Equation of tangent and normal to the circle

Consider an equation of circle

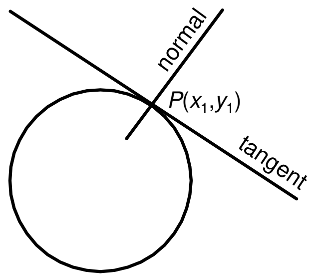
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Then equation of tangent at (x_1, y_1) is

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

The equation of normal at (x_1, y_1) is

$$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$$



(*See proof at page 257*)

Question # 1

Write down equations of tangent and normal to the circle

(i) $x^2 + y^2 = 25$ at $(4,3)$ and at $(5\cos\theta, 5\sin\theta)$

(ii) $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ at $\left(1, \dfrac{10}{3}\right)$

Solution $\quad x^2 + y^2 = 25$

Differentiating w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x \qquad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

At $(4,3)$:

Slope of tangent at $(4,3)$ $= m = \left.\dfrac{dy}{dx}\right|_{(4,3)} = -\dfrac{4}{3}$

Now equation of tangent at $(4,3)$ having slope $-\dfrac{4}{3}$

$$y - 3 = -\frac{4}{3}(x - 4) \qquad \Rightarrow \quad 3y - 9 = -4x + 16$$

$$\Rightarrow \quad 4x - 16 + 3y - 9 = 0 \qquad \Rightarrow \quad \boxed{4x + 3y - 25 = 0}$$

Since normal is $\perp$ ar to tangent therefore

Slope of normal at $(4,3)$ $= -\dfrac{1}{m} = -\dfrac{1}{-4/3} = \dfrac{3}{4}$

Now equation of normal at $(4,3)$ having slope $\dfrac{3}{4}$

$$y - 3 = \frac{3}{4}(x - 4)$$

$$\Rightarrow \quad 4y - 12 = 3x - 12 \qquad \Rightarrow \quad 3x - 12 - 4y + 12 = 0$$

$$\Rightarrow \quad \boxed{3x - 4y = 0}$$

At $(5\cos\theta, 5\sin\theta)$

Slope of tangent at $(5\cos\theta, 5\sin\theta) = m = \left.\dfrac{dy}{dx}\right|_{(5\cos\theta,5\sin\theta)} = -\dfrac{5\cos\theta}{5\sin\theta} = -\dfrac{\cos\theta}{\sin\theta}$

Now equation of tangent at $(5\cos\theta, 5\sin\theta)$ having slope $-\dfrac{\cos\theta}{\sin\theta}$

$$y-5\sin\theta = -\frac{\cos\theta}{\sin\theta}(x-5\cos\theta)$$
$$\Rightarrow\quad y\sin\theta-5\sin^2\theta = -x\cos\theta+5\cos^2\theta$$
$$\Rightarrow\quad x\cos\theta+y\sin\theta = 5\sin^2\theta+5\cos^2\theta$$
$$\Rightarrow\quad x\cos\theta+y\sin\theta = 5\left(\sin^2\theta+\cos^2\theta\right)$$
$$\Rightarrow\quad x\cos\theta+y\sin\theta = 5(1)$$
$$\Rightarrow\quad \boxed{x\cos\theta+y\sin\theta = 5}$$

Since normal are $\perp$ar to tangent therefore

Slope of normal $= -\dfrac{1}{m} = \dfrac{\sin\theta}{\cos\theta}$

Now equation of normal at $(5\cos\theta, 5\sin\theta)$ having slope $\dfrac{\sin\theta}{\cos\theta}$

$$y-5\sin\theta = \frac{\sin\theta}{\cos\theta}(x-5\cos\theta)$$
$$\Rightarrow\quad y\cos\theta-5\sin\theta\cos\theta = x\sin\theta-5\sin\theta\cos\theta$$
$$\Rightarrow\quad x\sin\theta-5\sin\theta\cos\theta-y\cos\theta+5\sin\theta\cos\theta = 0$$
$$\Rightarrow\quad \boxed{x\sin\theta-y\cos\theta = 0}$$

(ii) **[Alternative Method]**

$$3x^2+3y^2+5x-13y+2 = 0$$
$$\Rightarrow\quad x^2+y^2+\frac{5}{3}x-\frac{13}{3}y+\frac{2}{3} = 0$$

Comparing it with general equation of circle

$$2g = \frac{5}{3}\quad,\quad 2f = -\frac{13}{3}\quad,\quad c = \frac{2}{3}$$
$$\Rightarrow\quad g = \frac{5}{6}\quad,\quad f = -\frac{13}{6}$$

Now equation of tangent at (x_1, y_1)

$$x_1x+y_1y+g(x+x_1)+f(y+y_1)+c = 0$$

Here $(x_1, y_1) = \left(1, \dfrac{10}{3}\right)$

$$\Rightarrow\quad 1\cdot x+\frac{10}{3}\cdot y+\frac{5}{6}(x+1)-\frac{13}{6}\left(y+\frac{10}{3}\right)+\frac{2}{3} = 0$$
$$\Rightarrow\quad x+\frac{10}{3}y+\frac{5}{6}x+\frac{5}{6}-\frac{13}{6}y-\frac{130}{18}+\frac{2}{3} = 0$$

$$\Rightarrow \frac{11}{6}x+\frac{7}{6}y+\frac{157}{18} = 0$$
$$\Rightarrow 33x+21y+157 = 0$$

is the required tangent.

Now equation of normal at (x_1, y_1)

$$(y-y_1)(x_1+g) = (x-x_1)(y_1+f)$$

Here $(x_1, y_1) = \left(1, \frac{10}{3}\right)$

$$\Rightarrow \left(y-\frac{10}{3}\right)\left(1+\frac{5}{6}\right) = (x-1)\left(\frac{10}{3}-\frac{13}{6}\right)$$
$$\Rightarrow \left(y-\frac{10}{3}\right)\left(\frac{11}{6}\right) = (x-1)\left(\frac{7}{6}\right) \qquad \Rightarrow 11y-\frac{110}{3} = 7x-7$$
$$\Rightarrow 7x-7-11y+\frac{110}{3} = 0 \qquad \Rightarrow 7x-11y+\frac{89}{3} = 0$$
$$\Rightarrow 21x-33y+89 = 0$$

is required equation of normal.

Question # 2

Write down equations of tangent and normal to the circle

$4x^2+4y^2-16x+24y-117 = 0$

at the point on the circle whose abscissa is -4.

Solution

$$4x^2+4y^2-16x+24y-117 = 0 \;\ldots\ldots\; \text{(i)}$$

Since abscissa $= -4$, so putting $x=-4$ in given eq.

$$4(-4)^2+4y^2-16(-4)+24y-117 = 0$$
$$\Rightarrow 64+4y^2+64+24y-117 = 0$$
$$\Rightarrow 4y^2+24y+11 = 0$$
$$\Rightarrow y = \frac{-24\pm\sqrt{(24)^2-4(4)(11)}}{2(4)}$$
$$= \frac{-24\pm\sqrt{576-176}}{8} \qquad = \frac{-24\pm\sqrt{400}}{8}$$
$$\Rightarrow y = \frac{-24\pm 20}{8}$$
$$\Rightarrow y = \frac{-24+20}{8} \quad \text{or} \quad y = \frac{-24-20}{8}$$
$$\Rightarrow y = -\frac{1}{2} \quad \text{or} \quad y = -\frac{11}{2}$$

So we have points $\left(-4,-\frac{1}{2}\right)$ & $\left(-4,-\frac{11}{2}\right)$

$$\text{(i)} \Rightarrow x^2+y^2-4x+6y-\frac{117}{4} = 0$$

Comparing it with general equation of circle

$$2g = -4 \quad , \quad 2f = 6 \quad , \quad c = -\frac{117}{4}$$

$$\Rightarrow \; g = -2 \quad , \quad f = 3$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

For $(x_1, y_1) = \left(-4, -\frac{1}{2}\right)$

Solve yourself as Q # 1(ii)

For $(x_1, y_1) = \left(-4, -\frac{11}{2}\right)$

Solve yourself as Q # 1(ii)

Position of the point with a circle

Consider the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

The point $P(x_1, y_1)$ lies on the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

The point $P(x_1, y_1)$ lies outside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

And the point $P(x_1, y_1)$ lies inside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$$

Question # 3

Check the position of the point $(5,6)$ with respect to the circle

(i) $x^2 + y^2 = 81$ (ii) $2x^2 + 2y^2 + 12x - 8y + 1 = 0$

Solution

(i) $x^2 + y^2 = 81$

$$\Rightarrow \; x^2 + y^2 - 81 = 0 \; \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

To check the position of point $(5,6)$, Putting $x = 5$ & $y = 6$ on L.H.S of (i)

$$(5)^2 + (6)^2 - 81 = 25 + 36 - 81$$

$$= -20 < 0$$

Hence $(5,6)$ lies inside the circle.

(ii) $2x^2 + 2y^2 + 12x - 8y + 1 = 0$

$$\Rightarrow \; x^2 + y^2 + 6x - 4y + \frac{1}{2} = 0 \; \ldots\ldots\ldots\ldots \text{(i)}$$

To check the position of point put $x = 5$ & $y = 6$ on L.H.S of (i)

$$(5)^2 + (6)^2 + 6(5) - 4(6) + \frac{1}{2}$$

$$= 25+36+30-24+\frac{1}{2}$$

$$= \frac{135}{2} > 0$$

Hence $(5,6)$ lies outside the circle.

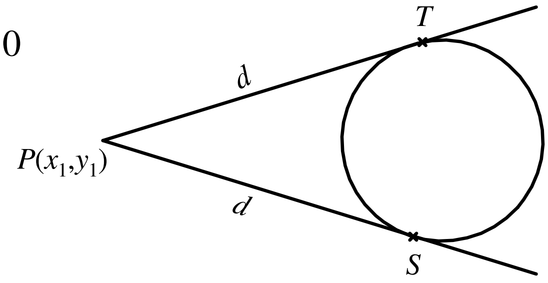
Length of tangent to the circle

Consider equation of circle

$$x^2+y^2+2gx+2fy+c = 0$$

If d denotes length of tangent from point $P(x_1, y_1)$ to the circle then

$$d = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$



Question # 4

Find the length of the tangent drawn from the point $(-5,4)$ to the circle $5x^2+5y^2-10x+15y-131 = 0$

Solution $\quad 5x^2+5y^2-10x+15y-131 = 0$

$$\Rightarrow x^2+y^2-2x+3y-\frac{131}{3} = 0$$

Now length of tangent from point $P(x_1, y_1)$ is

$$d = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

For $(x_1, y_1) = (-5,4)$

$$d = \sqrt{(-5)^2+(4)^2-2(-5)+3(4)-\frac{131}{5}}$$

$$= \sqrt{25+16+10+12-\frac{131}{5}}$$

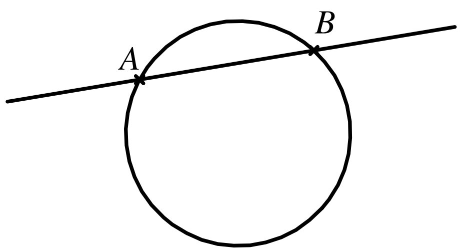
$$= \sqrt{\frac{184}{5}} = 2\sqrt{\frac{46}{5}} \text{ units}$$

Question # 5

Find the length of the chord cut off from the line $2x+3y = 13$ by the circle $x^2+y^2 = 26$

Solution $\quad 2x+3y = 13$ ……….. (i)

$x^2+y^2 = 26$ ……….. (ii)



From (i)

$$2x = 13-3y$$

$$\Rightarrow x = \frac{13-3y}{2} \quad \text{…………. (iii)}$$

Putting in (ii)

$$\left(\frac{13-3y}{2}\right)^2+y^2 = 26$$

$$\Rightarrow \frac{169-78y+9y^2}{4}+y^2 = 26$$

$$\Rightarrow \frac{169-78y+9y^2+4y^2}{4} = 26$$

$$\Rightarrow 13y^2-78y+169 = 104$$

$$\Rightarrow 13y^2-78y+169-104 = 0$$

$$\Rightarrow 13y^2-78y+65 = 0$$

$$\Rightarrow y^2-6y+5 = 0$$

$$\Rightarrow y^2-5y-y+5 = 0$$

$$\Rightarrow y(y-5)-1(y-5) = 0$$

$$\Rightarrow (y-5)(y-1) = 0$$

$$\Rightarrow y = 5 \quad \text{or} \quad y = 1$$

Putting in (iii)

$$x = \frac{13-3(5)}{2} = -1 \qquad \Bigg| \qquad x = \frac{13-3(1)}{2} = 5$$

$\Rightarrow (-1,5)$ and $(5,1)$ are end points of chord intercepted.

So length of chord $= \sqrt{(5+1)^2+(1-5)^2}$

$= \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

Question # 6

Find the coordinates of the points of intersection of the line $x+2y = 6$ with the circle : $x^2+y^2-2x-2y-39 = 0$

Solution $x^2+y^2-2x-2y-39 = 0$ .... (i)

$x+2y = 6$ .... (ii)

Just solve (i) & (ii) *to get the points*

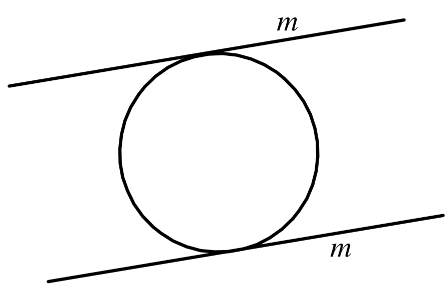
Equation of tangent to the circle having slope m

Consider an equation of circle

$$x^2+y^2 = a^2$$

Then equations of tangents parallel to the line having slope m are

$$y = mx \pm a\sqrt{1+m^2}$$



Question # 7

Find equations of the tangents to the circle $x^2 + y^2 = 2$

(i) parallel to the $x - 2y + 1 = 0$

(ii) perpendicular to the line $3x + 2y = 6$

Solution $x^2 + y^2 = 2$

Centre of circle is at origin with radius $a = \sqrt{2}$

i) Let l: $x - 2y + 1 = 0$

Slope of $l = m = -\dfrac{1}{-2} = \dfrac{1}{2}$

Since required tangent is parallel to l

$\therefore$ Slope of tangent $= m = \dfrac{1}{2}$

Now equations of tangents are

$$y = mx \pm a\sqrt{1+m^2}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\left(\frac{1}{2}\right)^2} \qquad \Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\frac{1}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{\frac{5}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{\frac{10}{4}} \qquad \Rightarrow y = \frac{1}{2}x \pm \frac{\sqrt{10}}{2}$$

$$\Rightarrow 2y = x \pm \sqrt{10}$$

$\Rightarrow x - 2y \pm \sqrt{10} = 0$ are the req. tangents.

(ii) *Do yourself as above*

Question # 8

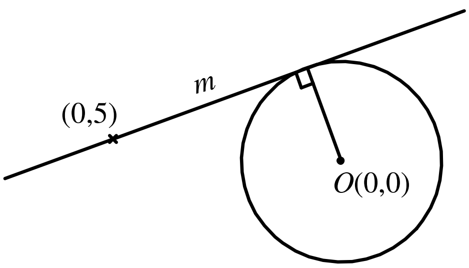
Find equations of the tangent drawn from

(i) $(0,5)$ to $x^2 + y^2 = 16$

(ii) $(-1,2)$ to $x^2 + y^2 + 4x + 2y = 0$

(iii) $(-1,2)$ to $(x+1)^2 + (y-2)^2 = 26$

Also find the point of contact.



Solution

(i) $x^2 + y^2 = 16$

$\Rightarrow$ radius of circle is 4 with centre $O(0,0)$

Let slope of required tangent be m, then eq. of tangent passing through $(0,5)$

$$y - 5 = m(x - 0)$$

$$\Rightarrow y - 5 = mx$$

$$\Rightarrow mx - y + 5 = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now since (i) is tangent to circle, therefore

Radius of circle = $\perp$ar distance of (i) from centre $O(0,0)$

$$\Rightarrow\ 4 = \frac{|m(0)-0+5|}{\sqrt{m^2+(-1)^2}}$$

$$\Rightarrow\ 4 = \frac{|5|}{\sqrt{m^2+1}} \qquad \Rightarrow\ 4\sqrt{m^2+1} = |5|$$

On squaring

$$\left(4\sqrt{1+m^2}\right)^2 = |5|^2$$

$$\Rightarrow\ 16\left(m^2+1\right) = 25 \qquad \Rightarrow\ 16m^2+16 = 25$$

$$\Rightarrow\ 16m^2 = 25-16 \qquad \Rightarrow\ 16m^2 = 9$$

$$\Rightarrow\ m^2 = \frac{9}{16} \qquad \Rightarrow\ m = \pm\frac{3}{4}$$

When $m=\frac{3}{4}$, putting in (i)

$$\frac{3}{4}x-y+5 = 0$$

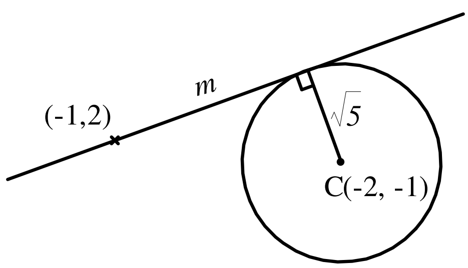
$$\Rightarrow\ 3x-4y+20 = 0$$

When $m=-\frac{3}{4}$, putting in (i)

$$-\frac{3}{4}x-y+5 = 0$$

$$\Rightarrow\ 3x+4y-2 = 0$$

(ii) $\qquad x^2+y^2+4x+2y = 0$



Comparing it with general equation of circle

We have $\quad 2g = 4\quad,\quad 2f = 2\quad,\quad c = 0$

$$\Rightarrow\ g = 2\quad,\quad f = 1$$

Centre $C(-g,-f) = C(-2,-1)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{(2)^2+(1)^2-0}$

$= \sqrt{4+1} = \sqrt{5}$

Let m be a slope of required tangent, so equation of tangent passing thorough $(-1,2)$

$$y-2 = m(x+1)$$

$$\Rightarrow\ y-2 = mx+m$$

$$\Rightarrow\ mx-y+(m+2) = 0 \ \ldots\ldots\ldots\ldots \text{(i)}$$

$\because$ (i) is tangent ot circle,

$\therefore$ $\perp$ar distance of tangent from centre $(-2,-1)$ = Radius of circle

$$\Rightarrow\ \frac{|m(-2)-(-1)+(m+2)|}{\sqrt{(m)^2+(-1)^2}} = \sqrt{5}$$

$$\Rightarrow\ \frac{|-2m+1+m+2|}{\sqrt{m^2+1}} = \sqrt{5}$$

$$\Rightarrow \left|-m+3\right| = \sqrt{5}\cdot\sqrt{m^2+1}$$

On squaring

$$m^2-6m+9 = 5\left(m^2+1\right)$$
$$\Rightarrow 5m^2+5-m^2+6m-9 = 0$$
$$\Rightarrow 4m^2+6m-4 = 0$$
$$\Rightarrow 2m^2+3m-2 = 0$$
$$\Rightarrow 2m^2+4m-m-2 = 0$$
$$\Rightarrow 2m(m+2)-1(m+2) = 0$$
$$\Rightarrow (m+2)(2m-1) = 0$$
$$\Rightarrow m+2 = 0 \quad \text{or} \quad 2m-1 = 0$$
$$\Rightarrow m=-2 \quad \text{or} \quad m=\frac{1}{2}$$

Putting value of m in (i)

$$-2x-y+(-2+2) = 0$$
$$\Rightarrow -2x-y+0 = 0$$
$$\Rightarrow 2x+y = 0$$

$$\frac{1}{2}x-y+\left(\frac{1}{2}+2\right) = 0$$
$$\Rightarrow \frac{1}{2}x-y+\frac{5}{2} = 0$$
$$\Rightarrow x-2y+5 = 0$$

(iii) $$(x+1)^2+(y-2)^2 = 26$$
$$\Rightarrow (x-(-1))^2+(y-2)^2 = \left(\sqrt{26}\right)^2$$

Centre of circle is $(-1,2)$ and radius $\sqrt{26}$

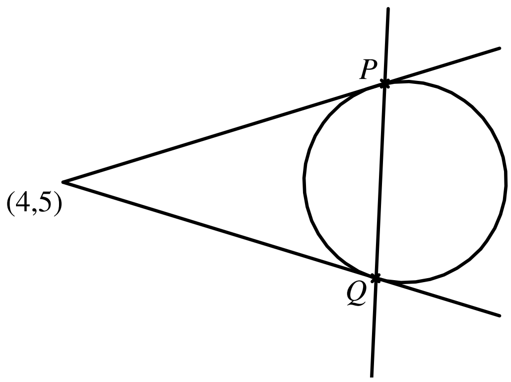
Now do yourself as above.

Note: *To find point of contact, solve equation of tangent and circle.*

Question # 9

Find an equation of the chord of contact of the tangents drawn from $(4,5)$ to the circle

$$2x^2+2y^2-8x+12y+21 = 0$$



Solution

Given: $2x^2+2y^2-8x+12y+21 = 0$

$$\Rightarrow x^2+y^2-4x+6y+\frac{21}{2} = 0$$

Comparing it with general equation of circle

$$2g = -4 \ , \quad 2f = 6 \ , \quad c = \frac{21}{2}$$
$$\Rightarrow g = -2 \ , \quad f = 3$$

Let the point of contact of two tangent be $P(x_1,y_1)$ and $Q(x_2,y_2)$

Eq. of tangent at $P(x_1, y_1)$

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

$$\Rightarrow x_1x + y_1y - 2(x + x_1) + 3(y + y_1) + \frac{21}{2} = 0$$

$$\Rightarrow x_1x + y_1y - 2x - 2x_1 + 3y + 3y_1 + \frac{21}{2} = 0$$

Since tangent is drawn from $(4,5)$, therefore

$$\Rightarrow x_1(4) + y_1(5) - 2(4) - 2x_1 + 3(5) + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 4x_1 + 5y_1 - 8 - 2x_1 + 15 + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 2x_1 + 8y_1 + \frac{35}{2} = 0$$

$$\Rightarrow 4x_1 + 16y_1 + 35 = 0 \quad \ldots\ldots\ldots \text{(i)}$$

Similarly equation of tangent passing through $Q(x_2, y_2)$ and $(4,5)$ gives

$$4x_2 + 16y_2 + 35 = 0 \quad \ldots\ldots\ldots \text{(ii)}$$

Eqs. (i) and (ii) show that both points P & Q lies on the line

$$4x + 16y + 35 = 0$$

So it is the required equation of chord of contact.

Book: **Exercise 6.2 (Page 263)**
Calculus and Analytic Geometry Mathematic 12
Punjab Textbook Board, Lahore.

Available online at http://www.MathCity.org *in PDF Format*
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