[201]

$$\underline{u}\times\underline{v} = a_1 b_2 \underline{k} - a_1 c_2 \underline{j} - b_1 a_2 \underline{k} + b_1 c_2 \underline{i} + c_1 a_2 \underline{j} - c_1 b_2 \underline{i}$$

$$\underline{u}\times\underline{v} = (b_1c_2 - b_2c_1)\underline{i} - (a_1c_2 - a_2c_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

$$\Rightarrow \underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

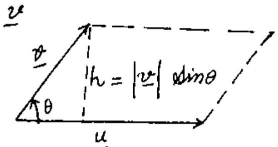
## Parallel Vectors :-

$\underline{u} \parallel \underline{v}$ iff $\underline{u}\times\underline{v} = \underline{0}$

**Note**:- For every vector $\underline{u}$, we know that
$\underline{0}\times\underline{u} = \underline{0}$
$\therefore$ we say that zero vector is parallel to every vector.

## Area of Parallelogram :-

Let $\underline{u}$ and $\underline{v}$ be along the adjacent sides of a //gm.




and $\theta$ is the angle between them.
Then $|\underline{u}|$ and $|\underline{v}|$ are the length of adjacent sides of the //gm.

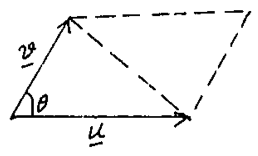
$$\text{Area of the //gm} = (\text{base})(\text{height}) = |\underline{u}|\,(|\underline{v}|\sin\theta) = |\underline{u}||\underline{v}|\sin\theta = |\underline{u}\times\underline{v}|$$

## Area of Triangle :-

From fig.

$$\text{Area of triangle} = \frac{1}{2}(\text{area of //gm}) = \frac{1}{2}|\underline{u}\times\underline{v}|$$

Where $\underline{u}$ and $\underline{v}$ are acting along the adjacent sides of the triangle.




## Example 1 :-

Find a vector perpendicular to each of the vectors $\underline{a} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{b} = 4\underline{i} + 2\underline{j} - \underline{k}$

**Sol**:- Required vector $= \underline{a}\times\underline{b}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$= \underline{i}(+1-2) - \underline{j}(-2-4) + \underline{k}(4+4)$$

$$= -\underline{i} + 6\underline{j} + 8\underline{k} \quad \text{Ans.}$$

## ★ EXERCISE 7.4 ★

**(1)** (i) Given that
$\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Now
$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\underline{a}\times\underline{b} = \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$

$$\boxed{\underline{a}\times\underline{b} = 0\underline{i} - 3\underline{j} - 3\underline{k}}$$

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$$

$$\boxed{\underline{b}\times\underline{a} = 0\underline{i} + 3\underline{j} + 3\underline{k}}$$

$$\underline{a}\cdot(\underline{a}\times\underline{b}) = (2\underline{i} + \underline{j} - \underline{k})\cdot(0\underline{i} - 3\underline{j} - 3\underline{k})$$
$$= (2)(0) + (1)(-3) + (-1)(-3) = 0 - 3 + 3$$
$$= 0 \Rightarrow \underline{a} \perp \underline{a}\times\underline{b}$$

$$\underline{a}\cdot(\underline{b}\times\underline{a}) = (2\underline{i} + \underline{j} - \underline{k})\cdot(0\underline{i} + 3\underline{j} + 3\underline{k})$$
$$= (2)(0) + (1)(3) + (-1)(3)$$
$$= 0 + 3 - 3 = 0 \Rightarrow \underline{a} \perp \underline{b}\times\underline{a}$$

$$\underline{b}\cdot(\underline{a}\times\underline{b}) = (\underline{i} - \underline{j} + \underline{k})\cdot(0\underline{i} - 3\underline{j} - 3\underline{k})$$
$$= (1)(0) + (-1)(-3) + (1)(-3)$$
$$= 0 + 3 - 3 = 0 \Rightarrow \underline{b} \perp \underline{a}\times\underline{b}$$

$$\underline{b}\cdot(\underline{b}\times\underline{a}) = (\underline{i} - \underline{j} + \underline{k})\cdot(0\underline{i} + 3\underline{j} + 3\underline{k})$$
$$= (1)(0) + (-1)(3) + (1)(3)$$
$$= 0 - 3 + 3 = 0 \Rightarrow \underline{b} \perp \underline{b}\times\underline{a}$$

(ii) Given that
$\underline{a} = \underline{i} + \underline{j} = \underline{i} + \underline{j} + 0\underline{k}$
$\underline{b} = \underline{i} - \underline{j} = \underline{i} - \underline{j} + 0\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$\Rightarrow \underline{a}\times\underline{b} = \underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(-1-1)$

$\Rightarrow \underline{a}\times\underline{b} = 0\underline{i}-0\underline{j}-2\underline{k} = -2\underline{k}$

$$\underline{b}\times\underline{a}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(1+1)$$

$\underline{b}\times\underline{a} = 0\underline{i}-0\underline{j}+2\underline{k} = 2\underline{k}$

$\underline{a}\cdot(\underline{a}\times\underline{b}) = (\underline{i}+\underline{j}+0\underline{k})\cdot(0\underline{i}-0\underline{j}-2\underline{k})$

$= (1)(0)+(1)(0)+(0)(-2) = 0+0+0$

$= 0 \Rightarrow \underline{a} \perp \underline{a}\times\underline{b}$

$\underline{a}\cdot(\underline{b}\times\underline{a}) = (\underline{i}+\underline{j}+0\underline{k})\cdot(0\underline{i}-0\underline{j}+2\underline{k})$

$= (1)(0)+(1)(0)+(0)(2) = 0+0+0$

$= 0 \Rightarrow \underline{a} \perp \underline{b}\times\underline{a}$

$\underline{b}\cdot(\underline{a}\times\underline{b}) = (\underline{i}-\underline{j}+0\underline{k})\cdot(0\underline{i}-0\underline{j}-2\underline{k})$

$= (1)(0)+(-1)(0)+(0)(-2) = 0+0+0$

$= 0 \Rightarrow \underline{b} \perp \underline{a}\times\underline{b}$

$\underline{b}\cdot(\underline{b}\times\underline{a}) = (\underline{i}-\underline{j}+0\underline{k})\cdot(0\underline{i}-0\underline{j}+2\underline{k})$

$= (1)(0)+(-1)(0)+(0)(2) = 0+0+0$

$= 0 \Rightarrow \underline{b} \perp \underline{b}\times\underline{a}$

---

(iii) Given that

$\underline{a} = 3\underline{i}-2\underline{j}+\underline{k}$ , $\underline{b} = \underline{i}+\underline{j}+0\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \underline{i}(0-1)-\underline{j}(0-1)+\underline{k}(3+2)$$

$\boxed{\underline{a}\times\underline{b} = -\underline{i}+\underline{j}+5\underline{k}}$

$$\underline{b}\times\underline{a}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$\underline{b}\times\underline{a} = \underline{i}(1-0)-\underline{j}(1-0)+\underline{k}(-2-3)$

$\boxed{\underline{b}\times\underline{a} = \underline{i}-\underline{j}-5\underline{k}}$

$\underline{a}\cdot(\underline{a}\times\underline{b}) = (3\underline{i}-2\underline{j}+\underline{k})\cdot(-\underline{i}+\underline{j}+5\underline{k})$

$= (3)(-1)+(-2)(1)+(1)(5) = -3-2+5$

$= 0 \Rightarrow \underline{a} \perp \underline{a}\times\underline{b}$

$\underline{b}\cdot(\underline{a}\times\underline{b}) = (\underline{i}+\underline{j}+0\underline{k})\cdot(-\underline{i}+\underline{j}+5\underline{k})$

$= (1)(-1)+(1)(1)+(0)(5) = -1+1+0$

$= 0 \Rightarrow \underline{b} \perp \underline{a}\times\underline{b}$

$\underline{b}\cdot(\underline{b}\times\underline{a}) = (\underline{i}+\underline{j}+0\underline{k})\cdot(\underline{i}-\underline{j}-5\underline{k})$

$= (1)(1)+(1)(-1)+(0)(-5) = 1-1+0$

$= 0 \Rightarrow \underline{b} \perp \underline{b}\times\underline{a}$

$\underline{a}\cdot(\underline{b}\times\underline{a}) = (3\underline{i}-2\underline{j}+\underline{k})\cdot(\underline{i}-\underline{j}-5\underline{k})$

$= (3)(1)+(-2)(-1)+(1)(-5)$

$= 3+2-5 = 0 \Rightarrow \underline{a} \perp \underline{b}\times\underline{a}$

---

(iv) Given that

$\underline{a} = -4\underline{i}+\underline{j}-2\underline{k}$ and

$\underline{b} = 2\underline{i}+\underline{j}+\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = \underline{i}(1+2)-\underline{j}(-4+4)+\underline{k}(-4-2)$$

$\Rightarrow \boxed{\underline{a}\times\underline{b} = 3\underline{i}-0\underline{j}-6\underline{k}}$

$$\underline{b}\times\underline{a}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix}$$

$\underline{b}\times\underline{a} = \underline{i}(-2-1)-\underline{j}(-4+4)+\underline{k}(2+4)$

$\boxed{\underline{b}\times\underline{a} = -3\underline{i}-0\underline{j}+6\underline{k}}$

$\underline{a}\cdot(\underline{a}\times\underline{b}) = (-4\underline{i}+\underline{j}-2\underline{k})\cdot(3\underline{i}-0\underline{j}-6\underline{k})$

$= (-4)(3)+(1)(0)+(-2)(-6) = -12+0+12$

$= 0 \Rightarrow \underline{a} \perp \underline{a}\times\underline{b}$

$\underline{a}\cdot(\underline{b}\times\underline{a}) = (-4\underline{i}+\underline{j}-2\underline{k})\cdot(-3\underline{i}-0\underline{j}+6\underline{k})$

$= (-4)(-3)+(1)(0)+(-2)(6) = 12+0-12$

$= 0 \Rightarrow \underline{a} \perp \underline{b}\times\underline{a}$

$\underline{b}\cdot(\underline{a}\times\underline{b}) = (2\underline{i}+\underline{j}+\underline{k})\cdot(3\underline{i}-0\underline{j}-6\underline{k})$

$= (2)(3)+(1)(0)+(1)(-6) = 6+0-6$

$= 0 \Rightarrow \underline{b} \perp \underline{a}\times\underline{b}$

$\underline{b}\cdot(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}+\underline{k})\cdot(-3\underline{i}-0\underline{j}+6\underline{k})$

$= (2)(-3)+(1)(0)+(1)(6) = -6+0+6$

$= 0 \Rightarrow \underline{b} \perp \underline{b}\times\underline{a}$

---

**To Find a Unit Vector perpendicular to The Plane Containing $\underline{a}$ and $\underline{b}$ and To find Sine of the angle between them.**

We know that

$\underline{a}\times\underline{b} = |\underline{a}|\,|\underline{b}|\sin\theta\ \hat{\underline{n}}$ ------ (1)

$|\underline{a}\times\underline{b}| = |\underline{a}|\,|\underline{b}|\sin\theta$ ------ (2) $\because |\hat{\underline{n}}| = 1$

Dividing (1) by (2), we get

$$\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \hat{\underline{n}}$$

or $\boxed{\hat{n} = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}}$

From (2) $\dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}|\,|\underline{b}|} = \sin\theta$ or

$$\boxed{\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}|\,|\underline{b}|}}$$

---

(2) (i) Given that

$\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k} \Rightarrow |\underline{a}| = \sqrt{4+36+9} = \sqrt{49} = 7$

$\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k} \Rightarrow |\underline{b}| = \sqrt{16+9+1} = \sqrt{26}$

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24)$

$\underline{a}\times\underline{b} = 15\underline{i} - 10\underline{j} + 30\underline{k}$

$\Rightarrow |\underline{a}\times\underline{b}| = \sqrt{(15)^2+(-10)^2+(30)^2} = \sqrt{225+100+900} = \sqrt{1225} = 35$

$\because \hat{n} = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$\Rightarrow \hat{n} = \dfrac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35} = \dfrac{15}{35}\underline{i} - \dfrac{10}{35}\underline{j} + \dfrac{30}{35}\underline{k}$

$\Rightarrow \boxed{\hat{n} = \frac{3}{7}\underline{i} - \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k}}$

If $\theta$ be the angle between $\underline{a}$ and $\underline{b}$, then $\sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}|\,|\underline{b}|}$

$\Rightarrow \sin\theta = \dfrac{35}{(7)(\sqrt{26})} \Rightarrow \boxed{\sin\theta = \frac{5}{\sqrt{26}}}$ Ans.

---

(ii) Given that

$\underline{a} = -\underline{i} - \underline{j} - \underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1+1} = \sqrt{3}$

$\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k} \Rightarrow |\underline{b}| = \sqrt{4+9+16} = \sqrt{29}$

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix} = \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$

$\Rightarrow \underline{a}\times\underline{b} = -7\underline{i} + 2\underline{j} + 5\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{(-7)^2+(2)^2+(5)^2} = \sqrt{49+4+25}$

$|\underline{a}\times\underline{b}| = \sqrt{78}$ Let $\hat{n}$ be the required unit vector

Then $\hat{n} = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$= \dfrac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$

$\hat{n} = -\dfrac{7}{\sqrt{78}}\underline{i} + \dfrac{2}{\sqrt{78}}\underline{j} + \dfrac{5}{\sqrt{78}}\underline{k}$ Ans.

Let $\theta$ be the angle between the vectors $\underline{a}$ and $\underline{b}$, then

$\sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}|\,|\underline{b}|} = \dfrac{\sqrt{78}}{\sqrt{3}\sqrt{29}} = \dfrac{\sqrt{78}}{\sqrt{3\times 29}}$

$= \dfrac{\sqrt{78}}{\sqrt{87}} = \sqrt{\dfrac{78}{87}} = \sqrt{\dfrac{26}{29}}$ Ans.

---

(iii) Given that

$\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k} \Rightarrow |\underline{a}| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

$\underline{b} = -\underline{i} + \underline{j} - 2\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1+4} = \sqrt{6}$

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2)$

$\underline{a}\times\underline{b} = 0\underline{i} - 0\underline{j} + 0\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{0^2+0^2+0^2} = \sqrt{0} = 0$

Let $\hat{n}$ be the required unit vector, then $\hat{n} = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \dfrac{0\underline{i} - 0\underline{j} + 0\underline{k}}{0}$

$\Rightarrow \hat{n}$ is arbitrary (not unique)

Let $\theta$ be the angle between the vectors $\underline{a}$ and $\underline{b}$, then

$\sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}|\,|\underline{b}|} = \dfrac{0}{(2\sqrt{6})(\sqrt{6})} = 0$ Ans.

---

(iv) Given that

$\underline{a} = \underline{i} + \underline{j} = \underline{i} + \underline{j} + 0\underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1+0} = \sqrt{2}$

$\underline{b} = \underline{i} - \underline{j} = \underline{i} - \underline{j} + 0\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1+0} = \sqrt{2}$

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(-1-1)$

$\Rightarrow \underline{a}\times\underline{b} = 0\underline{i} - 0\underline{j} - 2\underline{k}$

$\Rightarrow |\underline{a}\times\underline{b}| = \sqrt{0^2+0^2+(-2)^2} = \sqrt{4} = 2$

Let $\hat{n}$ be the required unit vector, then

$\hat{n} = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \dfrac{0\underline{i} - 0\underline{j} - 2\underline{k}}{2}$

$\hat{n} = \dfrac{0}{2}\underline{i} - \dfrac{0}{2}\underline{j} - \dfrac{2}{2}\underline{k} = -\underline{k}$ Ans.

Let $\theta$ be the angle between the vectors $\underline{a}$ and $\underline{b}$, then

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$$\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|}$$

$$\Rightarrow \sin\theta = \frac{2}{\sqrt{2}\,\sqrt{2}} = \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1 \quad \text{Ans}$$

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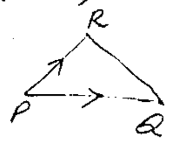
③ (i) Given that

$P(0,0,0),\ Q(2,3,2),\ R(-1,1,4)$

Then




$\overrightarrow{PQ} = (2-0)\underline{i} + (3-0)\underline{j} + (2-0)\underline{k}$

$\overrightarrow{PQ} = 2\underline{i} + 3\underline{j} + 2\underline{k}$

$\overrightarrow{PR} = (-1-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k} = -\underline{i} + \underline{j} + 4\underline{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3)$

$= 10\underline{i} - 10\underline{j} + 5\underline{k}$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(10)^2 + (-10)^2 + (5)^2}$

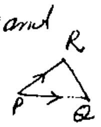
$= \sqrt{100 + 100 + 25} = \sqrt{225} = 15$

Now Area of $\triangle PQR = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$= \frac{1}{2}\cdot(15) = \frac{15}{2}$ Square units.

---

(ii) Given that $P(1,-1,-1), Q(2,0,-1)$ and $R(0,2,1)$. Then




$\overrightarrow{PQ} = (2-1)\underline{i} + (0+1)\underline{j} + (-1+1)\underline{k}$

$\Rightarrow \overrightarrow{PQ} = \underline{i} + \underline{j} + 0\underline{k}$

$\overrightarrow{PR} = (0-1)\underline{i} + (2+1)\underline{j} + (1+1)\underline{k} = -\underline{i} + 3\underline{j} + 2\underline{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$= \underline{i}(2-0) - \underline{j}(2-0) + \underline{k}(3+1)$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 2\underline{i} - 2\underline{j} + 4\underline{k}$

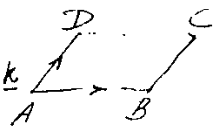
$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

Now area of $\triangle PQR = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$= \frac{1}{2}(2\sqrt{6}) = \sqrt{6}$ Square units.

---

④ (i) Given that $A(0,0,0)$, $B(1,2,3)$, $C(2,-1,1)$, $D(3,1,4)$




$\overrightarrow{AB} = (1-0)\underline{i} + (2-0)\underline{j} + (3-0)\underline{k}$

$\overrightarrow{AB} = \underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{AD} = (3-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k}$

$\overrightarrow{AD} = 3\underline{i} + \underline{j} + 4\underline{k}$

Now area of ||gm ABCD

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix}$$

$= \underline{i}(8-3) - \underline{j}(4-9) + \underline{k}(1-6)$

$\overrightarrow{AB} \times \overrightarrow{AD} = 5\underline{i} + 5\underline{j} - 5\underline{k}$

$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{25+25+25} = \sqrt{75} = 5\sqrt{3}$

Area of ||gm $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}| = 5\sqrt{3}$ Ans

---

(ii) Given that $A(1,2,-1)$, $B(4,2,-3)$, $C(6,-5,2)$ & $D(9,-5,0)$

$\overrightarrow{AB} = (4-1)\underline{i} + (2-2)\underline{j} + (-3+1)\underline{k} = 3\underline{i} + 0\underline{j} - 2\underline{k}$

$\overrightarrow{AD} = (9-1)\underline{i} + (-5-2)\underline{j} + (0+1)\underline{k} = 8\underline{i} - 7\underline{j} + \underline{k}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$$

$= \underline{i}(0-14) - \underline{j}(3+16) + \underline{k}(-21-0)$

$\overrightarrow{AB} \times \overrightarrow{AD} = -14\underline{i} - 19\underline{j} - 21\underline{k}$

$\therefore$ Area of ||gm $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

$= \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$

$= \sqrt{196 + 361 + 441} = \sqrt{998}$ Ans

---

(iii) Given that $A(-1,1,1)$, $B(-1,2,2)$, $C(-3,4,-5)$ and $D(-3,5,-4)$

$\overrightarrow{AB} = (-1+1)\underline{i} + (2-1)\underline{j} + (2-1)\underline{k} = 0\underline{i} + \underline{j} + \underline{k}$

$\overrightarrow{AD} = (-3+1)\underline{i} + (5-1)\underline{j} + (-4-1)\underline{k} = -2\underline{i} + 4\underline{j} - 5\underline{k}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 4 & -5 \end{vmatrix}$$

$= \underline{i}(-5-4) - \underline{j}(0+2) + \underline{k}(0+2)$

$\overrightarrow{AB} \times \overrightarrow{AD} = -9\underline{i} - 2\underline{j} + 2\underline{k}$

$\therefore$ Area of ||gm $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

$= \sqrt{81+4+4} = \sqrt{89}$ Ans

⑤ (i) Given that

$\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$

$\underline{v} = \underline{j} - 5\underline{k} = 0\underline{i} + \underline{j} - 5\underline{k}$

$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$\because \underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$

$\Rightarrow \underline{w} = -3\underline{u}$

$\Rightarrow \underline{w} \parallel \underline{u}$ in opposite direction.

---

(ii) Given that

$\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$

$\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$\because \underline{u}\cdot\underline{v} = (\underline{i} + 2\underline{j} - \underline{k})\cdot(-\underline{i} + \underline{j} + \underline{k})$

$= (1)(-1) + (2)(1) + (-1)(1)$

$= -1 + 2 - 1 = 0$

$\therefore \underline{u} \perp \underline{v}$

$\because \underline{v}\cdot\underline{w} = (-\underline{i} + \underline{j} + \underline{k})\cdot(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k})$

$= (-1)(-\frac{\pi}{2}) + (1)(-\pi) + (1)(\frac{\pi}{2})$

$= \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = \frac{0}{2}$

$\Rightarrow \underline{v}\cdot\underline{w} = 0$

$\therefore \underline{v} \perp \underline{w}$

Also $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$\Rightarrow \underline{w} = -\frac{\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$

$\Rightarrow \underline{w} = -\frac{\pi}{2}\underline{u}$

$\Rightarrow \underline{w} \parallel \underline{u}$ in opposite direction.

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⑥ Prove that $\underline{a}\times(\underline{b}+\underline{c}) + \underline{b}\times(\underline{c}+\underline{a}) + \underline{c}\times(\underline{a}+\underline{b}) = \underline{0}$

Sol:- $L.H.S. = \underline{a}\times(\underline{b}+\underline{c}) + \underline{b}\times(\underline{c}+\underline{a}) + \underline{c}\times(\underline{a}+\underline{b})$

$= \underline{a}\times\underline{b} + \underline{a}\times\underline{c} + \underline{b}\times\underline{c} + \underline{b}\times\underline{a} + \underline{c}\times\underline{a} + \underline{c}\times\underline{b}$

$= \underline{a}\times\underline{b} - \underline{c}\times\underline{a} + \underline{b}\times\underline{c} - \underline{a}\times\underline{b} + \underline{c}\times\underline{a} - \underline{b}\times\underline{c}$

$\because \underline{a}\times\underline{c} = -\underline{c}\times\underline{a}$

$\underline{b}\times\underline{a} = -\underline{a}\times\underline{b}$ and

$\underline{c}\times\underline{b} = -\underline{b}\times\underline{c}$

$\therefore L.H.S. = \underline{0} = R.H.S.$

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⑦ If $\underline{a} + \underline{b} + \underline{c} = \underline{0}$, then prove that

$\underline{a}\times\underline{b} = \underline{b}\times\underline{c} = \underline{c}\times\underline{a}$

Sol:- Given that

$\underline{a} + \underline{b} + \underline{c} = \underline{0}$ —— ①

Taking cross product of ① with $\underline{a}$ we get

$\underline{a}\times(\underline{a} + \underline{b} + \underline{c}) = \underline{a}\times\underline{0}$

$\Rightarrow \underline{a}\times\underline{a} + \underline{a}\times\underline{b} + \underline{a}\times\underline{c} = \underline{0}$

$\Rightarrow 0 + \underline{a}\times\underline{b} - \underline{c}\times\underline{a} = \underline{0}$ $\because \underline{a}\times\underline{c} = -\underline{c}\times\underline{a}$

$\Rightarrow \underline{a}\times\underline{b} = \underline{c}\times\underline{a}$ —— ② $\because \underline{a}\times\underline{a} = 0$

Taking cross product of ① with $\underline{b}$, we get

$\underline{b}\times(\underline{a} + \underline{b} + \underline{c}) = \underline{b}\times\underline{0}$

$\Rightarrow \underline{b}\times\underline{a} + \underline{b}\times\underline{b} + \underline{b}\times\underline{c} = \underline{0}$

$\Rightarrow -\underline{a}\times\underline{b} + 0 + \underline{b}\times\underline{c} = \underline{0}$ $\because \underline{b}\times\underline{a} = -\underline{a}\times\underline{b}$

$\Rightarrow \underline{b}\times\underline{c} = \underline{a}\times\underline{b}$

$\Rightarrow \underline{a}\times\underline{b} = \underline{b}\times\underline{c}$ —— ③

$\therefore$ From ② & ③, we get

$\underline{a}\times\underline{b} = \underline{b}\times\underline{c} = \underline{c}\times\underline{a}$ (Proved)

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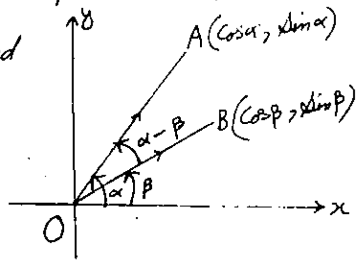
⑧ Prove that

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Sol:- Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the unit vectors making angles $\alpha$ and $\beta$ with x-axis respectively. Then




$\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\overrightarrow{OB} = \cos\beta\,\underline{i} + \sin\beta\,\underline{j}$

$$\overrightarrow{OB}\times\overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$|\overrightarrow{OB}|\,|\overrightarrow{OA}|\sin(\alpha-\beta)\,\underline{k} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(\cos\beta\sin\alpha - \sin\beta\cos\alpha)$

$(1)(1)\sin(\alpha-\beta)\underline{k} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\,\underline{k}$

$\sin(\alpha-\beta)\,\underline{k} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\,\underline{k}$

$\Rightarrow \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(Proved)

(9) $\underline{a} \times \underline{b} = \underline{0}$

$\Rightarrow \underline{a} \parallel \underline{b}$ or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$

or $\underline{a} = \underline{b} = \underline{0}$

Now $\underline{a} \cdot \underline{b} = 0$

$\Rightarrow \underline{a} \perp \underline{b}$ or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or

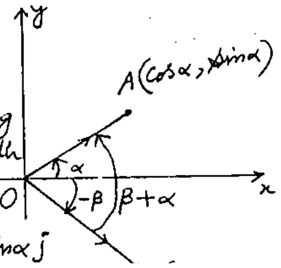
$\underline{a} = \underline{b} = \underline{0}$

# Example 3:- Prove that

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Sol:-

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be unit vectors making angles $\alpha$ and $\beta$ with x-axis resp.




Then $\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\overrightarrow{OB} = \cos\beta\,\underline{i} - \sin$

$\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} \\ \cos\beta & -\sin\beta \\ \cos\alpha & \sin\alpha \end{vmatrix}$

$\Rightarrow |\overrightarrow{OB}|\,|\overrightarrow{OA}|\,\sin(\beta+\alpha)\,\underline{k} = \underline{i}$ ...

$+A$ ... $\beta + \alpha$

$\Rightarrow (1)(1)\sin(\alpha+\beta)\underline{k} = (\sin\alpha\ldots$

$\Rightarrow \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\ldots$

(Proved)

# Scalar Triple ... of Vectors:-

The scalar triple product of vectors $\underline{u}, \underline{v}$ and $\underline{w}$ is defined by

$\underline{u} \cdot (\underline{v} \times \underline{w})$ or $(\underline{u} \times \underline{v}) \cdot \underline{w}$


Note that

$\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$

This can be written as

$[\underline{u}\ \underline{v}\ \underline{w}] = [\underline{v}\ \underline{w}\ \underline{u}] = [\underline{w}\ \underline{u}\ \underline{v}]$

# Component Form:

Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$

$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$

To find $\underline{u} \cdot (\underline{v} \times \underline{w})$, we have

$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\underline{v} \times \underline{w} = (b_2c_3 - b_3c_2)\underline{i} - (a_2c_3 - a_3c_2)\underline{j} + (a_2b_3 - a_3b_2)\underline{k}$

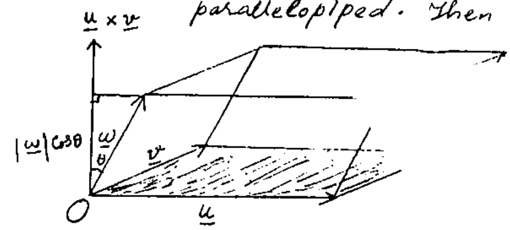
Now

$\underline{u} \cdot (\underline{v} \times \underline{w}) = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (Deter... formula ... scalar tr... of $\underline{u}, \underline{v}$ and ...)

# The Volume of parallelopiped:-

Let $\underline{u}, \underline{v}$ and $\underline{w}$ are along coterminous edges of parallelopiped. Then




Area of //opiped = (Area of //gm) (height)

$= |\underline{u} \times \underline{v}|\ |\underline{w}|\cos\theta$

$(\underline{u} \times \underline{v}) \cdot \underline{w} = [\underline{u}\ \underline{v}\ \underline{w}]$

# The Volume of Tetrahedron:-

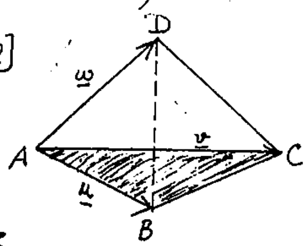
Volume of tetrahedron ABCD

$= \frac{1}{3}$(area of $\triangle ABC$)(height of D above the plane ABC)

$= \frac{1}{3} \times \frac{1}{2}|\underline{u} \times \underline{v}|\ (\quad \cos\theta)$

$= \frac{1}{6}[(\underline{u} \times \underline{v}) \cdot \underline{w}]$

$= \frac{1}{6}[\underline{u}\ \underline{v}\ \underline{w}]$




# Properties:-

(i) If $\underline{u}, \underline{v}$ and $\underline{w}$ are coplanar, then volume of the parallelopiped so formed is zero.