Question # 1
Find the cosine of the angle θ between $\underline{u}$ and $\underline{v}$:

(i) $\underline{u}=3\underline{\hat{i}}+\underline{\hat{j}}-\underline{\hat{k}}$, $\underline{v}=2\underline{\hat{i}}-\underline{\hat{j}}+\underline{\hat{k}}$ (ii) $\underline{u}=\underline{\hat{i}}-3\underline{\hat{j}}+4\underline{\hat{k}}$, $\underline{v}=4\underline{\hat{i}}-\underline{\hat{j}}+3\underline{\hat{k}}$

(iii) $\underline{u}=[-3,5]$, $\underline{v}=[6,-2]$ (iv) $\underline{u}=[2,-3,1]$, $\underline{v}=[2,4,1]$

Solution

(i) $\underline{u}=3\underline{\hat{i}}+\underline{\hat{j}}-\underline{\hat{k}}$, $\underline{v}=2\underline{\hat{i}}-\underline{\hat{j}}+\underline{\hat{k}}$

$$|\underline{u}|=\sqrt{(3)^2+(1)^2+(-1)^2}=\sqrt{9+1+1}=\sqrt{11}$$

$$|\underline{v}|=\sqrt{(2)^2+(-1)^2+(1)^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$\underline{u}\cdot\underline{v}=\left(3\underline{\hat{i}}+\underline{\hat{j}}-\underline{\hat{k}}\right)\cdot\left(2\underline{\hat{i}}-\underline{\hat{j}}+\underline{\hat{k}}\right)$$

$$=(3)(2)+(1)(-1)+(-1)(1)=6-1-1=4$$

Now $\underline{u}\cdot\underline{v}=|\underline{u}||\underline{v}|\cos\theta$

$$\Rightarrow\ \cos\theta=\frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}=\frac{4}{\sqrt{11}\times\sqrt{6}}\qquad\Rightarrow\ \boxed{\cos\theta=\frac{4}{\sqrt{66}}}$$

(ii) *Do yourself as above*

(iii) $\underline{u}=[-3,5]=-3\underline{\hat{i}}+5\underline{\hat{j}}$, $\underline{v}=[6,-2]=6\underline{\hat{i}}-2\underline{\hat{j}}$
Now do yourself as above

(iv) $\underline{u}=[2,-3,1]=2\underline{\hat{i}}-3\underline{\hat{j}}+\underline{\hat{k}}$, $\underline{v}=2\underline{\hat{i}}+4\underline{\hat{j}}+\underline{\hat{k}}$ *Now do yourself as* (i)

Question # 2
Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when:

(i) $\underline{a}=\underline{\hat{i}}-\underline{\hat{k}}$, $\underline{b}=\underline{\hat{j}}+\underline{\hat{k}}$ (ii) $\underline{a}=3\underline{\hat{i}}+\underline{\hat{j}}-\underline{\hat{k}}$, $\underline{b}=-2\underline{\hat{i}}-\underline{\hat{j}}+\underline{\hat{k}}$

Solution

(i) $\underline{a}=\underline{\hat{i}}-\underline{\hat{k}}$, $\underline{b}=\underline{\hat{j}}+\underline{\hat{k}}$

$$|\underline{a}|=\sqrt{(1)^2+(0)^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}$$

$$|\underline{b}|=\sqrt{(0)^2+(1)^2+(1)^2}=\sqrt{1+1}=\sqrt{2}$$

$$\underline{a}\cdot\underline{b}=\left(\underline{\hat{i}}-\underline{\hat{k}}\right)\cdot\left(\underline{\hat{j}}+\underline{\hat{k}}\right)=(1)(0)+(0)(1)+(-1)(1)=0+0-1=-1$$

Since $\underline{a}\cdot\underline{b}=|\underline{a}||\underline{b}|\cos\theta$

So projection of $\underline{a}$ along $\underline{b}$ $=|\underline{a}|\cos\theta=\dfrac{\underline{a}\cdot\underline{b}}{|\underline{b}|}=\dfrac{-1}{\sqrt{2}}$

Also projection of $\underline{b}$ along $\underline{a}$ $=|\underline{b}|\cos\theta=\dfrac{\underline{a}\cdot\underline{b}}{|\underline{a}|}=\dfrac{-1}{\sqrt{2}}$

(ii) *Do yourself as above*

Question # 3

Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular.

(i) $\underline{u} = 2\alpha\hat{\underline{i}} + \hat{\underline{j}} - \hat{\underline{k}},\ \underline{v} = \hat{\underline{i}} + \alpha\hat{\underline{j}} + 4\hat{\underline{k}}$ (ii) $\underline{u} = \alpha\hat{\underline{i}} + 2\alpha\hat{\underline{j}} - \hat{\underline{k}},\ \underline{v} = \hat{\underline{i}} + \alpha\hat{\underline{j}} + 3\hat{\underline{k}}$

Solution

(i) *Do yourself as* (ii) *below*

(ii) $\underline{u} = \alpha\hat{\underline{i}} + 2\alpha\hat{\underline{j}} - \hat{\underline{k}}$, $\underline{v} = \hat{\underline{i}} + \alpha\hat{\underline{j}} + 3\hat{\underline{k}}$

Since $\underline{u}$ and $\underline{v}$ are perpendicular therefore $\underline{u}.\underline{v} = 0$

$$\Rightarrow \left(\alpha\hat{\underline{i}} + 2\alpha\hat{\underline{j}} - \hat{\underline{k}}\right)\cdot\left(\hat{\underline{i}} + \alpha\hat{\underline{j}} + 3\hat{\underline{k}}\right) = 0$$

$$\Rightarrow (\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) = 0$$

$$\Rightarrow \alpha + 2\alpha^2 - 3 = 0 \quad \Rightarrow 2\alpha^2 + \alpha - 3 = 0$$

$$\Rightarrow 2\alpha^2 + 3\alpha - 2\alpha - 3 = 0 \quad \Rightarrow \alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$$

$$\Rightarrow (2\alpha + 3)(\alpha - 1) = 0$$

$$\Rightarrow 2\alpha + 3 = 0 \quad \text{or} \quad \alpha - 1 = 0$$

$$\Rightarrow \alpha = -\frac{3}{2} \quad \text{or} \quad \alpha = 1$$

Question # 4

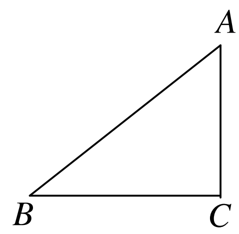
Find the number z so that the triangle with vertices $A(1,-1,0), B(-2,2,1)$ and $C(0,2,z)$ is a right triangle with right angle at C.

Solution

Given vertices: $A(1,-1,0)$, $B(-2,2,1)$ and $C(0,2,z)$

$$\overrightarrow{CA} = (1-0)\hat{\underline{i}} + (-1-2)\hat{\underline{j}} + (0-z)\hat{\underline{k}} = \hat{\underline{i}} - 3\hat{\underline{j}} - z\hat{\underline{k}}$$

$$\overrightarrow{CB} = (-2-0)\hat{\underline{i}} + (2-2)\hat{\underline{j}} + (1-z)\hat{\underline{k}} = -2\hat{\underline{i}} + (1-z)\hat{\underline{k}}$$



Now $\overrightarrow{CA}$ is $\perp$ to $\overrightarrow{CB}$ therefore $\overrightarrow{CA}\cdot\overrightarrow{CB} = 0$

$$\Rightarrow \left(\hat{\underline{i}} - 3\hat{\underline{j}} - z\hat{\underline{k}}\right)\cdot\left(-2\hat{\underline{i}} + (1-z)\hat{\underline{k}}\right) = 0$$

$$\Rightarrow (1)(-2) + (-3)(0) + (-z)(1-z) = 0$$

$$\Rightarrow -2 + 0 - z + z^2 = 0 \quad \Rightarrow z^2 - z - 2 = 0$$

$$\Rightarrow z^2 - 2z + z - 2 = 0 \quad \Rightarrow z(z-2) + 1(z-2) = 0$$

$$\Rightarrow (z-2)(z+1) = 0$$

$$\Rightarrow z - 2 = 0 \quad \text{or} \quad z + 1 = 0$$

$$\Rightarrow z = 2 \quad \text{or} \quad z = -1$$

Question # 5

If $\underline{v}$ is a vector for which

$\underline{v}\cdot\hat{\underline{i}} = 0,\quad \underline{v}\cdot\hat{\underline{j}} = 0,\quad \underline{v}\cdot\hat{\underline{k}} = 0$, find $\underline{v}$.

Solution

Suppose $\underline{v} = a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}$

Since $\underline{v}\cdot\hat{\underline{i}} = 0 \Rightarrow \left(a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}\right)\cdot\hat{\underline{i}} = 0$

$\Rightarrow a_1\hat{\underline{i}}\cdot\hat{\underline{i}} + a_2\hat{\underline{j}}\cdot\hat{\underline{i}} + a_3\hat{\underline{k}}\cdot\hat{\underline{i}} = 0$

$\Rightarrow a_1(1) + a_2(0) + a_3(0) = 0 \qquad \Rightarrow a_1 = 0$

Also $\underline{v}\cdot\hat{\underline{j}} = 0 \Rightarrow \left(a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}\right)\cdot\hat{\underline{j}} = 0$

$\Rightarrow a_1\hat{\underline{i}}\cdot\hat{\underline{j}} + a_2\hat{\underline{j}}\cdot\hat{\underline{j}} + a_3\hat{\underline{k}}\cdot\hat{\underline{j}} = 0$

$\Rightarrow a_1(0) + a_2(1) + a_3(0) = 0 \qquad \Rightarrow a_2 = 0$

Also $\underline{v}\cdot\hat{\underline{k}} = 0 \Rightarrow \left(a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}\right)\cdot\hat{\underline{k}} = 0$

$\Rightarrow a_1\hat{\underline{i}}\cdot\hat{\underline{k}} + a_2\hat{\underline{j}}\cdot\hat{\underline{k}} + a_3\hat{\underline{k}}\cdot\hat{\underline{k}} = 0$

$\Rightarrow a_1(0) + a_2(0) + a_3(1) = 0 \qquad \Rightarrow a_3 = 0$

Hence

$$\underline{v} = (0)\hat{\underline{i}} + (0)\hat{\underline{j}} + (0)\hat{\underline{k}} = 0$$

Question # 6

(i) Show that the vectors $3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}$, $\hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}$ and $2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}$ from a right angle.

(ii) Show that the set of points $P = (1,3,2)$, $Q = (4,1,4)$ and $R(6,5,5)$ from a right triangle.

Solution

(i) Let $\underline{a} = 3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}$, $\underline{b} = \hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}$ and $\underline{c} = 2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}$

Now $\underline{b} + \underline{c} = \hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}} + 2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}$

$= 3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}} = \underline{a}$

Hence $\underline{a}$, $\underline{b}$ and $\underline{c}$ form a triangle.

Now $\underline{a}\cdot\underline{b} = \left(3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}\right)\cdot\left(\hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}\right)$

$= (3)(1) + (-2)(-3) + (1)(5) = 3 + 6 + 5 = 14$

$\underline{b}\cdot\underline{c} = \left(\hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}\right)\cdot\left(2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}\right)$

$= (1)(2) + (-3)(1) + (5)(-4) = 2 - 3 - 20 = -21$

$\underline{c}\cdot\underline{a} = \left(2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}\right)\cdot\left(3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}\right)$

$= (2)(3) + (1)(-2) + (-4)(1) = 6 - 2 - 4 = 0$

Since $\underline{c}\cdot\underline{a} = 0$ therefore $\underline{c} \perp \underline{a}$

Hence $\underline{a}$, $\underline{b}$ and $\underline{c}$ represents sides of right triangle.

(ii) Given: $P(1,3,2)$, $Q(4,1,4)$ and $R(6,5,5)$

$$\overrightarrow{PQ} = (4-1)\hat{\underline{i}} + (1-3)\hat{\underline{j}} + (4-2)\hat{\underline{k}} = 3\hat{\underline{i}} - 2\hat{\underline{j}} + 2\hat{\underline{k}}$$

$$\overrightarrow{QR} = (6-4)\,\underline{\hat{i}} + (5-1)\,\underline{\hat{j}} + (5-4)\underline{\hat{k}} \;= 2\underline{\hat{i}} + 4\underline{\hat{j}} + \underline{\hat{k}}$$

$$\overrightarrow{RP} = (1-6)\,\underline{\hat{i}} + (3-5)\,\underline{\hat{j}} + (2-5)\underline{\hat{k}} \;= -5\underline{\hat{i}} - 2\underline{\hat{j}} - 3\underline{\hat{k}}$$

Now

$$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$$

$$= 3\underline{\hat{i}} - 2\underline{\hat{j}} + 2\underline{\hat{k}} + 2\underline{\hat{i}} + 4\underline{\hat{j}} + \underline{\hat{k}} - 5\underline{\hat{i}} - 2\underline{\hat{j}} - 3\underline{\hat{k}} \;= 0$$

Hence P, Q and R are vertices of triangle.

Now

$$\overrightarrow{PQ} \cdot \overrightarrow{QR} = \left(3\underline{\hat{i}} - 2\underline{\hat{j}} + 2\underline{\hat{k}}\right) \cdot \left(2\underline{\hat{i}} + 4\underline{\hat{j}} + \underline{\hat{k}}\right)$$

$$= (3)(2) + (-2)(4) + (2)(1) \;= 6 - 8 + 2 \;= 0$$

$$\Rightarrow \overrightarrow{PQ} \perp \overrightarrow{QR}$$

Hence P, Q and R are vertices of right triangle.

Question # 7

Show that mid point of hypotenuse a right triangle is equidistant from its vertices.

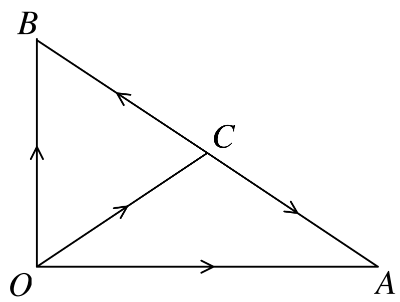
Solution

Suppose a right triangle OAB. Let C be a midpoint of hypotenuse AB, then

$$\overrightarrow{CA} = -\overrightarrow{CB} \quad \Rightarrow \left|\overrightarrow{CA}\right| = \left|\overrightarrow{CB}\right| \;\ldots\ldots\ldots\ldots\text{(i)}$$

Now $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$



Since $\overrightarrow{OA} \perp \overrightarrow{OB}$ therefore $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$

$$\Rightarrow \left(\overrightarrow{OC} + \overrightarrow{CA}\right) \cdot \left(\overrightarrow{OC} + \overrightarrow{CB}\right) = 0$$

$$\Rightarrow \left(\overrightarrow{OC} - \overrightarrow{CB}\right) \cdot \left(\overrightarrow{OC} + \overrightarrow{CB}\right) = 0 \qquad \because \; \overrightarrow{CA} = -\overrightarrow{CB}$$

$$\Rightarrow \overrightarrow{OC} \cdot \left(\overrightarrow{OC} + \overrightarrow{CB}\right) - \overrightarrow{CB} \cdot \left(\overrightarrow{OC} + \overrightarrow{CB}\right) = 0$$

$$\Rightarrow \overrightarrow{OC} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{CB} - \overrightarrow{CB} \cdot \overrightarrow{OC} - \overrightarrow{CB} \cdot \overrightarrow{CB} = 0$$

$$\Rightarrow \left|\overrightarrow{OC}\right|^2 + \overrightarrow{OC} \cdot \overrightarrow{CB} - \overrightarrow{OC} \cdot \overrightarrow{CB} - \left|\overrightarrow{CB}\right|^2 = 0 \qquad \because \; \overrightarrow{OC} \cdot \overrightarrow{CB} = \overrightarrow{CB} \cdot \overrightarrow{OC}$$

$$\Rightarrow \left|\overrightarrow{OC}\right|^2 - \left|\overrightarrow{CB}\right|^2 = 0$$

$$\Rightarrow \left|\overrightarrow{OC}\right|^2 = \left|\overrightarrow{CB}\right|^2 \qquad \Rightarrow \left|\overrightarrow{OC}\right| = \left|\overrightarrow{CB}\right| \;\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Combining (i) and (ii), we have

$$\left|\overrightarrow{OC}\right| = \left|\overrightarrow{CA}\right| = \left|\overrightarrow{CB}\right|$$

Hence midpoint of hypotenuse of right triangle is equidistant from its vertices.

Question # 8

Prove that perpendicular bisectors of the sides of a triangle are concurrent.

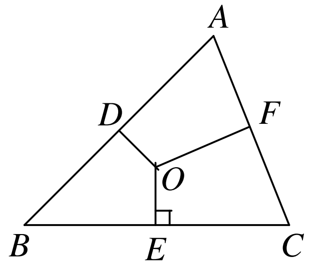
Solution

Let A, B and C be a vertices of a triangle having position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively.

Also consider D, E and F are midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, then p.v of $D = \overrightarrow{OD} = \dfrac{\underline{a}+\underline{b}}{2}$

p.v of $E = \overrightarrow{OE} = \dfrac{\underline{b}+\underline{c}}{2}$

p.v of $F = \overrightarrow{OF} = \dfrac{\underline{c}+\underline{a}}{2}$



Let right bisector on $\overline{AB}$ and $\overline{BC}$ intersect at point O, which is an origin.

Since $\overrightarrow{OD}$ is $\perp$ to $\overrightarrow{AB}$

Therefore $\overrightarrow{OD}\cdot\overrightarrow{AB}=0$

$\Rightarrow \left(\dfrac{\underline{a}+\underline{b}}{2}\right)\cdot(\underline{b}-\underline{a})=0 \qquad \Rightarrow \dfrac{1}{2}(\underline{b}+\underline{a})\cdot(\underline{b}-\underline{a})=0$

$\Rightarrow (\underline{b}+\underline{a})\cdot(\underline{b}-\underline{a})=0 \qquad \Rightarrow \underline{a}\cdot(\underline{b}-\underline{a})+\underline{b}\cdot(\underline{b}-\underline{a})=0$

$\Rightarrow \underline{a}\cdot\underline{b}-\underline{a}\cdot\underline{a}+\underline{b}\cdot\underline{b}-\underline{b}\cdot\underline{a}=0$

$\Rightarrow \underline{a}\cdot\underline{b}-|\underline{a}|^2+|\underline{b}|^2-\underline{a}\cdot\underline{b}=0 \qquad \because\ \underline{a}\cdot\underline{b}=\underline{b}\cdot\underline{a}$

$\Rightarrow |\underline{b}|^2-|\underline{a}|^2=0$ ………….. (i)

Also $\overrightarrow{OE}$ is $\perp$ to $\overrightarrow{BC}$

Therefore $\overrightarrow{OE}\cdot\overrightarrow{BC}=0 \quad \Rightarrow \left(\dfrac{\underline{b}+\underline{c}}{2}\right)\cdot(\underline{c}-\underline{b})=0$

Similarly solving as above, we get

$|\underline{c}|^2-|\underline{b}|^2=0$…………(ii)

Adding (i) and (ii), we have

$|\underline{b}|^2-|\underline{a}|^2+|\underline{c}|^2-|\underline{b}|^2=0+0$

$\Rightarrow |\underline{c}|^2-|\underline{a}|^2=0$

$\Rightarrow (\underline{c}+\underline{a})\cdot(\underline{c}-\underline{a})=0$

$\Rightarrow \left(\dfrac{\underline{c}+\underline{a}}{2}\right)\cdot(\underline{c}-\underline{a})=0$

$\Rightarrow \overrightarrow{OF}\cdot\overrightarrow{AC}=0 \quad \Rightarrow \overrightarrow{OF}$ is $\perp$ to $\overrightarrow{AC}$

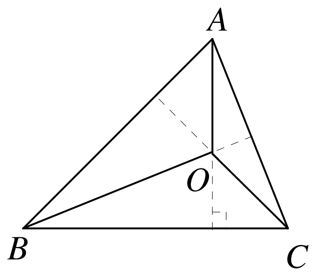
i.e. $\overrightarrow{OF}$ is also right bisector of $\overrightarrow{AC}$.

Hence perpendicular bisector of the sides of the triangle are concurrent.

Question # 9

Prove that the altitudes of a triangle are concurrent.

Solution



Consider A, B and C are vertices of triangle having position vectors $\underline{a}, \underline{b}$ and $\underline{c}$ respectively.

Let altitude on $\overrightarrow{AB}$ and $\overrightarrow{BC}$ intersect at origin $O(0,0)$.

Since $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$

$\Rightarrow \overrightarrow{OC}\cdot\overrightarrow{AB}=0$

$\Rightarrow \underline{c}\cdot(\underline{b}-\underline{a})=0$

$\Rightarrow \underline{c}\cdot\underline{b}-\underline{c}\cdot\underline{a}=0$ ... (i)

Also $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$

$\Rightarrow \overrightarrow{OA}\cdot\overrightarrow{BC}=0$

$\Rightarrow \underline{a}\cdot(\underline{c}-\underline{b})=0$

$\Rightarrow \underline{a}\cdot\underline{c}-\underline{a}\cdot\underline{b}=0$ ... (ii)

Adding (i) and

$\underline{c}\cdot\underline{b}-\underline{c}\cdot\underline{a}+\underline{a}\cdot\underline{c}-\underline{a}\cdot\underline{b}=0+0$

$\Rightarrow \underline{c}\cdot\underline{b}-\underline{c}\cdot\underline{a}+\underline{c}\cdot\underline{a}-\underline{a}\cdot\underline{b}=0$ $\qquad\because\ \underline{a}\cdot\underline{c}=\underline{c}\cdot\underline{a}$

$\Rightarrow \underline{c}\cdot\underline{b}-\underline{a}\cdot\underline{b}=0$

$\Rightarrow (\underline{c}-\underline{a})\cdot\underline{b}=0$

$\Rightarrow \overrightarrow{AC}\cdot\overrightarrow{OB}=0$ $\qquad\because\ \overrightarrow{AC}=\underline{c}-\underline{a}$

$\Rightarrow \overrightarrow{AC}$ is perpendicular to $\overrightarrow{OB}$.

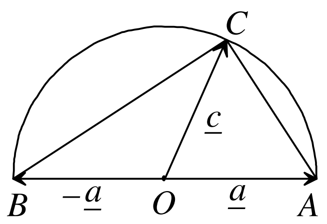
Hence altitude of the triangle are concurrent.

Question # 10

Prove that the angle in a semi circle is a right angle.

Solution

Consider a semicircle having centre at origin $O(0,0)$ and A, B are end points of diameter having position vectors $\underline{a}$, $-\underline{a}$ respectively. Let C be any point on a circle having position vector $\underline{c}$.



Clearly radius of semicircle $=|\underline{a}|=|-\underline{a}|=|\underline{c}|$

Now $\overrightarrow{AC}=\underline{c}-\underline{a}$

$\overrightarrow{BC}=\underline{c}-(-\underline{a})=\underline{c}+\underline{a}$

Consider

$$\overrightarrow{AC}\cdot\overrightarrow{BC}=(\underline{c}-\underline{a})\cdot(\underline{c}+\underline{a})$$
$$=\underline{c}\cdot(\underline{c}+\underline{a})-\underline{a}\cdot(\underline{c}+\underline{a})$$
$$=\underline{c}\cdot\underline{c}+\underline{c}\cdot\underline{a}-\underline{a}\cdot\underline{c}-\underline{a}\cdot\underline{a}$$
$$=|\underline{c}|^2+\underline{a}\cdot\underline{c}-\underline{a}\cdot\underline{c}+|\underline{a}|^2 \qquad \because\ \underline{a}\cdot\underline{c}=\underline{c}\cdot\underline{a}$$
$$=|\underline{c}|^2-|\underline{a}|^2$$
$$=|\underline{c}|^2-|\underline{c}|^2=0 \qquad \because\ |\underline{a}|=|\underline{c}|$$

This show $\overrightarrow{AC}$ is $\perp$ to $\overrightarrow{BC}$ i.e. $\angle ACB=90^\circ$

Hence angle in a semi circle is a right angle.

Question # 11

Prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

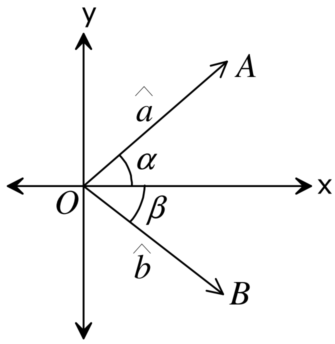
Solution

Consider two unit vectors $\underline{\hat{a}}$ and $\underline{\hat{b}}$ making angle α and $-\beta$ with $+$ive $x-$axis.

Then $\hat{\underline{a}} = OA = \cos\alpha\,\hat{\underline{i}} + \sin\alpha\,\hat{\underline{j}}$

and $\hat{\underline{b}} = OB = \cos(-\beta)\,\hat{\underline{i}} + \sin(-\beta)\,\hat{\underline{j}}$

$= \cos\beta\,\hat{\underline{i}} - \sin\beta\,\hat{\underline{j}}$



Now

$$\hat{\underline{a}}\cdot\hat{\underline{b}} = \left(\cos\alpha\,\hat{\underline{i}} + \sin\alpha\,\hat{\underline{j}}\right)\cdot\left(\cos\beta\,\hat{\underline{i}} - \sin\beta\,\hat{\underline{j}}\right)$$

$\Rightarrow\ \hat{\underline{a}}\cdot\hat{\underline{b}} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (i)

But we have $\angle AOB = \alpha + \beta$

$\Rightarrow\ \hat{\underline{a}}\cdot\hat{\underline{b}} = |\hat{\underline{a}}|\,|\hat{\underline{b}}|\cos(\alpha+\beta)$

$= (1)(1)\cos(\alpha+\beta)$ $\quad\because\ |\hat{\underline{a}}| = |\hat{\underline{b}}| = 1$

$\Rightarrow\ \hat{\underline{a}}\cdot\hat{\underline{b}} = \cos(\alpha+\beta)$ … (ii)

Comparing (i) and(ii), we have

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Question # 12

Prove that in any triangle ABC .

(i) $b = a\cos C + c\cos A$ (ii) $c = a\cos B + b\cos A$

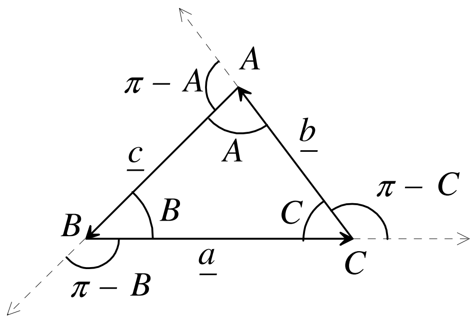
(ii) $b^2 = c^2 + a^2 - 2ca\cos B$ (iv) $c^2 = a^2 + b^2 - 2ab\cos C$

Solution

(i) Consider $\underline{a}, \underline{b}$ and $\underline{c}$ are vectors along the sides of triangle BC, CA and AB, also let $|\underline{a}| = a$, $|\underline{b}| = b$ and $|\underline{c}| = c$

then form triangle,

$\underline{a} + \underline{b} + \underline{c} = 0$ … (i)



(i) $\Rightarrow\ \underline{b} = -\underline{a} - \underline{c}$

Taking dot product of above with $\underline{b}$, we have

$\underline{b}\cdot\underline{b} = (-\underline{a} - \underline{c})\cdot\underline{b}$

$|\underline{b}|^2 = -\underline{a}\cdot\underline{b} - \underline{c}\cdot\underline{b}$

$= -|\underline{a}|\,|\underline{b}|\cos(\pi - C) - |\underline{c}|\,|\underline{b}|\cos(\pi - A)$

$= |\underline{a}|\,|\underline{b}|\cos C + |\underline{c}|\,|\underline{b}|\cos A$ $\quad\because\ \cos(\pi - B) = -\cos B$

$\Rightarrow\ b^2 = ab\cos C + cb\cos A$

$\Rightarrow\ b = a\cos C + c\cos A$ $\quad\div$ing by b

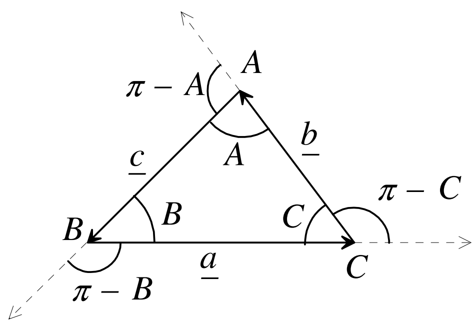
(ii) From equation (i)

$\underline{c} = -\underline{a} - \underline{b}$

Taking dot product of above equation with $\underline{c}$.

$\underline{c}\cdot\underline{c} = (-\underline{a} - \underline{b})\cdot\underline{c}$

Now do yourself as above.



(iii) From equation (i)

$\underline{b} = -\underline{a} - \underline{c}$

Taking dot product of above equation with $\underline{b}$

$$\underline{b}\cdot\underline{b}=(-\underline{a}-\underline{c})\cdot\underline{b}$$

$$=(-\underline{a}-\underline{c})\cdot(-\underline{a}-\underline{c}) \qquad \because\ \underline{b}=-\underline{a}-\underline{c}$$

$$|\underline{b}|^2=-\underline{a}\cdot(-\underline{a}-\underline{c})-\underline{c}\cdot(-\underline{a}-\underline{c})$$

$$=\underline{a}\cdot\underline{a}+\underline{a}\cdot\underline{c}+\underline{c}\cdot\underline{a}+\underline{c}\cdot\underline{c}$$

$$=\underline{a}\cdot\underline{a}+\underline{a}\cdot\underline{c}+\underline{a}\cdot\underline{c}+\underline{c}\cdot\underline{c} \qquad \because\ \underline{a}\cdot\underline{b}=\underline{b}\cdot\underline{a}$$

$$=\underline{a}\cdot\underline{a}+2\underline{a}\cdot\underline{c}+\underline{c}\cdot\underline{c}$$

$$=|\underline{a}|^2+2|\underline{a}||\underline{c}|\cos(\pi-B)+|\underline{c}|^2$$

$$\Rightarrow\ b^2=a^2+ac(-\cos B)+c^2 \qquad \because\ \cos(\pi-B)=-\cos B$$

Hence $$b^2=c^2+a^2-2ca\cos B$$

(iv) From equation (i)

$$\underline{c}=-\underline{a}-\underline{b}$$

Taking dot product of above equation with $\underline{c}$

$$\underline{c}\cdot\underline{c}=(-\underline{a}-\underline{b})\cdot\underline{c}$$

$$=(-\underline{a}-\underline{b})\cdot(-\underline{a}-\underline{b}) \qquad \because\ \underline{c}=-\underline{a}-\underline{b}$$

Now do yourself as above (iii)

Error Analyst		
Saqib Aleem (2015)	-	Punjab College of Sciences
Uzair Amin (2016)		

Book: ***Exercise 7.3, page 349***
Calculus and Analytic Geometry Mathematic 12
Punjab Textbook Board, Lahore.

Available online at http://www.MathCity.org *in PDF Format*
(Picture format to view online).
Updated: october,5,2017.