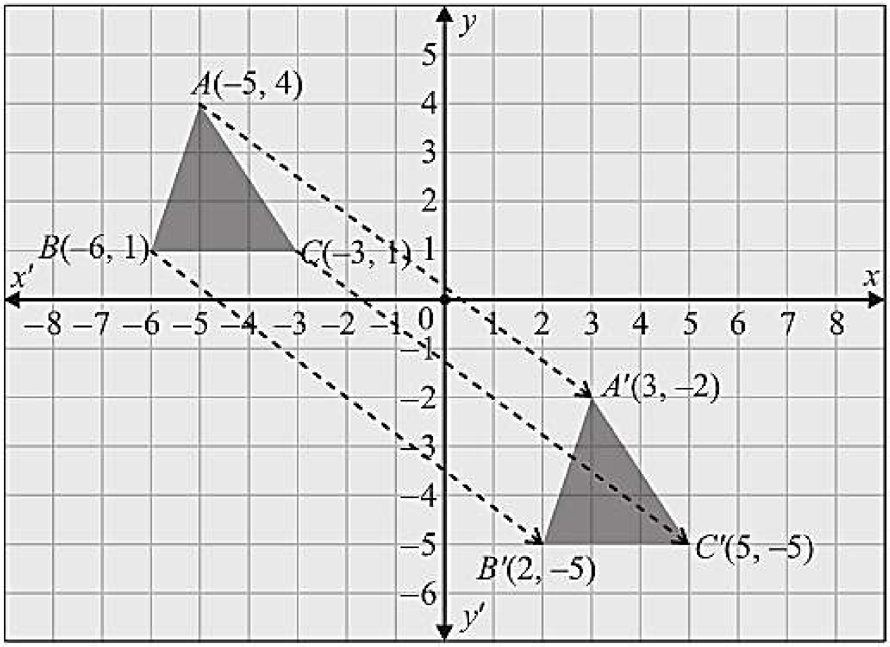
$$m = 5 - \frac{188}{37}$$

$$m = \frac{185 - 188}{37} = \frac{-3}{37}$$

EXERCISE 6.3

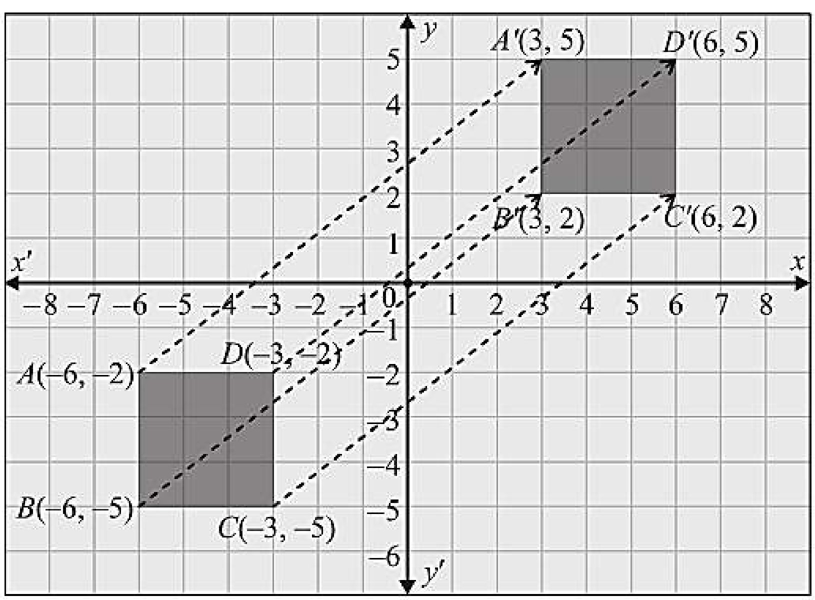
1. Plot $A(-5, 4)$, $B(-6, 1)$ and $C(-3, 1)$ to form a ΔABC. Also translate ΔABC to $\Delta A'B'C'$ by translation vector $8\underline{i} - 6\underline{j}$.

Solution



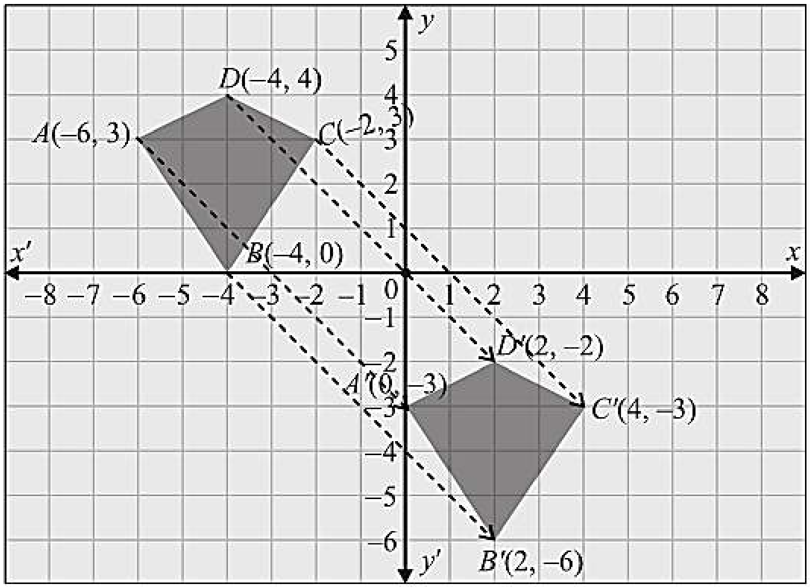
2. Plot $A(-6, -2)$, $B(-6, -5)$, $C(-3, -5)$ and $D(-3, -2)$ to form a square $ABCD$. Also translate square $ABCD$ to square $A'B'C'D'$ by translation vector $9\underline{i} + 7\underline{j}$.

Solution



3. Plot $A(-6, 3)$, $B(-4, 0)$, $C(-2, 3)$ and $D(-4, 4)$ to form a kite $ABCD$. Also translate kite $ABCD$ to kite $A'B'C'D'$ by translation vector $6\underline{i} - 6\underline{j}$.

Solution



4. The coordinates of A, B and D are (1,2), (6,3) and (2,8) respectively. Find the coordinates of C by using vector method if $ABCD$ is a parallelogram.

Solution

$$\overrightarrow{AB} = \overrightarrow{DC}.$$

$$\overrightarrow{AB} = (6-1, 3-2) = (5, 1).$$

$$\overrightarrow{DC} = (x_C - 2, y_C - 8).$$

$$x_C - 2 = 5$$
$$x_C = 5 + 2$$
$$x_C = 7.$$

$$y_C - 8 = 1$$
$$y_C = 1 + 8$$
$$y_C = 9.$$

$$x_C = 7,\ y_C = 9.$$

Answer is $C(7, 9)$

5. In parallelogram $ABCD$, the vectors representing two opposite sides are $\overrightarrow{AB} = 6\underline{i} + 2\underline{j}$, $\overrightarrow{DC} = -6\underline{i} - 2\underline{j}$. Show that the opposite sides are equal in magnitude and parallel.

Solution

$$|\overrightarrow{AB}| = \sqrt{6^2 + 2^2} = \sqrt{40},$$
$$|\overrightarrow{DC}| = \sqrt{(-6)^2 + (-2)^2} = \sqrt{40}.$$

$$\overrightarrow{AB} = -\overrightarrow{DC}.$$

Opposite sides are equal in magnitude and parallel.

6. Points $A(1, 2)$, $B(4, 6)$ and $C(7, 2)$ form a triangle. Check whether triangle ABC is an isosceles by using vector magnitude.

Solution

$$\overrightarrow{AB} = (4 - 1, 6 - 2) = (3, 4),$$
$$\overrightarrow{AC} = (7 - 1, 2 - 2) = (6, 0),$$
$$\overrightarrow{BC} = (7 - 4, 2 - 6) = (3, -4).$$

$$|\overrightarrow{AB}| = \sqrt{3^2 + 4^2} = 5,$$
$$|\overrightarrow{AC}| = 6,$$
$$|\overrightarrow{BC}| = \sqrt{3^2 + (-4)^2} = 5.$$

Since $|\overrightarrow{AB}| = |\overrightarrow{BC}|$, triangle ABC is **isosceles**.

7. Use vectors to show that $PQRS$ is a parallelogram, where the points P, Q, R and S have coordinates (1, 2), (5, 2), (7, 6) and (3, 6) respectively.

Solution

$$\overrightarrow{PQ} = (5-1, 2-2) = (4, 0),$$
$$\overrightarrow{SR} = (7-3, 6-6) = (4, 0).$$

$$\overrightarrow{PS} = (3-1, 6-2) = (2, 4),$$
$$\overrightarrow{QR} = (7-5, 6-2) = (2, 4).$$

Since $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\overrightarrow{PS} = \overrightarrow{QR}$, $PQRS$ is a parallelogram.

8. Use vectors to show that triangle XYZ is an isosceles, where the points X, Y, Z have the coordinates (0, 0), (2, 0) and (1, 3) respectively.

Solution

$$\overrightarrow{XY} = (2-0, 0-0) = (2, 0),$$
$$\overrightarrow{XZ} = (1-0, 3-0) = (1, 3),$$
$$\overrightarrow{YZ} = (1-2, 3-0) = (-1, 3).$$

$$|\overrightarrow{XY}| = 2,$$
$$|\overrightarrow{XZ}| = \sqrt{1^2 + 3^2} = \sqrt{10},$$
$$|\overrightarrow{YZ}| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}.$$

Since $|\overrightarrow{XZ}| = |\overrightarrow{YZ}|$, triangle XYZ is **isosceles**.

9. A ball is projected with an initial velocity vector $\underline{v}_0 = 10\underline{i} + 20\underline{j}$. The horizontal component is in the x-direction and gravity is $\underline{g} = 0\underline{i} - 10\underline{j}$. Find the maximum height and horizontal range.

Solution

Given $\underline{v}_0 = 10\underline{i} + 20\underline{j}$ and $\underline{g} = -10\underline{j}$.

Maximum height $H = \frac{v_{0y}^2}{2g} = \frac{20^2}{2 \cdot 10} = 20.$

Horizontal range $R = \frac{2v_{0x}v_{0y}}{g} = \frac{2 \cdot 10 \cdot 20}{10} = 40.$

10. A car enters a loop with velocity vector $\underline{v} = 30\underline{j}$ and exits with velocity $\underline{v}' = 30\underline{i}$. What is the change in velocity vector?

Solution

Given $\underline{v} = 30\underline{j}$ and $\underline{v}' = 30\underline{i}$.

Change in velocity $\Delta\underline{v} = \underline{v}' - \underline{v} = 30\underline{i} - 30\underline{j}.$

Magnitude of change

$|\Delta\underline{v}| = \sqrt{30^2 + (-30)^2} = 30\sqrt{2}.$

11. An aeroplane has airspeed $\underline{v}_p = 20\underline{i}$ and there is a crosswind $\underline{v}_w = 50\,\underline{j}$. Find the resultant velocity and its magnitude.

Solution

Given $\underline{v}_p = 20\underline{i}$ and $\underline{v}_w = 50\underline{j}$.

Resultant velocity $\underline{v}_r = 20\underline{i} + 50\underline{j}.$

Magnitude $|\underline{v}_r| = \sqrt{20^2 + 50^2} = \sqrt{2900} = 53.85.$

REVIEW EXERCISE 6

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) x-axis and y-axis divide a coordinate plane into ________ parts.
(a) one (b) two (c) three ✓(d) four

(ii) $P(4,-4)$ lies in ________ quadrant.
(a) first (b) second (c) third ✓(d) fourth

(iii) A vector having magnitude 1, is called:
(a) equal vector (b) parallel vector
✓(c) unit vector (d) zero vector

(iv) What is the value of $|3\underline{i}+4\underline{j}|$?
(a) 3 (b) 4 ✓(c) 5 (d) 7

(v) If $\underline{a}=\lambda\underline{b}$, then $\underline{a}$ and $\underline{b}$ are:
(a) equal ✓(b) parallel (c) perpendicular (d) non-parallel

(vi) If $\overrightarrow{OA}=\underline{a}, \overrightarrow{OB}=\underline{b}$, then $\overrightarrow{AB}$ is:
✓(a) $\underline{b}-\underline{a}$ (b) $\underline{a}-\underline{b}$ (c) $\underline{a}+\underline{b}$ (d) $\underline{b}+\underline{a}$

(vii) Translation vector shows:
(a) deformation (b) rotation ✓(c) movement (d) enlargement

(viii) Sum of two vectors is:
(a) a triangle ✓(b) a vector (c) a scalar (d) a length

(ix) The position vector of point P(3, −2) with respect to O is:
(a) $3\underline{i}+2\underline{j}$ ✓(b) $3\underline{i}-2\underline{j}$ (c) $-3\underline{i}+2\underline{j}$ (d) $2\underline{i}-3\underline{j}$

(x) Vector from point P(3, 4) to origin is:
(a) $3\underline{i}+4\underline{j}$ (b) $-3\underline{i}+4\underline{j}$ ✓(c) $-3\underline{i}-4\underline{j}$ (d) $3\underline{i}-4\underline{j}$

2. Find magnitude of the $\overrightarrow{AB}$:

(i) $A(7, 7), B(-12, 0)$ (ii) $A(9, 3), B(2, 11)$

Solution

(i)	(ii)
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$	$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (-12, 0) - (7, 7)$	$= (2, 11) - (9, 3)$
$= (-12 - 7, 0 - 7)$	$= (2 - 9, 11 - 3)$
$= (-19, -7).$	$= (-7, 8).$
$\lvert\overrightarrow{AB}\rvert = \sqrt{(-19)^2 + (-7)^2}$	$\lvert\overrightarrow{AB}\rvert = \sqrt{(-7)^2 + 8^2}$
$= \sqrt{361 + 49} = \sqrt{410}.$	$= \sqrt{49 + 64} = \sqrt{113}.$

3. Find a unit vector in the direction of $\underline{a} = \frac{5}{3}\underline{i} + \frac{1}{3}\underline{j}$.

Solution

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

$$= \frac{\frac{5}{3}\underline{i} + \frac{1}{3}\underline{j}}{\sqrt{(\frac{5}{3})^2 + (\frac{1}{3})^2}}$$

$$= \frac{\frac{5}{3}\underline{i} + \frac{1}{3}\underline{j}}{\sqrt{\frac{25}{9} + \frac{1}{9}}}$$

$$= \frac{\frac{5}{3}\underline{i} + \frac{1}{3}\underline{j}}{\sqrt{\frac{26}{9}}}$$

$$= \frac{1}{\sqrt{26}}(5\underline{i} + \underline{j}).$$

$$= \frac{5}{\sqrt{26}}\underline{i} + \frac{1}{\sqrt{26}}\underline{j}$$

4. If $\underline{a} = 2\underline{i} - \underline{j}$, $\underline{b} = 3\underline{i} + \underline{j}$ and $\underline{c} = 4\underline{i} + \underline{j}$, then find the following vectors:

(i) $5\underline{b} - \underline{a} + \underline{c}$ (ii) $8\underline{a} + \underline{b} + 5\underline{c}$ (iii) $\underline{c} + \underline{b} - 4\underline{a}$

Solution

(i)

$$\begin{aligned} 5\underline{b} - \underline{a} + \underline{c} &= 5(3\underline{i} + \underline{j}) - (2\underline{i} - \underline{j}) + (4\underline{i} + \underline{j}) \\ &= 15\underline{i} + 5\underline{j} - 2\underline{i} + \underline{j} + 4\underline{i} + \underline{j} \\ &= 17\underline{i} + 7\underline{j}. \end{aligned}$$

(ii)

$$\begin{aligned} 8\underline{a} + \underline{b} + 5\underline{c} &= 8(2\underline{i} - \underline{j}) + (3\underline{i} + \underline{j}) + 5(4\underline{i} + \underline{j}) \\ &= 16\underline{i} - 8\underline{j} + 3\underline{i} + \underline{j} + 20\underline{i} + 5\underline{j} \\ &= 39\underline{i} - 2\underline{j}. \end{aligned}$$

(iii)

$$\begin{aligned} \underline{c} + \underline{b} - 4\underline{a} &= (4\underline{i} + \underline{j}) + (3\underline{i} + \underline{j}) - 4(2\underline{i} - \underline{j}) \\ &= 4\underline{i} + \underline{j} + 3\underline{i} + \underline{j} - 8\underline{i} + 4\underline{j} \\ &= -\underline{i} + 6\underline{j}. \end{aligned}$$

5. Find the values of x and y in the following equation.

$$(2x\underline{i} + y\underline{j}) + (-\underline{i} + 5\underline{j}) = \frac{1}{4}\underline{i} - 8\underline{j}$$

Solution

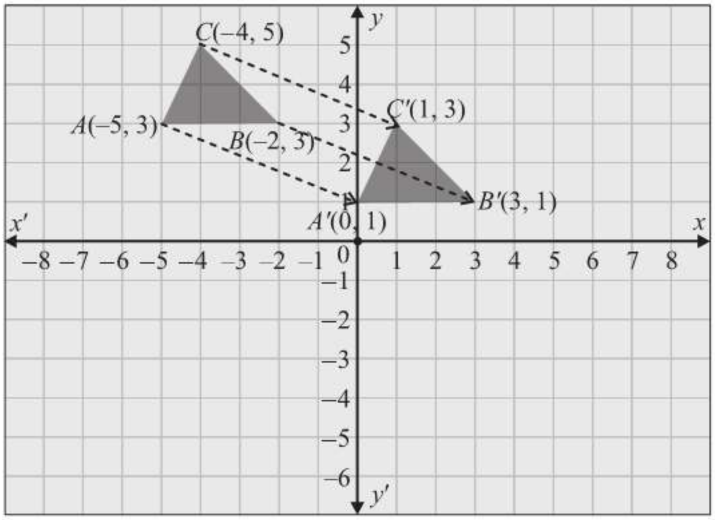
$$\begin{aligned} (2x\underline{i} + y\underline{j}) + (-\underline{i} + 5\underline{j}) &= \frac{1}{4}\underline{i} - 8\underline{j} \\ (2x - 1)\underline{i} + (y + 5)\underline{j} &= \frac{1}{4}\underline{i} - 8\underline{j}. \end{aligned}$$

Equating components,

$$\begin{aligned} 2x - 1 = \frac{1}{4} \quad &\Rightarrow \quad 2x = \frac{5}{4} \quad \Rightarrow \quad x = \frac{5}{8}, \\ y + 5 = -8 \quad &\Rightarrow \quad y = -13. \end{aligned}$$

6. Plot $A(-5, 3)$, $B(-2, 3)$ and $C(-4, 5)$ to form a triangle ABC. Also, translate ΔABC to $\Delta A'B'C'$ by translation vector $5\underline{i} - 2\underline{j}$.

Solution



7. Use vectors to show that $ABCD$ is a parallelogram, where the points are $A(2, 3)$, $B(6, 3)$, $C(7, 6)$ and $D(3, 6)$.

Solution

$$\overrightarrow{AB} = (6-2, 3-3) = (4, 0),$$
$$\overrightarrow{DC} = (7-3, 6-6) = (4, 0).$$

Since $\overrightarrow{AB} = \overrightarrow{DC}$, $ABCD$ is a parallelogram.

8. Use vectors to show that triangle ABC is an isosceles triangle, where the points A, B and C have coordinates $(1, 2)$, $(4, 6)$ and $(7, 2)$ respectively.

Solution

$$\overrightarrow{AB} = (4-1, 6-2) = (3, 4),$$
$$\overrightarrow{AC} = (7-1, 2-2) = (6, 0),$$
$$\overrightarrow{BC} = (7-4, 2-6) = (3, -4).$$

$$|\overrightarrow{AB}| = \sqrt{3^2 + 4^2} = 5,$$
$$|\overrightarrow{AC}| = \sqrt{6^2 + 0^2} = 6,$$
$$|\overrightarrow{BC}| = \sqrt{3^2 + (-4)^2} = 5.$$

Since $|\overrightarrow{AB}| = |\overrightarrow{BC}|$, triangle ABC is isosceles.

9. A ball is projected with velocity vector $\underline{v} = 6\underline{i} + 8\underline{j}$. What is the magnitude of velocity?

Solution

$$\begin{aligned}|\underline{v}| &= |6\underline{i} + 8\underline{j}| \\ &= \sqrt{6^2 + 8^2} \\ &= \sqrt{36 + 64} = \sqrt{100} = 10.\end{aligned}$$

10. An aircraft is flying due east with an airspeed of 200 km/h. There is a wind blowing due north at 60 km/h. Find the resultant velocity and its magnitude.

Solution

$$\begin{aligned}\underline{v}_{\text{resultant}} &= 200\underline{i} + 60\underline{j}, \\ |\underline{v}_{\text{resultant}}| &= \sqrt{200^2 + 60^2} \\ &= \sqrt{40000 + 3600} = \sqrt{43600} = 208.8 \text{ k}\end{aligned}$$

UNIT 7 Trigonometry

EXERCISE 7.1

1. Find the signs of the following:

(i)	$\sin 55^\circ$	(ii)	$\cos 145^\circ$	(iii)	$\tan 111^\circ$
(iv)	$\sec 179^\circ$	(v)	$\text{cosec } 88^\circ$	(vi)	$\cot 14^\circ$

Solution

1. **(i)** Positive **(ii)** Negative **(iii)** Negative **(iv)** Negative **(v)** Positive **(vi)** Positive

2. Fill in the blanks:

Solution

(i)	$\tan(180^\circ - \theta) = \underline{\;-\;}\tan\theta$	(ii)	$\sin(180^\circ - \theta) = \underline{\;+\;}\sin\theta$
(iii)	$\tan(90^\circ + \theta) = \underline{\;-\;}\cot\theta$	(iv)	$\cos(90^\circ + \theta) = \underline{\;-\;}\sin\theta$
(v)	$\text{cosec}(180^\circ - \theta) = \underline{\;+\;}\text{cosec}\,\theta$	(vi)	$\sec(90^\circ - \theta) = \underline{\;+\;}\text{cosec}\,\theta$

3. Without using calculator, find the exact values of the following trigonometric functions:

(i)	$\sin 150^\circ$	(ii)	$\tan 150^\circ$	(iii)	$\sec 150^\circ$
(iv)	$\text{cosec } 120^\circ$	(v)	$\cos 120^\circ$	(vi)	$\cot 120^\circ$
(vii)	$\sin 135^\circ$	(viii)	$\sec 135^\circ$	(ix)	$\cot 135^\circ$

Solution

$\boldsymbol{sin150}^\circ = sin(1\times 90^\circ + 60^\circ) = cos60^\circ = \frac{1}{2}$	$\boldsymbol{tan150}^\circ = tan(1\times 90^\circ + 60^\circ) = -cot60^\circ = -\frac{1}{\sqrt{3}}$
$\boldsymbol{sec150}^\circ = sec(1\times 90^\circ + 60^\circ) = -cosec60^\circ = -\frac{2}{\sqrt{3}}$	$\boldsymbol{cosec120}^\circ = cosec(1\times 90^\circ + 30^\circ) = sec30^\circ = \frac{2}{\sqrt{3}}$
$\boldsymbol{cos120}^\circ = cos(1\times 90^\circ + 30^\circ) = -sin30^\circ = -\frac{1}{2}$	$\boldsymbol{cot120}^\circ = cot(1\times 90^\circ + 30^\circ) = -tan30^\circ = -\frac{1}{\sqrt{3}}$
$\boldsymbol{sin135}^\circ = sin(1\times 90^\circ + 45^\circ) = cos45^\circ = \frac{1}{\sqrt{2}}$	$\boldsymbol{sec135}^\circ = sec(1\times 90^\circ + 45^\circ) = -cosec45^\circ = -\sqrt{2}$
$\boldsymbol{cot135}^\circ = cot(1\times 90^\circ + 45^\circ) = -tan45^\circ - 1$	

EXERCISE 7.2

1. Solve the triangle ABC, in which

(i) $a = 6.1$ cm, $b = 8.4$ cm, $\alpha = 42°$

(ii) $a = 12.2$ cm, $c = 15.8$ cm, $\gamma = 50°$

(iii) $b = 5.2$ cm, $c = 5$ cm, $\gamma = 48°$ (iv) $b = 4.8$ cm, $a = 4$ cm, $\beta = 71°$

(v) $\beta = 70°$, $b = 8$cm, $\alpha = 100°$ (vi) $a = 6$ cm, $\alpha = 55°$, $\gamma = 60°$

(vii) $c = 7$ cm, $\beta = 34°$, $\gamma = 64°$ (viii) $b = 12$cm, $\alpha = 92°$, $\beta = 77°$

Solution

(i) $a = 6.1$, $b = 8.4$, $\alpha = 42°$

Law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{8.4\sin 42°}{6.1} = 0.921.$$

$$\beta = \sin^{-1}(0.921) = 67.1°.$$

$$\gamma = 180° - \alpha - \beta = 180° - 42° - 67.1° = 70.9$$

$$c = \frac{a\sin\gamma}{\sin\alpha} = \frac{6.1\sin 70.9°}{\sin 42°} = 8.8.$$

(ii) $a = 12.2$, $c = 15.8$, $\gamma = 50°$

Law of sines:

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}.$$

$$\sin\alpha = \frac{a\sin\gamma}{c} = \frac{12.2\sin 50°}{15.8} = 0.591.$$

$$\alpha = \sin^{-1}(0.591) = 36.2°.$$

$$\beta = 180° - \alpha - \gamma = 180° - 36.2° - 50° = 93.8$$

$$b = \frac{c\sin\beta}{\sin\gamma} = \frac{15.8\sin 93.8°}{\sin 50°} = 20.6.$$

(iii) $b = 5.2$, $c = 5$, $\gamma = 48°$

Law of sines:

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

$$\sin\beta = \frac{b\sin\gamma}{c} = \frac{5.2\sin 48°}{5} = 0.773.$$

$$\beta = \sin^{-1}(0.773) = 50.6°.$$

$$\alpha = 180° - \beta - \gamma = 180° - 50.6° - 48° = 81.4$$

$$a = \frac{b\sin\alpha}{\sin\beta} = \frac{5.2\sin 81.4°}{\sin 50.6°} = 6.61.$$

(iv) $b = 4.8$, $a = 4$, $\beta = 71°$

Law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$$

$$\sin\alpha = \frac{a\sin\beta}{b} = \frac{4\sin 71°}{4.8} = 0.788.$$

$$\alpha = \sin^{-1}(0.788) = 52.0°.$$

$$\gamma = 180° - \alpha - \beta = 180° - 52.0° - 71° = 57.0$$

$$c = \frac{a\sin\gamma}{\sin\alpha} = \frac{4\sin 57.0°}{\sin 52.0°} = 4.20.$$

(v) $\beta = 70^\circ$, $b = 8$, $\alpha = 100^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 100^\circ - 70^\circ = 10^\circ.$$

Law of sines:

$$a = \frac{b\sin\alpha}{\sin\beta} = \frac{8\sin 100^\circ}{\sin 70^\circ} = 8.39.$$

$$c = \frac{b\sin\gamma}{\sin\beta} = \frac{8\sin 10^\circ}{\sin 70^\circ} = 1.48.$$

(vi) $a = 6$, $\alpha = 55^\circ$, $\gamma = 60^\circ$

$$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 55^\circ - 60^\circ = 65^\circ.$$

Law of sines:

$$b = \frac{a\sin\beta}{\sin\alpha} = \frac{6\sin 65^\circ}{\sin 55^\circ} = 6.64.$$

$$c = \frac{a\sin\gamma}{\sin\alpha} = \frac{6\sin 60^\circ}{\sin 55^\circ} = 6.34.$$

(vii) $c = 7$, $\beta = 34^\circ$, $\gamma = 64^\circ$

$$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 34^\circ - 64^\circ = 82^\circ.$$

Law of sines:

$$a = \frac{c\sin\alpha}{\sin\gamma} = \frac{7\sin 82^\circ}{\sin 64^\circ} = 7.72.$$

$$b = \frac{c\sin\beta}{\sin\gamma} = \frac{7\sin 34^\circ}{\sin 64^\circ} = 4.35.$$

(viii) $b = 12$, $\alpha = 92^\circ$, $\beta = 77^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 92^\circ - 77^\circ = 11^\circ.$$

Law of sines:

$$a = \frac{b\sin\alpha}{\sin\beta} = \frac{12\sin 92^\circ}{\sin 77^\circ} = 12.3.$$

$$c = \frac{b\sin\gamma}{\sin\beta} = \frac{12\sin 11^\circ}{\sin 77^\circ} = 2.29.$$

2. Calculate area of each triangle ABC.

(i) $a = 7$ cm, $b = 8$ cm, $\gamma = 38^\circ$ (ii) $a = 11$ cm, $c = 14$ cm, $\beta = 51^\circ$

(iii) $b = 3$ cm, $c = 9$ cm, $\alpha = 78^\circ$ (iv) $a = 10$ cm, $\alpha = 62^\circ$, $\beta = 69^\circ$

(v) $c = 4$ cm, $\beta = 36^\circ$, $\gamma = 80^\circ$ (vi) $c = 6.6$ cm, $\alpha = 23^\circ$, $\gamma = 89^\circ$

(vii) $a = 5.3$ cm, $b = 4.7$ cm, $c = 8.2$ cm (viii) $a = 6.12$ cm, $b = 8.34$ cm, $c = 7.12$ cm

Solution

(i) $a = 7$, $b = 8$, $\gamma = 38^\circ$

Area formula:

$$\text{Area} = \frac{1}{2}ab\sin\gamma.$$

$$\text{Area} = \frac{1}{2}(7)(8)\sin 38^\circ = 17.24\ \text{cm}^2$$

(ii) $a = 11$, $c = 14$, $\beta = 51^\circ$

$$\text{Area} = \frac{1}{2}ac\sin\beta.$$

$$\text{Area} = \frac{1}{2}(11)(14)\sin 51^\circ = 59.82\ \text{cm}^2$$

(iii) $b = 3$, $c = 4$, $\alpha = 78^\circ$

$$\text{Area} = \frac{1}{2}bc\sin\alpha.$$

$$\text{Area} = \frac{1}{2}(3)(4)\sin 78^\circ = 13.21\ \text{cm}^2$$

(iv) $a = 10$, $\alpha = 62^\circ$, $\beta = 69^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 62^\circ - 69^\circ = 49^\circ.$$

Area formula:

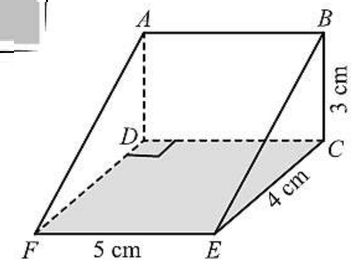
$$\text{Area} = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha}.$$

$$\text{Area} = \frac{10^2\sin 69^\circ\sin 49^\circ}{2\sin 62^\circ} = 39.89\ \text{cm}^2$$

(v) $c=4$, $\beta=36^\circ$, $\gamma=80^\circ$ $\alpha=180^\circ-\beta-\gamma=180^\circ-36^\circ-80^\circ=64^\circ$ $\text{Area}=\dfrac{c^2\sin\alpha\sin\beta}{2\sin\gamma}.$ $\text{Area}=\dfrac{4^2\sin 64^\circ\sin 36^\circ}{2\sin 80^\circ}=4.29\text{ cm}^2$	**(vi)** $c=6.6$, $\alpha=23^\circ$, $\gamma=89^\circ$ $\beta=180^\circ-\alpha-\gamma=180^\circ-23^\circ-89^\circ=68^\circ.$ $\text{Area}=\dfrac{c^2\sin\alpha\sin\beta}{2\sin\gamma}.$ $\text{Area}=\dfrac{6.6^2\sin 23^\circ\sin 68^\circ}{2\sin 89^\circ}=7.89\text{ cm}^2$
(vii) $a=5.3\text{ cm}, b=4.7\text{ cm}, c=8.2\text{ cm}$ $S=\frac{a+b+c}{2}=\frac{5.3+4.7+8.2}{2}=9.1$ $\Delta=\sqrt{S(S-a)(S-b)(S-c)}$ $\Delta=\sqrt{9.1(9.1-5.3)(9.1-4.7)(9.1-8.2)}$ $\Delta=\sqrt{9.1(3.8)(4.4)(0.9)}$ $\Delta=\sqrt{136.94}$ $\Delta=11.70\text{cm}^2$	(viii) $a=6.12\text{ cm}, b=8.34\text{ cm}, c=7.12\text{ cm}$ $S=\frac{a+b+c}{2}=\frac{6.12+8.34+7.12}{2}=10.79$ $\Delta=\sqrt{S(S-a)(S-b)(S-c)}$ $\Delta=$ $\sqrt{10.79(10.79-6.12)(10.79-8.34)(10.79-7.12)}$ $\Delta=\sqrt{10.79(4.67)(2.45)(3.67)}$ $\Delta=\sqrt{453.1}$ $\Delta=21.29\text{cm}^2$

EXERCISE 7.3

1. In the triangular prism, find
   (i) the length $\overline{CF}$.
   (ii) the length $\overline{BF}$.
   (iii) the angle BFC, correct to one decimal place.

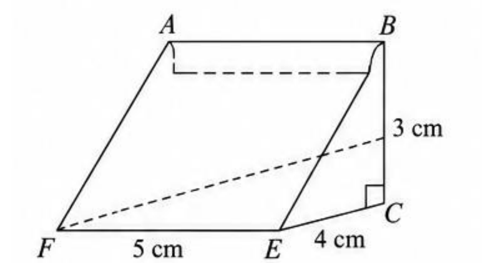


Solution

(i) In right ΔFEC,

$$CF^2 = FE^2 + EC^2 = 5^2 + 4^2 = 25 + 16 = 41$$

$$CF = \sqrt{41} \approx \boxed{6.4\text{ cm}}$$



(ii) In right ΔBCF,

$$BF^2 = BC^2 + CF^2 = 3^2 + 41 = 9 + 41 = 50$$

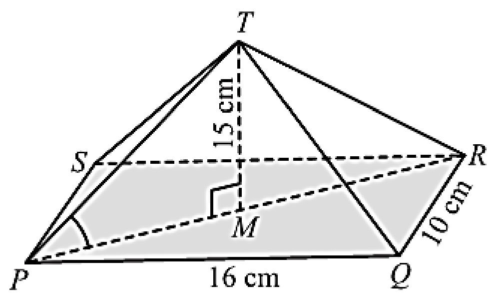
$$BF = \sqrt{50} = 5\sqrt{2} \approx \boxed{7.1\text{ cm}}$$

(iii) In right ΔBCF,

$$\tan \angle BFC = \frac{BC}{CF} = \frac{3}{6.4} = 0.46875$$

$$\angle BFC = \tan^{-1}(0.46875) \approx \boxed{25.1^\circ}$$

2. The diagram shows a triangular pyramid with a horizontal rectangular base $PQRS$, in which $m\overline{PQ} = 16$ cm, $m\overline{QR} = 10$ cm. M is the midpoint of the line PR. The vertex, T, is vertically above M and $m\overline{MT} = 15$ cm. Calculate the size of the angle between TP and the base $PQRS$. Give your answer correct to 1 decimal place.



Solution

In rectangle $PQRS$,

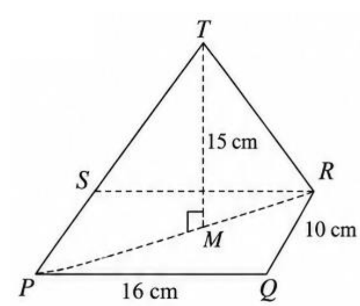
$$PR = \sqrt{PQ^2 + QR^2} = \sqrt{16^2 + 10^2} = \sqrt{356} \approx 18.9\text{ cm}$$

Since M is midpoint of PR, $PM = \frac{PR}{2} \approx \frac{18.9}{2} = 9.45$ cm

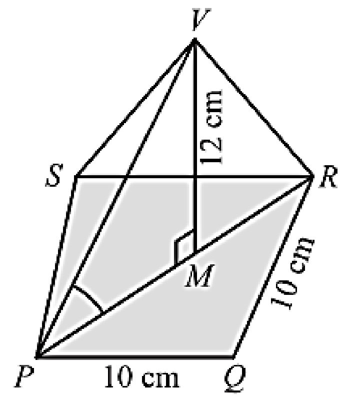
In right ΔTPM,

$$\tan\theta = \frac{MT}{PM} = \frac{15}{9.45} = 1.587$$

$$\theta = \tan^{-1}(1.587) \approx \boxed{57.8^\circ}$$



3. The diagram shows a pyramid. The base, *PQRS*, is a horizontal square of side 10 cm. The vertex, *V*, is vertically above the midpoint, *M* and $m\overline{VM} = 12$ cm. Calculate the size of angle *VPM*.



Solution

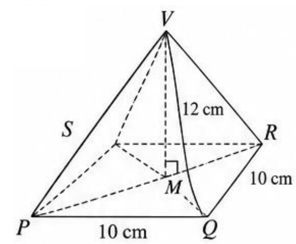
In square *PQRS*, diagonal $PR = \sqrt{10^2 + 10^2} = \sqrt{200} = 14.14$ cm

Since *M* is midpoint of *PR*, $PM = \dfrac{PR}{2} = \dfrac{14.14}{2} = 7.07$ cm

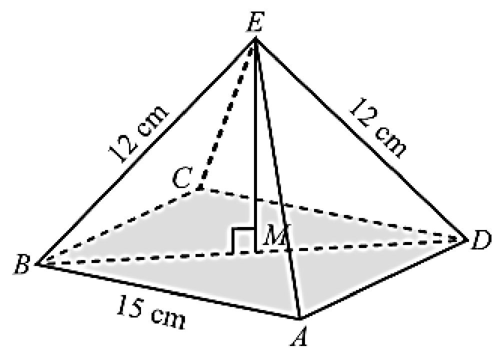
In right ΔVPM,

$$\tan\theta = \frac{VM}{PM} = \frac{12}{7.07} = 1.697$$

$$\theta = \tan^{-1}(1.697) \approx \boxed{59.5^\circ}$$



4. *ABCDE* is a square based pyramid, in which $m\overline{AE} = m\overline{BE} = m\overline{CE} = m\overline{DE} = 12$ cm and $m\overline{AB} = 15$ cm. Calculate the size of angle *DEB*. Give your answer in degree (whole numbers).



Solution

In square base *ABCD*, diagonal $BD = \sqrt{15^2 + 15^2} = \sqrt{450} = 15\sqrt{2} \approx 21.21$ cm

In ΔDEB, $DE = BE = 12$ cm and $BD \approx 21.21$ cm

By cosine rule,

$$BD^2 = DE^2 + BE^2 - 2(DE)(BE)\cos\angle DEB$$

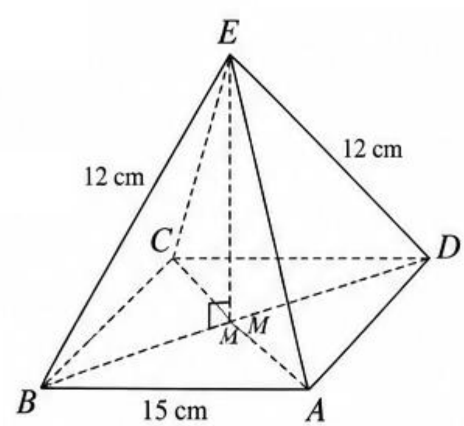
$$(21.21)^2 = 12^2 + 12^2 - 2(12)(12)\cos\angle DEB$$

$$450 = 288 - 288\cos\angle DEB$$

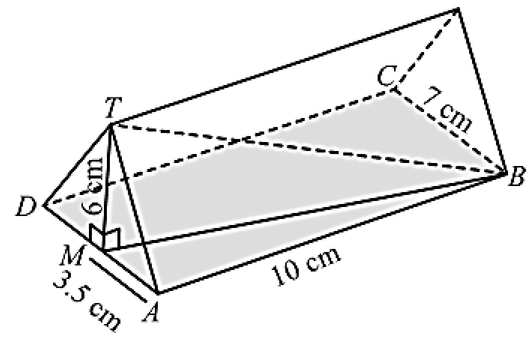
$$288\cos\angle DEB = 288 - 450 = -162$$

$$\cos\angle DEB = -\frac{162}{288} = -0.5625$$

$$\angle DEB = \cos^{-1}(-0.5625) \approx \boxed{124^\circ}$$



5. The diagram shows a triangular prism with a horizontal rectangular base $ABCD$. $m\overline{AB} = 10$ cm, $m\overline{BC} = 7$ cm, M is the midpoint of AD. The vertex T is vertically above M and $m\overline{MT} = 6$ cm. Calculate the size of the angle between $\overline{TB}$ and the base.



Solution

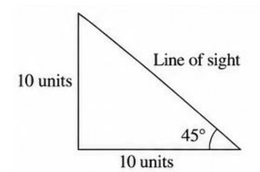
In right ΔTMB

$MB = \sqrt{AM^2 + AB^2} = \sqrt{3.5^2 + 10^2} = 10.60cm$

$tan\theta = \frac{MT}{MB} = \frac{6}{10.60} = 0.5668$

$\theta = tan^{-1}(0.5668) = 29.5°$

6. In an isometric game, the camera is placed at a 45° angle from the ground. If the player is 10 units in front and 10 units above, what is the direct line of sight distance?



Solution

Using Pythagoras theorem,

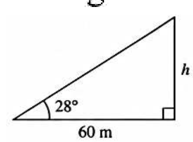
Distance $= \sqrt{10^2 + 10^2} = \sqrt{200} = 10\sqrt{2} \approx$ **14.14 units**

7. A surveyor spots the top of a tower at an elevation angle of 28°. He is standing 60 metres from the base. Find the height of the tower.

Solution

$\tan 28° = \frac{height}{60}$

height $= 60 \tan 28° = 60 \times 0.5317 \approx$ **31.90 m**



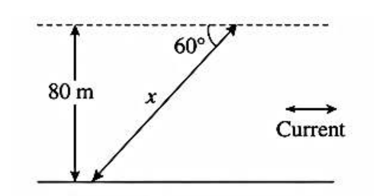
8. A boat crosses a river 80 m wide, at a 60° angle to the current. What distance does the boat actually travel?

Solution

$cos60° = \frac{80}{x}$

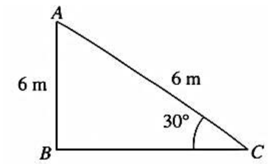
$x = \frac{80}{cos60°}$

$x = \frac{80}{\frac{1}{2}} = 160m$



9. A listener hears a sound from two speakers. One is 6 m directly ahead and the other is at 30° to the side, 6 m away. Find the distance between the speakers.

Solution



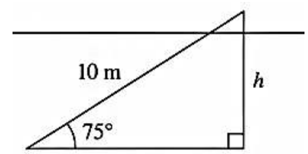
Using cosine rule,

$$d^2 = 6^2 + 6^2 - 2(6)(6)\cos 30^\circ$$
$$= 72 - 72\left(\frac{\sqrt{3}}{2}\right) = 72\left(1 - \frac{\sqrt{3}}{2}\right)$$

$d^2 = 9.65 \Rightarrow d = \sqrt{9.65} = 3.1m$

10. A 10 m ladder leans against a wall making an angle of 75° with the ground. How high does it reach up the wall?

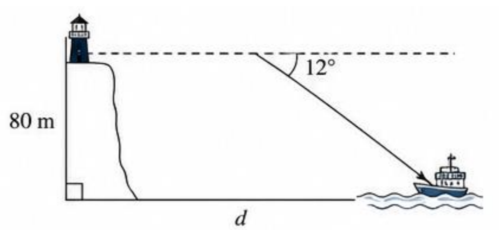
Solution



$$\text{Height} = 10\sin 75^\circ = 10 \times 0.9659 \approx \boxed{9.66 \text{ m}}$$

11. A lighthouse is located on a cliff 80 m above sea level. A ship is spotted at an angle of depression of 12°. How far is the ship from the base of the cliff?

Solution



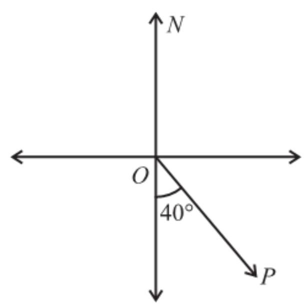
$$\tan 12^\circ = \frac{80}{d}$$
$$d = \frac{80}{\tan 12^\circ} = \frac{80}{0.2126} \approx \boxed{376.5 \text{ m}}$$

EXERCISE 7.4

1. Find bearing of point P in each of the following:

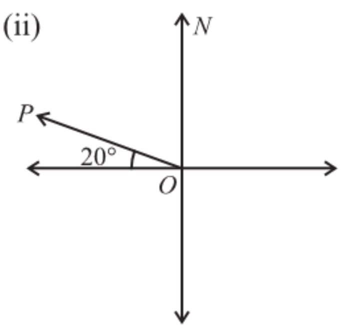
Solution

(i)



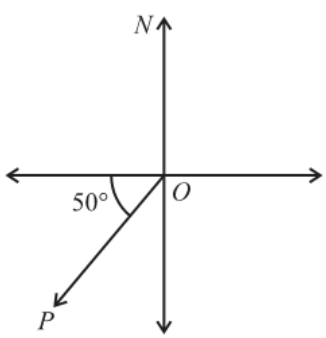
Bearing of $\overrightarrow{OP} = 180° - 40° = 140°$

(ii)



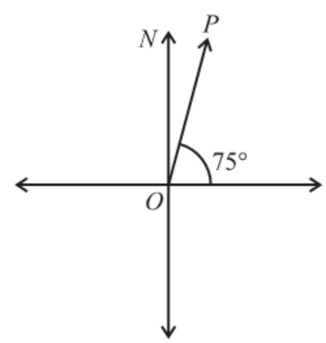
Bearing of $\overrightarrow{OP} = 270° + 20° = 290°$

(iii)



Bearing of $\overrightarrow{OP} = 180° + 40° = 220°$

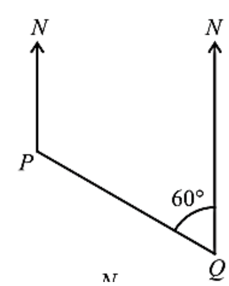
(iv)



Bearing of $\overrightarrow{OP} = 090° - 75° = 015°$

Or Bearing of $\overrightarrow{OP} = 270° - 50° = 220°$

2. The diagram shows the positions of two ships P and Q.
   (i) What is the bearing of ship P from ship Q?
   (ii) What is bearing of ship Q from ship P?



Solution

i. Bearing of ship $\vec{P}$ from ship $\vec{Q} = 360° - 60° = 300°$
ii. Bearing of ship $\vec{Q}$ from ship $\vec{P} = 300° - 180° = 120°$

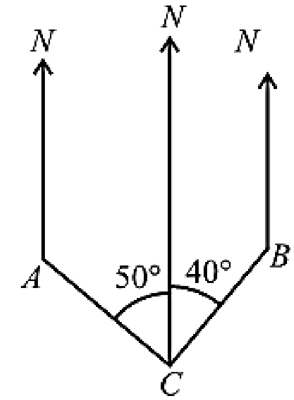
3. The diagram shows 3 places A, B and C. Find the bearing of:

(i) A from C

(ii) C from A

(iii) C from B

(iv) B from C



Solution

i. Bearing of $\vec{A}$ from $\vec{C} = 360° - 50° = 310°$

ii. Bearing of $\vec{C}$ from $\vec{A} = 180° - 50° = 130°$

iii. Bearing of $\vec{C}$ from $\vec{B} = 180° + 40° = 220°$

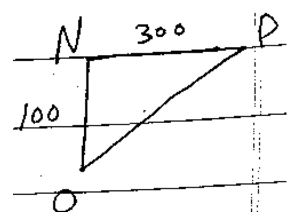
iv. Bearing of $\vec{B}$ from $\vec{C} = 040°$

4. Abdul Hadi walks 100 m North and then 300 m East.

(i) How far is he from his starting position?

(ii) On what bearing should he walk to get back to his starting position?

Solution



(i) Distance from starting position:

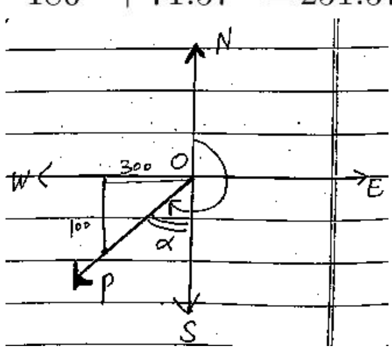
$$\sqrt{100^2 + 300^2} = \sqrt{10000 + 90000} = 316.2 \text{ m}$$

(ii) Bearing to return to start:

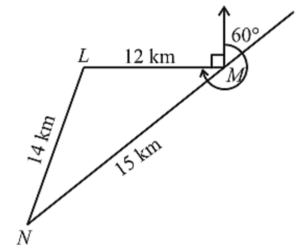
$$\tan\theta = \frac{300}{100} \implies \theta = \tan^{-1}(3) = 71.57°.$$

Bearing is **S** $71.57°$ **W** (or $252°$

$180° + 71.57° = 251.57°$).



5. Three ships L, M, N are in the position shown in the diagram. Ship M is North East of ship N. Find the bearing of L from M.



Solution

Bearing of $\vec{L}$ from $\vec{M} = 270°$

Or another solution

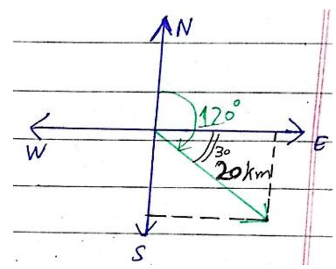
At M, MN is on a bearing $225°$ (from M to N).

ML is $90°$ to the left of MN.

Bearing of L from M

$= 225° - 90° = 135°$

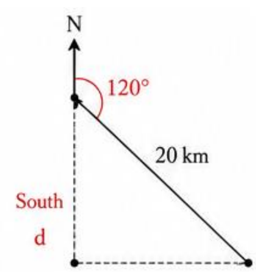
6. A ship sails 20 km on a bearing of 120°. How far to the South has the ship moved from its original position?



Solution

Ship movement from its original position from the south $= 20Sin30° = 10\text{km}$

Or another solution



Southward component

$= 20 \times \cos(120° - 180°)$

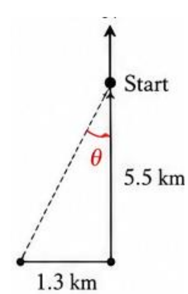
$= 20 \times \cos 60°$

$= 20 \times \frac{1}{2}$

$= 10$ km

7. Fatima walked South for 5.5 km and then turned West for 1.3 km. Calculate Huria's bearing from her starting point.

Solution



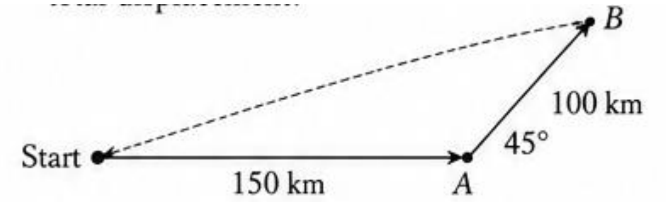
In ΔOPS

$tan \propto = \frac{1.3}{5.5} \Rightarrow \propto = tan^{-1}(0.236) \Rightarrow \propto = 13.29°$

Horia's bearing from Start $= 180° + 13.29° = 193.2°$

8. An aircraft flies 150 km east, then 100 km northeast (45° from East). What is the total displacement?

Solution



Resultant components:

East $= 150 + 100\cos 45° = 220.71$ km

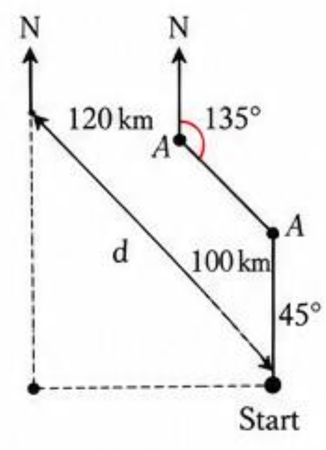
North $= 100\sin 45° = 70.71$ km

Displacement $= \sqrt{(220.71)^2 + (70.71)^2}$

$= 231.91$ km

9. A ship sails 100 km on a bearing of 045°, then changes course and sails 120 km on a bearing of 135°. Find the distance between the starting point and the final position.

Solution



Angle between the two courses at $A = 135° - 45° = 90°$

Using cosine rule:

$d^2 = 100^2 + 120^2 - 2(100)(120)\cos 90°$

$= 10000 + 14400 - 0$

$d = \sqrt{24400} = 156.2$ km

REVIEW EXERCISE 7

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) What is the value of cot 60°?

(a) 0 (b) 1 (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

(ii) All trigonometric ratios are positive in:

(a) I-quadrant (b) II-quadrant
(c) III-quadrant (d) IV-quadrant

(iii) cosec θ is positive in:

(a) I & III-quadrants (b) II & IV-quadrants
(c) I & II-quadrants (d) I & IV-quadrants

(iv) $\sin(90^\circ + \theta) =$

(a) $\sin\theta$ (b) $-\sin\theta$ (c) $\cos\theta$ (d) $-\cos\theta$

(v) $\tan(180^\circ - \theta) =$

(a) $\tan\theta$ (b) $\cot\theta$ (c) $-\cot\theta$ (d) $-\tan\theta$

(vi) The law of sines is:

(a) $\frac{b}{\sin\alpha} = \frac{a}{\sin\beta} = \frac{c}{\sin\gamma}$ (b) $\frac{a}{\sin\alpha} = \frac{c}{\sin\beta} = \frac{b}{\sin\gamma}$

(c) $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$ (d) $\frac{a}{\sin\alpha} = \frac{a}{\sin\beta} = \frac{a}{\sin\gamma}$

c

(vii) The law of cosines is:

(a) $\cos\alpha = \frac{b^2 + a^2 - c^2}{2ab}$ (b) $\cos\beta = \frac{c^2 + b^2 - a^2}{2cb}$

(c) $\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$ (d) $\cos\gamma = \frac{ca^2 + b^2 - c^2}{2ab}$

(viii) Area of $\Delta ABC =$

(a) $\frac{1}{2}ab\sin\alpha$ (b) $\frac{1}{2}bc\sin\beta$ (c) $\frac{1}{2}\sin\gamma$ (d) $\frac{1}{2}ac\sin\beta$

(ix) Bearing is measured from:

(a) East (b) West (c) North (d) South

(x) Bearing is written as a:

(a) 1 figure (b) 2 figures (c) 3 figures (d) 4 figures

2. Calculate the area of ΔABC, in which

(i) $a = 4$ cm, $b = 6$ cm, $c = 8$ cm (ii) $b = 2.1$ cm, $c = 5$ cm, $\gamma = 45°$

(iii) $c = 3.1$ cm, $\gamma = 44°$, $\alpha = 36°$

Solution

Q2 (i)

$$S = \frac{a+b+c}{2} = \frac{4\ +\ 6\ +\ 8}{2} = 9$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$\text{Area} = \sqrt{[9(9-4)(9-6)(9-8)]}$$

$$= \sqrt{135}$$

$$\approx 11.62 \text{ cm}^2$$

Q2 (ii)

Area = ½ × 2.1 × 5 × sin45°

= 5.25 × 0.707

≈ 3.71 cm^2

Q2 (iii)

β = 180° − 44° − 36° = 100°

$$\boldsymbol{a} = \frac{\mathbf{3.1} \times \boldsymbol{sin}\mathbf{36}°}{\boldsymbol{sin}\mathbf{44}°} = \mathbf{2.623}$$

$$\boldsymbol{b} = \frac{\mathbf{3.1} \times \boldsymbol{sin}\mathbf{100}°}{\boldsymbol{sin}\mathbf{44}°} = \mathbf{4.394}$$

$$\text{Area} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{absin}\mathbf{44}° = \mathbf{4.01cm}^2$$

3. Solve the triangle ABC, in which

(i) $a = 5.4$ cm, $b = 3.4$ cm, $\alpha = 49°$ (ii) $\alpha = 32°$, $\gamma = 48°$, $c = 81$cm

Solution

Q3 (i)

(i) $a = 5.4$ cm, $b = 3.4$ cm, $\alpha = 49°$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{3.4\sin 49°}{5.4} = 0.4755$$

$$\beta = 28.39°$$

$$\gamma = 180° - \alpha - \beta = 102.61°$$

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha} = 6.48 \text{ cm}$$

Q3 (ii)

β = 180° − 32° − 48° = 100°

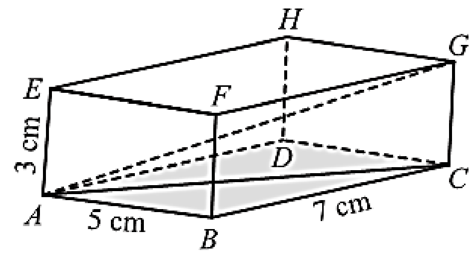
a / sin32° = 81 / sin48°

a ≈ 57.8 cm

b / sin100° = 81 / sin48°

b ≈ 107.3 cm

4. The diagram shows a cuboid $ABCDEFGH$ in which $m\overline{AB} = 5$ cm, $m\overline{BC} = 7$ cm and $m\overline{AE} = 3$ cm.



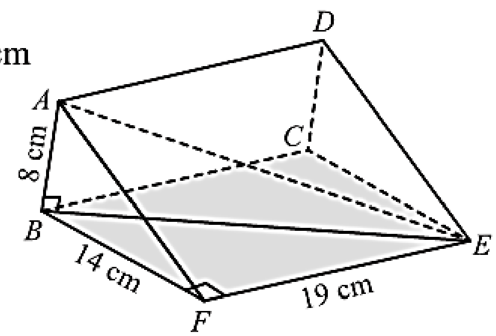
(i) Calculate the length of $\overline{AG}$.
Give your answer correct to 3 significant figures.

(ii) Calculate the size of the angle between $\overline{AG}$ and the plane $ABCD$.
Give your answer correct to 1 decimal place.

Solution

In a cuboid, $\overline{AG}$ is the space diagonal. $\overline{AG} = \sqrt{(AB)^2 + (BC)^2 + (AE)^2}$ $= \sqrt{5^2 + 7^2 + 3^2} = \sqrt{83} = 9.11$ cm	Let D be the foot of the perpendicular from G to the plane $ABCD$. In right triangle ADG, $\tan\theta = \frac{AE}{AD} = \frac{3}{\sqrt{5^2+7^2}} = \frac{3}{\sqrt{74}}$ $\theta = 20.9°$

5. The diagram shows a triangular prism $ABCDEF$ in which $m\overline{AB} = 8$ cm, $m\overline{BF} = 14$ cm and $m\overline{EF} = 19$ cm.



(i) Calculate the distance between A and F.

(ii) Calculate the angle between $\overline{AF}$ and the plane $BCEF$.

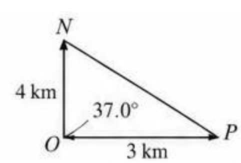
Solution

In right triangle ABF (right-angled at B), $AF = \sqrt{AB^2 + BF^2} = \sqrt{8^2 + 14^2} = \sqrt{260} = 16.1$ cm	In right triangle AFC (right-angled at F), $\sin\theta = \frac{AF}{AC} = \frac{AB}{AC} = \frac{8}{16.1}$ $\theta = 29.7°$

6. Hashim walks 4 km due North, then turns and walks 3km due East. What is the bearing from the starting point to his final position?

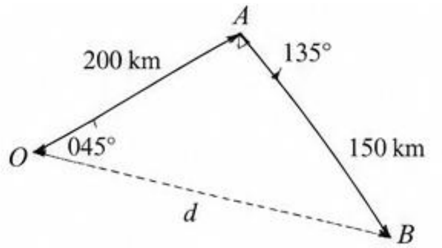
Solution

Let the starting point be O, final position be P. Then $OP = \sqrt{4^2 + 3^2} = 5$ km.
The direction of OP is 37.0° east of North.
Bearing = 037°



7. A pilot flies 200 km on a bearing of 045°, then turns and flies 150 km on a bearing of 135°. How far is the plane from its original position?

Solution



The angle between the two bearings is $135° - 45° = 90°$.

Using the cosine rule,

$$d^2 = 200^2 + 150^2 - 2(200)(150)\cos 90° = 40000 + 22500 - 0 = 62500$$

$$d = \sqrt{62500} = 250 \text{ km}$$

UNIT 8

Chords and Arcs of a Circle

EXERCISE 8.1

1. Calculate the length of a chord which stands at a distance of 5 cm from the centre of a circle whose radius is 13 cm.

Solution

$$\begin{aligned}\text{Chord} &= 2\sqrt{r^2 - d^2}\\ &= 2\sqrt{13^2 - 5^2}\\ &= 2\sqrt{169 - 25}\\ &= 2\sqrt{144}\\ &= 24 \quad \text{cm}\end{aligned}$$

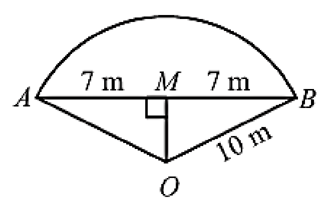
2. In construction, three steel rods are fixed at points A, B, and C (not in a straight line). A circular hoop needs to pass through all three. How many hoops can be used?

Solution

Three non-collinear points determine a unique circle passing through them.

$$\text{Number of hoops} = 1.$$

3. In a park, lamp posts are 14 m apart on the edge of a circular part of radius 10 m as shown in figure. Find the distance of the chord from the centre of the park.



Solution

$$\begin{aligned}d &= \sqrt{r^2 - (\frac{\text{chord}}{2})^2}\\ &= \sqrt{10^2 - 7^2}\\ &= \sqrt{100 - 49}\\ &= \sqrt{51}\\ &\approx 7.14 \text{ m}\end{aligned}$$

4. In a circle, chords AB and CD both have length 10 cm.
If the distance from the centre to $\overline{AB}$ is 6 cm, what is the distance from centre to $\overline{CD}$?

Solution

Chords AB and CD have equal length
they are equidistant from the centre.

$$d_{AB} = 6$$
$$d_{CD} = d_{AB} = 6 \text{ cm}$$

5. Two holes A and B are drilled 12 cm apart on a circular tabletop of radius 10 cm. Find the perpendicular distance from the centre to AB.

Solution

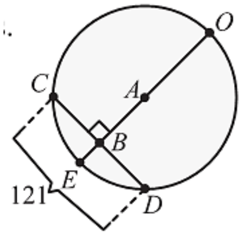
$$d = \sqrt{r^2 - (\frac{\text{chord}}{2})^2}$$
$$= \sqrt{10^2 - 6^2}$$
$$= \sqrt{100 - 36}$$
$$= \sqrt{64}$$
$$= 8 \text{ cm}$$

6. A chord 8 cm long is at a distance of 3 cm from the centre. Calculate the radius of the circle.

Solution

$$r = \sqrt{d^2 + (\frac{\text{chord}}{2})^2}$$
$$= \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5 \text{ cm}$$

7. In the given figure, $m\overline{CD} = 121$ units and $m\overline{BC} = 3x$ units. Find the value of x.



Solution

Since $\overline{CD}$ and $\overline{CB}$ are chords in the circle with a diameter $\overline{OCD}$, and angles in the same segment are equal, we have

$m\widehat{BC} = \frac{1}{2}m\widehat{CD}$

$3x = \frac{1}{2}(121°)$

$x = \frac{121}{6} = 20.17$units

8. In a circle with centre at O, the perpendicular distance of the each chord PQ and RS from the centre is 6 cm. If the length of chord PQ is 18 cm, find the length of the other chord.

Solution

$d = 6, \quad PQ = 18.$

$r^2 = d^2 + \left(\frac{PQ}{2}\right)^2.$

$r^2 = 6^2 + 9^2.$

$r^2 = 36 + 81 = 117.$

$RS = 2\sqrt{r^2 - d^2}.$

$RS = 2\sqrt{117 - 6^2}.$

$RS = 2\sqrt{117 - 36}.$

$RS = 2\sqrt{81}.$

$RS = 18$ cm

9. A line from the centre of a circle cuts a 10 cm chord at right angle where radius of the circle is 6cm. What is the length from the centre to the chord?

Solution

$r = 6, \quad \text{chord} = 10.$

$d^2 = r^2 - \left(\frac{10}{2}\right)^2.$

$d^2 = 6^2 - 5^2.$

$d^2 = 36 - 25.$

$d = \sqrt{11}. \qquad = 3.32 \text{ cm}$

10. In a circle, a perpendicular is drawn from the centre to chord AB. If $m\overline{AB} = 12\,\text{cm}$, what is the length of each segment after bisecting?

Solution

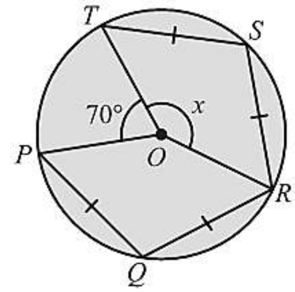
$m\overline{AB} = 12.$

$\text{segment length} = \frac{12}{2}.$

$= 6 \text{ cm}$

EXERCISE 8.2

1. In the given figure, if $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{ST}$ and $m\angle POT = 70°$, then find the value of x.



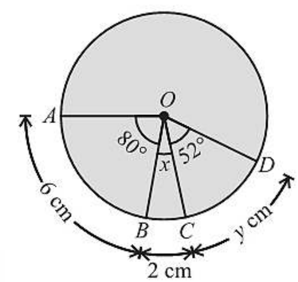
Solution

Since $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{ST}$,
the corresponding central angles are equal.
Let each of them be $x°$.

$m\angle POT + 4x = 360°$
$70° + 4x = 360°$
$4x = 290°$
$x = 145°$

$x = 145°$

2. In the given figure, find the values of x and y.



Solution

Let the radius be r.
Arc AB = 6 cm corresponds to 80°.

$$\frac{6}{80} = \frac{2}{x} \Rightarrow x = \frac{80 \times 2}{6} = 26.67° \approx 77°$$

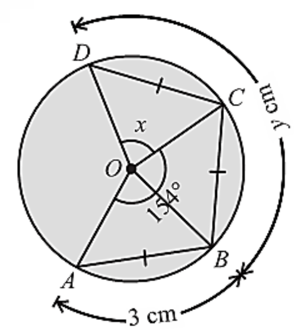
Also, $\angle AOB + \angle BOC + \angle COD = 180°$
$80° + x + 52° = 180°$
$x = 180° - 132° = 77°$

Arc AB = 6 cm corresponds to 80°.
Arc CD = y cm corresponds to 52°.

$$\frac{6}{80} = \frac{y}{52} \Rightarrow y = \frac{6 \times 52}{80} = 3.9 \text{ cm} \approx 6 \text{ cm}$$

$x = 77°$, $y = 6$ cm

3. Find the values of x and y in the given figure, such that $m\widehat{AB} = m\widehat{BD}$



Solution

Given $m\widehat{AB} = m\widehat{BD}$
$\Rightarrow \angle AOB = \angle BOD$
From the figure, $\angle AOB = 140°$
$\Rightarrow \angle BOD = 140°$
Now, $\angle AOD + \angle DOB + \angle BOA = 360°$
$x + 140° + 140° = 360°$
$x = 360° - 280° = 80°$
For arcs, equal central angles subtend equal arcs, so arc BD = arc AB = 3 cm.
Hence, $y = 4$ cm.

$x = 28°$, $y = 4$ cm

4. Two congruent arcs in a circular track subtend angles of 60° each at the centre. If the length of one chord of the circle is 10 metres, what is the length of other chord?

Solution

Congruent arcs subtend equal chords.

Both arcs subtend $60^\circ \Rightarrow$ their chords are equal.

So, the other chord also has length 10 metres.

Length of other chord = 10 metres

5. In a circular fountain, two water jets are installed such that they spray water along arcs of equal length. If one jet sprays between points P and Q and the straight-line distance (chord PQ) is 12 metres, what is the straight-line distance (chord RS) covered by the second jet spraying along arc RS, which is congruent to arc PQ?

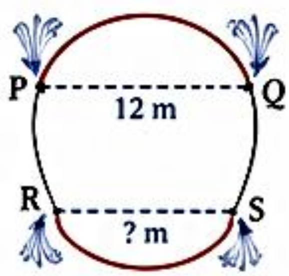
Solution:

Equal arcs subtend equal chords.

Since arc $\overline{RS} \cong$ arc $PQ \Rightarrow$ chord RS = chord PQ

Given chord PQ = 12 metres.

Chord RS = 12 metres



6. In a circular park, two walkways $\overline{AB}$ and $\overline{CD}$ are both straight paths (chords) of length 14 metres. What can be said about the minor arcs subtended by these walkways?

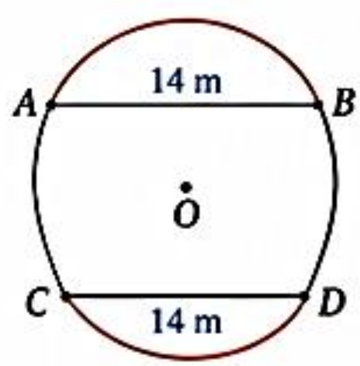
Solution:

Equal chords of a circle subtend equal (minor) arcs.

Since $\overline{AB} = \overline{CD} = 14$ metres,

the minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent.

Minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent.



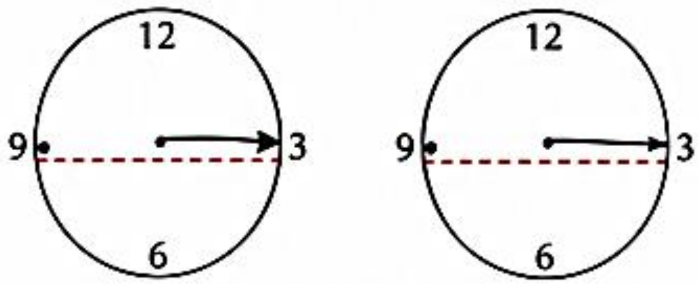
7. In two congruent circular clocks, the minute hand points from the centre to the 3 on both. A decorative string connects 3 to 9 on both clocks. Are the arcs from 3 to 9 on both clocks congruent?

Solution:

Both clocks are congruent circles, and points 3 and 9 are in corresponding positions.

Therefore, arcs from 3 to 9 on both clocks are congruent.

Yes, the arcs from 3 to 9 on both clocks are congruent.



REVIEW EXERCISE 8

1. Four possible answers are given for the following questions. Choose the correct answer.

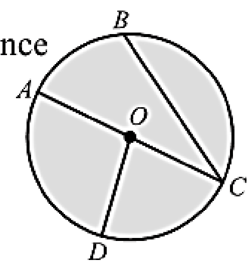
(i) Distance of a point on the circumference to the centre of the circle is called:
(a) radius (b) arc (c) chord (d) tangent

(ii) Radii of same circles are:
(a) all unequal (b) all equal
(c) half of each chord (d) double of the diameter

(iii) The boundary of the circle is called:
(a) chord (b) segment (c) circumference (d) diameter

(iv) Any part of a circumference is called:
(a) chord (b) diameter (c) radius (d) arc

(v) A chord passing through the centre of the circle is called:
(a) radius (b) diameter (c) secant (d) circumference



(vi) In the given figure, what is AB?
(a) diameter (b) tangent (c) chord (d) arc

(vii) In the given figure, major arc is:
(a) $m\widehat{AB}$ (b) $m\widehat{BC}$ (c) $m\widehat{BDC}$ (d) $m\widehat{AD}$

(viii) ________ circle(s) can pass through the three non-collinear points.
(a) one (b) two (c) three (d) many

(ix) Perpendicular bisector of a chord always passes through the _____ of circle.
(a) arc (b) radius (c) centre (d) circumference

(x) The sum of the measures of central angles of a circle is:
(a) 90° (b) 180° (c) 270° (d) 360°

2. On a circular clock with a radius of 6cm, the points from 2 to 10 form a chord that is 10cm long. Find the perpendicular distance from the centre of clock to the chord.

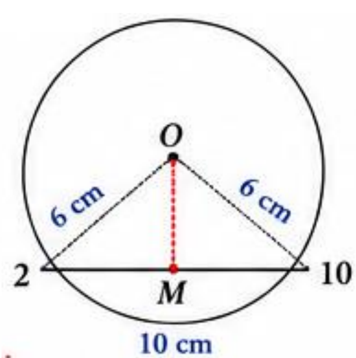
Solution

$OM \perp$ chord and M is midpoint.

So, $PM = 5$ cm, $OP = 6$ cm.

$$OP^2 = OM^2 + PM^2$$
$$6^2 = OM^2 + 5^2$$
$$OM^2 = 36 - 25 = 11$$
$$OM = \sqrt{11} \text{ cm}$$

Perpendicular distance $= \sqrt{11}$ cm



3. A chord 6 cm long is at a distance of 4 cm from the centre. Calculate the radius of the circle.

Solution:

$OM \perp$ chord and M is midpoint.

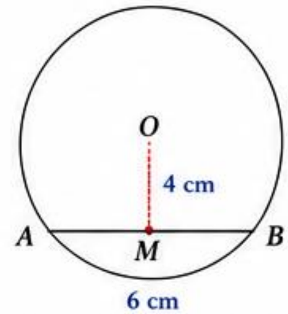
$AM = 3$ cm, $OM = 4$ cm.

$OA^2 = OM^2 + AM^2$

$r^2 = 4^2 + 3^2 = 16 + 9 = 25$

$r = 5$ cm

Radius = 5 cm



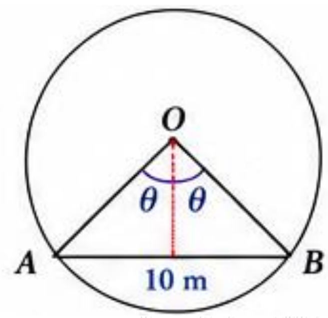
4. In a circular park, two benches are placed so that the chords formed by their positions are both 10 m long. What can you say about the angles subtended at the centre by each bench?

Solution:

Equal chords in a circle subtend equal angles at the centre.

Therefore, $\angle AOB = \angle BOA = \theta$.

The angles subtended at the centre by each bench are equal.

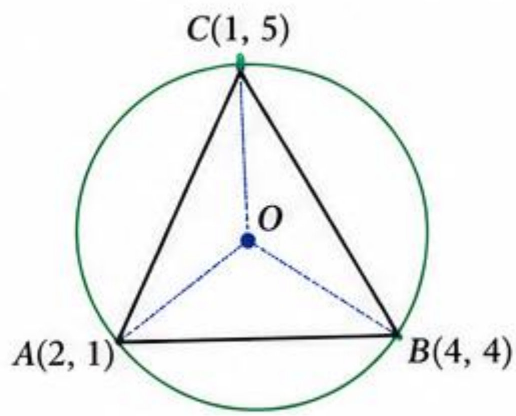


5. Jannat is designing a triangular garden with corners at points $A(2, 1)$, $B(4, 4)$ and $C(1, 5)$. Can she install a circular fountain that touches all three corners?

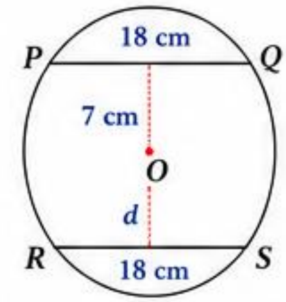
Solution:

The perpendicular bisectors of all sides intersect at one point O (circumcentre). Hence, a circle with centre O passes through A, B and C.

Yes, she can install the fountain.



6. Two chords, PQ and RS, each measure 18cm in length. If the distance of PQ from the centre of circle is 7cm. Find the distance of RS from the centre.



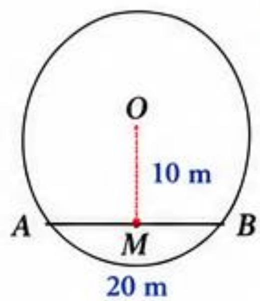
Solution:

Equal chords are equidistant from the centre.

So, $d = 7$ cm

Distance of RS from the centre $= 7$ cm

7. A tree branch lies across a circular pond, forming a 20 m chord. A measuring rod having length 10 m is perpendicular to chord from the centre. What is the radius of the pond?



Solution:

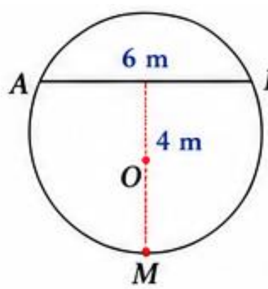
$AM = 10$ m (half of 20 m),

$OM = 10$ m.

$$OA^2 = OM^2 + AM^2$$
$$r^2 = 10^2 + 10^2 = 100 + 100$$
$$r^2 = 200$$
$$r = 10\sqrt{2} \text{ m}$$

Radius $= 10\sqrt{2}$ m

8. A steel bar 6 m long lies inside a circular structure with ends on the circle. It is bisected by a perpendicular rod from the centre. Find the radius of the circular structure if the perpendicular distance from the centre to the bar is 4 m.



Solution:

$AM = 3$ m (half of 6 m),

$OM = 4$ m.

$$OA^2 = OM^2 + AM^2$$
$$r^2 = 4^2 + 3^2 = 16 + 9 = 25$$
$$r = 5 \text{ m}$$

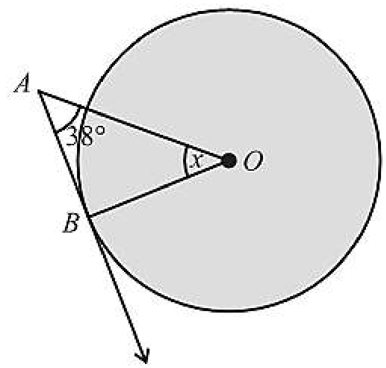
Radius $= 5$ m

UNIT 9
Tangent and Angles of a Circle

EXERCISE 9.1

1. Find the value of x.

Solution



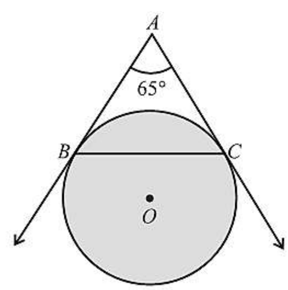
Given:

- Tangent AB
- Radius OB
- $\angle BAO = 38°$

Since radius is perpendicular to tangent, so:
$\angle ABO = 90°$
In triangle AOB
$x + 38° + 90° = 180°$
$x = 180° - 128°$
$x = 52°$

2. Find the angle ABC.

Solution

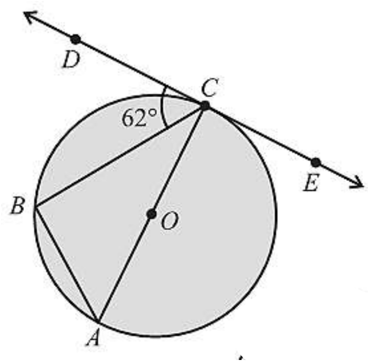


Concept:

- Tangent drawn from an external point to a circle are equal i making $\triangle ABC$ an isosceles triangles
- In an isosceles triangle the angle opposite to equal side are equal ($\angle ABC = \angle ACB$)
- The sum of angles in a triangles is 180°

Given:
$\angle BAC = 65°$
Calculation:
Let $\angle ABC = y$,since $\angle ABC = \angle ACB$
$y + y + 65° = 180°$
$2y = 180° - 65°$
$y = \frac{115°}{2}$
$y = 57.5°$

3. A, B and C are points on the circumference of a circle with centre at O. $\overline{AC}$ is the diameter of the circle and $\overleftrightarrow{DE}$ is the tangent to the circle at the point C and $m\angle BCD = 62°$. Find (i) $m\angle BCA$ (ii) $m\angle BAC$



Solution

Given: $\overleftrightarrow{DE}$ is a tangent at point c and $m\angle BCD = 62°$, AC the diameter.

(i) Find $m\angle BCD$

Concept:

- A tangent is perpendicular to the radius (or diameter) at the point of tangency, therefore $m\angle ACD = 90°$

Calculation:

$\angle BCD = \angle ACD - \angle BCD = 90° - 62° = 28°$

(ii) Find m∠BAC

Concept:

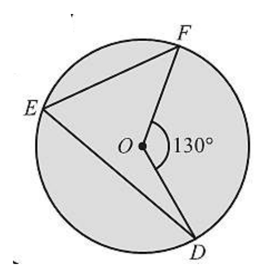
- The angle in a semicircle is a right angle $\angle ABC = 90°$
- Use the alternate segment theorem OR tangent – chord theorem: the angle between a tangent and a chord(∠BCD) is equal to the angle in the alternate segment (∠BAC)

Calculation:

By the alternate theorem: $\angle BAC = \angle BCD = 62°$

$m\angle BAC = 62°$

4. D, E and F are points on the circumference of a circle with centre at O and $m\angle DOF = 130°$. Find $m\angle DEF$.



Solution

Concept:

- The angle subtended an arc at the center of a circle is double the angle subtended by the same arc at any point on the remaining circumference.

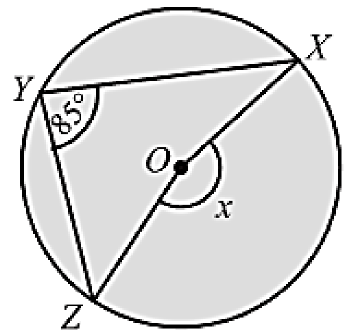
Given: Central angle $m\angle DOF = 130°$

Calculation:

The angle of the circumference is half of the central angle ($m\angle DOF$)

$m\angle DEF = \frac{m\angle DOF}{2} = \frac{130°}{2} = 65°$

5. X, Y and Z are points on the circumference of a circle with centre at O and $m\angle XYZ = 85°$. Find x.



Solution

Concept:

- **Using the standard theorem:** the central angle x($m\angle$XOZ) subtended by the major arc is twice the angle at the circumference $m\angle$XYZ.

Given: $m\angle \text{XYZ} = 85°$

Calculation:

$x = 2 \times m\angle \text{XYZ}$

$\quad = 2 \times 85° = 170°$

6. In a historical monument, a circular fountain with a radius of 3 m is built. A flagpole is erected 7 m away from the centre of the fountain. Two ropes from the pole are tied to the edge of the fountain, just touching it. Find the length of each rope.

Solution

Concept:

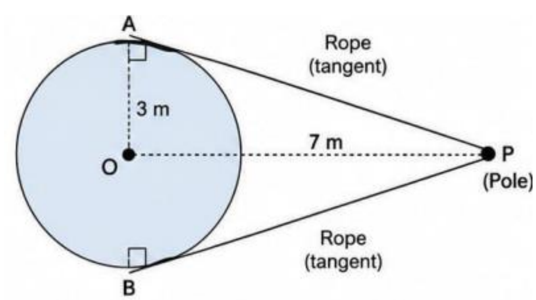
- A line touching a circle at just one point is a tangent
- The radius of a circle is perpendicular to the tangent at the point of creating a right-angled triangle
- We can use the Pythagorean theorem

Given:

- Radius of the fountain(r) = 3cm (perpendicular)
- Distance from the center to the flag pole (d) = 7m(hypotenuse)
- Length of the rope (L)= tangent (base)

Calculation:

By using Pythagorean Theorem



$d^2 = r^2 + L^2$

$7^2 = 3^2 + L^2$

$L^2 = 49 - 9$

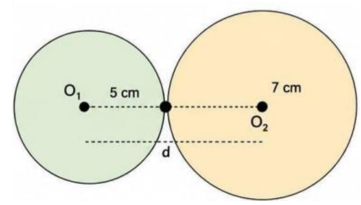
$L^2 = 40$, L= 6.32m

7. Two circular gears touch each other externally for proper rotation in a machine. The radii of the two circular gears are 5cm and 7cm. What is the distance between their centres if they touch externally?

Solution

Concept:

- When two circles touch each other externally, the distance between their centers is equal to the sum of their radii.



Given:

- Radius of the first gear (r_1) = 5cm
- Radius of the second gear (r_2) =7cm

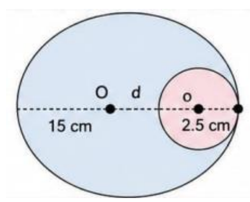
$D = r_1 + r_2 = 5+7 = 12cm$

8. A small sensor lies inside a satellite dish and touches its wall internally. If the dish has radius 15 cm and the sensor has radius 2.5 cm, find the distance between their centres.

Solution

Concept:

- When one circle lies inside the another and touches it internally, the distance between their centers is equal to the different between the radii.



Given:

- Radius of the satellite dish (R)= 15cm
- Radius of the sensor (r) = 2.5 cm

Calculation:

Distance = R – r = 15cm - 2.5cm = 12.5cm

9. An inner holder touches the outer cylindrical container internally. If the outer container has radius 8 cm and the inner holder has radius 6 cm, find the distance between their centres.

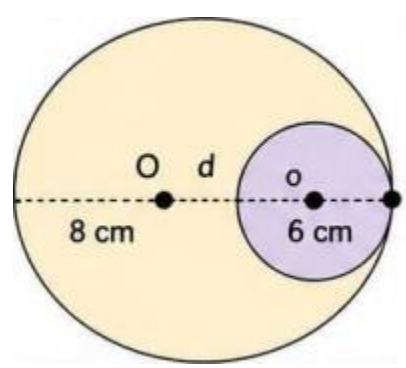
Solution

Concept:

- When one circle lies inside another and touches it internally, the distance between their centers is equal to the different between the radii.

Given:

- Radius of the outer container (R)= 8cm
- Radius of the inner container (r)= 6cm



Calculation:

Distance = R – r

= 8cm - 6cm = 2cm

10. A pyramid-shaped sculpture is placed in the center of a circular plaza with a radius of 10 m. A decorative pole stands 26 m from the center of the circle. Two guide wires are attached from the pole to the plaza edge, just touching the circle. Find the length of each wire.

Solution

Concept:

- A wire that just touches the edge of a circular plaza acts as a tangent to the circle.
- The radius connecting the center of the plaza to the point of tangency from a right angle with the wire.
- This creates a right angle triangle where we can apply the Pythagorean Theorem.

Given:

- Radius of the circular plaza (r) =10m (perpendicular)
- Distance from the pole to the center (d)= 26 m (hypotenuse)
- Length of each guide wire (W) = tangent (base)

Calculation:

Using the Pythagorean Theorem

$d^2 = r^2 + w^2$

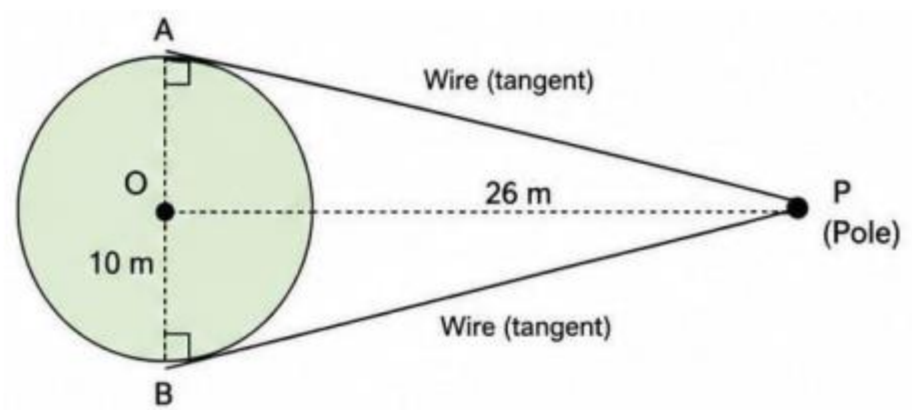
$26^2 = 10^2 + w^2$

$676 = 100 + w^2$

$w^2 = 676 - 100$

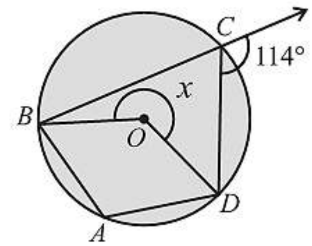
$w^2 = 576$

$w = 24m$



EXERCISE 9.2

1. A, B, C and D are points on circumference of a circle with centre at O as shown in figure. Find the angle x.



Solution

114° is the exterior angle at C.

Interior angle at C:

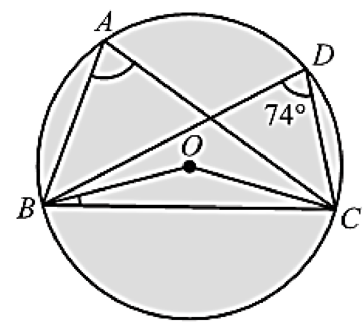
$\angle BCD = 180° - 114° = 66°$

Angle at centre is twice the angle at circumference on the same arc:

$x = 2 \times 66° = 132°$

Or $\quad x = 360° - 132° = 228°$

2. In the adjoining figure $m\angle BDC = 74°$, find $m\angle BAC$, $m\angle BOC$ and $m\angle OBC$.



Solution

Given:

$\angle BDC = 74°$

Angles standing on the same chord are equal: $\mathbf{\angle BAC = 74°}$

Central angle:

$\angle BOC = 2 \times 74° = \mathbf{148°}$

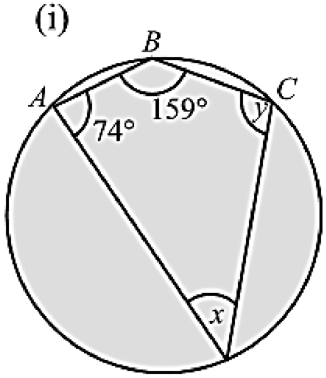
In triangle BOC:

$OB = OC$

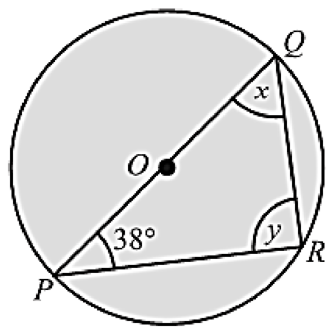
$\angle OBC = (180° - 148°) / 2 = \mathbf{16°}$

3. Find the angles x and y in the following figures.

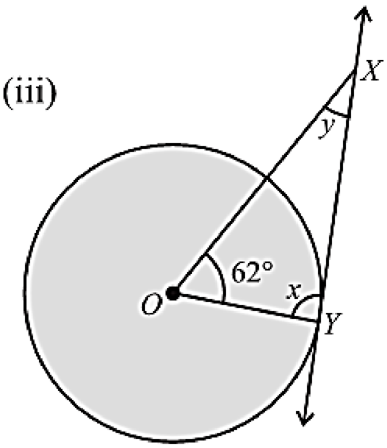
(i)



(ii)



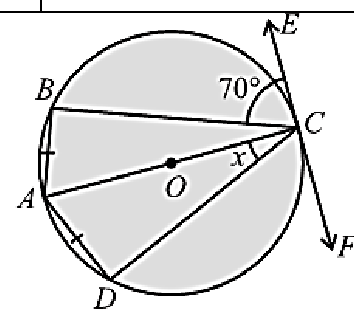
(iii)



Solution

(i)	**(ii)**	**(iii)**
Opposite angles of cyclic quadrilateral are supplementary. x + 74° = 180° x = **106**° y + 159° = 180° y = **21**°	PQ is diameter. Angle in semicircle = 90° y **= 90**° In triangle PQR: x + 90° + 38° = 180° x = **52**°	Radius is perpendicular to tangent. x = **90**° 62° + 90° + y = 180° y = **28**°

4. Find the angle x in the given figure.



Solution: using alternate segment theorem

Angle between tangent and chord equals angle in alternate segment.

∠BCE = ∠BAC = 70° and ∠OCE = 90°

∠OCB = ∠OBC = 90° - 70° = 20° also ∠BOC =180° - 20° - 20° = 140°

∠ACD = ∠ ABD = x = 20° since $\widehat{AB} = \widehat{AD}$

5. In a circular arch over a monument entrance, two spotlight fixtures are placed such that they each shine from two different points on the arch to the same chord PQ on the base of the arch. If the angle formed at one light is 55°, what is the angle at the second light on the same side of chord PQ?

Solution

Angles standing on the same chord and same side are equal.

Second angle = 55°

6. A circular garden has a walking path forming a quadrilateral inscribed in it. If one angle is 87°, what is the opposite angle?

Solution

Opposite angles of cyclic quadrilateral are supplementary.

Opposite angle = 180° − 87° = 93°

7. A ferris wheel has a radius of 12 m. If a passenger travels a distance of 18 m along the circumference of the ferris wheel, what is the angle (in radians) swept by the passenger's position from the starting point?

Solution

$l = r\theta \Rightarrow \theta = \frac{l}{r}$

θ = 18 / 12 = 1.5 radian

8. Find θ, when

(i) $\ell = 3$ cm, $r = 2.2$ cm (ii) $\ell = 5.6$ cm, $r = 2$ cm

Solution

(i)	(ii)
$l = r\theta \Rightarrow \theta = \frac{l}{r}$ θ = 3 / 2.2 = 1.36 rad	$l = r\theta \Rightarrow \theta = \frac{l}{r}$ θ = 5.6 / 2 = 2.8 rad

9. Find r, when

(i) $\ell = 5.5$ cm, $\theta = 40°20'$ (ii) $\ell = 13$ cm, $\theta = 70°$

Solution

(i)	(ii)
θ = 40°20' = 0.704 rad $l = r\theta \Rightarrow r = \frac{l}{\theta}$ r = 5.5 / 0.704 = 7.81 cm	θ = 70° = 1.222 rad $l = r\theta \Rightarrow r = \frac{l}{\theta}$ r = 13 / 1.222 = 10.64 cm

10. Find ℓ and area of sector, when

(i) $r = 1.7$ cm, $\theta = 0.25$ radian (ii) $r = 3$ cm, $\theta = 45°$

Solution

(i)	(ii)
$l = r\theta$ l =1.7 × 0.25 = 0.425 cm $A = \frac{1}{2}r^2\theta$ $A = \frac{1}{2} \times (1.7)^2 \times 0.25 = 0.361$ cm^2	$l = r\theta$ l =3π / 4 = 2.36cm $A = \frac{1}{2}r^2\theta$ $A = 9\pi / 8$ cm^2 = 3.53 cm^2

11. Uzma cut a pizza of radius 14 cm into 8 equal slices. What is the area of one slice (sector).

Solution

Radius of pizza: $r = 14$ cm

Area of whole pizza = πr^2

$A = \pi \times (14)^2 = \pi \times 196$

$A = (22/7) \times 196 = 616\ \text{cm}^2$ Using $\pi = 22/7$

Since the pizza is divided into 8 equal slices:

Area of one slice = $616 \div 8 =$ **77 cm²**

12. The perimeter and area of a sector are 14 cm and 10 cm^2 respectively. Find the radius of the circle and the central angle of the sector.

Solution

Perimeter of sector = $2r + l$

$14 = 2r + l$

$l = 14 - 2r$

Area of sector = $1/2 \times r \times l$

$10 = 1/2 \times r \times (14 - 2r)$

$20 = r(14 - 2r)$

$20 = 14r - 2r^2$

$2r^2 - 14r + 20 = 0$

$r^2 - 7r + 10 = 0$

$(r - 5)(r - 2) = 0$

Therefore, $r = 5$ cm or $r = 2$ cm

Case I: r = **5 cm** $l = 14 - 2(5) = 4$ $\theta = \frac{l}{r} = 4/5 =$ **0.8 radian**	Case II: r = **2 cm** $l = 14 - 2(2) = 10$ $\theta = \frac{l}{r} = 10/2 =$ **5 radians**

REVIEW EXERCISE 9

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) Tangent is a line that touches the circumference of the circle at:
(a) many points (b) three points
(c) two points (d) one point

(ii) ________ tangents can be drawn to a circle from a point outside the circle.
(a) one (b) two (c) three (d) many

(iii) ________ tangents can be drawn to a circle from centre of the circle.
(a) zero (b) one (c) two (d) three

(iv) In circle, radius and tangent are:
(a) parallel (b) equal
(c) perpendicular (d) zero

(v) An angle in a segment greater than a semicircle is ______ angle:
(a) an acute (b) right (c) an obtuse (d) straight

(vi) The angle subtended by the arc at the centre of the circle is called:
(a) acute angle (b) central angle
(c) right angle (d) complete angle

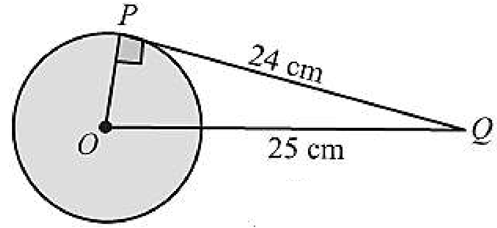
(vii) What is the measure of an angle inscribed in a semi-circle?
(a) 45° (b) 60°
(c) 90° (d) 120°

(viii) Any two angles in the same segment of the circle are:
(a) supplementary (b) complementary
(c) equal (d) zero

(ix) Area of the sector of the circle = ________
(a) $\frac{1}{2}r\theta$ (b) $\frac{3}{2}r^2\theta$ (c) $\frac{1}{2}r^2\theta$ (d) $\frac{3}{2}r\theta$

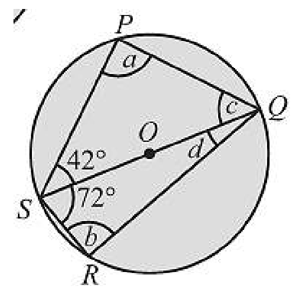
(x) If $r = 6$cm and $\theta = 2$ radians, then arc length is:
(a) 12 cm (b) 8 cm (c) 6 cm (d) 3 cm

2. In the adjoining figure, find the radius of the circle.



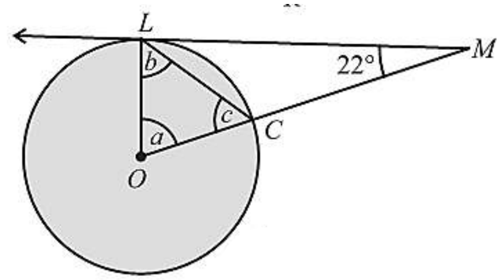
Solution
Given Data from Figure:
• Hypotenuse (OQ) = 25 cm
• Tangent segment (PQ) = 24 cm
• ∠OPQ = 90° (Since a radius is always perpendicular to a tangent at its point of contact).
Step-by-step Solution:
Applying the Pythagorean theorem in the right-angled triangle ΔOPQ:
$OQ^2 = OP^2 + PQ^2$
$(25)^2 = OP^2 + (24)^2$
$625 = OP^2 + 576$
$OP^2 = 625 - 576$
$OP^2 = 49$
$OP = \sqrt{49} = 7$ cm

3. Find the angles a, b, c and d in the given figure, if O is the centre of the circle.



Solution
Angle at the center = 180° because it's a straight line across the circle.
Angle at the circumference = 1/2 × angle at the center.
a = 1/2 × 180° = **90**°
b = 1/2 × 180° = **90**°
c is the angle at the circumference for arc PS.
Angle at the center for arc PS = 96°.
c = 1/2 × 96° = **48**°
Arc PS = 2 × c = 2 × 48°
Arc PS = 96°
∠POQ = 180° - Arc PS
∠POQ = 180° - 96° = 96°
d + d + ∠POQ = 180°
2d + 96° = 180°
2d = 180° - 96°
2d = 84°
d = 84° / 2
d = **18**°

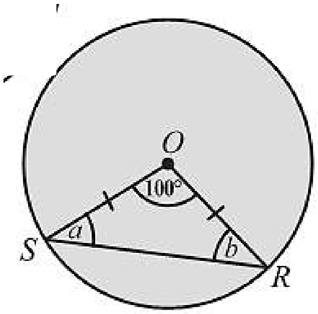
4. Find the angles a, b and c, if $\overrightarrow{ML}$ is a tangent line and O is the centre of the circle.



Solution

Using Alternate Segment Theorem:
Alternate Segment Theorem: Angle between tangent and chord = angle in alternate segment.
Angle between tangent ML and chord LC = ∠MLC = 56°
So ∠OLC = ∠MLC = 56°
b = 56°
OC = OL = radius, so triangle OLC is isosceles.
Therefore ∠OCL = ∠OLC
c = b = 56°
Sum of angles in triangle OLC:
∠LOC + ∠OLC + ∠OCL = 180°
a + 56° + 56° = 180°
a + 112° = 180°
a = 180° - 112°
a = 68°

5. Find the angles marked in the given figure, where O is the centre of the circle.



Solution

In triangle ΔSOR, segments OS and OR are both radii of the circle (OS = OR). Since two sides are equal, it forms an isosceles triangle where its base angles are equivalent: $a = b$
Summing up the internal angles of ΔSOR:
∠SOR + a + b = 180°
100° + a + a = 180° (Since a = b)
100° + 2a = 180°
2a = 180° – 100°
2a = 80°
a = 40°
Since a = b, then: b = 40°

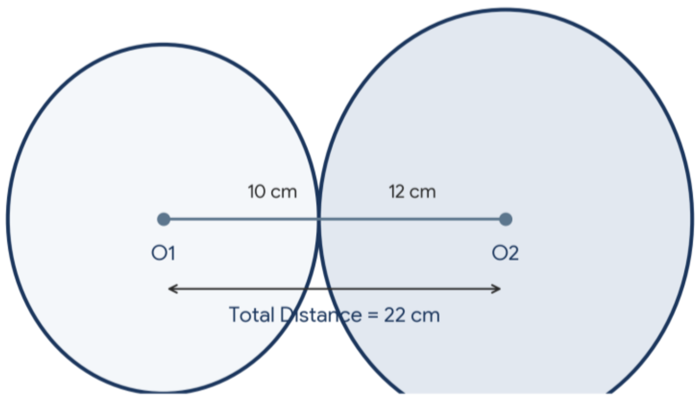
6. Two round dining tables are arranged to touch at their edges. The radii of the two round dining tables are 10 cm and 12 cm. What is the distance between their centres if they touch externally?

Solution

When two distinct circles touch each other externally, the straight-line distance (d) between their centers is exactly equal to the sum of their individual radii ($r_1 + r_2$).

$d = r_1 + r_2$

d = 10 cm + 12 cm = 22 cm.



7. A circular mosaic pattern is laid on the floor beneath a modern pyramid structure. Two tiles are placed such that they both connect to the same chord EF and reach two points G and H on the arc above EF. If $m\angle EGF = 65°$, find $m\angle EHF$.

Solution

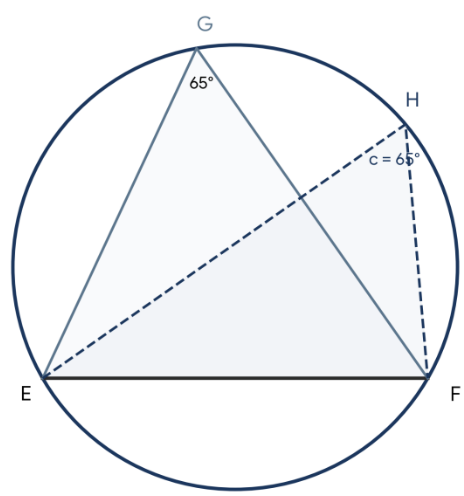
According to the properties of cyclic domains and the circle segment theorem, angles subtended by the same base chord across the same major segment region are perfectly uniform.

Since both vertex points G and H rely natively on the identical base chord baseline EF, it implies that:

$m\angle EHF = m\angle EGF$

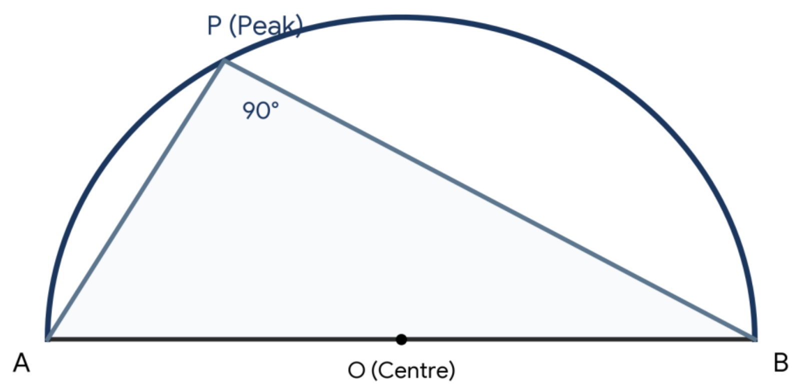
Given: $m\angle EGF = 65°$

Therefore: $m\angle EHF = 65°$.



8. A triangular frame within a semicircular gate joins the ends of the diameter to the peak. What is the angle formed at the peak?

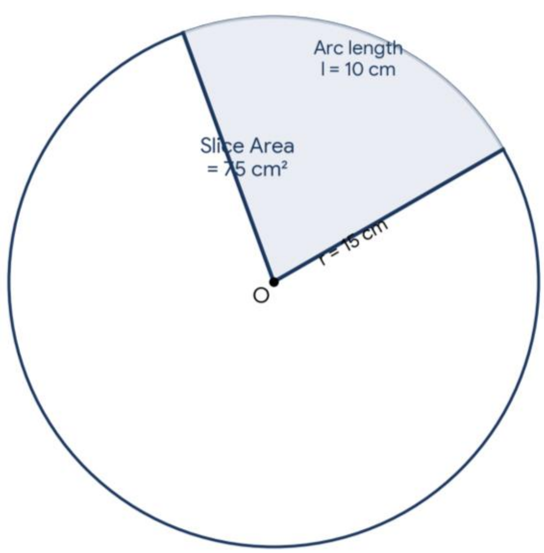
Solution



This system relies directly on Thales's Semicircular Apex Principle. The theorem demonstrates that whenever the diameter of a circle is projected as baseline chords matching an arbitrary node along its higher outer arc circumference; it produces a perfect right angle. Because both end elements of the diameter line extend up to the high peak terminal along the curved path of the semicircle, the interior vertex angle maps natively to a right angle (90°).

9. A circular birthday cake has a radius of 15cm. A slice has an arc length of 10cm. What is the area of this slice?

Solution



The physical area (A) of a sector slice cut out from a circle structure is derived smoothly using the active radius (r) along with its arc length span (l):

$A = \frac{1}{2} \times l \times r$

$A = \frac{1}{2} \times 10 \times 15$

$A = 5 \times 15 = 75 \text{ cm}^2$.

10. A sector cut from a circle of radius 5cm has a perimeter of 16cm. Find area of this sector.

Solution

Step 1: Isolate the sector's edge arc dimension (l).

The geometric perimeter boundaries (P) of any standard sector slice wrap two straight radial borders plus the curved perimeter arc length segment:

$P = 2r + l$

$16 = 2(5) + l$

$16 = 10 + l$

$l = 6$ cm

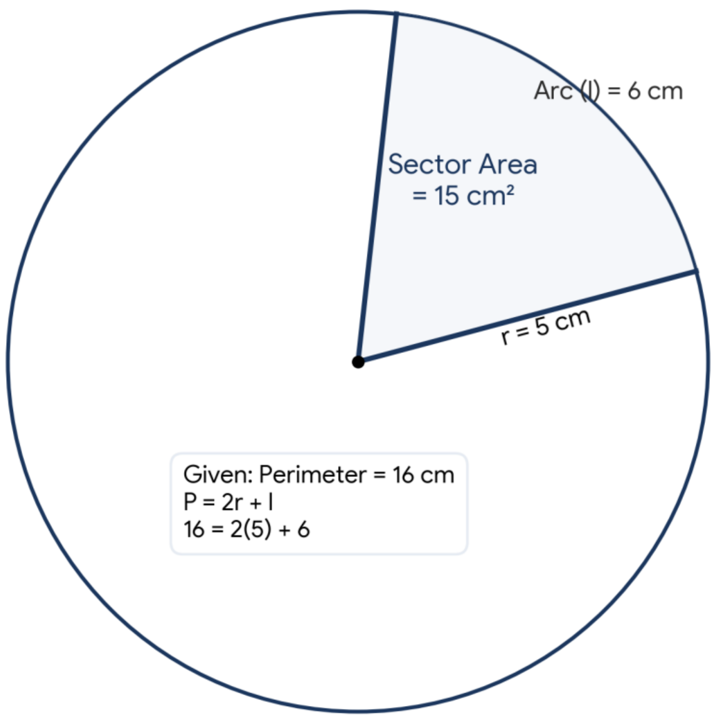
Step 2: Solve the total area equation.

Using the arc area relationship:

$A = \frac{1}{2} \times l \times r$

$A = \frac{1}{2} \times 6 \times 5$

$A = 3 \times 5 = 15$ cm².



UNIT 10 Practical Geometry of Circles

EXERCISE 10.1

1. Construct a circle with the help of given radius and verify its centre by construction:

(i) $r = 1.5$ cm (ii) $r = 1.7$ cm (iii) $r = 2$ cm

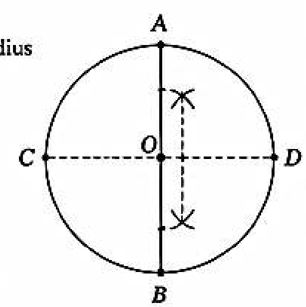
Solution

Given : Radius r

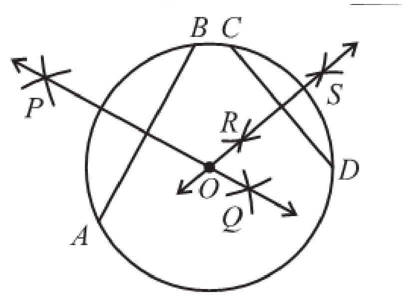
Required : To construct a circle of given radius and verify its centre.

Steps of Construction :

1. Mark a point O on the paper.
2. Open the compass equal to the given radius.
3. Place the needle at O and draw a circle.
4. Draw any chord AB of the circle.
5. Draw another chord CD.
6. Construct perpendicular bisectors of chords AB and CD.
7. Let the perpendicular bisectors intersect at point O.
8. Point O is the centre of the circle.



(i)



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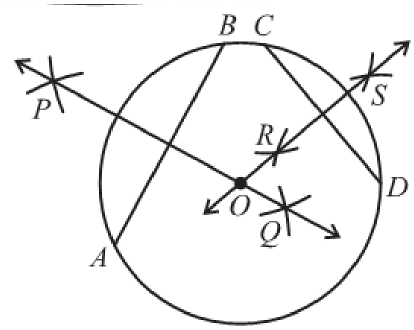
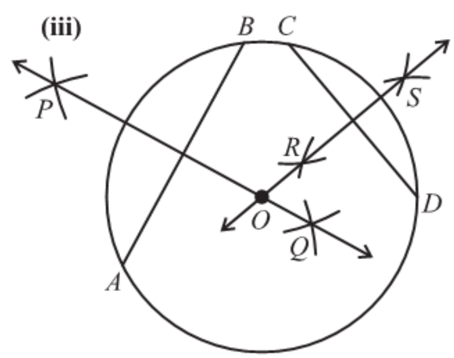


Or according to book

(iii)



2. Take any three non-collinear points P, Q, R and construct a circle passing through these points.

Solution

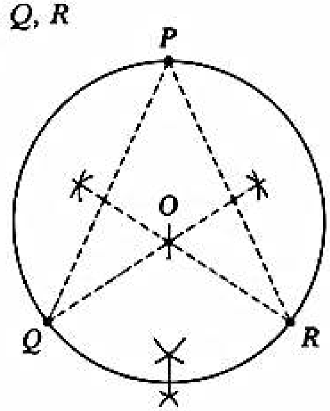
Or according to book

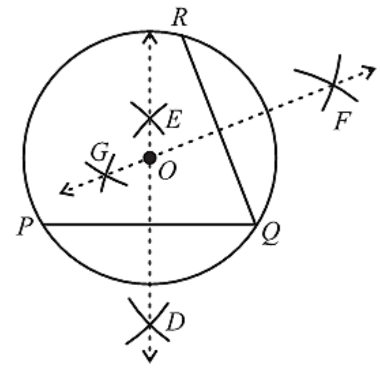
Given : Three non-collinear points P, Q, R

Required : To construct a circle passing through P, Q, R.

Steps of Construction :

1. Join PQ and QR.
2. Construct perpendicular bisector of PQ.
3. Construct perpendicular bisector of QR.
4. Let both bisectors intersect at point O.
5. With centre O and radius OP, draw a circle.
6. The circle passes through P, Q and R.





3. Draw an arc ABC and complete a circle by finding its centre.

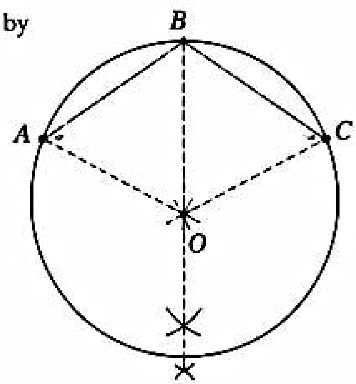
Solution

Given : An arc ABC

Required : To complete the circle by finding its centre.

Steps of Construction :

1. Join AB and BC.
2. Construct perpendicular bisector of chord AB.
3. Construct perpendicular bisector of chord BC.
4. Let the bisectors intersect at point O.
5. With centre O and radius OA, draw the complete circle.



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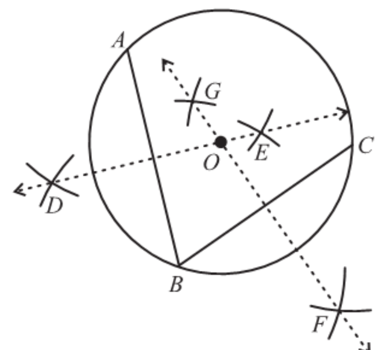


4. Draw an arc PQR and complete a circle without finding its centre.

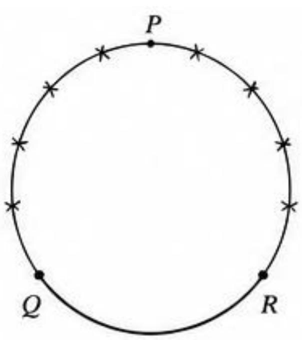
Solution

Given : An arc PQR

Required : To complete the circle without finding its centre.

Steps of Construction :

1. Choose point P and Q on the arc.
2. Draw equal chords on both sides of the arc.
3. Continue marking equal distances along the boundary using compass.
4. Join the successive points smoothly.
5. Thus the complete circle is obtained without locating the centre.



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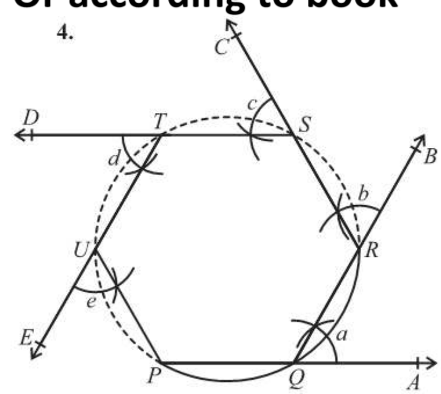


5. Take any three non-collinear points (locations of the lamp posts in the park), construct a circle passing through all these points.

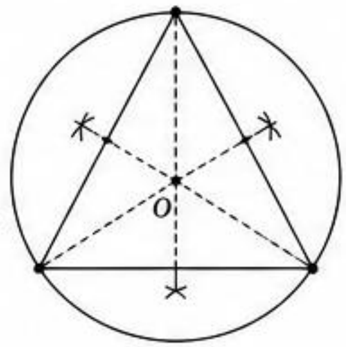
Solution

Given : Three non-collinear points (lamp posts)

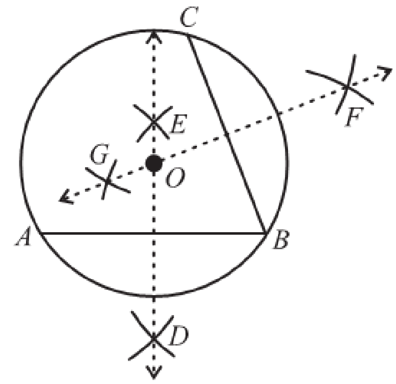
Required : To construct a circle passing through all three points.

Steps of Construction :

1. Join the three points to form a triangle.
2. Draw perpendicular bisector of one side.
3. Draw perpendicular bisector of another side.
4. Let the bisectors meet at point O.
5. With centre O and radius equal to distance from O to any point, draw a circle.
6. The circle passes through all three lamp posts.



Or according to book



6. A part of the Ferris wheel rim is visible as an arc. Using any three points on the arc, construct the circle by finding its centre.

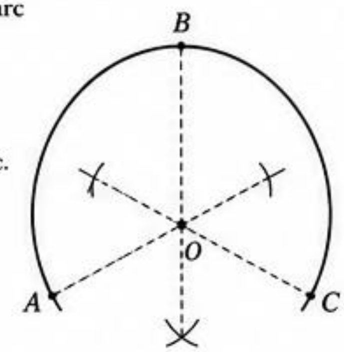
Solution

Given : Three points A, B, C on the arc

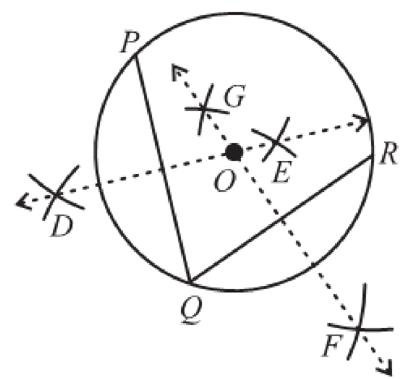
Required : To construct the circle and find its centre.

Steps of Construction :

1. Mark any three points A, B, C on the arc.
2. Join AB and BC.
3. Draw perpendicular bisector of AB.
4. Draw perpendicular bisector of BC.
5. Let the bisectors intersect at O.
6. With centre O and radius OA, draw the complete circle.



Or according to book



7. ABC an arc of a fountain, complete a circle without finding its centre.

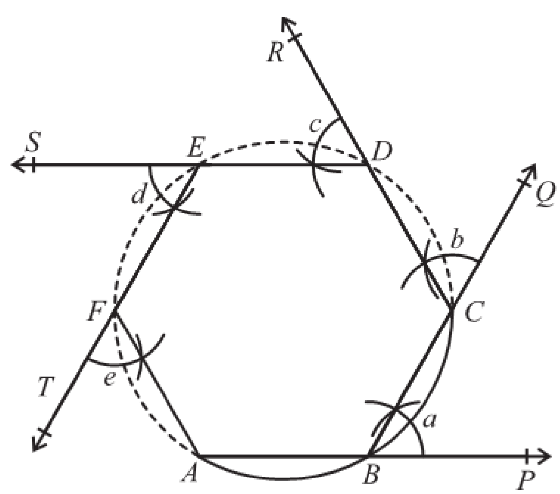
Solution

Given : Arc ABC

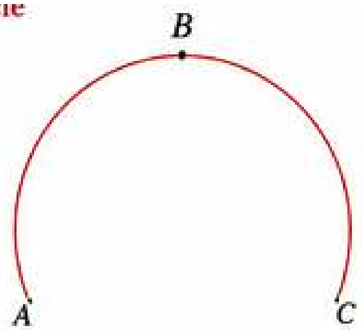
Required : To complete the circle without finding its centre.

Steps of Construction :

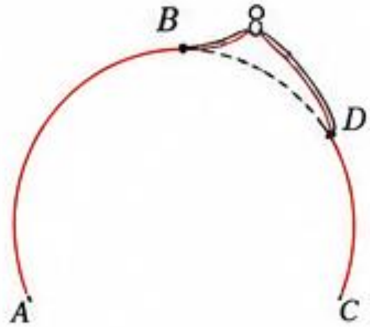
1. Select equal points on the given arc.
2. Using compass, continue taking of equal chord lengths around the arc.
3. Mark successive points carefully.
4. Join all points smoothly to form the circle.
5. Hence the complete circle is obtained without locating the centre.



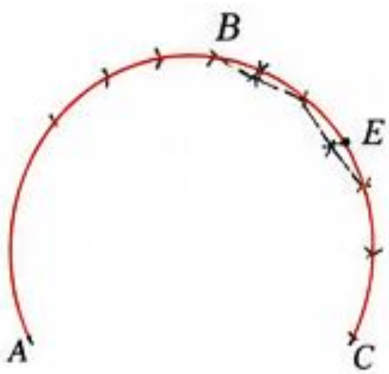
Explanation to Question 7



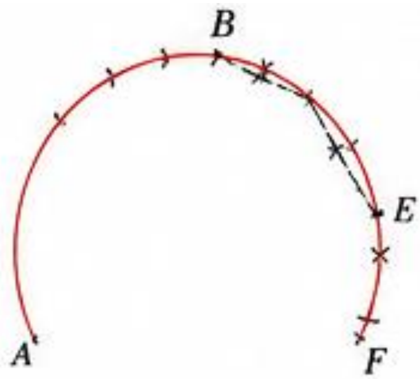
Given arc ABC



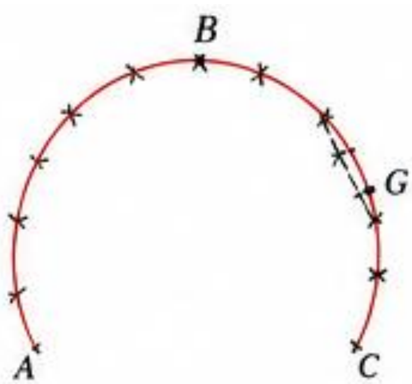
1. *Take any point D on the arc. With centre A, mark off AD on the arc.*



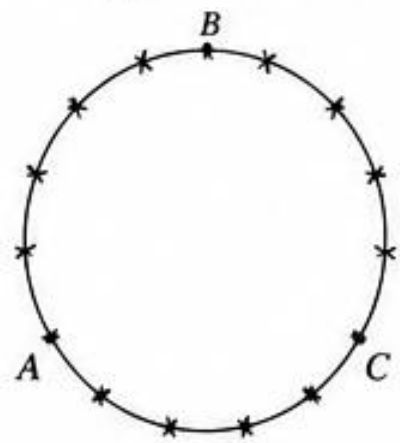
2. *With centre D, mark off DE equal to AD.*



3. *With centre E, mark off EF equal to AD.*



4. *Continue in the same way until you reach near point A.*



5. Join all the points smoothly to get the complete circle.

EXERCISE 10.2

1. Draw tangent to $\widehat{APB}$ at point P, when P is midpoint of the arc.

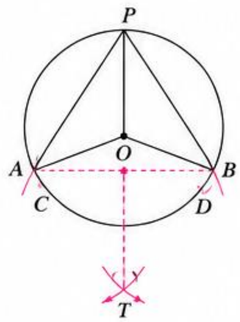
Solution

Given: A circle with centre O. Arc APB with P ndpoint of the arc.

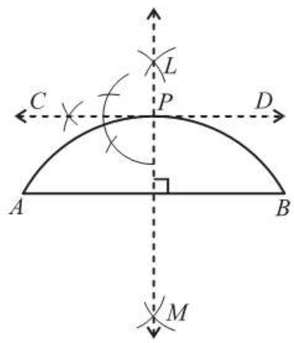
Required: Tangent to $\widehat{APB}$ at P.

Steps of Construction:

1. Join OA and OB.
2. Join OP.
3. With O as centre, draw a circle with any radius to cut OA at C and OB at D.
4. With C and D as centres and equal radius (greater than $\frac{1}{2}CD$), draw arcs to intersect at T.
5. Join PT.
   PT is the required tangent.



Or according to book



2. Draw tangent to $\widehat{PQR}$ at point P, when P is endpoint of the arc.

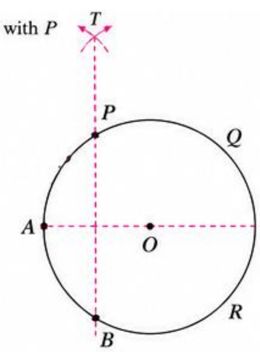
Solution

Given: A circle with centre O. Arc PQR with P endpoint of the arc.

Required: Tangent to $\widehat{PQR}$ at P.

Steps of Construction:

1. Join OP.
2. With O as centre, draw a circle with any radius to cut the given circle at A and B.
3. With A and B as centres and equal radius (greater than $\frac{1}{2}AB$), draw arcs to intersect at T.
4. Join PT.
   PT is the required tangent.



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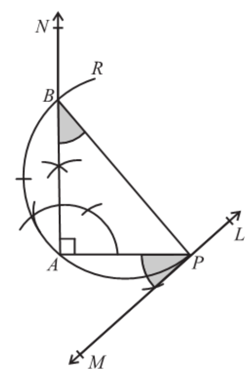


3. Draw tangent to $\widehat{ABC}$ from a point P, when P is outside the arc.

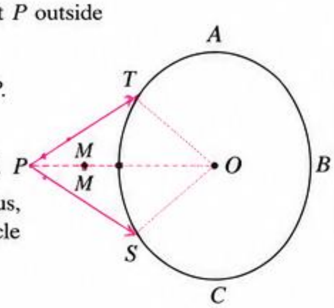
Solution

Given: A circle with centre O. Point P outside the circle.

Required: Tangent to $\widehat{ABC}$ from P.

Steps of Construction:

1. Join OP and draw its midpoint M.
2. With M as centre and MO as radius, draw a circle to cut the given circle at T and S.
3. Join PT and PS.
   PT and PS are the required tangents.



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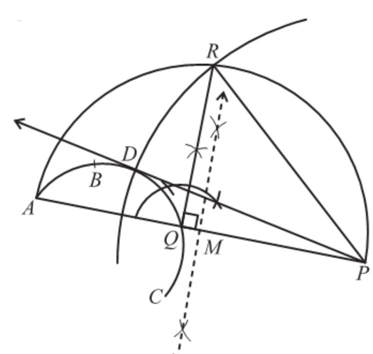


4. Draw a circle of radius 1.3 cm and draw a tangent at point P, when P lies on its circumference.

Solution

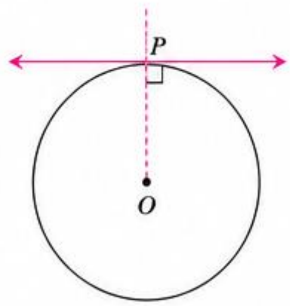
Given: Radius $r = 1.3$ cm.
Point P on the circle.

Required: Tangent at P.

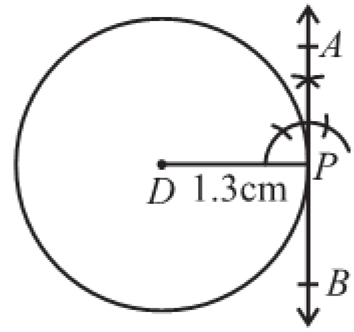
Steps of Construction:

1. With O as centre, draw a circle of radius 1.3 cm.
2. Mark any point P on the circumference.
3. Draw OP.
4. Through P, draw a line perpendicular to OP.

The line is the required tangent.



Or according to book



5. Draw a circle of radius 1.5 cm and draw a tangent at point P, when P is at a distance of 8 cm from its centre.

Solution

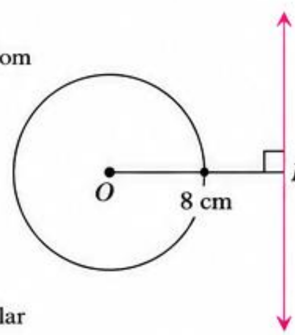
Given: Radius $r = 1.5$ cm.
Point P at a distance 8 cm from centre O.

Required: Tangent at P.

Steps of Construction:

1. Draw $OP = 8$ cm.
2. With O as centre, draw a circle of radius 1.5 cm.
3. Through P, draw a line perpendicular to OP.

The line is the required tangent.



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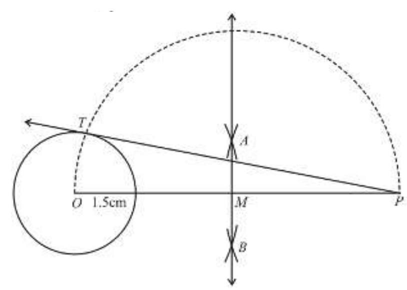


6. Draw a circle of radius 1.6 cm. Draw two tangents that meet an angle of 30°.

Solution

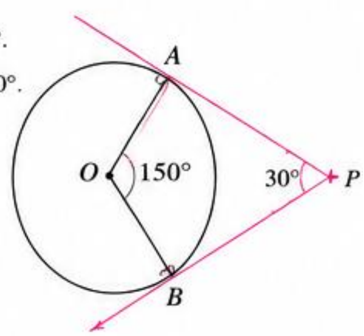
Given: Radius $r = 1.6$ cm.
Angle between tangents = 30°.

Required: Two tangents meeting at 30°.

Steps of Construction:

1. Draw a circle with centre O and radius 1.6 cm.
2. At O, construct $\angle AOB = 150°$.
3. Bisect $\angle AOB$ to get OC.
4. Through A and B, draw lines perpendicular to OA and OB respectively. Let them meet at P.

Then $\angle APB = 30°$.



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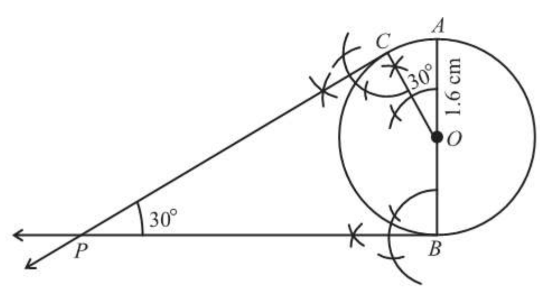


7. Draw a tangent to any point on a circular part of the track having radius = 2 cm.

Solution

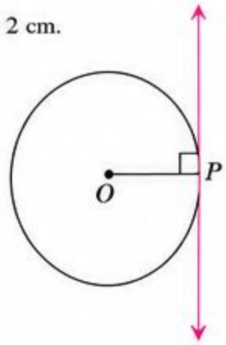
Given: Circular part of the track with radius 2 cm.
Point P on the track.

Required: Tangent at P.

Steps of Construction:

1. Draw OP, where O is the centre of the track.
2. Through P, draw a line perpendicular to OP.

The line is the required tangent.



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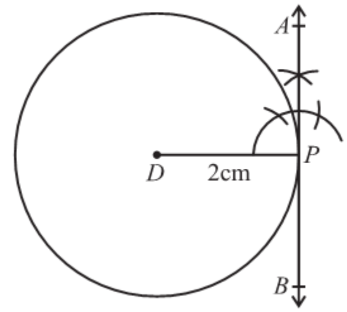


8. From a pulley point, construct two tangents to a machine wheel having $r = 2.1\text{cm}$, such that the angle between them is $30°$.

Solution

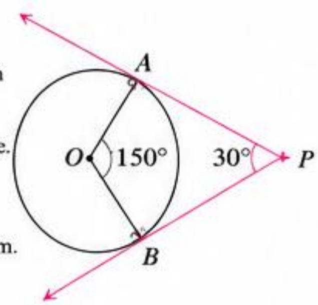
Given: Radius $r = 2.1$ cm.
Pulley point P such that angle between tangents = $30°$.

Required: Two tangents from P to the circle.

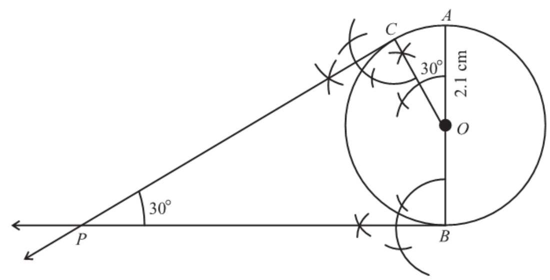
Steps of Construction:

1. Draw a circle with centre O and radius 2.1 cm.
2. At O, construct $\angle AOB = 150°$.
3. Through A and B, draw lines perpendicular to OA and OB respectively. Let them meet at P.

Then PA and PB are the required tangents and $\angle APB = 30°$.



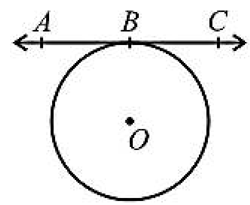
Or according to book



REVIEW EXERCISE 10

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) In the adjacent figure, $\overleftrightarrow{ABC}$ is:



(a) a chord (b) an arc
(c) a tangent (d) a secant

(ii) There are __________ types of arcs.
(a) 2 (b) 3
(c) 4 (d) 5

(iii) Right bisector of the chord of a circle always passes through the:
(a) diameter (b) non-collinear points (c) radius (d) centre

(iv) A circle has only one:
(a) chord (b) centre (c) diameter (d) secant

(v) The point where two tangents meet outside a circle forms:
(a) a semicircle (b) a diameter (c) a radius (d) an angle

(vi) Two equal tangents from a point to a circle can be drawn when the point is ________ the circle.
(a) on (b) inside (c) outside (d) at centre of

(vii) Tangents drawn from a single external point to a circle are:
(a) unequal in length (b) perpendicular to each other
(c) equal in length (d) inside the circle

(viii) At least how many chords are needed to locate the centre of the circle?
(a) 1 (b) 2 (c) 3 (d) 4

(ix) To draw a tangent at middle point of an arc, first step is to:
(a) join endpoints of an arc
(b) draw radius
(c) draw a line perpendicular to chord
(d) draw an angle

(x) The angle between the radius and a tangent at the point of contact is:
(a) 30° (b) 60° (c) 90° (d) 120°

2. Draw a circle of radius 1.4 cm and draw a tangent at point P, when P lies on its circumference.

Solution

Or according to book

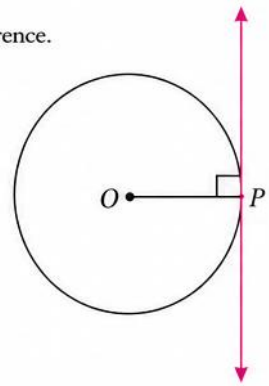
Given: Radius $r = 1.4$ cm
Point P on the circumference.

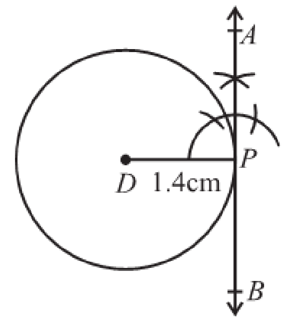
Required: Tangent at P.

Steps of Construction:

1. With O as centre, draw a circle of radius 1.4 cm.
2. Mark any point P on the circumference.
3. Draw OP.
4. Through P, draw a line perpendicular to OP.

The line is the required tangent.





3. Draw a circle of radius 1.2 cm and draw a tangent at point P, when P is at a distance of 5 cm from the centre.

Solution

Or according to book

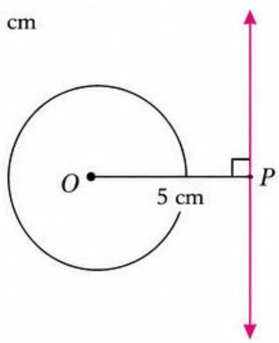
Given: Radius $r = 1.2$ cm
Point P is at a distance 5 cm from the centre O.

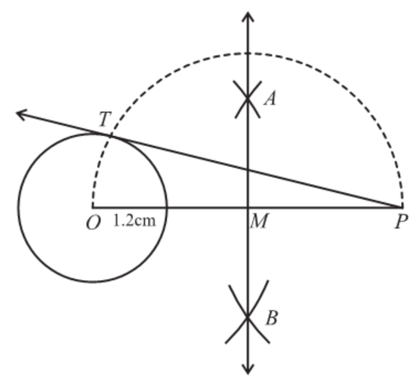
Required: Tangent at P.

Steps of Construction:

1. Draw $OP = 5$ cm.
2. With O as centre, draw a circle of radius 1.2 cm.
3. Through P, draw a line perpendicular to OP.

The line is the required tangent.





4. Draw a circle of radius 1.7 cm. Draw two tangents that meet an angle of 30°.

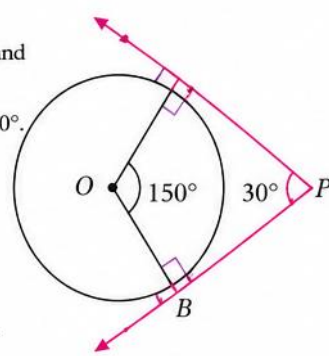
Solution

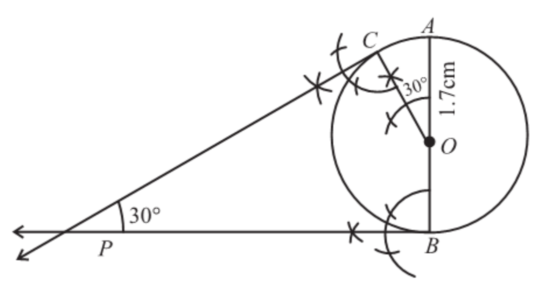
Or according to book

Required: Two tangents meeting at 30°.

Steps of Construction:

1. Draw a circle with centre O and radius 1.7 cm.
2. At O, construct $\angle AOB = 150°$.
3. Draw OA and OB.
4. Through A, draw a line perpendicular to OA. Through B, draw a line perpendicular to OB. Let the two lines meet at P. Then $\angle APB = 30°$.





5. Take a part of the circular track, use chords and perpendicular bisectors to complete a circular track.

Solution

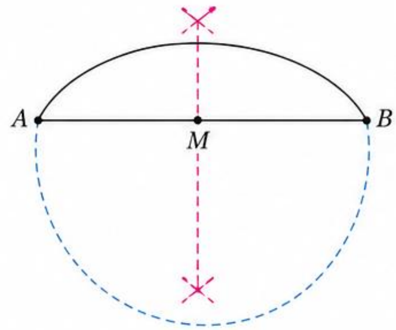
Or according to book

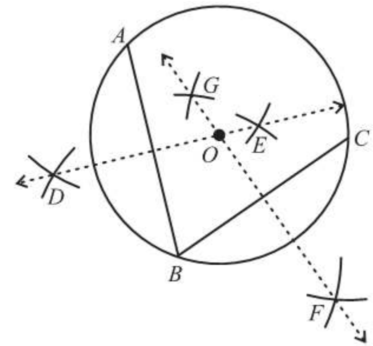
Given: A part of a circular track.

Required: Complete the circular track.

Steps of Construction:

1. Mark two points A and B on the given part of the track.
2. Join AB (chord).
3. Construct the perpendicular bisector of AB. Let it meet AB at M.
4. Taking M as centre and MA as radius, draw an arc on the side of the given track. This arc completes the circular track.
5. Darken the required circle.





6. Two decorative fences touch the circular flower bed of radius 2.1 cm and meet outside it at an angle of 30° to form an entrance arch. Draw two tangents to the flower bed from the point where the fences meet at an angle of 30°.

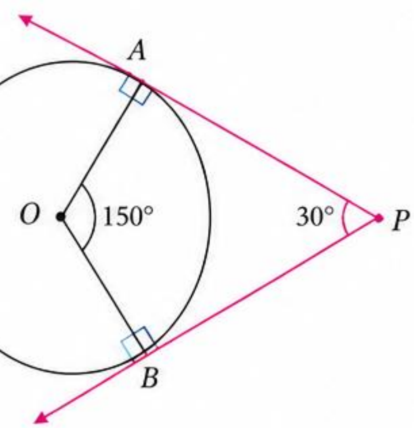
Solution

Given: Radius $r = 2.1$ cm
Angle between the decorative fences = 30°.

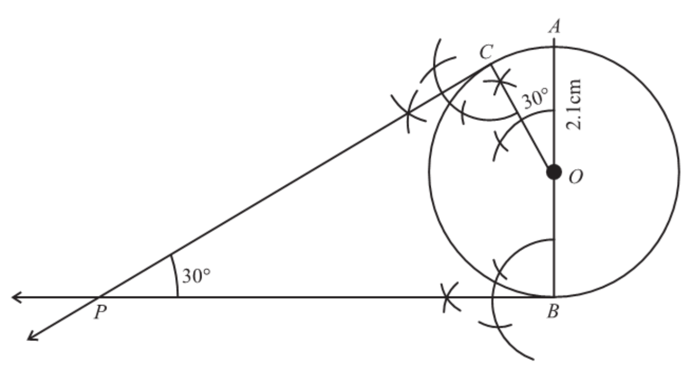
Required: Two tangents from the point where fences meet at 30°.

Steps of Construction:

1. Draw a circle with centre O and radius 2.1 cm.
2. At O, construct $\angle AOB = 150°$.
3. Draw OA and OB.
4. Through A, draw a line perpendicular to OA. Through B, draw a line perpendicular to OB. Let the two lines meet at P.
5. Then PA and PB are the required tangents and $\angle APB = 30°$.



Or according to book



UNIT 11
Information Handling

Some useful Formulae

Correlation Coefficient Formulae

- $r = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\ \text{Var}(Y)}}$
- $r = \frac{n\sum XY-(\sum X)(\sum Y)}{\sqrt{[n\sum X^2-(\sum X)^2][n\sum Y^2-(\sum Y)^2]}}$
- $r = \frac{n\sum XY-nXY}{\sqrt{[\sum X^2-nX^2][\sum Y^2-nY^2]}}$
- $r = \frac{\sum(X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum(X-\bar{X})^2\sum(Y-\bar{Y})^2}}$
- $r = \frac{\sum XY-nXY}{\sqrt{[\sum X^2-nX^2][\sum Y^2-nY^2]}}$
- $r = \frac{\sum XY-\frac{(\sum X)(\sum Y)}{n}}{\sqrt{\left[\sum X^2-\frac{(\sum X)^2}{n}\right]\left[\sum Y^2-\frac{(\sum Y)^2}{n}\right]}}$

Types of Correlation

Positive or direct correlation: If r is positive, the relationship between the domain and range values has a **positive or direct correlation**. In this case, if the domain value increases, the range value also tends to increase and vice versa. In this case both variables move in the same direction. The value of correlation coefficient for positive correlation is between 0 and 1. i.e. $0 < r < 1$.

Negative or inverse correlation: If r is negative, the linear relationship between the domain and range values has a **negative or inverse correlation**. In this case, if the domain value increases, the range value tends to decrease. In this case both variables move in the opposite direction. The value of correlation coefficient for negative correlation is between -1 and 0. i.e. $-1 < r < 0$.

Zero or null correlation: The absence of any relation between the variables is called zero correlation. In this case variables are independent to each other. i.e. $r = 0$.

Estimated Regression Line

The estimated regression line of Y on X; $Y = a_{YX} + b_{YX}X$	The estimated regression line of X on Y; $X = a_{XY} + b_{XY}Y$
$b_{YX} = \dfrac{\sum(X-\overline{X})(Y-\overline{Y})}{\sum(X-\overline{X})^2}$	$b_{XY} = \dfrac{\sum(X-\overline{X})(Y-\overline{Y})}{\sum(Y-\overline{Y})^2}$
$b_{YX} = \dfrac{\sum(XY - n\overline{X}\,\overline{Y})}{\sum(X-\overline{X})^2}$	$b_{XY} = \dfrac{\sum(XY - n\overline{X}\,\overline{Y})}{\sum(Y-\overline{Y})^2}$
$b_{YX} = \dfrac{\sum XY - \frac{(\sum X)(\sum Y)}{n}}{\sum X^2 - \frac{(\sum X)^2}{n}}$	$b_{XY} = \dfrac{\sum XY - \frac{(\sum X)(\sum Y)}{n}}{\sum Y^2 - \frac{(\sum Y)^2}{n}}$
$b_{YX} = \dfrac{n\sum XY - (\sum X)(\sum Y)}{n\sum X^2 - (\sum X)^2}$	$b_{XY} = \dfrac{n\sum XY - (\sum X)(\sum Y)}{n\sum Y^2 - (\sum Y)^2}$
$b_{YX} = \dfrac{S_{XY}}{S_X^2} = r\dfrac{S_Y}{S_X}$	$b_{XY} = \dfrac{S_{XY}}{S_Y^2} = r\dfrac{S_X}{S_Y}$
$a_{YX} = \dfrac{\sum Y - b_{YX}\sum X}{n}$	$a_{XY} = \dfrac{\sum X - b_{XY}\sum Y}{n}$
$a_{YX} = \overline{Y} - b_{YX}\overline{X}$	$a_{XY} = \overline{X} - b_{XY}\overline{Y}$

EXERCISE 11.1

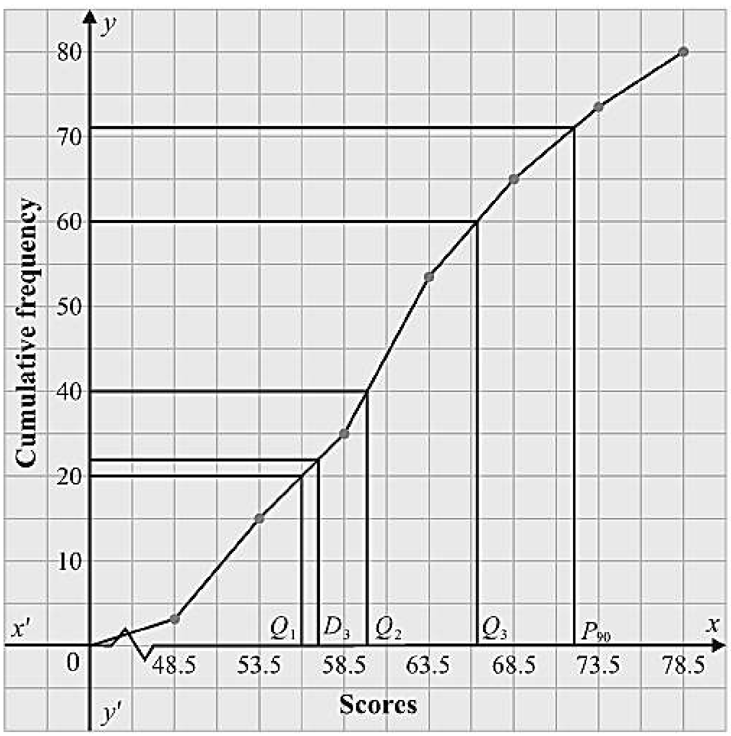
1. The following frequency distribution represents the achievement of a student's group upon a memory test:
   (i) Plot cumulative frequency graph of group scores.
   (ii) Determine median, lower and upper quartiles, D_3, P_{90} and interquartile range graphically.

Note: Write your answers in whole numbers.

Scores	44 – 48	49 – 53	54 – 58	59 – 63	64 – 68	69 – 73	74 – 78
No. of Students	3	12	15	23	12	8	7

Solution

(i)



(ii) Median = 61,
$Q_1 = 55$,
$Q_3 = 66$,
$D_3 = 57$,
$P_{90} = 73$,
$IQR = 11$

(ii) $N = 80$, Class width, $h = 5$

Formulae: Median $= L + \dfrac{\frac{N}{2} - c_f}{f} \times h$,

$$Q_1 = L + \frac{\frac{N}{4} - c_f}{f} \times h, \quad Q_3 = L + \frac{\frac{3N}{4} - c_f}{f} \times h$$

$$D_k = L + \frac{\frac{kN}{10} - c_f}{f} \times h, \quad P_k = L + \frac{\frac{kN}{100} - c_f}{f} \times h$$

Median: $\dfrac{N}{2} = \dfrac{80}{2} = 40$

Median class $= 59 - 63$

$L = 58.5, \quad c_f = 30, \quad f = 23$

$$\text{Median} = 58.5 + \frac{40 - 30}{23} \times 5 = 58.5 + \frac{10}{23} \times 5 = 58.5 + 2.1739 = 60.67 \approx 61$$

Q_1: $\dfrac{N}{4} = \dfrac{80}{4} = 20$

Class $= 54 - 58$

$L = 53.5, \quad c_f = 15, \quad f = 15$

$$Q_1 = 53.5 + \frac{20 - 15}{15} \times 5 = 53.5 + \frac{5}{15} \times 5 = 53.5 + 1.6667 = 55.17 \approx 55$$

Q_3: $\dfrac{3N}{4} = \dfrac{3(80)}{4} = 60$

Class $= 64 - 68$

$L = 63.5, \quad c_f = 53, \quad f = 12$

$$Q_1 = 63.5 + \frac{60 - 53}{12} \times 5 = 63.5 + \frac{7}{12} \times 5 = 63.5 + 2.9167 = 66.42 \approx 66$$

D_3: $\dfrac{3N}{10} = \dfrac{3(80)}{10} = 24$

Class $= 54 - 58$

$L = 53.5, \quad c_f = 15, \quad f = 15$

$$D_3 = 53.5 + \frac{24 - 15}{15} \times 5 = 53.5 + \frac{9}{15} \times 5 = 53.5 + 3 = 56.5 \approx 57$$

P_{90}: $\dfrac{90N}{100} = \dfrac{90(80)}{100} = 72$

Class $= 69 - 73$

$L = 68.5, \quad c_f = 65, \quad f = 8$

$$P_{90} = 68.5 + \frac{72 - 65}{8} \times 5 = 68.5 + \frac{7}{8} \times 5 = 68.5 + 4.375 = 72.875 \approx 73$$

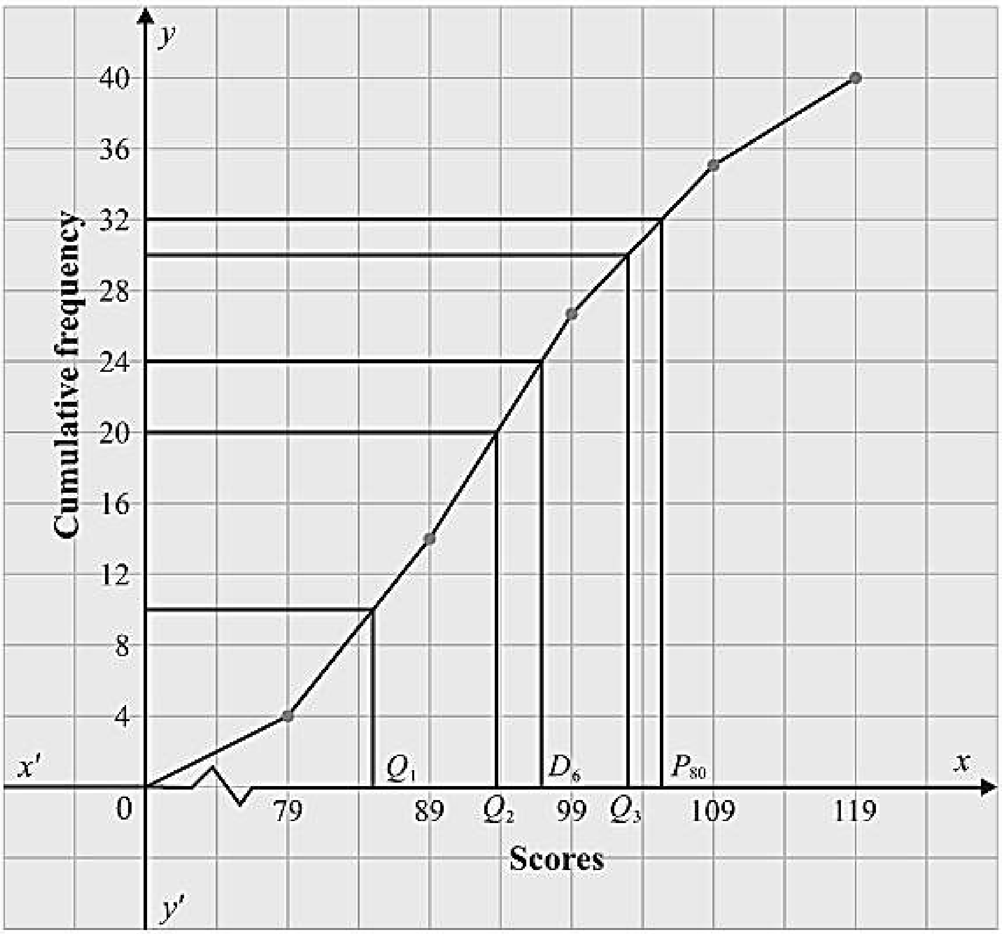
$$\text{IQR} = Q_3 - Q_1 = 66.42 - 55.17 = 11.25 \approx 11$$

2. Construct an ogive for the following distribution of scores:

Scores	69 – 79	79 – 89	89 – 99	99 – 109	109 – 119
f	4	10	13	8	5

Determine median, lower and upper quartiles, D_6, P_{80} and interquartile range graphically.

Solution



Median = 94,

$Q_1 = 85,$

$Q_3 = 103,$

$D_6 = 97,$

$P_{80} = 105$

$IQR = 18$

Scores	69–79	79–89	89–99	99–109	109–119
f	4	10	13	8	5
C.F.	4	14	27	35	40

$N = 40$, $h = 10$

Formulae: $\text{Median} = L + \dfrac{\frac{N}{2} - c_f}{f} \times h$, $Q_1 = L + \dfrac{\frac{N}{4} - c_f}{f} \times h$,

$Q_3 = L + \dfrac{\frac{3N}{4} - c_f}{f} \times h$, $D_6 = L + \dfrac{\frac{6N}{10} - c_f}{f} \times h$,

$P_{80} = L + \dfrac{\frac{80N}{100} - c_f}{f} \times h$

Median: $\dfrac{N}{2} = \dfrac{40}{2} = 20$

Class = 89–99

$L = 88.5$, $c_f = 14$, $f = 13$

$$\begin{aligned}\text{Median} &= 88.5 + \frac{20-14}{13} \times 10\\ &= 88.5 + \frac{6}{13} \times 10\\ &= 88.5 + 4.6154\\ &= 93.12 \approx 93\end{aligned}$$

Q_1: $\dfrac{N}{4} = \dfrac{40}{4} = 10$

Class = 79–89

$L = 78.5$, $c_f = 4$, $f = 10$

$$\begin{aligned}Q_1 &= 78.5 + \frac{10-4}{10} \times 10\\ &= 78.5 + \frac{6}{10} \times 10\\ &= 78.5 + 6\\ &= 84.5 \approx 85\end{aligned}$$

Q_3: $\dfrac{3N}{4} = \dfrac{3(40)}{4} = 30$

Class = 99–109

$L = 98.5$, $c_f = 27$, $f = 8$

$$\begin{aligned}Q_3 &= 98.5 + \frac{30-27}{8} \times 10\\ &= 98.5 + \frac{3}{8} \times 10\\ &= 98.5 + 3.75\\ &= 102.25 \approx 102\end{aligned}$$

D_6: $\dfrac{6N}{10} = \dfrac{6(40)}{10} = 24$

Class = 89–99

$L = 88.5$, $c_f = 14$, $f = 13$

$$\begin{aligned}D_6 &= 88.5 + \frac{24-14}{13} \times 10\\ &= 88.5 + \frac{10}{13} \times 10\\ &= 88.5 + 7.6923\\ &= 96.19 \approx 96\end{aligned}$$

P_{80}: $\dfrac{80N}{100} = \dfrac{80(40)}{100} = 32$

Class = 99–109

$L = 98.5$, $c_f = 27$, $f = 8$

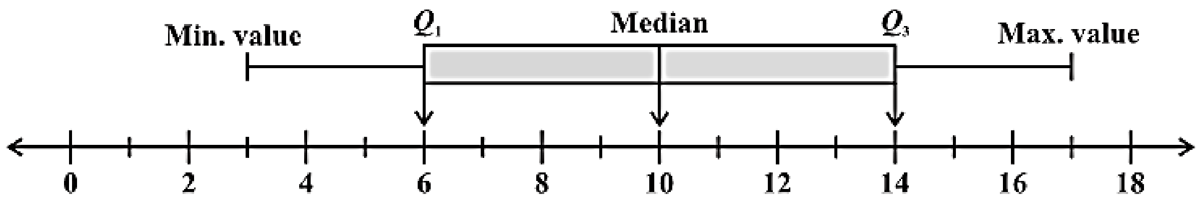
$$\begin{aligned}P_{80} &= 98.5 + \frac{32-27}{8} \times 10\\ &= 98.5 + \frac{5}{8} \times 10\\ &= 98.5 + 4.25\\ &= 104.75 \approx 105\end{aligned}$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 102.25 - 84.5\\ &= 17.75 \approx 17\end{aligned}$$

Answers are in approximation so both of above true either from book or my given solution

3. Find Q_1, Q_3, median, range and IQR for the dataset given below:
3, 6, 8, 4, 7, 5, 10, 11, 13, 9, 14, 12, 15, 16, 17
Also draw a box-and-whisker plot.

Solution



Ordered data:

3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 $\quad n = 15$

Median $= \dfrac{n+1}{2} = \dfrac{15+1}{2} = 8^{th}$ item
Median $= 10$

Q_1 position $= \dfrac{n+1}{4} = \dfrac{16}{4} = 4^{th}$ item
$Q_1 = 6$

Q_3 position $= \dfrac{3(n+1)}{4} = \dfrac{3(16)}{4} = 12^{th}$ item
$Q_3 = 14$

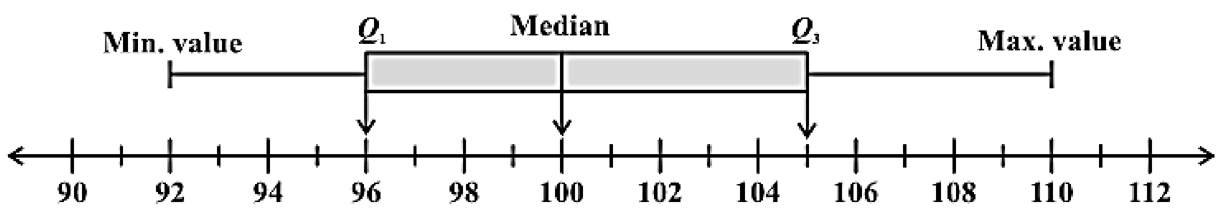
Range = Max - Min = 17 − 3 = 14

IQR $= Q_3 - Q_1 = 14 - 6 = 8$

Five-number summary: (3, 6, 10, 14, 17)

4. Find Q_1, Q_3, median, range, IQR and extreme values plot for the following data:
102, 98, 95, 100, 93, 110, 108, 104, 97, 96, 92, 101, 99, 105, 107
Also draw a box-and-whisker plot.

Solution



Ordered data:

92, 93, 95, 96, 97, 98, 99, 100, 101, 102, 104, 105, 107, 108, 110 $\quad n = 15$

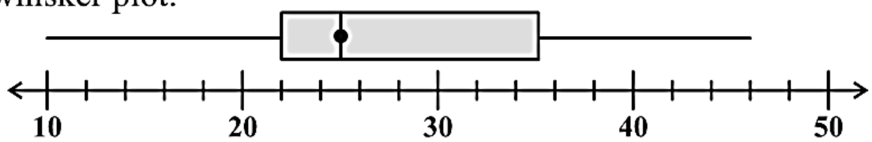
Median $= \dfrac{n+1}{2} = \dfrac{15+1}{2} = 8^{th}$ item
Median $= 100$

Q_1 position $= \dfrac{n+1}{4} = \dfrac{16}{4} = 4^{th}$ item
$Q_1 = 96$

Q_3 position $= \dfrac{3(n+1)}{4} = \dfrac{3(16)}{4} = 12^{th}$ item
$Q_3 = 105$

Five-number summary: (92, 96, 100, 105, 110)

5. Find Q_1, Q_3, median, minimum and maximum values for the following box-and-whisker plot:

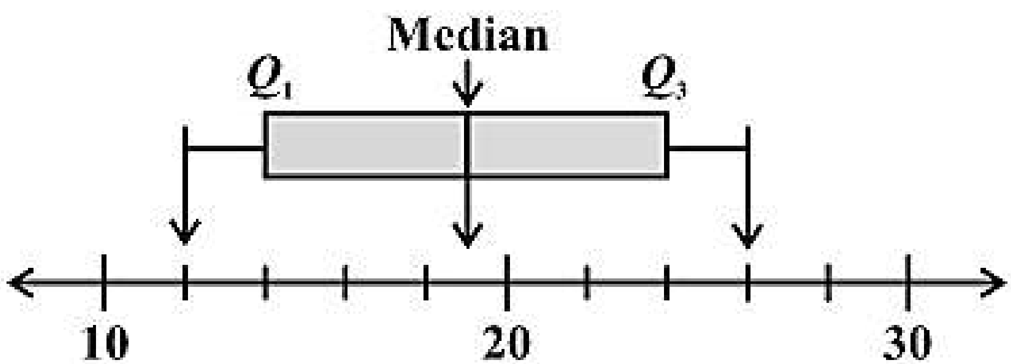


Solution

Minimum	=	10
Q_1	=	22
Median	=	25
Q_3	=	35
Maximum	=	46

6. Draw the box-and-whisker plot, if min. value = 12, max. value = 26, median = 19, Q_1 = 14 and Q_3 = 24.

Solution

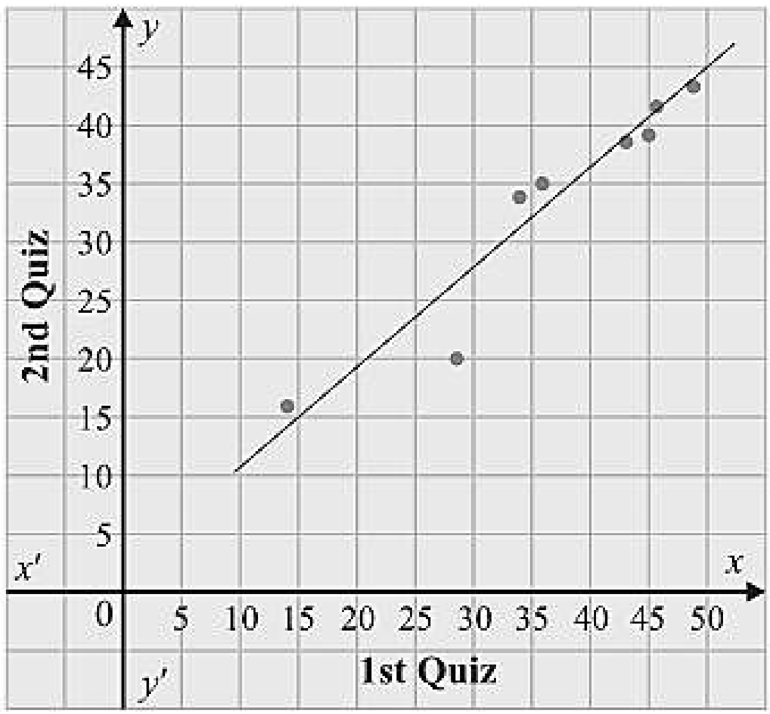


7. Construct a scatter diagram and draw a line of best fit for the following quiz scores for 8 students in a class:

1st quiz	14	28	34	36	43	45	46	49
2nd quiz	16	20	33	35	38	39	42	43

Describe correlation between 1st and 2nd quiz scores also.

Solution



Strong positive correlation between 1st and 2nd quiz.

1st quiz	14	28	34	36	43	45	46	49
2nd quiz	16	20	33	35	38	39	42	43

To describe correlation:
As the 1st quiz scores increase, the 2nd quiz scores also increase.
The points would lie close to an upward sloping line.
Therefore, there is a **strong positive correlation.**

8. The following table shows the test scores (one measuring speed and the other measuring strength) of football players out of 200:

1[st] Test	20	38	70	40	100	110	50	136	150	80
2[nd] Test	160	142	130	120	70	50	140	44	36	100

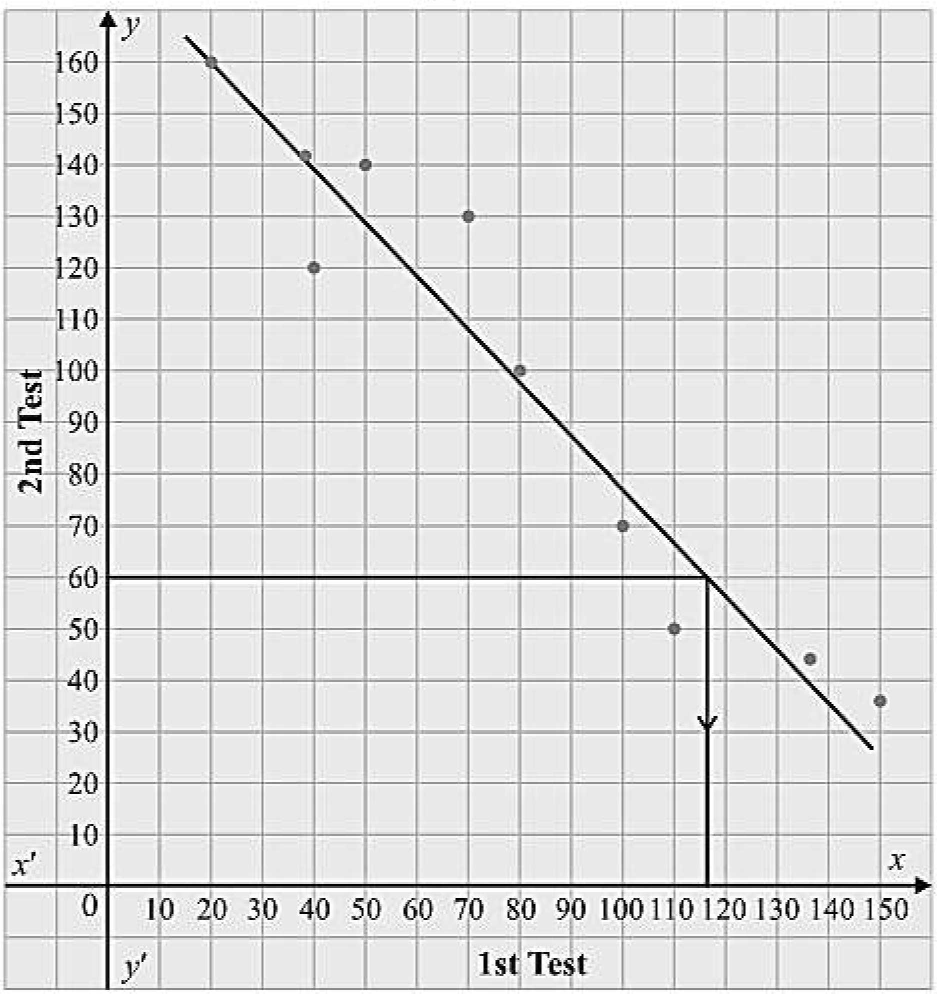
(i) Construct a scatter diagram and draw a line of best fit.

(ii) Describe correlation between the 1[st] and 2[nd] tests.

(iii) Abdullah scores 60 in the 2[nd] test. Estimate his score in the 1[st] test.

Solution

8. (i)



(ii) Strong negative correlation. **(iii)** 115.5

(iii)

x	y	xy	y²
20	160	3200	25600
38	142	5396	20164
70	130	9100	16900
40	120	4800	14400
100	70	7000	4900
110	50	5500	2500
50	140	7000	19600
136	44	5984	1936
150	36	5400	1296
80	100	8000	10000
794	**992**	**62280**	**114500**

$$b=\frac{n\sum xy-\sum x\sum y}{n\sum y^2-(\sum y)^2}=\frac{622800-787648}{1145000-984064}$$

$$b=\frac{-164848}{160936}=-1.0243$$

$$a=\bar{x}-b\bar{y}=79.4+1.0243(99.2)=181.01$$

$$x=181.01-1.0243y$$

$$x=181.01-1.0243(60)=115.5$$

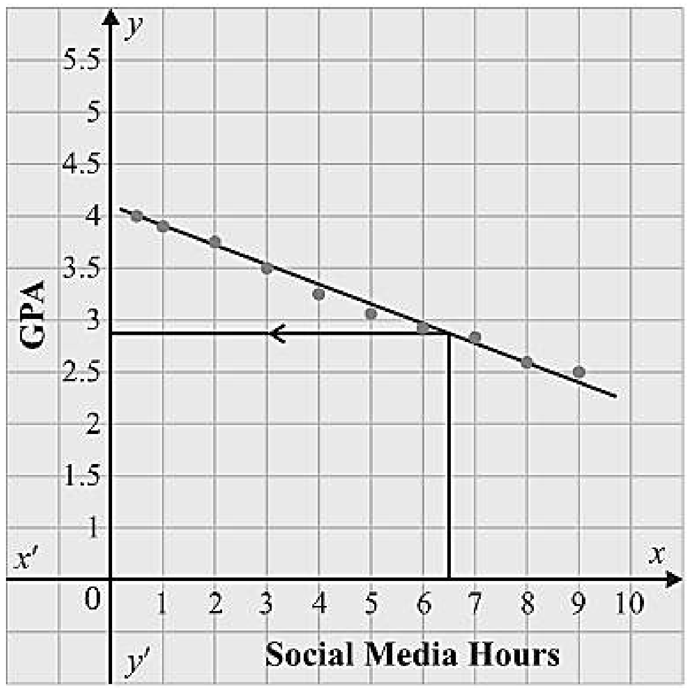
9. The following table shows the number of daily social media hours and GPA of university students.

Social Media Hours	0.5	1	2	3	4	5	6	7	8	9
GPA	4.0	3.9	3.7	3.5	3.3	3.1	2.9	2.8	2.6	2.5

(i) Construct a scatter diagram and draw a line of best fit.
(ii) Identify the nature of correlation between social media use and GPA.
(iii) Estimate the GPA for a student who spends 6.5 hours on social media.

Solution

(i)



(ii) Strong negative correlation. **(iii)** 2.9

(iii) Estimate GPA for $x = 6.5$ hours

n	$\sum x$	$\sum y$	$\sum xy$	$\sum x^2$
10	45.5	32.3	135.55	295.25

Least squares regression of y on x:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}, \quad a = \frac{\sum y - b\sum x}{n}$$

$$b = \frac{10(135.55) - (45.5)(32.3)}{10(295.25) - (45.5)^2} = \frac{1,355.5 - 1,470.65}{2,952.5 - 2,070.25} = \frac{-115.15}{882.25} = -0.1305$$

$$a = \frac{32.3 - (-0.1305)(45.5)}{10} = \frac{32.3 + 5.939}{10} = \frac{38.239}{10} = 3.8239$$

Regression equation: $y = 3.8239 - 0.1305x$

For $x = 6.5$, $y = 3.8239 - 0.1305(6.5) = 3.8239 - 0.84825 = 2.97565 \approx 2.98$

Estimated GPA for 6.5 hours is about 2.98.

EXERCISE 11.2

1. Find the range of the following data sets:

(i) 63, 89, 98, 125, 79, 108, 117, 60 (ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.4

Solution

(i) Data: 63, 89, 98, 125, 79, 108, 117, 60

Highest value = 125, Lowest value = 60

Range = Highest – Lowest = 125 – 60 = 65

Therefore, $\boxed{\text{Range} = 65}$

(ii) Data: 43.5, 13.6, 18.9, 38.4, 61.4, 29.4

Highest value = 61.4, Lowest value = 13.6

Range = Highest – Lowest = 61.4 – 13.6 = 47.8

Therefore, $\boxed{\text{Range} = 47.8}$

2. If the range and the lowest value of a set of data are 46.7 and 13.4 respectively, then find the highest value.

Solution

Given, Range = 46.7 and Lowest value = 13.4

Range = Highest – Lowest

46.7 = Highest – 13.4

Highest = $\boxed{46.7 + 13.4 = 60.1}$

Therefore, $\boxed{\text{Highest value} = 60.1}$

3. Calculate the range of the following data:

Income (in Rs.)	4000 – 4500	4500 – 5000	5000 – 5500	5500 – 6000	6000 – 6500
No. of workers	8	12	30	21	6

Solution

Highest income = 6500, Lowest income = 4000

Range = Highest – Lowest = 6500 – 4000 = 2500

Therefore, **Range = Rs. 2500**

4. A group of 7 workers reported the number of items they assembled in a day as:

52, 55, 50, 53, 54, 56, 52

Find the standard deviation and variance of the items assembled.

Solution

Data: 52, 55, 50, 53, 54, 56, 52 $(n = 7)$

Mean, $\bar{x} = \dfrac{\sum x}{n} = \dfrac{52+55+50+53+54+56+52}{7} = \dfrac{372}{7} = 53.14$

Variance, $\sigma^2 = \dfrac{\sum (x-\bar{x})^2}{n}$

$$= \frac{(52-53.14)^2+(55-53.14)^2+(50-53.14)^2+(53-53.14)^2+(54-53.14)^2+(56-53.14)^2+(52-53.14)^2}{7}$$

$$= \frac{24.86}{7} = 3.55$$

Standard deviation, $\sigma = \sqrt{\sigma^2} = \sqrt{3.55} = 1.88$

Variance = 3.55, Standard Deviation = 1.88

5. A librarian recorded the number of visitors during 5 days of a week.

120, 135, 130, 125, 140

Calculate the variance and standard deviation of visitors.

Solution

Data: 120, 135, 130, 125, 140 $(n = 5)$

Mean, $\bar{x} = \frac{120 + 135 + 130 + 125 + 140}{5} = \frac{650}{5} = 130$

Variance, $\sigma^2 = \frac{\sum(x - \bar{x})^2}{n}$

$$= \frac{(120-130)^2 + (135-130)^2 + (130-130)^2 + (125-130)^2 + (140-100)^2}{7}$$

$$= \frac{100 + 25 + 0 + 25 + 100}{5} = \frac{250}{5} = 50$$

Standard deviation, $\sigma = \sqrt{50} = 7.07$

Variance = 50, Standard Deviation = 7.07

6. Find the range, variance and standard deviation of first 23 odd numbers.

Solution

First 23 odd numbers are 1,3,5,…,45

Range $= 45 - 1 = 44$

$\overline{X} = \frac{\text{First+Last}}{2} = \frac{1+45}{2} = 23$

Variance $= \sigma^2 = \frac{n^2-1}{n} = \frac{23^2-1}{3} = \frac{529-1}{3} = \frac{528}{3} = 176$

Standard Deviation $= \sigma = \sqrt{\frac{n^2-1}{n}} = \sqrt{176} = 13.27$

7. The rainfall recorded in various places of five districts in a week is given below. Find its variance and standard deviation.

Rainfall (in mm)	42	51	54	61	63	71
Number of places	5	13	4	9	5	4

Solution

x	f	x - $\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
42	5	-13.925	194.006	970.03
51	13	-4.925	24.256	315.33
54	4	-1.925	3.706	14.82
61	9	5.075	25.756	231.80
63	5	7.075	50.056	250.28
71	4	15.075	227.256	909.02
Total	**40**			**2691.28**

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2237}{40} = 55.925$$

$$\sigma^2 = \frac{\sum f(x-\bar{x})^2}{N} = \frac{2691.28}{40} = 67.282$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{67.282} \approx 8.20$$

8. Machine A Output (units): 98, 100, 102, 101, 99
Machine B Output (units): 95, 100, 105, 90, 110
(i) Which machine has better performance?
(ii) Which machine is more consistent?

Solution

$$\bar{x}_A = \frac{98+100+102+101+99}{5} = 100$$

$$\sigma_A^2 = \frac{(98-100)^2+(100-100)^2+(102-100)^2+(101-100)^2+(99-100)^2}{5} = \frac{10}{5} = 2$$

$$\sigma_A = \sqrt{2} = 1.41$$

Machine B: 95, 100, 105, 90, 110

$$\bar{x}_B = \frac{95+100+105+90+110}{5} = 100$$

$$\sigma_B^2 = \frac{(95-100)^2+(100-100)^2+(105-100)^2+(90-100)^2+(110-100)^2}{5} = \frac{250}{5} = 50$$

$$\sigma_B = \sqrt{50} = 7.07$$

(i) Both machines have equal performance.
(ii) Machine A is more consistant.

9. The monthly sales (rupees in lacs) for two salespersons over 6 months are:
Person A: 5.5, 5.7, 5.4, 5.6, 5.8, 5.6
Person B: 5.5, 6.5, 4.5, 6.0, 5.0, 6.0
Compare their performance and consistency.

Solution

Person A: 5.5, 5.7, 5.4, 5.6, 5.8, 5.6

$\bar{x}_A = 5.60$, $\sigma_A = 0.14$

Person B: 5.5, 6.5, 4.5, 6.0, 5.0, 6.0

$\bar{x}_B = 5.58$, $\sigma_B = 0.72$

Comparison:
(i) Mean: A (5.60) > B (5.58) $\Rightarrow$ Person A has slightly better average sales.
(ii) Standard deviation: A (0.14) < B (0.72) $\Rightarrow$ Person A is more consistent.

10. The table given below shows the daily wages of workers in a textile mill, grouped into six income brackets:

Daily Wage (Rs)	800 – 1000	1000 – 1200	1200 – 1400	1400 – 1600	1600 – 1800	1800 – 2000
Frequency	2	4	6	8	2	1

Calculate the mean, variance and standard deviation of the wages.

Solution

Wage (Rs)	f_i	Midpoint x_i	f_ix_i	$x_i - \bar{x}$	$f_i(x_i - \bar{x})^2$
800-1000	2	900	1800	-460.87	425,025.8
1000-1200	4	1100	4400	-260.87	272,300.5
1200-1400	6	1300	7800	-60.87	22,227.6
1400-1600	8	1500	12,000	139.13	154,938.3
1600-1800	2	1700	3400	339.13	230,045.2
1800-2000	1	1900	1900	539.13	290,663.5
Total	**23**		**31,300**		**1,395,200.9**

Mean

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{31300}{23} = 1360.87$$

Variance

$$\sigma^2 = \frac{\sum f_i (x_i - \bar{x})^2}{N} = \frac{1,395,200.9}{23} = 60642.72$$

Standard Deviation

$$\sigma = \sqrt{\sigma^2} = \sqrt{60642.72} = 246.26$$

11. A company forecasts monthly sales (rupees in millions): 15, 18, 14, 20, 13. Find variability in sales predictions.

Solution

Mean $\mu = \frac{15+18+14+20+13}{5} = \frac{80}{5} = 16$

x	$x-\mu$	$(x-\mu)^2$
15	-1	1
18	2	4
14	-2	4
20	4	16
13	-3	9
Total		**34**

Variance $= \sigma^2 = \frac{\sum(x-\mu)^2}{N} = 6.8 \text{ million}^2$

Standard Deviation $= \sigma = \sqrt{\frac{\sum(x-\mu)^2}{N}} = \sqrt{6.8} = 2.61 \text{ million}$

12. Unemployment rates (%) in five provinces are 5.2, 6.0, 4.8, 5.5, 6.2. Calculate standard deviation and describe is there balanced unemployment rate?

Solution

Mean $\mu = \frac{5.2+6.0+4.8+5.5+6.2}{5} = 5.54$

x	$x-\mu$	$(x-\mu)^2$
5.2	-0.34	0.1156
6.0	0.46	0.2116
4.8	-0.74	0.5476
5.5	-0.04	0.0016
6.2	0.66	0.4356
Total		**1.312**

Variance $= \sigma^2 = \frac{\sum(x-\mu)^2}{N} = 0.2624$

Standard Deviation $= \sigma = \sqrt{0.2624} = 0.51$

Unemployment rates are fairly balanced

13. Find variance and standard deviation:

(i) $\Sigma x = 45, \Sigma x^2 = 421, n = 5$ (ii) $\Sigma fx = 210, \Sigma fx^2 = 7560, \Sigma f = 6$

(iii) $\overline{x} = 18, \Sigma fx^2 = 1670, \Sigma f = 5$

Solution

(i) $\sum x = 45, \sum x^2 = 421, n = 5$

$$\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \frac{421}{5} - \left(\frac{45}{5}\right)^2 = 84.2 - 81 = 3.2$$

$$\sigma = \sqrt{3.2} = 1.79$$

(ii) $\sum fx = 210, \sum fx^2 = 7560, \sum f = 6$

$$\sigma^2 = \frac{7560}{6} - \left(\frac{210}{6}\right)^2 = 1260 - 1225 = 35$$

$$\sigma = \sqrt{35} = 5.92$$

(iii) $\bar{x} = 18, \sum fx^2 = 1670, \sum f = 5$

$$\sigma^2 = \frac{1670}{5} - 18^2 = 334 - 324 = 10$$

$$\sigma = \sqrt{10} = 3.16$$

REVIEW EXERCISE 11

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) ________ is used to get the cumulative frequencies.

✓(a) Addition (b) square root (c) multiplication (d) division

(ii) Second quartile represents:

(a) mean (b) mode ✓(c) median (d) variance

(iii) First quartile divides the data into ________ equal parts.

(a) Two (b) Three ✓(c) four (d) ten

(iv) Difference between the highest and the lowest values is called:

(a) mean (b) variance ✓(c) range (d) standard deviation

(v) Scatter diagram represents a relationship between ________ variables.

(a) five ✓(b) two (c) three (d) four

(vi) ________ is measure of dispersion:

(a) mean (b) median (c) mode ✓(d) variance

(vii) Positive square root of variance is called:

(a) mean (b) median
✓(c) standard deviation (d) range

(viii) ________ is not measure of dispersion.

(a) range ✓(b) arithmetic mean
(c) variance (d) standard deviation

(ix) Variance of the data 8, 8, 8, 8, 8, 8 is:

✓(a) 0 (b) 16 (c) 8 (d) 48

(x) Range of first 20 natural numbers is:

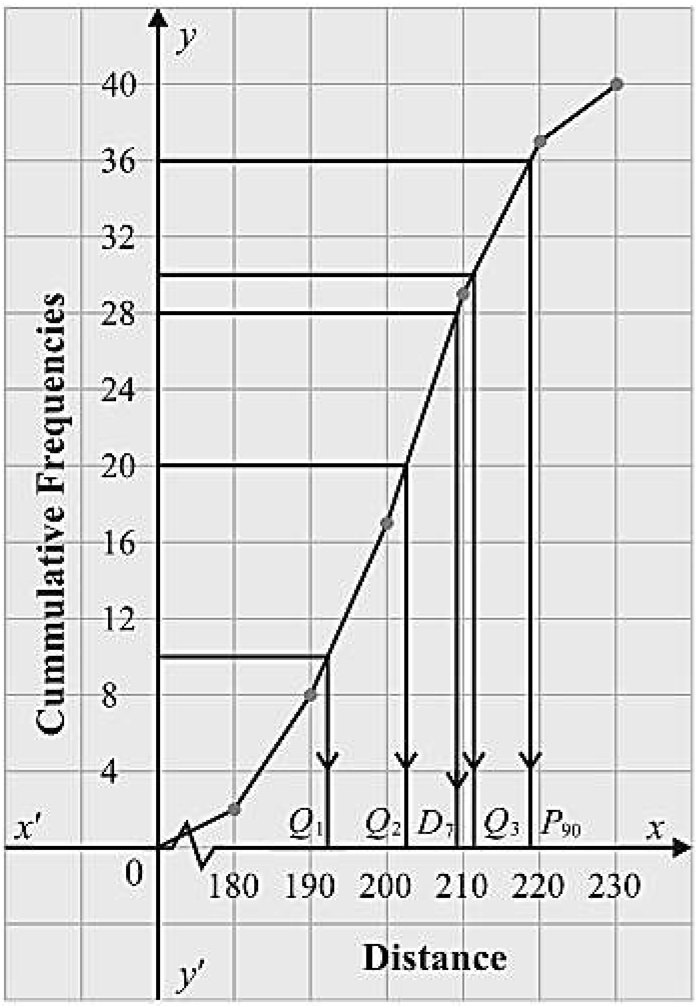
(a) 20 (b) 10 ✓(c) 19 (d) 30

2. The following results for the long jump were recorded:

Distance (in cm)	170 – 180	180 – 190	190 – 200	200 – 210	210 – 220	220 – 230
f	2	6	9	12	8	3

Construct the cumulative frequency polygon and locate median, Q_1, Q_3, D_7, P_{90} and interquartile range on it.

Solution



Median $= 202.5$, $Q_1 = 192.2$, $Q_3 = 211.3$,

$D_7 = 209.2$, $P_{90} = 218.8$, $IQR = 19.1$

(ii)

Distance (in cm)	f	cf
170 – 180	2	2
180 – 190	6	8
190 – 200	9	17
200 – 210	12	29
210 – 220	8	37
220 – 230	3	40

Total $N = 40$

Cumulative frequency (less-than type): (upper class boundary, c.f.)

$(170, 0), (180, 2), (190, 8), (200, 17), (210, 29), (220, 37), (230, 40)$

$$\text{Median} = l + \left(\frac{N/2 - c.f._{prev}}{f_m}\right)h = 200 + \left(\frac{20-17}{12}\right)10 = 202.50\ cm$$

$$Q_1 = l + \left(\frac{N/4 - c.f._{prev}}{f}\right)h = 190 + \left(\frac{10-8}{9}\right)10 = 192.22\ cm$$

$$Q_3 = l + \left(\frac{3N/4 - c.f._{prev}}{f}\right)h = 210 + \left(\frac{30-29}{8}\right)10 = 211.25\ cm$$

$$D_7 = l + \left(\frac{7N/10 - c.f._{prev}}{f}\right)h = 200 + \left(\frac{28-17}{12}\right)10 = 209.17 cm$$

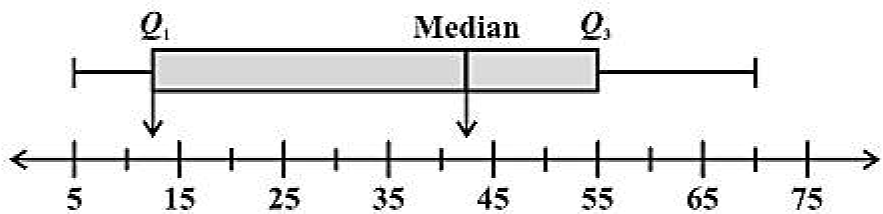
$$P_{90} = l + \left(\frac{90N/100 - c.f._{prev}}{f}\right)h = 210 + \left(\frac{36-29}{8}\right)10 = 218.8\ cm$$

$$\text{Interquartile range} = Q_3 - Q_1 = 211.25 - 192.22 = 19.03\ cm$$

3. The summary statistics for a data set is given below. Show it with a box-and-whisker plot.

Min. Value	Max. Value	Q_1	Median	Q_3
5	70	12.6	43	55.6

Solution



4. The following table shows the weekly hours spent exercising and resting heart rate (beats per minute) for ten individuals:

Exercise Hours	0	1	2	3	4	5	6	7	8	9
Heart Rate	90	85	83	80	76	74	72	70	68	65

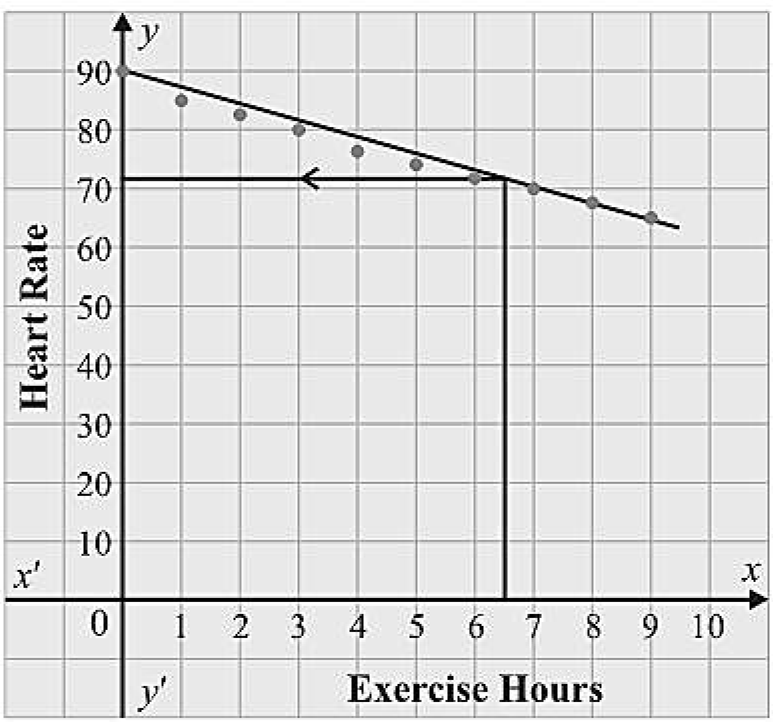
(i) Plot the data on a scatter diagram and draw a line of best fit.

(ii) State the type of correlation observed.

(iii) Predict the heart rate of Sakeena who exercises for 6.5 hours per week.

Solution

(i)



(ii) Strong negative correlation.

(iii) 71

x	y	xy	x^2	y^2
0	90	0	0	8100
1	85	85	1	7225
2	83	166	4	6889
3	80	240	9	6400
4	76	304	16	5776
5	74	370	25	5476
6	72	432	36	5184
7	70	490	49	4900
8	68	544	64	4624
9	65	585	81	4225
45	**763**	**3216**	**285**	**58799**

$b = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2} = -2.64$ and $a = \frac{y - b\sum x}{n} = 88.16$

The best fitted line is $\hat{y} = a + bx = 88.16 - 2.64x$

(ii)

Type of correlation: $r = \frac{n\sum XY - (\sum X)(\sum Y)}{\sqrt{[n\sum X^2 - (\sum X)^2][n\sum Y^2 - (\sum Y)^2]}} = -0.993$

Strong negative correlation

(iii)

Prediction for 6.5 hours

$$\hat{y} = a + bx = 88.16 - 2.64(6.5) = 71.03$$

5. Find the range for the given data, 25 , 30 , 35 , 40 , 50 , 60 , 65 , 75

Solution

Range = 75 – 25 = 50

6. Calculate range, variance and standard deviation for the following data set:

Class Interval	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25	25 – 30
f	4	12	20	24	16	4

Solution

Class Interval	f	Midpoint x	fx	$x-\bar{x}$	$f(x-\bar{x})^2$
0 – 5	4	2.5	10	-14.75	870.25
5 – 10	12	7.5	90	-9.75	1140.75
10 – 15	20	12.5	250	-4.75	451.25
15 – 20	24	17.5	420	0.25	1.50
20 – 25	16	22.5	360	5.25	441.00
25 – 30	4	27.5	110	10.25	420.25
Total	**80**		**1240**		**3325.00**

Range

$$\text{Range} = 30 - 0 = 30$$

Mean

$$\bar{x} = \frac{\sum fx}{N} = \frac{1240}{80} = 15.5$$

Variance

$$\sigma^2 = \frac{\sum f(x-\bar{x})^2}{N} = \frac{3080}{80} = 38.5$$

Standard deviation

$$\sigma = \sqrt{38.5} \approx 6.2$$

7. The sum of 5 numbers is 45 and the sum of their squares is 421. Find the mean and standard deviation of the data.

Solution

Sum of numbers, $\sum x = 45$ (given)

Sum of squares, $\sum x^2 = 421$ (given)

$$\text{Mean } \bar{x} = \frac{\sum x}{n} = \frac{45}{5} = 9$$

$$\text{Variance } \sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \frac{421}{5} - 9^2 = 3.2$$

$$\text{Standard deviation } \sigma = \sqrt{3.2} = 1.789$$

8. The monthly household expenses (rupees in thousands) of two families for 6 months were:

Family A: 45, 47, 46, 48, 46, 47

Family B: 38, 52, 40, 50, 42, 49

Calculate the mean and standard deviation of the monthly expenses. Which family spends more on average? Which family has more stable expenses?

Solution

Family A: 45, 47, 46, 48, 46, 47

$$\bar{x}_A = \frac{279}{6} = 46.50$$

$$\sum x_A^2 = 13031 \Rightarrow s_A^2 = \frac{13031}{6} - (46.5)^2 = 0.9167$$

$$s_A = 0.9574$$

Family B: 38, 52, 40, 50, 42, 49

$$\bar{x}_B = \frac{271}{6} = 45.17$$

$$\sum x_B^2 = 12521 \Rightarrow s_B^2 = \frac{12521}{6} - (45.17)^2 = 32.8056$$

$$s_B = 5.7285$$

9. The daily wages of 40 workers in a factory are grouped as follows:

Daily Wages (Rs.)	1000-1200	1200-1400	1400-1600	1600-1800	1800-2000
No. of Workers	6	10	12	8	4

Find the mean, variance and standard deviation of the daily wages.

Solution

Daily Wages (Rs.)	f	Midpoint x	fx	$x-\bar{x}$	$f(x-\bar{x})^2$
1000–1200	6	1100	6600	-370	822600
1200–1400	10	1300	13000	-170	289000
1400–1600	12	1500	18000	30	10800
1600–1800	8	1700	13600	230	423200
1800–2000	4	1900	7600	430	739600
Total	**40**		**58800**		**2285200**

Midpoint $x = \frac{\text{lower} + \text{upper}}{2}$

2. Mean

$$\bar{x} = \frac{\sum fx}{N} = \frac{58800}{40} = 1470$$

3. Variance

$$\sigma^2 = \frac{\sum f(x-\bar{x})^2}{N} = \frac{2285200}{40} = 57130$$

Using the given value: $\sigma^2 = 57100$

4. Standard deviation

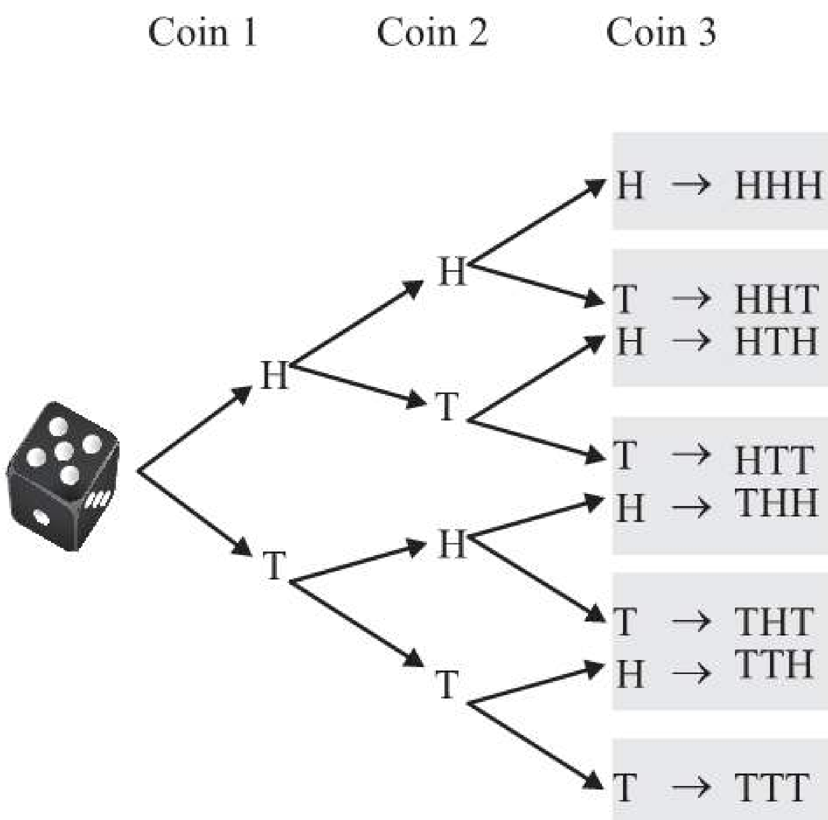
$$\sigma = \sqrt{57100} \approx 238.96$$

UNIT 12 Probability

EXERCISE 12.1

1. Find the sample space for tossing three coins using tree diagram.

Solution

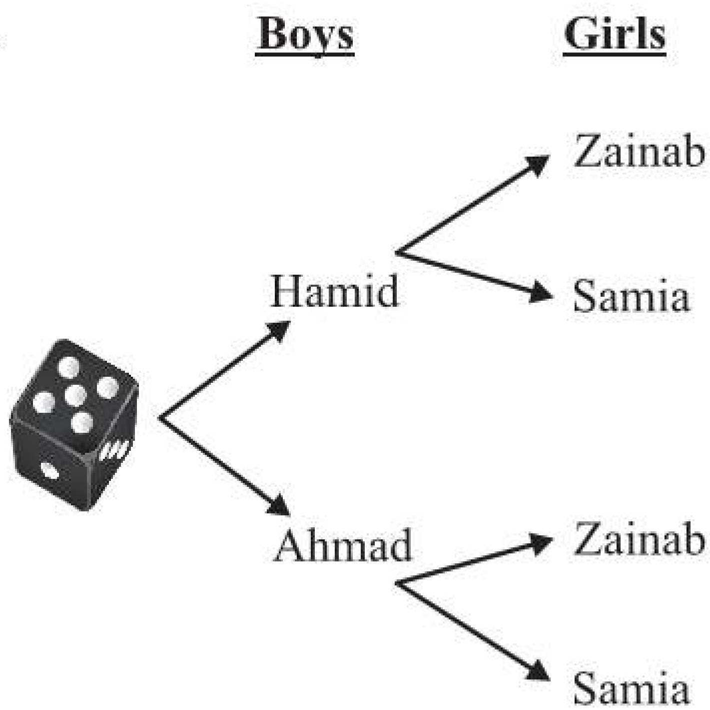


Sample space = {(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)}

Total outcomes: $n(S) = 8$

2. A teacher randomly selects one boy and one girl from a group of 2 boys (Hamid, Ahmad) and 2 girls (Zainab, Samia). Draw a tree diagram and list the sample space for all possible outcomes.

Solution



Sample Space = {(Hamid, Zainab),
(Hamid, Samia),
(Ahmad, Zainab),
(Ahmad, Samia)}

Total outcomes: n(S) = 4

3. If A is an event of a random experiment such that $P(A):P(\overline{A}) = 17:15$ and $n(S) = 640$, then find (i) $P(\overline{A})$ (ii) $n(A)$

Solution

$P(A) : P(\bar{A}) = 17 : 15$ and $n(S) = 640$ and Let $P(A) = 17x$, $P(\bar{A}) = 15x$.

Since $P(A) + P(\bar{A}) = 1 \Rightarrow 17x + 15x = 1 \Rightarrow 32x = 1 \Rightarrow x = \frac{1}{32}$

(i) $P(\bar{A}) = 15x \Rightarrow P(\bar{A}) = \frac{15}{32}$

(ii) $P(A) = P(A) \times n(S) = \frac{17}{32} \times 640 = 340$

4. A coin is tossed thrice. What is the probability of getting two consecutive tails?

Solution

Sample Space (S): {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} $\Rightarrow$ n(S) = 8

Event E (two consecutive tails): {HTT, TTH, TTT} $\Rightarrow$ n(E) = 3

P(E) = $\frac{3}{8}$

5. A dice is rolled and coin is tossed together.

(i) Find sample space by drawing possibility diagram.

(ii) Find sample space by sketching tree diagram.

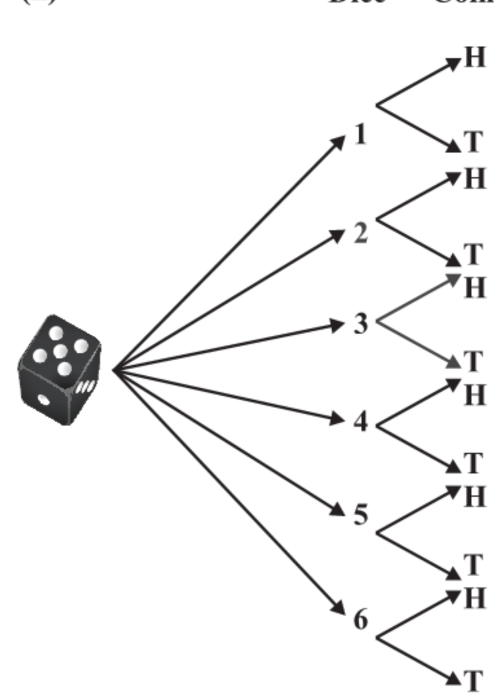
(iii) What is a probability of getting a tail and an even number?

Solution

(i)

Dice	Coin: *H*	Coin: *T*
1	(1, H)	(1, T)
2	(2, H)	(2, T)
3	(3, H)	(3, T)
4	(4, H)	(4, T)
5	(5, H)	(5, T)
6	(6, H)	(6, T)

(ii)



(iii) Probability of getting a tail and an even number:

Total outcomes: n(S) = 12

Event E = {(2,T), (4,T), (6,T)} $\Rightarrow$ n(E) = 3

P(E) = $\frac{3}{12} = \frac{1}{4}$

6. Two unbiased dice are rolled once.

(i) Find sample space by sketching tree diagram.

(ii) Find sample space by drawing possibility diagram.

(iii) Find the probability of getting

(a) same number on both dice.

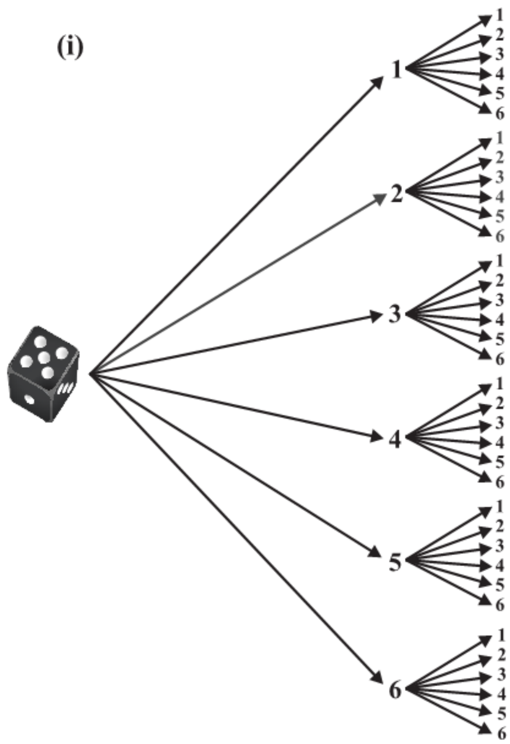
(b) the product as a prime number.

(c) the sum as an even number.

(d) the sum as 13.

Solution

(i)



(ii)

Dice 2 / Dice 1	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Total outcomes: $n(S) = 36$

(iii) Find the probability of getting:

(a) Same number on both dice:

$E = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \Rightarrow n(E) = 6 \Rightarrow P(E) = \frac{6}{36} = \frac{1}{6}$

(b) The product as a prime number: Products must be 2, 3, or 5.

$E = \{(1,2), (2,1), (1,3), (3,1), (1,5), (5,1)\} \Rightarrow n(E) = 6 \Rightarrow P(E) = \frac{6}{36} = \frac{1}{6}$

(c) The sum as an even number:

$n(E) = 18$ (half of the total outcomes) $\Rightarrow P(E) = \frac{18}{36} = \frac{1}{2}$

(d) The sum as 13:

Maximum possible sum is $6 + 6 = 12 \Rightarrow n(E) = 0 \Rightarrow P(E) = \frac{0}{36} = 0$

7. Three fair coins are tossed together. Find the probability of getting

(i) all tails (ii) at least one head
(iii) at most two tails (iv) 2 heads
(v) at most 2 heads (vi) no head

Solution

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} ⇒ n(S) = 8

(i) All tails: E = {TTT} ⇒ P(E) = $\frac{1}{8}$

(ii) At least one head: E = S − {TTT} ⇒ n(E) = 7 ⇒ P(E) = $\frac{7}{8}$

(iii) At most two tails: Everything except TTT ⇒ n(E) = 7 ⇒ P(E) = $\frac{7}{8}$

(iv) 2 heads: E = {HHT, HTH, THH} ⇒ P(E) = $\frac{3}{8}$

(v) At most 2 heads: Everything except HHH ⇒ n(E) = 7 ⇒ P(E) = $\frac{7}{8}$

(vi) No head: E = {TTT} ⇒ P(E) = $\frac{1}{8}$

8. A bag contains 4 red balls, 5 white balls, 6 green balls and 3 black balls. Ali draws a ball at random from the bag. Find the probability that the ball drawn is

(i) white (ii) red
(iii) not white (iv) not black

Solution

Total balls = 4 Red + 5 White + 6 Green + 3 Black ⇒ n(S) = 18

(i) White: P(White) = $\frac{5}{18}$ (ii) Red: P(Red) = $\frac{4}{18} = \frac{2}{9}$

(iii) Not white: P(Not White) = $1 - \frac{5}{18} = \frac{13}{18}$

(iv) Not black: P(Not Black) = $\frac{18-3}{18} = \frac{15}{18} = \frac{5}{6}$

9. A number is selected at random from the set of whole numbers 1 to 15, both inclusive. Find the probability that the number selected is:

(i) odd (ii) a multiple of 5 (iii) the square of 2
(iv) prime (v) 20

Solution

S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15} ⇒ n(S) = 15

(i) Odd: E = {1, 3, 5, 7, 9, 11, 13, 15} ⇒ P(E) = $\frac{8}{15}$

(ii) A multiple of 5: E = {5, 10, 15} ⇒ P(E) = $\frac{3}{15} = \frac{1}{5}$

(iii) The square of 2: $2^2 = 4$ ⇒ E = {4} ⇒ P(E) = $\frac{1}{15}$

(iv) Prime: E = {2, 3, 5, 7, 11, 13} ⇒ P(E) = $\frac{6}{15} = \frac{2}{5}$

(v) 20: No such element ⇒ P(E) = $\frac{0}{15} = 0$

10. If the probability of an event A is $\frac{7}{10}$, then find the probability of the event "not A".

Solution

Given P(A) = $\frac{7}{10}$

P(not A) = P($\bar{A}$) = 1 - P(A) = 1 - $\frac{7}{10} = \frac{3}{10}$

11. A dice is rolled twice. Find the probability of having a number greater than 4 on each roll.

Solution

A die is rolled twice. n(S) = 36

Numbers greater than 4 on a single die are {5, 6\}.

Event E (number > 4 on both rolls):

E = {(5,5), (5,6), (6,5), (6,6)}

⇒ n(E) = 4

P(E) = $\frac{4}{36} = \frac{1}{9}$

EXERCISE 12.2

1. If A and B are two events such that $P(A)=\frac{1}{4}$, $P(B)=\frac{1}{2}$ and $P(A \cap B)=\frac{1}{8}$, find $P(A \cup B)$.

Solution

$P(A) = \frac{1}{4}, P(B) = \frac{1}{2}, P(A \cap B) = \frac{1}{8}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{8}$

2. In an apartment, selecting a house from door numbers 1 to 50 randomly, find the probability of getting the door number of the house to be an even number or a perfect square number.

Solution

$S = \{1, 2, 3, \ldots, 50\} \Rightarrow n(S) = 50$

Let $A = \{\text{Even numbers}\} \Rightarrow n(A) = 25 \Rightarrow P(A) = \frac{25}{50}$

Let $B = \{\text{Perfect squares}\} = \{1, 4, 9, 16, 25, 36, 49\} \Rightarrow n(B) = 7 \Rightarrow P(B) = \frac{7}{50}$

$A \cap B = \{\text{Even perfect squares}\} = \{4, 16, 36\} \Rightarrow n(A \cap B) = 3 \Rightarrow P(A \cap B) = \frac{3}{50}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{25}{50} + \frac{7}{50} - \frac{3}{50} = \frac{29}{50}$

3. The probability of a team winning any match is $\frac{3}{10}$ and the probability of losing any match is $\frac{2}{10}$. What is the probability that
   (i) the team wins or loses a particular match.
   (ii) the team neither wins nor loses a match.

Solution

$P(\text{Win}) = \frac{3}{10}, P(\text{Lose}) = \frac{2}{10}, P(A \cap B) = \frac{1}{8}$

(i) $P\ (\text{Win or Lose}) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$

(ii) $P\ (\text{neither Win nor Lose}) = P(\text{Draw}) = 1 - \frac{5}{10} = \frac{5}{10} = \frac{1}{2}$

4. In a single throw of two dice, find the probability of having sum of 7 or 11.

Solution: $n(S) = 36$

Let $A = \{\text{Sum of } 7\} = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \Rightarrow n(A) = 6$

Let $B = \{\text{Sum of } 11\} = \{(5,6), (6,5)\} \Rightarrow n(B) = 2$

Mutually exclusive events: $\Rightarrow n(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$

5. Find the probability of getting a sum of 5 or 7 in a throw of two dice.

Solution

n(S) = 36

Let A = {Sum of 5} = {(1,4), (2,3), (3,2), (4,1)} ⇒ n(A) = 4

Let B = {Sum of 7} = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)} ⇒ n(B) = 6

$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{6}{36} = \frac{10}{36} = \frac{5}{18}$

6. A card is taken out at random from a standard pack of 52 cards. Find the probability of taking out.

(i) A king or a Jack. (ii) Neither a king nor a Jack.

Solution

n(S) = 52, Kings = 4, Jacks = 4

(i) A king or a Jack:

$P(K \cup J) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

(ii) Neither a king nor a Jack:

$P(\text{Neither}) = 1 - \frac{2}{13} = \frac{11}{13}$

7. A dice is thrown twice. What is the probability that at least one of the two throws comes up with number 3.

Solution

n(S) = 36

Event E (at least one 3):

E = {(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(1,3),(2,3),(4,3),(5,3),(6,3)}

$\Rightarrow n(E) = 11 \Rightarrow P(E) = \frac{11}{36}$

8. There are 15 cards in a bag marked as 1, 2, 3, …., 15. Find the probability of picking a card at random, the number written on which is a multiple of 5 or of 7.

Solution

S = {1, 2, 3,…, 15} ⇒ n(S) = 15

Let A = {Multiple of 5} = {5, 10, 15} ⇒ n(A) = 3

Let B = {Multiple of 7} = {7, 14} ⇒ n(B) = 2

$\Rightarrow n(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) = \frac{3}{15} + \frac{2}{15} = \frac{5}{15} = \frac{1}{3}$

9. Two fair coins are tossed once. What is the probability of getting at least one head or two heads.

Solution

S = {HH, HT, TH, TT} ⇒ n(S) = 4

Let A = {At least one head} = {HH, HT, TH} ⇒ n(A) = 3

Let B = {Two heads} = {HH} ⇒ n(B) = 1

Since B is a subset of A, "at least one head or two heads" is just event A.

$P(A \cup B) = \frac{3}{4}$

10. At a busy intersection, 50% of vehicles turn right, 30% turn left and 20% go straight. What is the probability that a randomly selected vehicle turn left or right?

Solution

P(Right) = 50%, P(Left) = 30%, P(Straight) = 20%

$P(\text{Left} \cup \text{Right}) = 30\% + 50\% = 80\% = \frac{80}{100} = \frac{4}{5}$ (or 0.8)

11. Two fair coins are tossed. What is the probability of getting either two heads or two tails?

Solution

S = {HH, HT, TH, TT} ⇒ n(S) = 4

Event E (two heads or two tails) = {HH, TT} ⇒ n(E) = 2

$P(E) = \frac{2}{4} = \frac{1}{2}$

12. If A and B are two mutually exclusive events of a random experiment and $P(\text{not } A) = 0.45$, $P(A \cup B) = 0.65$, then find $P(B)$.

Solution

Mutually exclusive ⇒ $P(A \cap B) = 0$

$P(\text{not } A) = 0.45$, $P(A \cup B) = 0.65$

$P(A) = 1 - 0.45 = 0.55$

$P(A \cup B) = P(A) + P(B)$

$0.65 = 0.55 + P(B)$

$P(B) = 0.65 - 0.55 = 0.10$

13. If $P(A) = \frac{2}{3}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{1}{3}$ then find $P(A \cap B)$.

Solution

$P(A) = \frac{2}{3}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{1}{3}$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$P(A \cap B) = \frac{2}{3} + \frac{2}{5} - \frac{1}{3} = \frac{11}{15}$

14. A and B are two events such that, $P(A) = 0.42$, $P(B) = 0.48$, and $P(A \cap B) = 0.16$.

Find: (i) $P(\bar{A})$ (ii) $P(\bar{B})$ (iii) $P(A \cup B)$

Solution

$P(A) = 0.42$, $P(B) = 0.48$, $P(A \cap B) = 0.16$

(i) $P(\bar{A}) = 1 - 0.42 = 0.58$

(ii) $P(\bar{B}) = 1 - 0.48 = 0.52$

(iii) $P(A \cup B) = 0.42 + 0.48 - 0.16 = 0.74$

EXERCISE 12.3

1. Two numbers are randomly chosen from 1 to 10 with replacement. Find the probability that:
   (i) both are prime (ii) their product is even

Solution

Numbers chosen from 1 to 10 with replacement.

Total sample space outcomes: $n(S) = 10 \times 10 = 100$

Prime numbers in range: $\{2, 3, 5, 7\} \Rightarrow 4$ numbers

Even numbers in range: $\{2, 4, 6, 8, 10\} \Rightarrow 5$ numbers

Odd numbers in range: $\{1, 3, 5, 7, 9\} \Rightarrow 5$ numbers

(i) Both are prime:

Outcomes: $4 \times 4 = 16$

$P(\text{Both Prime}) = \frac{16}{100} = \frac{4}{25}$

(ii) Their product is even:

Odd outcomes: $5 \times 5 = 25$

P(Product Even) = 1 - P(Both Odd)

$P(\text{Product Even}) = 1 - \frac{25}{100} = 1 - \frac{1}{4} = \frac{3}{4}$

2. One letter is chosen from the word PUNJAB and another from LAHORE. What is the probability that:
   (i) both are vowels
   (ii) one is consonant and other is vowel

Solution

Words: PUNJAB (6 letters: Vowels = {U, A}, Consonants = {P, N, J, B}),
LAHORE (6 letters: Vowels = {A, O, E}, Consonants = {L, H, R})

Total sample space outcomes: $n(S) = 6 \times 6 = 36$

(i) Both are vowels:

Outcomes: $2 \times 3 = 6$

$P(\text{Both Vowels}) = \frac{6}{36} = \frac{1}{6}$

(ii) One is consonant and other is vowel:

Case 1: Consonant from PUNJAB (4) × Vowel from LAHORE (3) = 12

Case 2: Vowel from PUNJAB (2) × Consonant from LAHORE (3) = 6

Total favorable outcomes: 12 + 6 = 18

$P(\text{One Consonant, One Vowel}) = \frac{18}{36} = \frac{1}{2}$

3. A single dice is rolled twice. Find the probability that one roll is a multiple of 3 and the other is a 5.

Solution

A single die is rolled twice ⇒ n(S) = 36

Let M_3 = Multiple of 3 = {3, 6} and 5 = {5}.

Favorable outcomes: (M_3, 5) or (5, M_3)

E = {(3,5), (6,5), (5,3), (5,6)} ⇒ n(E) = 4

P(E) = $\frac{4}{36} = \frac{1}{9}$

4. Two dice are rolled. Find the probability of getting an odd number on one and a multiple of 2 on other.

Solution

Two dice are rolled ⇒ n(S) = 36.

Let O = Odd = {1, 3, 5} and E_v = Multiple of 2 = {2, 4, 6}.

Case 1: First die is odd, second die is even ⇒ 3 × 3 = 9 outcomes

Case 2: First die is even, second die is odd ⇒ 3 × 3 = 9 outcomes

Total favorable outcomes: 9 + 9 = 18

P(E) = $\frac{18}{36} = \frac{1}{2}$

5. From a pack of well shuffled cards, two cards are drawn at random one by one with replacement. Find the probability that the first is heart and second is king.

Solution

Two cards drawn one by one "with replacement" ⇒ n(S) = 52 × 52.

Hearts = 13, Kings = 4

P(First Heart and Second King) = $\frac{13}{52} \times \frac{4}{52} = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$

6. If two cards are selected from a standard deck of 52 cards without replacement, find the probability that

(i) Both are black. (ii) Both are queens.
(iii) Both are spades. (iv) Both are diamonds.

Solution

Two cards selected "without replacement".
Total cards = 52.

(i) Both are black:

Black cards = 26

P(Both Black) = $\frac{26}{52} \times \frac{25}{51} = \frac{1}{2} \times \frac{25}{51} = \frac{25}{102}$

(ii) Both are queens:

Queens = 4

P(Both Queens) = $\frac{4}{52} \times \frac{3}{51} = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$

(iii) Both are spades:

Spades = 13

P(Both Spades) = $\frac{13}{52} \times \frac{12}{51} = \frac{1}{4} \times \frac{4}{17} = \frac{1}{17}$

(iv) Both are diamonds:

Diamonds = 13

P(Both Diamonds) = $\frac{13}{52} \times \frac{12}{51} = \frac{1}{4} \times \frac{4}{17} = \frac{1}{17}$

7. Saleem draws two cards one by one without replacement from a well shuffled pack of 52 playing cards. What is the probability that first card is jack and the second card is queen.

Solution

Two cards drawn one by one "without replacement".
Total cards = 52.
Jacks = 4, Queens = 4

P(First Jack and Second Queen) = $\frac{4}{52} \times \frac{4}{51} = \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$

REVIEW EXERCISE 12

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) The probability of impossible event is:
(a) 0 (b) 1 (c) 2 (d) -1

(ii) What is the probability of getting a head when a fair coin is tossed once?
(a) 0 (b) 0.25 (c) 0.5 (d) 1

(iii) The sum of probabilities of all possible outcomes of an experiment is:
(a) 0.5 (b) 0.25 (c) 1 (d) 0.4

(iv) If $P(A) = 0.6$, then the probability of event A not happening is:
(a) 0.4 (b) 0.6 (c) 1 (d) 1.6

(v) Two events are said to be mutually exclusive if:
(a) they can happen at the same time.
(b) one affects the other.
(c) they cannot happen together.
(d) they are always equal.

(vi) If one coin is tossed and one dice is rolled, then the number of sample point are:
(a) 3 (b) 6 (c) 2 (d) 12

(vii) What is the probability of sure event?
(a) 1 (b) 0 (c) $\frac{2}{3}$ (d) $\frac{4}{5}$

(viii) The probability of getting a number greater than 6 on dice is:
(a) 0.33 (b) 1 (c) 0 (d) 0.5

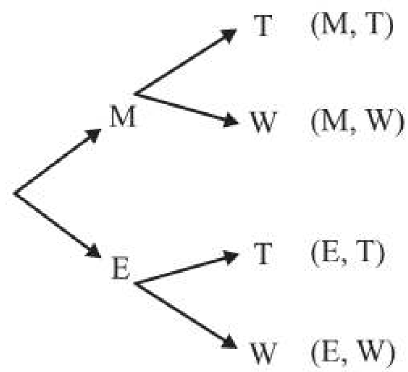
(ix) ________ outcomes are possible when we draw a card from deck of cards.
(a) 13 (b) 1 (c) 52 (d) 26

(x) If $P(E) = 0.07$, then the probability of 'not E' is:
(a) 0.95 (b) 0.89 (c) 0.93 (d) 0.90

2. Arshia selects a day from (Tuesday, Wednesday) and a time from (Morning, Evening). Draw a possibility diagram. Also sketch a tree diagram.

Solution

	Tuesday	Wednesday
Morning	(M, T)	(M, W)
Evening	(E, T)	(E, W)



3. A card is drawn from a deck of cards. Find the probability of getting an ace or a spade card.

Solution

Total cards: n(S) = 52

Let A = {Ace} ⇒ n(A) = 4

Let B = {Spade} ⇒ n(B) = 13

$A \cap B$ = {Ace of Spades} ⇒ $n(A \cap B) = 1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

4. Fatima dealt two cards successively (without replacement) from a shuffled deck of 52 playing cards. Find the probability that both cards are red.

Solution

Two cards dealt “without replacement”.

Total cards = 52

Red cards = 26

$P(\text{Both Red}) = \frac{26}{52} \times \frac{25}{51} = \frac{1}{2} \times \frac{25}{51} = \frac{25}{102}$

5. If two dice are rolled, then find the probability of getting the product of face values 6 or the difference of face values 5.

Solution

Two dice are rolled ⇒ n(S) = 36

Let A = {Product is 6} = {(1,6), (2,3), (3,2), (6,1)} ⇒ n(A) = 4

Let B = {Difference is 5} = {(1,6), (6,1)} ⇒ n(B) = 2

$A \cap B = \{(1,6), (6,1)\} \Rightarrow n(A \cap B) = 2$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{4}{36} + \frac{2}{36} - \frac{2}{36} = \frac{4}{36} = \frac{1}{9}$

6. A bag contains 7 orange and 5 purple marbles. Two marbles are drawn one after the other without replacement. Find the probability that the first marble is orange and the second marble is also orange.

Solution

Total marbles = 7 Orange + 5 Purple = 12

Drawn "without replacement"

P(First Orange and Second Orange) = $\frac{7}{12} \times \frac{6}{11} = \frac{42}{132} = \frac{7}{22}$

حرفِ آخر (26-05-2026)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (آمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

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