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EXERCISE 6.8

1. Find an equation of each of the following with respect to new parallel axes obtained by shifting the origin to the indicated points.

(i) $x^2 + 16y - 16 = 0$, $O' (0, 1)$

Solution. $x^2 + 16y - 16 = 0$... (1), $O' (0, 1) = (h, k)$

Equations of transformation are

$$x = X + h = X + 0 = X, \quad y = Y + k = Y + 1$$

Substituting these values of x, y into (1), we have

$$X^2 + 16(Y + 1) - 16 = 0$$

or $X^2 + 16Y + 16 - 16 = 0 \Rightarrow X^2 + 16Y = 0$

is the required transformed equation.

(ii) $4x^2 + y^2 + 16x - 10y + 37 = 0$, $O' (-2, 5)$

Solution. $4x^2 + y^2 + 16x - 10y + 37 = 0$... (1), $O' (-2, 5) = (h, k)$

Equations of transformation are

$$x = X + h = X - 2, \quad y = Y + k = Y + 5$$

Substituting these values of x, y into (1), we have

$$4(X-2)^2 + (Y+5)^2 + 16(X-2) - 10(Y+5) + 37 = 0$$

$$\text{or } 4(X^2 - 4X + 4) + (Y^2 + 10Y + 25) + 16(X-2) - 10(Y+5) + 37 = 0$$

$$\text{or } 4X^2 - 16X + 16 + Y^2 + 10Y + 25 + 16X - 32 - 10Y - 50 + 37 = 0$$

$$\text{or } 4X^2 + Y^2 - 4 = 0 \text{ is the required transformed equation.}$$

$$(iii) \quad 9x^2 + 4y^2 + 18x - 16y - 11 = 0, \quad O'(-1, 2)$$

$$\text{Solution. } 9x^2 + 4y^2 + 18x - 16y - 11 = 0 \quad \dots (1), \quad O'(-1, 2) = (h, k)$$

Equations of transformation are

$$x = X + h = X - 1, \quad y = Y + k = Y + 2$$

Putting these values of x and y into (1), we have

$$9(X-1)^2 + 4(Y+2)^2 + 18(X-1) - 16(Y+2) - 11 = 0$$

$$\text{or } 9(X^2 - 2X + 1) + 4(Y^2 + 4Y + 4) + 18(X-1) - 16(Y+2) - 11 = 0$$

$$\text{or } 9X^2 - 18X + 9 + 4Y^2 + 16Y + 16 + 18X - 18 - 16Y - 32 - 11 = 0$$

$$\text{or } 9X^2 + 4Y^2 - 36 = 0 \text{ is the required transformed equation.}$$

$$(iv) \quad x^2 - y^2 + 4x + 8y - 11 = 0, \quad O'(-2, 4)$$

$$\text{Solution. } x^2 - y^2 + 4x + 8y - 11 = 0 \quad \dots (1), \quad O'(-2, 4) = (h, k)$$

Equations of transformation are

$$x = X + h = X - 2, \quad y = Y + k = Y + 4$$

Substituting these values of x and y into (1), we have

$$(X-2)^2 - (Y+4)^2 + 4(X-2) + 8(Y+4) - 11 = 0$$

$$\text{or } X^2 - 4X + 4 - (Y^2 + 8Y + 16) + 4X - 8 + 8Y + 32 - 11 = 0$$

$$\text{or } X^2 - 4X + 4 - Y^2 - 8Y - 16 + 4X - 8 + 8Y + 32 - 11 = 0$$

$$\text{or } X^2 - Y^2 + 1 = 0 \text{ is the required transformed equation.}$$

$$(v) \quad 9x^2 - 4y^2 + 36x + 8y - 4 = 0, \quad O'(-2, 1)$$

$$\text{Solution. } 9x^2 - 4y^2 + 36x + 8y - 4 = 0 \quad \dots (1) \quad O'(-2, 1) = (h, k)$$

Equations of transformation are

$$x = X + h = X - 2, \quad y = Y + k = Y + 1$$

Substituting these values of x and y into (1), we have

$$9(X-2)^2 - 4(Y+1)^2 + 36(X-2) + 8(Y+1) - 4 = 0$$

$$\text{or } 9(X^2 - 4X + 4) - 4(Y^2 + 2Y + 1) + 36(X-2) + 8(Y+1) - 4 = 0$$

$$\text{or } 9X^2 - 36X + 36 - 4Y^2 - 8Y - 4 + 36X - 72 + 8Y + 8 - 4 = 0$$

or $9X^2 - 4Y^2 - 36 = 0$ is the required transformed equation.

2. Find coordinates of the new origin (axes remaining parallel) so that first degree terms are removed from the transformed equation of each of the following. Also find the transformed equation.

(i) $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

Solution. $3x^2 - 2y^2 + 24x + 12y + 24 = 0 \quad \dots (1)$

Let the coordinates of the new origin be $O'(h, k)$.

Equations of transformation are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$3(X+h)^2 - 2(Y+k)^2 + 24(X+h) + 12(Y+k) + 24 = 0$$

$$\text{or } 3(X^2 + 2Xh + h^2) - 2(Y^2 + 2kY + k^2) + 24(X+h) + 12(Y+k) + 24 = 0$$

$$\text{or } 3X^2 + 6hX + 3h^2 - 2Y^2 - 4kY - 2k^2 + 24X + 24h + 12Y + 12k + 24 = 0$$

$$\text{or } 3X^2 - 2Y^2 + X(6h + 24) + Y(-4k + 12) + 3h^2 + 24h - 2k^2 + 12k + 24 = 0 \quad \dots (2)$$

To remove first degree terms, we have

$$6h + 24 = 0 \Rightarrow h = -4$$

$$\text{and } -4k + 12 = 0 \Rightarrow k = 3$$

New origin is $O'(-4, 3)$. Putting $h = -4$ and $k = 3$ into (2), the transformed equation is

$$3X^2 - 2Y^2 + X[6(-4) + 24] + Y[-4(3) + 12] + 3(-4)^2 + 24(-4) - 2(3)^2 + 12(3) + 24 = 0$$

$$\text{or } 3X^2 - 2Y^2 + X(-24 + 24) + Y(-12 + 12) + 3(16) - 96 - 2(9) + 36 + 24 = 0$$

$$\text{or } 3X^2 - 2Y^2 + X(0) + Y(0) + 48 - 96 - 18 + 36 + 24 = 0$$

$$\text{or } 3X^2 - 2Y^2 - 6 = 0$$

which is the required transformed equation with new origin at $(-4, 3)$

(ii) $25x^2 + 9y^2 + 50x - 36y - 164 = 0$

Solution. $25x^2 + 9y^2 + 50x - 36y - 164 = 0 \quad \dots (1)$

Let the coordinates of the new origin be $O'(h, k)$. Equations of transformed are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$25(X+h)^2 + 9(Y+k)^2 + 50(X+h) - 36(Y+k) - 164 = 0$$

$$\text{or } 25(X^2 + 2Xh + h^2) + 9(Y^2 + 2Yk + k^2) + 50X + 50h - 36Y - 36k - 164 = 0$$

$$\text{or } 25X^2 + 50Xh + 25h^2 + 9Y^2 + 18Yk + 9k^2 + 50X + 50h - 36Y - 36k - 164 = 0$$

$$\text{or } 25X^2 + 9Y^2 + (50h + 50)X + (18k - 36)Y + 25h^2 + 9k^2 + 50h - 36k - 164 = 0 \dots (2)$$

To remove the first degree terms, we put

$$50h + 50 = 0 \Rightarrow 50h = -50 \Rightarrow h = -1$$

$$\text{and } 18k - 36 = 0 \Rightarrow 18k = 36 \Rightarrow k = 2$$

$\therefore$ New origin is $O'(-1, 2)$.

Putting $h = -1, k = 2$ into (2), the transformed equation is

$$25X^2 + 9Y^2 + (-50 + 50)X + (36 - 36)Y + 25(-1)^2 + 9(2)^2 + 50(-1) - 36(2) - 164 = 0$$

$$\text{or } 25X^2 + 9Y^2 + (0)X + (0)Y + 25(1) + 9(4) - 50 - 72 - 164 = 0$$

$$\text{or } 25X^2 + 9Y^2 + 25 + 36 - 50 - 72 - 164 = 0$$

$$\text{or } 25X^2 + 9Y^2 - 225 = 0$$

which is the required equation with new origin at $(-1, 2)$.

$$(iii) \quad x^2 - y^2 - 6x + 2y + 7 = 0$$

$$\text{Solution. } x^2 - y^2 - 6x + 2y + 7 = 0 \dots (1)$$

Let the coordinates of the new origin be $O'(h, k)$. Equations of transformation are

$$x = X + h, \quad y = Y + k$$

Substituting these values of x, y into (1), we get

$$(X+h)^2 - (Y+k)^2 - 6(X+h) + 2(Y+k) + 7 = 0$$

$$\text{or } X^2 + 2Xh + h^2 - (Y^2 + 2Yk + k^2) - 6X - 6h + 2Y + 2k + 7 = 0$$

$$\text{or } X^2 - Y^2 + X(2h - 6) + Y(-2k + 2) + h^2 - k^2 - 6h + 2k + 7 = 0 \dots (2)$$

To remove the first degree terms, we have

$$2h - 6 = 0 \Rightarrow h = 3$$

$$\text{and } -2k + 2 = 0 \Rightarrow k = 1$$

New origin is $O'(3, 1)$. Putting $h = 3, k = 1$ into (2) the transformed equation is

$$X^2 - Y^2 + X(0) + Y(0) + (3)^2 - (1)^2 - 6(3) + 2(1) + 7 = 0$$

$$\text{or } X^2 - Y^2 + 9 - 1 - 18 + 2 + 7 = 0$$

$$\text{or } X^2 - Y^2 - 1 = 0$$

which is the required equation with new origin at (3, 1).

3. In each of the following, find an equation referred to the new axes obtained by rotation of axes about the origin through the given angle:

(i) $xy = 1$, $\theta = 45^\circ$

Solution. Here $\theta = 45^\circ$. Equations of transformation are

$$x = X \cos 45^\circ - Y \sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X \sin 45^\circ + Y \cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

Substituting these values for x, y into the given equation, we have

$$\left(\frac{X - Y}{\sqrt{2}} \right) \left(\frac{X + Y}{\sqrt{2}} \right) = 1$$

$$\text{or } \frac{X^2 - Y^2}{2} = 1 \quad \text{or } X^2 - Y^2 = 2$$

is the required transformed equation

(ii) $7x^2 - 8xy + y^2 - 9 = 0$, $\theta = \tan^{-1} 2$

Solution. Here $\theta = \tan^{-1} 2 \Rightarrow \tan \theta = 2 = \frac{2}{1}$

$$\Rightarrow \text{base} = 2, \perp = 1 \text{ and hypotenuse} = \sqrt{4+1} = \sqrt{5}$$

$$\therefore \sin \theta = \frac{2}{\sqrt{5}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{5}}$$

Equations of transformation are

$$\left. \begin{aligned} x &= X \cos \theta - Y \sin \theta \\ y &= X \sin \theta + Y \cos \theta \end{aligned} \right\}$$

Using the above values

$$x = X \cdot \frac{1}{\sqrt{5}} - Y \cdot \frac{2}{\sqrt{5}} = \frac{X - 2Y}{\sqrt{5}}$$

$$y = X \cdot \frac{2}{\sqrt{5}} + Y \cdot \frac{1}{\sqrt{5}} = \frac{2X + Y}{\sqrt{5}}$$

Substituting these values for x, y into the given equation, we have

$$7\left(\frac{X-2Y}{\sqrt{5}}\right)^2 - 8\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right) + \left(\frac{2X+Y}{\sqrt{5}}\right)^2 - 9 = 0$$

By multiplying by 5, we get

$$7\left(\frac{X^2 - 4XY + 4Y^2}{5}\right) - 8\left(\frac{2X^2 - 3XY - 2Y^2}{5}\right) + \left(\frac{4X^2 + 4XY + Y^2}{5}\right) - 9 = 0$$

$$\text{or } 7X^2 - 28XY + 28Y^2 - 16X^2 + 24XY + 16Y^2 + 4X^2 + 4XY + Y^2 - 45 = 0$$

$$\text{or } -5X^2 + 45Y^2 - 45 = 0$$

$$\text{or } X^2 - 9Y^2 + 9 = 0$$

which is the required transformed equation.

$$(iii) \quad 9x^2 + 12xy + 4y^2 - x - y = 0, \quad \theta = \tan^{-1} \frac{2}{3}$$

$$\text{Solution. Here } \theta = \tan^{-1} \frac{2}{3} \Rightarrow \tan \theta = \frac{2}{3}$$

$$\Rightarrow \text{base} = 3, \text{perp} = 2 \text{ and hypotenuse} = \sqrt{9+4} = \sqrt{13}$$

$$\therefore \sin \theta = \frac{2}{\sqrt{13}} \quad \text{and} \quad \cos \theta = \frac{3}{\sqrt{13}}$$

Equations of transformation are

$$\left. \begin{aligned} x &= X \cos \theta - Y \sin \theta \\ y &= X \sin \theta + Y \cos \theta \end{aligned} \right\}$$

Putting values, we have

$$x = X \cdot \frac{3}{\sqrt{13}} - Y \cdot \frac{2}{\sqrt{13}} = \frac{3X - 2Y}{\sqrt{13}}$$

$$y = X \cdot \frac{2}{\sqrt{13}} + Y \cdot \frac{3}{\sqrt{13}} = \frac{2X + 3Y}{\sqrt{13}}$$

Substituting these values for x, y in to the given equations, we have

$$9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2 + 12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right) + 4\left(\frac{2X+3Y}{\sqrt{13}}\right)^2 - \left(\frac{3X-2Y}{\sqrt{13}}\right) - \left(\frac{2X+3Y}{\sqrt{13}}\right) = 0$$

$$\text{or } 9\left(\frac{9X^2 - 12XY + 4Y^2}{13}\right) + 12\left(\frac{6X^2 + 5XY - 6Y^2}{13}\right)$$

$$+ 4 \left(\frac{4X^2 + 12XY + 9Y^2}{13} \right) - \left(\frac{3X - 2Y}{\sqrt{13}} \right) - \left(\frac{2X + 3Y}{\sqrt{13}} \right) = 0$$

$$\text{or } 81X^2 - 108XY + 36Y^2 + 72X^2 + 60XY - 72Y^2 + 16X^2 + 48XY + 36Y^2 - 3\sqrt{13}X + 2\sqrt{13}Y - 2\sqrt{13}X - 3\sqrt{13}Y = 0$$

$$\text{or } 169X^2 - 5\sqrt{13}X - \sqrt{13}Y = 0$$

$$\text{or } 13\sqrt{13}X^2 - 5X - Y = 0$$

is the required transformed equation

$$\text{(iv) } x^2 - xy + y^2 - 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0, \theta = 45^\circ$$

Solution. Here $\theta = 45^\circ$

Equations of transformation are

$$\left. \begin{aligned} x &= X \cos \theta - Y \sin \theta \\ y &= X \sin \theta + Y \cos \theta \end{aligned} \right\}$$

Putting values, we have

$$x = X \cos 45^\circ - Y \sin 45^\circ = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X \sin 45^\circ + Y \cos 45^\circ = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

Substituting these expressions for x, y into the given equation, we have

$$\begin{aligned} \left(\frac{X - Y}{\sqrt{2}} \right)^2 - 2 \left(\frac{X - Y}{\sqrt{2}} \right) \left(\frac{X + Y}{\sqrt{2}} \right) + \left(\frac{X + Y}{\sqrt{2}} \right)^2 - 2\sqrt{2} \left(\frac{X - Y}{\sqrt{2}} \right) \\ - 2\sqrt{2} \left(\frac{X + Y}{\sqrt{2}} \right) + 2 = 0 \end{aligned}$$

$$\begin{aligned} \text{or } \left(\frac{X^2 - 2XY + Y^2}{2} \right) - 2 \left(\frac{X^2 - Y^2}{2} \right) + \left(\frac{X^2 + 2XY + Y^2}{2} \right) \\ - 2(X - Y) - 2(X + Y) + 2 = 0 \end{aligned}$$

$$(\text{by } \times 2): X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 4X + 4Y - 4X - 4Y + 4 = 0$$

$$\text{or } 4Y^2 - 8X + 4 = 0 \quad \text{or } Y^2 - 2X + 1 = 0$$

which is the required transformed equation.

4. Find measure of the angle through which the axes be rotated so that the product term XY is removed from the transformed equation. Also find the transformed equation.

$$(i) \quad 2x^2 + 6xy + 10y^2 - 11 = 0$$

Solution. Let the axes be rotated through an angle θ .

Equations of transformation are

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta$$

Substituting into the given equation, we get

$$2(X \cos \theta - Y \sin \theta)^2 + 6(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + 10(X \sin \theta + Y \cos \theta)^2 - 11 = 0$$

$$\begin{aligned} \text{or} \quad & 2(X^2 \cos^2 \theta - 2XY \sin \theta \cos \theta + Y^2 \sin^2 \theta) \\ & + 6(X^2 \sin \theta \cos \theta - Y^2 \sin \theta \cos \theta + XY \cos^2 \theta - XY \sin^2 \theta) \\ & + 10(X^2 \sin^2 \theta + 2XY \sin \theta \cos \theta + Y^2 \cos^2 \theta) - 11 = 0 \end{aligned}$$

$$\begin{aligned} \text{or} \quad & X^2(2 \cos^2 \theta + 6 \sin \theta \cos \theta + 10 \sin^2 \theta) \\ & + XY(-4 \sin \theta \cos \theta + 6 \cos^2 \theta - 6 \sin^2 \theta + 20 \sin \theta \cos \theta) \\ & + Y^2(2 \sin^2 \theta - 6 \sin \theta \cos \theta + 10 \cos^2 \theta) - 11 = 0 \end{aligned} \quad (i)$$

Since this equation is to be free from the product term, i.e.;

$$-4 \sin \theta \cos \theta + 6 \cos^2 \theta - 6 \sin^2 \theta + 20 \sin \theta \cos \theta = 0$$

$$\text{or} \quad -6 \sin^2 \theta + 16 \sin \theta \cos \theta + 6 \cos^2 \theta = 0$$

$$\text{or} \quad 3 \tan^2 \theta - 8 \tan \theta - 3 = 0 \quad (\text{by } + \cos^2 \theta)$$

$$\begin{aligned} \therefore \tan \theta &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-3)}}{2(3)} = \frac{8 \pm \sqrt{64 + 36}}{6} \\ &= \frac{8 \pm \sqrt{100}}{6} = \frac{8 \pm 10}{6} = \frac{8+10}{6}, \frac{8-10}{6} = \frac{18}{6}, \frac{-2}{6} \\ &= 3, \frac{-1}{3} \Rightarrow \tan \theta = 3 \quad (\text{as } \theta \text{ is in the first quadrant}) \end{aligned}$$

$$\Rightarrow \text{base} = 1, \perp = 3 \text{ and hypotenuse} = \sqrt{1+9} = \sqrt{10}$$

$$\therefore \sin \theta = \frac{3}{\sqrt{10}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{10}}$$

Substituting $\sin \theta = \frac{3}{\sqrt{10}}$ and $\cos \theta = \frac{1}{\sqrt{10}}$ in (i), then

$$\begin{aligned} & X^2 \left(\left(2 \left(\frac{1}{\sqrt{10}} \right) \right)^2 + 6 \left(\frac{3}{\sqrt{10}} \right) \left(\frac{1}{\sqrt{10}} \right) + 10 \left(\frac{3}{\sqrt{10}} \right)^2 \right) \\ & + XY \left(-4 \left(\frac{3}{\sqrt{10}} \right) \left(\frac{1}{\sqrt{10}} \right) + 6 \left(\frac{1}{\sqrt{10}} \right)^2 - 6 \left(\frac{3}{\sqrt{10}} \right)^2 + 20 \left(\frac{3}{\sqrt{10}} \right) \left(\frac{1}{\sqrt{10}} \right) \right) \end{aligned}$$

$$+ Y^2 \left(2 \left(\frac{3}{\sqrt{10}} \right)^2 - 6 \left(\frac{3}{\sqrt{10}} \right) \left(\frac{1}{\sqrt{10}} \right) + 10 \left(\frac{1}{\sqrt{10}} \right)^2 \right) - 11 = 0$$

$$\text{or } X^2 \left(\frac{2}{10} + \frac{18}{10} + \frac{90}{10} \right) + XY \left(-\frac{12}{10} + \frac{6}{10} - \frac{54}{10} + \frac{60}{10} \right) + Y^2 \left(\frac{18}{10} - \frac{18}{10} + \frac{10}{10} \right) - 11 = 0$$

$$\text{or } 11X^2 + Y^2 - 11 = 0$$

$$(ii) \quad xy + 4x - 3y - 10 = 0 \quad \dots (1)$$

Solution. Let the axes be rotated through an angle θ .

Equation of transformation are

$$x = X \cos \theta - Y \sin \theta ; y = X \sin \theta + Y \cos \theta$$

Substituting into the given equation (1), we have

$$\begin{aligned} (X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + 4(X \cos \theta - Y \sin \theta) \\ + XY \cos^2 \theta - XY \sin^2 \theta - 3(X \sin \theta + Y \cos \theta) - 10 = 0 \\ X^2 \sin \theta \cos \theta + XY \cos^2 \theta - XY \sin^2 \theta - Y^2 \sin \theta \cos \theta + 4X \cos \theta \\ - 4Y \sin \theta - 3X \sin \theta - 3Y \cos \theta - 10 = 0 \end{aligned}$$

$$\text{or } X^2 \sin \theta \cos \theta + XY (\cos^2 \theta - \sin^2 \theta) - Y^2 \sin \theta \cos \theta + X(4 \cos \theta - 3 \sin \theta) - Y(4 \sin \theta + 3 \cos \theta) - 10 = 0 \quad \dots (2)$$

Since this equation is to be free from the product term XY

$$\text{i.e., } \cos^2 \theta - \sin^2 \theta = 0$$

$$\text{or } \cos^2 \theta = \sin^2 \theta \Rightarrow \tan^2 \theta = 1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = 45^\circ$$

Thus axes be rotated through an angle of 45° . So that XY term is removed from the transformed equation.

Setting $\theta = 45^\circ$ in (i), the transformed equation is

$$\begin{aligned} X^2 \sin 45^\circ \cos 45^\circ + XY (\cos^2 45^\circ - \sin^2 45^\circ) - Y^2 \sin 45^\circ \cos 45^\circ \\ + X(4 \cos 45^\circ - 3 \sin 45^\circ) - Y(4 \sin 45^\circ + 3 \cos 45^\circ) - 10 = 0 \\ X^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + XY(0) - Y^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + X \left(4 \cdot \frac{1}{\sqrt{2}} - 3 \cdot \frac{1}{\sqrt{2}} \right) \\ - Y \left(4 \cdot \frac{1}{\sqrt{2}} + 3 \cdot \frac{1}{\sqrt{2}} \right) - 10 = 0 \end{aligned}$$

$$\text{or } X^2 \cdot \frac{1}{2} - Y^2 \cdot \frac{1}{2} + X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{7}{\sqrt{2}} - 10 = 0$$

$$\text{or } X^2 - Y^2 + \sqrt{2} X - 7\sqrt{2} Y - 20 = 0$$

is the required equation.

$$(iii) \quad 5x^2 - 6xy + 5y^2 - 8 = 0 \quad \dots (1)$$

Solution. Let the axes be rotated through an angle θ .

Equation of transformation are

$$x = X \cos \theta - Y \sin \theta ; y = X \sin \theta + Y \cos \theta$$

Substituting into the given equation, we get

$$5(X \cos \theta - Y \sin \theta)^2 - 6(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) \\ + 5(X \sin \theta + Y \cos \theta)^2 - 8 = 0$$

$$\text{or } 5(X^2 \cos^2 \theta - 2XY \sin \theta \cos \theta + Y^2 \sin^2 \theta) - 6(X^2 \sin \theta \cos \theta \\ + XY \cos^2 \theta - XY \sin^2 \theta - Y^2 \sin \theta \cos \theta) \\ + 5(X^2 \sin^2 \theta + 2XY \sin \theta \cos \theta + Y^2 \cos^2 \theta) - 8 = 0$$

$$\text{or } X^2(5 \cos^2 \theta - 6 \sin \theta \cos \theta + 5 \sin^2 \theta) + XY(-10 \sin \theta \cos \theta - 6 \cos^2 \theta \\ + 6 \sin^2 \theta + 10 \sin \theta \cos \theta) + Y^2(5 \sin^2 \theta + 6 \sin \theta \cos \theta + 5 \cos^2 \theta) - 8 = 0$$

Since this equation is to be free from the product term XY

$$\text{i.e., } -6 \cos^2 \theta - 6 \sin^2 \theta = 0$$

$$\text{or } \sin^2 \theta = \cos^2 \theta \Rightarrow \tan^2 \theta = 1 \Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = 45^\circ \quad (\text{as } \theta \text{ is in the first quadrant})$$

Setting $\theta = 45^\circ$ in (1), the transformed equation is

$$X^2 \left(5 \cdot \left(\frac{1}{\sqrt{2}} \right)^2 - 6 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 5 \cdot \left(\frac{1}{\sqrt{2}} \right)^2 \right) \\ + XY \left(-10 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - 6 \cdot \left(\frac{1}{\sqrt{2}} \right)^2 + 6 \cdot \left(\frac{1}{\sqrt{2}} \right)^2 + 10 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \right) \\ + Y^2 \left(5 \cdot \left(\frac{1}{\sqrt{2}} \right)^2 + 6 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 5 \cdot \left(\frac{1}{\sqrt{2}} \right)^2 \right) - 8 = 0$$

$$\text{or } X^2 \left(\frac{5}{2} - \frac{6}{2} + \frac{5}{2} \right) + XY(0) + Y^2 \left(\frac{5}{2} + \frac{6}{2} + \frac{5}{2} \right) - 8 = 0$$

$$\text{or } 2X^2 + 8Y^2 - 8 = 0 \Rightarrow X^2 + 4Y^2 - 4 = 0$$

which is the required equation.