

$$\begin{aligned} \underline{u} \times \underline{v} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \underline{j} \\ &\quad + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \underline{k} \end{aligned}$$

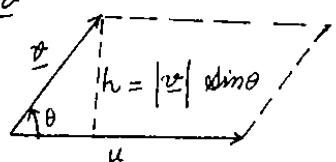
Parallel Vectors :-

$\underline{u} \parallel \underline{v}$ iff $\underline{u} \times \underline{v} = \underline{0}$

Note:- For every vector \underline{u} , we know that $\underline{0} \times \underline{u} = \underline{0}$
 \therefore we say that zero vector is parallel to every vector.

Area of Parallelogram :-

Let \underline{u} and \underline{v} be along the adjacent sides of a //gm. and θ is the angle between them.



Then $|\underline{u}|$ and $|\underline{v}|$ are the length of adjacent sides of the //gm.

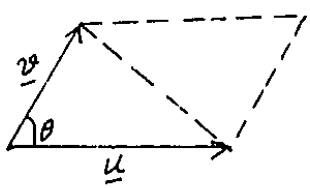
$$\begin{aligned} \text{Area of the //gm} &= (\text{base}) (\text{height}) \\ &= |\underline{u}| (|\underline{v}| \sin \theta) \\ &= |\underline{u}| |\underline{v}| \sin \theta \\ &= |\underline{u} \times \underline{v}| \end{aligned}$$

Area of Triangle :-

From fig.

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2} (\text{area of //gm}) \\ &= \frac{1}{2} |\underline{u} \times \underline{v}| \end{aligned}$$

Where \underline{u} and \underline{v} are acting along the adjacent sides of the triangle.



J201

Example 1 :-

Find a vector perpendicular to each of the vectors

$\underline{a} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{b} = 4\underline{i} + 2\underline{j} - \underline{k}$

Sol:- Required vector = $\underline{a} \times \underline{b}$

$$\begin{aligned} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix} \\ &= \underline{i} (+1-2) - \underline{j} (-2-4) + \underline{k} (4+4) \\ &= -\underline{i} + 6\underline{j} + 8\underline{k} \text{ Ans.} \end{aligned}$$

*** EXERCISE 7.4 ***

(i) Given that

$\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Now $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$\underline{a} \times \underline{b} = \underline{i} (1-1) - \underline{j} (2+1) + \underline{k} (-2-1)$

$\underline{a} \times \underline{b} = \underline{0i} - 3\underline{j} - 3\underline{k}$

$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$= \underline{i} (1-1) - \underline{j} (-1-2) + \underline{k} (1+2)$

$\underline{b} \times \underline{a} = \underline{0i} + 3\underline{j} + 3\underline{k}$

$\underline{a} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{0i} - 3\underline{j} - 3\underline{k})$
 $= (2)(0) + (1)(-3) + (-1)(-3) = 0 - 3 + 3 = 0$
 $\Rightarrow \underline{a} \perp \underline{a} \times \underline{b}$

$\underline{a} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{0i} + 3\underline{j} + 3\underline{k})$
 $= (2)(0) + (1)(3) + (-1)(3) = 0 + 3 - 3 = 0 \Rightarrow \underline{a} \perp \underline{b} \times \underline{a}$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (\underline{0i} - 3\underline{j} - 3\underline{k})$
 $= (1)(0) + (-1)(-3) + (1)(-3) = 0 + 3 - 3 = 0 \Rightarrow \underline{b} \perp \underline{a} \times \underline{b}$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (\underline{0i} + 3\underline{j} + 3\underline{k})$
 $= (1)(0) + (-1)(3) + (1)(3) = 0 - 3 + 3 = 0 \Rightarrow \underline{b} \perp \underline{b} \times \underline{a}$

(ii) Given that

$\underline{a} = \underline{i} + \underline{j} = \underline{i} + \underline{j} + \underline{0k}$

$\underline{b} = \underline{i} - \underline{j} = \underline{i} - \underline{j} + \underline{0k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\Rightarrow \underline{a} \times \underline{b} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$\Rightarrow \underline{a} \times \underline{b} = 0\underline{i} - 0\underline{j} - 2\underline{k} = -2\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(1+1)$$

$$\underline{b} \times \underline{a} = 0\underline{i} - 0\underline{j} + 2\underline{k} = 2\underline{k}$$

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = (\underline{i} + \underline{j} + 0\underline{k}) \cdot (0\underline{i} - 0\underline{j} - 2\underline{k})$$

$$= (1)(0) + (1)(0) + (0)(-2) = 0 + 0 + 0$$

$$= 0 \Rightarrow \underline{a} \perp \underline{a} \times \underline{b}$$

$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (\underline{i} + \underline{j} + 0\underline{k}) \cdot (0\underline{i} - 0\underline{j} + 2\underline{k})$$

$$= (1)(0) + (1)(0) + (0)(2) = 0 + 0 + 0$$

$$= 0 \Rightarrow \underline{a} \perp \underline{b} \times \underline{a}$$

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (\underline{i} - \underline{j} + 0\underline{k}) \cdot (0\underline{i} - 0\underline{j} - 2\underline{k})$$

$$= (1)(0) + (-1)(0) + (0)(-2) = 0 + 0 + 0$$

$$= 0 \Rightarrow \underline{b} \perp \underline{a} \times \underline{b}$$

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + 0\underline{k}) \cdot (0\underline{i} - 0\underline{j} + 2\underline{k})$$

$$= (1)(0) + (-1)(0) + (0)(2) = 0 + 0 + 0$$

$$= 0 \Rightarrow \underline{b} \perp \underline{b} \times \underline{a}$$

(iii) Given that

$$\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}, \quad \underline{b} = \underline{i} + \underline{j} + 0\underline{k}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \underline{i}(0-1) - \underline{j}(0-1) + \underline{k}(3+2)$$

$$\underline{a} \times \underline{b} = -\underline{i} + \underline{j} + 5\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$$\underline{b} \times \underline{a} = \underline{i}(1-0) - \underline{j}(1-0) + \underline{k}(-2-3)$$

$$\underline{b} \times \underline{a} = \underline{i} - \underline{j} - 5\underline{k}$$

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (-\underline{i} + \underline{j} + 5\underline{k})$$

$$= (3)(-1) + (-2)(1) + (1)(5) = -3 - 2 + 5$$

$$= 0 \Rightarrow \underline{a} \perp \underline{a} \times \underline{b}$$

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (\underline{i} + \underline{j} + 0\underline{k}) \cdot (-\underline{i} + \underline{j} + 5\underline{k})$$

$$= (1)(-1) + (1)(1) + (0)(5) = -1 + 1 + 0$$

$$= 0 \Rightarrow \underline{b} \perp \underline{a} \times \underline{b}$$

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} + \underline{j} + 0\underline{k}) \cdot (\underline{i} - \underline{j} - 5\underline{k})$$

$$= (1)(1) + (1)(-1) + (0)(-5) = 1 - 1 + 0$$

$$= 0 \Rightarrow \underline{b} \perp \underline{b} \times \underline{a}$$

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$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (\underline{i} - \underline{j} - 5\underline{k})$$

$$= (3)(1) + (-2)(-1) + (1)(-5)$$

$$= 3 + 2 - 5 = 0 \Rightarrow \underline{a} \perp \underline{b} \times \underline{a}$$

(iv) Given that

$$\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k} \quad \text{and}$$

$$\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = \underline{i}(1+2) - \underline{j}(-4+1) + \underline{k}(-4-2)$$

$$\underline{a} \times \underline{b} = 3\underline{i} - 0\underline{j} - 6\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix}$$

$$\underline{b} \times \underline{a} = \underline{i}(-2-1) - \underline{j}(-4+4) + \underline{k}(2+4)$$

$$\underline{b} \times \underline{a} = -3\underline{i} - 0\underline{j} + 6\underline{k}$$

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (3\underline{i} - 0\underline{j} - 6\underline{k})$$

$$= (-4)(3) + (1)(0) + (-2)(-6) = -12 + 0 + 12$$

$$= 0 \Rightarrow \underline{a} \perp \underline{a} \times \underline{b}$$

$$\underline{a} \cdot (\underline{b} \times \underline{a}) = (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (-3\underline{i} - 0\underline{j} + 6\underline{k})$$

$$= (-4)(-3) + (1)(0) + (-2)(6) = 12 + 0 - 12$$

$$= 0 \Rightarrow \underline{a} \perp \underline{b} \times \underline{a}$$

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (3\underline{i} - 0\underline{j} - 6\underline{k})$$

$$= (2)(3) + (1)(0) + (1)(-6) = 6 + 0 - 6$$

$$= 0 \Rightarrow \underline{b} \perp \underline{a} \times \underline{b}$$

$$\underline{b} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (-3\underline{i} - 0\underline{j} + 6\underline{k})$$

$$= (2)(-3) + (1)(0) + (1)(6) = -6 + 0 + 6$$

$$= 0 \Rightarrow \underline{b} \perp \underline{b} \times \underline{a}$$

* To Find a Unit Vector perpendicular to The Plane Containing \underline{a} and \underline{b} and To find sine of the angle between them.

We know that

$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \hat{n} \quad \text{--- (1)}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \theta \quad \text{--- (2)} \quad \because \left| \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} \right| = 1$$

Dividing (1) by (2), we get

$$\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \hat{n}$$

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

From (2) $\frac{|\underline{a} \times \underline{b}|}{|\underline{a}| |\underline{b}|} = \sin \theta$

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}| |\underline{b}|}$$

(2) (i) Given that

$$\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k} \Rightarrow |\underline{a}| = \sqrt{4+36+9} = \sqrt{49} = 7$$

$$\underline{b} = 4\underline{i} + 3\underline{j} \quad \Rightarrow |\underline{b}| = \sqrt{16+9} = 5$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & 0 \end{vmatrix} = \underline{i}(6+9) - \underline{j}(6+12) + \underline{k}(6+24)$$

$$\underline{a} \times \underline{b} = 15\underline{i} - 18\underline{j} + 30\underline{k}$$

$$\therefore \hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{15\underline{i} - 18\underline{j} + 30\underline{k}}{\sqrt{225+324+900}} = \frac{15\underline{i} - 18\underline{j} + 30\underline{k}}{\sqrt{1449}}$$

$$\Rightarrow \hat{n} = \frac{15\underline{i} - 18\underline{j} + 30\underline{k}}{\sqrt{1449}} = \frac{15}{\sqrt{1449}}\underline{i} - \frac{18}{\sqrt{1449}}\underline{j} + \frac{30}{\sqrt{1449}}\underline{k}$$

$$\Rightarrow \hat{n} = \frac{3}{\sqrt{1449}}\underline{i} - \frac{2}{\sqrt{1449}}\underline{j} + \frac{6}{\sqrt{1449}}\underline{k}$$

If θ is the angle between \underline{a} and \underline{b} ,

then $\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}| |\underline{b}|}$

$$\Rightarrow \sin \theta = \frac{35}{(7)(26)} \Rightarrow \sin \theta = \frac{5}{\sqrt{26}}$$

Ans

(ii) Given that

$$\underline{a} = -\underline{i} - \underline{j} - \underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k} \Rightarrow |\underline{b}| = \sqrt{4+9+16} = \sqrt{29}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix} = \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$$

$$\Rightarrow \underline{a} \times \underline{b} = -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2} = \sqrt{49+4+25}$$

$$|\underline{a} \times \underline{b}| = \sqrt{78}$$

Let \hat{n} be the required unit vector.

$$\text{Then } \hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$= \frac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$$

$$\hat{n} = \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}$$

Ans

[22]

Let θ be the angle between the vectors \underline{a} and \underline{b} , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}| |\underline{b}|} = \frac{\sqrt{78}}{\sqrt{3} \sqrt{29}} = \frac{\sqrt{78}}{\sqrt{87}}$$

$$= \frac{\sqrt{78}}{\sqrt{87}} = \sqrt{\frac{78}{87}} = \sqrt{\frac{26}{29}} \quad \text{Ans}$$

(iii) Given that

$$\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k} \Rightarrow |\underline{a}| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

$$\underline{b} = -\underline{i} + \underline{j} - 2\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1+4} = \sqrt{6}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \underline{i}(-4-4) - \underline{j}(-4+4) + \underline{k}(2-2)$$

$$\underline{a} \times \underline{b} = 0\underline{i} - 0\underline{j} + 0\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{0^2+0^2+0^2} = 0 = 0$$

Let \hat{n} be the required unit vector,

$$\text{then } \hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{0\underline{i} - 0\underline{j} + 0\underline{k}}{0}$$

$\Rightarrow \hat{n}$ is arbitrary (not unique)

Let θ be the angle between the vectors \underline{a} and \underline{b} , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}| |\underline{b}|} = \frac{0}{(2\sqrt{6})(\sqrt{6})} = 0 \quad \text{Ans}$$

(iv) Given that

$$\underline{a} = \underline{i} + \underline{j} = \underline{i} + \underline{j} + 0\underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1+0} = \sqrt{2}$$

$$\underline{b} = \underline{i} - \underline{j} = \underline{i} - \underline{j} + 0\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1+0} = \sqrt{2}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$\Rightarrow \underline{a} \times \underline{b} = 0\underline{i} - 0\underline{j} - 2\underline{k}$$

$$\Rightarrow |\underline{a} \times \underline{b}| = \sqrt{0^2+0^2+(-2)^2} = \sqrt{4} = 2$$

Let \hat{n} be the required unit vector, then

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{0\underline{i} - 0\underline{j} - 2\underline{k}}{2}$$

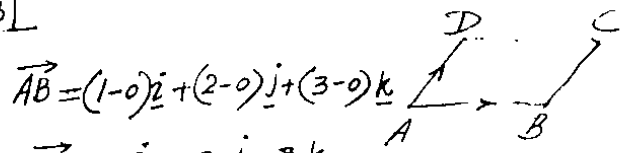
$$\hat{n} = \frac{0}{2}\underline{i} - \frac{0}{2}\underline{j} - \frac{2}{2}\underline{k} = -\underline{k} \quad \text{Ans}$$

Let θ be the angle between the vectors \underline{a} and \underline{b} , then

J 231

$$\sin \theta = \frac{|a \times b|}{|a| |b|}$$

$$\rightarrow \sin \theta = \frac{2}{\sqrt{2} \sqrt{2}} = \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1 \quad \text{Ans}$$



$$\begin{aligned} \vec{AB} &= (1-0)\underline{i} + (2-0)\underline{j} + (3-0)\underline{k} \\ \vec{AB} &= \underline{i} + 2\underline{j} + 3\underline{k} \\ \vec{AD} &= (3-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k} \\ \vec{AD} &= 3\underline{i} + \underline{j} + 4\underline{k} \end{aligned}$$

Now area of $\parallel\text{gm } ABCD$

$$\begin{aligned} \vec{AB} \times \vec{AD} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix} \\ &= \underline{i}(8-3) - \underline{j}(4-9) + \underline{k}(1-6) \end{aligned}$$

$$\vec{AB} \times \vec{AD} = 5\underline{i} + 5\underline{j} - 5\underline{k}$$

$$|\vec{AB} \times \vec{AD}| = \sqrt{25+25+25} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area of } \parallel\text{gm } ABCD = |\vec{AB} \times \vec{AD}| = 5\sqrt{3} \quad \text{Ans}$$

③ (i) Given that

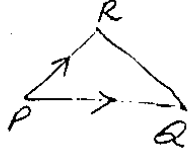
$$P(0,0,0), Q(2,3,2), R(-1,1,4)$$

Then

$$\vec{PQ} = (2-0)\underline{i} + (3-0)\underline{j} + (2-0)\underline{k}$$

$$\vec{PQ} = 2\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\vec{PR} = (-1-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k} = -\underline{i} + \underline{j} + 4\underline{k}$$



$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

$$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3)$$

$$= 10\underline{i} - 10\underline{j} + 5\underline{k}$$

$$\begin{aligned} |\vec{PQ} \times \vec{PR}| &= \sqrt{(10)^2 + (-10)^2 + (5)^2} \\ &= \sqrt{100 + 100 + 25} = \sqrt{225} = 15 \end{aligned}$$

$$\begin{aligned} \text{Now Area of } \Delta PQR &= \frac{1}{2} |\vec{PQ} \times \vec{PR}| \\ &= \frac{1}{2} (15) = \frac{15}{2} \text{ square units.} \end{aligned}$$

(ii) Given that $P(1,-1,-1), Q(2,0,-1)$ and $R(0,2,1)$. Then

$$\vec{PQ} = (2-1)\underline{i} + (0+1)\underline{j} + (-1+1)\underline{k}$$

$$\Rightarrow \vec{PQ} = \underline{i} + \underline{j} + 0\underline{k}$$

$$\vec{PR} = (0-1)\underline{i} + (2+1)\underline{j} + (1+1)\underline{k} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= \underline{i}(2-0) - \underline{j}(2-0) + \underline{k}(3+1)$$

$$\vec{PQ} \times \vec{PR} = 2\underline{i} - 2\underline{j} + 4\underline{k}$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

$$\begin{aligned} \text{Now area of } \Delta PQR &= \frac{1}{2} |\vec{PQ} \times \vec{PR}| \\ &= \frac{1}{2} (2\sqrt{6}) = \sqrt{6} \text{ square units.} \end{aligned}$$

④ (i) Given that $A(0,0,0), B(1,2,3)$

$$C(2,-1,1), D(3,1,4)$$

(ii) Given that $A(1,2,-1), B(4,2,-3)$

$$C(6,-5,2) \text{ \& } D(9,-5,0)$$

$$\vec{AB} = (4-1)\underline{i} + (2-2)\underline{j} + (-3+1)\underline{k} = 3\underline{i} + 0\underline{j} - 2\underline{k}$$

$$\vec{AD} = (9-1)\underline{i} + (-5-2)\underline{j} + (0+1)\underline{k} = 8\underline{i} - 7\underline{j} + \underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$$

$$= \underline{i}(0-14) - \underline{j}(3+16) + \underline{k}(-21-0)$$

$$\vec{AB} \times \vec{AD} = -14\underline{i} - 19\underline{j} - 21\underline{k}$$

$$\therefore \text{Area of } \parallel\text{gm } ABCD = |\vec{AB} \times \vec{AD}|$$

$$= \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$$

$$= \sqrt{196 + 361 + 441} = \sqrt{998} \quad \text{Ans}$$

(iii) Given that $A(-1,1,1), B(-1,2,2)$

$$C(-3,4,-5) \text{ \& } D(-3,5,-4)$$

$$\vec{AB} = (-1+1)\underline{i} + (2-1)\underline{j} + (2-1)\underline{k} = 0\underline{i} + \underline{j} + \underline{k}$$

$$\vec{AD} = (-3+1)\underline{i} + (5-1)\underline{j} + (-4-1)\underline{k} = -2\underline{i} + 4\underline{j} - 5\underline{k}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 4 & -5 \end{vmatrix}$$

$$= \underline{i}(-5-4) - \underline{j}(0+2) + \underline{k}(0+2)$$

$$\vec{AB} \times \vec{AD} = -9\underline{i} - 2\underline{j} + 2\underline{k}$$

$$\therefore \text{Area of } \parallel\text{gm } ABCD = |\vec{AB} \times \vec{AD}|$$

$$= \sqrt{81+4+4} = \sqrt{89} \quad \text{Ans}$$

⑤ (i) Given that

$$\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$$

$$\underline{v} = \underline{j} - 5\underline{k} = 0\underline{i} + \underline{j} - 5\underline{k}$$

$$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$$

$$\therefore \underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$$

$$\Rightarrow \underline{w} = -3\underline{u}$$

$\Rightarrow \underline{w} \parallel \underline{u}$ in opposite direction.

(ii) Given that

$$\underline{u} = \underline{i} + 2\underline{j} - \underline{k}, \quad \underline{v} = -\underline{i} + \underline{j} + \underline{k}$$

$$\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$$

$$\therefore \underline{u} \cdot \underline{v} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\underline{i} + \underline{j} + \underline{k})$$

$$= (1)(-1) + (2)(1) + (-1)(1)$$

$$= -1 + 2 - 1 = 0$$

$\therefore \underline{u} \perp \underline{v}$

$$\therefore \underline{v} \cdot \underline{w} = (-\underline{i} + \underline{j} + \underline{k}) \cdot \left(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}\right)$$

$$= (-1)\left(-\frac{\pi}{2}\right) + (1)(-\pi) + (1)\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = \frac{0}{2}$$

$$\Rightarrow \underline{v} \cdot \underline{w} = 0$$

$\therefore \underline{v} \perp \underline{w}$

Also $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$$\Rightarrow \underline{w} = -\frac{\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$$

$$\Rightarrow \underline{w} = -\frac{\pi}{2}\underline{u}$$

$\Rightarrow \underline{w} \parallel \underline{u}$ in opposite direction.

⑥ Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = \underline{0}$

Sol.:- L.H.S. = $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$

$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$$

$$= \underline{a} \times \underline{b} - \underline{c} \times \underline{a} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} + \underline{c} \times \underline{a} - \underline{b} \times \underline{c}$$

$$\therefore \underline{a} \times \underline{c} = -\underline{c} \times \underline{a}$$

$$\underline{b} \times \underline{a} = -\underline{a} \times \underline{b} \text{ and}$$

$$\underline{c} \times \underline{b} = -\underline{b} \times \underline{c}$$

$$\therefore \text{L.H.S.} = \underline{0} = \text{R.H.S.}$$

⑦ If $\underline{a} + \underline{b} + \underline{c} = \underline{0}$, then prove that

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

Sol.:- Given that

$$\underline{a} + \underline{b} + \underline{c} = \underline{0} \quad \text{--- (1)}$$

Taking cross product of (1) with \underline{a} we get

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times \underline{0}$$

$$\Rightarrow \underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = \underline{0}$$

$$\Rightarrow 0 + \underline{a} \times \underline{b} - \underline{c} \times \underline{a} = \underline{0} \therefore \underline{a} \times \underline{b} = -\underline{c} \times \underline{a}$$

$$\Rightarrow \underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad \text{--- (2)} \neq \underline{a} \times \underline{a} = 0$$

Taking cross product of (1) with \underline{b} , we get

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times \underline{0}$$

$$\Rightarrow \underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = \underline{0}$$

$$\Rightarrow -\underline{a} \times \underline{b} + 0 + \underline{b} \times \underline{c} = \underline{0} \therefore \underline{b} \times \underline{c} = \underline{a} \times \underline{b}$$

$$\Rightarrow \underline{b} \times \underline{c} = \underline{a} \times \underline{b}$$

$$\Rightarrow \underline{a} \times \underline{b} = \underline{b} \times \underline{c} \quad \text{--- (3)}$$

\therefore From (2) & (3), we get

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a} \text{ (Proved)}$$

⑧ Prove that

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Sol.:- Let \vec{OA} and

\vec{OB} be the unit vectors making angles α and

β with x-axis respectively. Then

$$\vec{OA} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

$$\vec{OB} = \cos \beta \underline{i} + \sin \beta \underline{j}$$

$$\vec{OB} \times \vec{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix}$$

$$|\vec{OB}| |\vec{OA}| \sin(\alpha - \beta) \underline{k} = \underline{i}(0-0) - \underline{j}(0-0)$$

$$+ \underline{k}(\cos \beta \sin \alpha - \sin \beta \cos \alpha)$$

$$(1)(1) \sin(\alpha - \beta) \underline{k} = (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \underline{k}$$

$$\sin(\alpha - \beta) \underline{k} = (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \underline{k}$$

$$\Rightarrow \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

(Proved)

Q) $\underline{a} \times \underline{b} = \underline{0}$
 $\Rightarrow \underline{a} \parallel \underline{b}$ or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$
 or $\underline{a} = \underline{b} = \underline{0}$

Now $\underline{a} \cdot \underline{b} = 0$

$\Rightarrow \underline{a} \perp \underline{b}$ or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or
 $\underline{a} = \underline{b} = \underline{0}$

Proof To find $\underline{u} \cdot (\underline{v} \times \underline{w})$, we have

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now $\underline{v} \times \underline{w} = (b_2 c_3 - b_3 c_2) \underline{i} - (a_2 c_3 - a_3 c_2) \underline{j} + (a_2 b_3 - a_3 b_2) \underline{k}$

$\underline{u} \cdot (\underline{v} \times \underline{w}) = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$

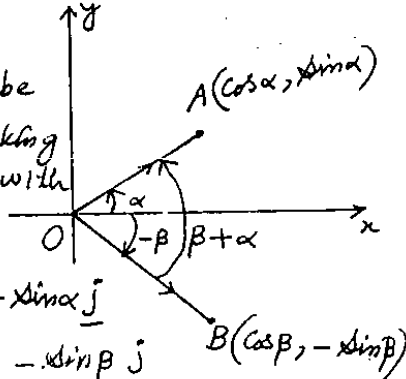
$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (Determinant formula for scalar triple product of \underline{u} , \underline{v} and \underline{w})

Example 3 Prove that

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

Sol.

Let \vec{OA} and \vec{OB} be unit vectors making angles α and β with x-axis resp.



Then $\vec{OA} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$

$\vec{OB} = \cos \beta \underline{i} - \sin \beta \underline{j}$

$\vec{OB} \times \vec{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \beta & -\sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix}$

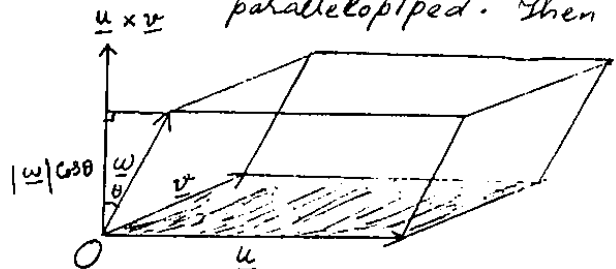
$\Rightarrow |\vec{OB}| |\vec{OA}| \sin(\beta + \alpha) \underline{k} = \underline{i} (0 - 0) - \underline{j} (0 - 0) + \underline{k} (\sin \alpha \cos \beta + \cos \alpha \sin \beta)$

$\Rightarrow (1)(1) \sin(\alpha + \beta) \underline{k} = (\sin \alpha \cos \beta + \cos \alpha \sin \beta) \underline{k}$

$\Rightarrow \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ (Proved)

The Volume of parallelopiped

Let \underline{u} , \underline{v} and \underline{w} are along coterminal edges of parallelopiped. Then



Area of //opiped = (Area of //gm) (height)
 $= |\underline{u} \times \underline{v}| |\underline{w}| \cos \theta$
 $= (\underline{u} \times \underline{v}) \cdot \underline{w} = [\underline{u} \ \underline{v} \ \underline{w}]$

Scalar Triple Product of Vectors

The scalar triple product of vectors \underline{u} , \underline{v} and \underline{w} is defined by

$\underline{u} \cdot (\underline{v} \times \underline{w})$ or $(\underline{u} \times \underline{v}) \cdot \underline{w}$

Note that

$\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$

This can be written as

$[\underline{u} \ \underline{v} \ \underline{w}] = [\underline{v} \ \underline{w} \ \underline{u}] = [\underline{w} \ \underline{u} \ \underline{v}]$

Component Form

Let $\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$

$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$

$\underline{w} = a_3 \underline{i} + b_3 \underline{j} + c_3 \underline{k}$

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