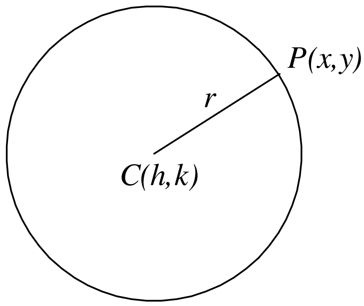
Circle

The set of all point in the plane that are equally distant from a fixed point is called a *circle*.

The fixed point is called *centre* of the circle and the distance from the centre of the circle to any point on the circle is called the *radius* of circle.

Equation of Circle

Let r be radius and $C(h,k)$ be centre of circle. Let $P(x,y)$ be any point on circle then



$$|PC| = r$$

$$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$$

On squaring

$$(x-h)^2+(y-k)^2 = r^2$$

This is equation of circle in standard form.

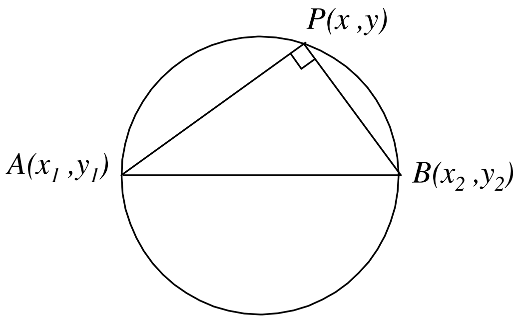
If centre of circle is at origin i.e. $C(h,k)=C(0,0)$ then equation of circle becomes

$$x^2+y^2 = r^2$$

Equation of circle with end points of diameter

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be end points of diameter.

Let $P(x,y)$ be any point on circle then



$m\angle APB = 90^\circ$

(***Note**: An angle in a semi circle is a right angle – see Theorem 4 at page 270*)

Thus the line AP and BP are $\perp$ar to each other and we have

$$(\text{Slope of } AP)\,(\text{Slope of } BP) = -1$$

$$\Rightarrow \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$\Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

is the required equation of circle with end points of diameter $A(x_1,y_1)$ & $B(x_2,y_2)$.

General form of an equation of a circle

The equation

$$x^2+y^2+2gx+2fy+c = 0$$

represents a circle.

$$\Rightarrow x^2+2gx+g^2+y^2+2fy+f^2+c = g^2+f^2$$

$$\Rightarrow (x+g)^2+(y+f)^2 = g^2+f^2-c$$

$$\Rightarrow \quad (x-(-g))^2+(y-(-f))^2 = \left(\sqrt{g^2+f^2-c}\right)^2$$

This is equation of circle in standard form with

centre at $(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$

Question # 1

In each of the following, find an equation of the circle with

(a) centre $(5,-2)$ and radius 4

(b) centre $\left(\sqrt{2},-3\sqrt{3}\right)$ and radius $2\sqrt{2}$

(c) ends of a diameter at $(-3,2)$ and $(5,-6)$.

Solution

(a) Given: centre $C(h,k) = (5,-2)$, radius $= r = 4$

Equation of circle:

$$(x-h)^2+(y-k)^2 = r^2$$
$$\Rightarrow (x-5)^2+(y+2)^2 = (4)^2$$
$$\Rightarrow x^2-10x+25+y^2+4y+4 = 16$$
$$\Rightarrow x^2+y^2-10x+4y+25+4-16 = 0$$
$$\Rightarrow x^2+y^2-10x+4y+13 = 0$$

(b) Given: centre $C(h,k) = \left(\sqrt{2},-3\sqrt{3}\right)$, radius $= r = 2\sqrt{2}$

Equation of circle:

$$(x-h)^2+(y-k)^2 = r^2$$
$$\Rightarrow \left(x-\sqrt{2}\right)^2+\left(y+3\sqrt{3}\right)^2 = \left(2\sqrt{2}\right)^2$$
$$\Rightarrow x^2-2\sqrt{2}x+2+y^2+6\sqrt{3}y+27 = 8$$
$$\Rightarrow x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+2+27-8 = 0$$
$$\Rightarrow x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+21 = 0$$

(c) Given end points of diameter:

$$A(x_1,y_1) = (-3,2) \quad , \quad B(x_2,y_2) = (5,-6)$$

Equation of circle with ends of diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$
$$\Rightarrow (x-(-3))(x-5)+(y-2)(y-(-6)) = 0$$
$$\Rightarrow (x+3)(x-5)+(y-2)(y+6) = 0$$
$$\Rightarrow x^2+3x-4x-15+y^2-2y+6y-12 = 0$$
$$\Rightarrow x^2+y^2-2x+4y-27 = 0$$

Question # 2

Find the centre and radius of the circle with the given equation

(a) $x^2+y^2+12x-10y = 0$

(b) $5x^2+5y^2+14x+12y-10 = 0$

(c) $x^2+y^2-6x+4y+13 = 0$ (d) $4x^2+4y^2-8x+12y-25 = 0$

Solution

(a) $x^2+y^2+12x-10y = 0$

Here $2g=12$, $2f=-10$, $c=0$

$\Rightarrow g=6$, $f=-5$

So centre $= (-g,-f) = (-6,5)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{(6)^2+(-5)^2-0}$

$= \sqrt{36+25} = \sqrt{61}$

(b) $5x^2+5y^2+14x+12y-10 = 0$

$\Rightarrow x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2 = 0$ ÷ing by 5

Here $2g=\frac{14}{5}$, $2f=\frac{12}{5}$, $c=-2$

$\Rightarrow g=\frac{7}{5}$, $f=\frac{6}{5}$

Centre $= (-g,-f) = \left(-\frac{7}{5},-\frac{6}{5}\right)$

Radius $= \sqrt{g^2+f^2-c} = \sqrt{\left(\frac{7}{5}\right)^2+\left(\frac{6}{5}\right)^2-(-2)}$

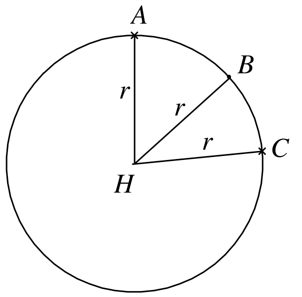
$= \sqrt{\frac{49}{25}+\frac{36}{25}+2} = \sqrt{\frac{27}{5}} = 3\sqrt{\frac{3}{5}}$

(c) *Do yourself as above.*

(d) *Do yourself as above.*

Question # 3

Written an equation of the circle that passes through the given points



(a) $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

(b) $A(-7,7)$, $B(5,-1)$, $C(10,0)$

(c) $A(a,0), B(0,b), C(0,0)$ (d) $A(5,6), B(-3,2), C(3,-4)$

Solution

Given: $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

Let $H(h,k)$ be centre and r be radius of circle, then

$|\overline{AH}| = |\overline{BH}| = |\overline{CH}| = r$

$\Rightarrow |\overline{AH}|^2 = |\overline{BH}|^2 = |\overline{CH}|^2 = r^2$

$\Rightarrow (h-4)^2+(k-5)^2=(h+4)^2+(k+3)^2=(h-8)^2+(k+3)^2=r^2$ …… (i)

From eq. (i)

$$(h-4)^2+(k-5)^2 = (h+4)^2+(k+3)^2$$
$$\Rightarrow h^2-8h+16+k^2-10k+25 = h^2+8h+16+k^2+6k+9$$
$$\Rightarrow h^2-8h+16+k^2-10k+25-h^2-8h-16-k^2-6k-9 = 0$$
$$\Rightarrow -16h-16k+16 = 0 \quad \Rightarrow h+k-1 = 0 \;\ldots\ldots\ldots\ldots\; \text{(ii)}$$

Again from (i)

$$(h+4)^2+(k+3)^2 = (h-8)^2+(k+3)^2$$
$$\Rightarrow (h+4)^2 = (h-8)^2 \Rightarrow h^2+8h+16 = h^2-16h+64$$
$$\Rightarrow h^2+8h+16-h^2+16h-64 = 0$$
$$\Rightarrow 24h-48 = 0 \quad \Rightarrow 24h = 48 \quad \Rightarrow \boxed{h = 2}$$

Putting value of h in (ii)

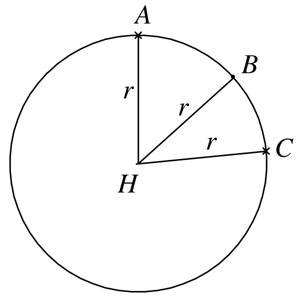
$$2+k-1 = 0 \quad \Rightarrow k+1 = 0 \quad \Rightarrow \boxed{k = -1}$$

Again from (i)

$$r^2 = (h-4)^2+(k-5)^2$$
$$= (2-4)^2+(-1-5)^2 \qquad \because h=2,\ k=-1$$
$$= (-2)^2+(-6)^2 = 4+36 = 40 \qquad \Rightarrow r = \sqrt{40}$$

Now equation of circle with centre at $H(2,-1)$ & $r=\sqrt{40}$

$$(x-2)^2+(y+1)^2 = \left(\sqrt{40}\right)^2$$
$$\Rightarrow x^2-4x+4+y^2+2y+1 = 40 \Rightarrow x^2+y^2-4x+2y+4+1-40 = 0$$
$$\Rightarrow x^2+y^2-4x+2y-35 = 0 \qquad Ans.$$



(b) Given: $A(-7,7)$, $B(5,-1)$, $C(10,0)$

Let $H(h,k)$ be centre and r be radius of circle, then

$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$
$$\Rightarrow \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$
$$\Rightarrow (h+7)^2+(k-7)^2=(h-5)^2+(k+1)^2=(h-10)^2+(k-0)^2=r^2 \;\ldots\ldots\; \text{(i)}$$

From equation (i) we have

$$(h+7)^2+(k-7)^2 = (h-10)^2+(k-0)^2$$
$$\Rightarrow h^2+14h+49+k^2-14k+49 = h^2-20h+100+k^2$$
$$\Rightarrow h^2+14h+49+k^2-14k+49-h^2+20h-100-k^2 = 0$$
$$\Rightarrow 34h-14k-2 = 0 \quad \Rightarrow 17h-7k-1 = 0 \;\ldots\ldots\ldots\; \text{(ii)}$$

Again from (i)

$$(h-5)^2+(k+1)^2 = (h-10)^2+(k-0)^2$$
$$\Rightarrow h^2-10h+25+k^2+2k+1 = h^2-20h+100+k^2$$

$$\Rightarrow \quad h^2-10h+25+k^2+2k+1-h^2+20h-100-k^2 = 0$$
$$\Rightarrow \quad 10h+2k-74 = 0$$
$$\Rightarrow \quad 5h+k-37 = 0 \;\ldots\ldots\ldots\ldots \text{(iii)}$$

Multiplying eq. (iii) by 7 and subtracting from (ii)

$$\begin{array}{l} 17h - 7k - 1 = 0 \\ 35h + 7k - 259 = 0 \\ \hline 52h \qquad - 260 = 0 \end{array}$$

$$\Rightarrow \quad 52h = 260 \quad \Rightarrow \quad \boxed{h = 5}$$

Putting value of h in eq. (iii)

$$5(5)+k-37 = 0 \Rightarrow 25+k-37 = 0 \Rightarrow k-12 = 0 \Rightarrow \boxed{k = 12}$$

Again from eq. (i), we have

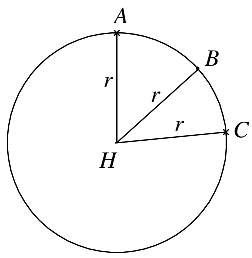
$$r^2 = (h+7)^2+(k-7)^2$$
$$= (5+7)^2+(12-7)^2 = (12)^2+(5)^2 = 144+25 = 169$$
$$\Rightarrow \quad r = 13$$

Now equation of circle with centre $(5,12)$ and radius 13:

$$(x-5)^2+(y-12)^2 = (13)^2$$
$$\Rightarrow \quad x^2-10x+25+y^2-24y+144 = 169$$
$$\Rightarrow \quad x^2+y^2-10x-24y+25+144-169 = 0$$
$$\Rightarrow \quad x^2+y^2-10x-24y = 0$$

(c) Given: $A(a,0)$, $B(0,b)$, $C(0,0)$

Let $H(h,k)$ be centre and r be radius of circle, then



$$\left|\overline{AH}\right| = \left|\overline{BH}\right| = \left|\overline{CH}\right| = r$$
$$\Rightarrow \quad \left|\overline{AH}\right|^2 = \left|\overline{BH}\right|^2 = \left|\overline{CH}\right|^2 = r^2$$

$$\Rightarrow \quad (h-a)^2+(k-0)^2=(h-0)^2+(k-b)^2=(h-0)^2+(k-0)^2=r^2$$
$$\Rightarrow \quad (h-a)^2+k^2=h^2+(k-b)^2=h^2+k^2=r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

From equation (i)

$$(h-a)^2+k^2 = h^2+k^2 \Rightarrow h^2-2ha+a^2+k^2 = h^2+k^2$$
$$\Rightarrow \quad -2ha+a^2 = 0 \quad \Rightarrow \quad -2ha = -a^2 \quad \Rightarrow \quad h = \frac{a^2}{2a} \quad \Rightarrow \quad \boxed{h = \frac{a}{2}}$$

Again from equation (i)

$$h^2+(k-b)^2 = h^2+k^2$$
$$\Rightarrow \quad h^2+k^2-2bk+b^2 = h^2+k^2 \quad \Rightarrow \quad -2bk+b^2 = 0$$
$$\Rightarrow \quad 2bk = b^2 \quad \Rightarrow \quad k = \frac{b^2}{2b} \quad \Rightarrow \quad \boxed{k = \frac{b}{2}}$$

Again from equation (i)

$$r^2 = h^2 + k^2$$

$$= \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} \Rightarrow r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

Now equation of circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\frac{\sqrt{a^2+b^2}}{2}$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2$$

$$\Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} = \frac{a^2}{4} + \frac{b^2}{4}$$

$$\Rightarrow x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} - \frac{a^2}{4} - \frac{b^2}{4} = 0 \Rightarrow x^2 + y^2 - ax - by = 0$$

Alternative Method

Given point on circle: $A(a,0)$, $B(0,b)$, $C(0,0)$

Consider an equation of circle in standard form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots \text{(i)}$$

Since $A(a,0)$ lies on circle, therefore

$$(a)^2 + (0)^2 + 2g(a) + 2f(0) + c = 0$$

$$\Rightarrow a^2 + 2ga + c = 0 \quad \ldots\ldots \text{(ii)}$$

Also $B(0,b)$ lies on the circle, then

$$(0)^2 + (b)^2 + 2g(0) + 2f(b) + c = 0$$

$$\Rightarrow b^2 + 2fb + c = 0 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Also $C(0,0)$ lies on the circle, therefore

$$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0 \Rightarrow c = 0$$

Putting value of c in (ii)

$$a^2 + 2ga + 0 = 0 \Rightarrow 2ga = -a^2 \Rightarrow g = -\frac{a^2}{2a} \Rightarrow g = -\frac{a}{2}$$

Putting value of c in (iii)

$$b^2 + 2fb + 0 = 0 \Rightarrow 2fb = -b^2$$

$$\Rightarrow f = -\frac{b^2}{2b} \Rightarrow f = -\frac{b}{2}$$

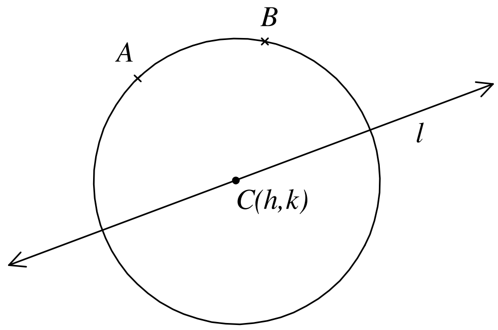
Putting value of g, f and c in (i)

$$x^2 + y^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 0 = 0$$

$$\Rightarrow x^2 + y^2 - ax - by = 0$$

Question # 4

In each of the following, find an equation of the circle passes through



(a) $A(3,-1)$, $B(0,1)$ and have centre at $4x-3y-3=0$

(b) $A(-3,1)$ with radius 2 and centre at $2x-3y+3=0$

(c) $A(5,1)$ and tangent to the line $2x-y-10=0$ at $B(3,-4)$

(d) $A(1,4)$, $B(-1,8)$ and tangent to the line $x+3y-3=0$

Solution

(a) Given: $A(3,-1)$, $B(0,1)$

$l:\ 4x-3y-3 = 0$

Let $C(h,k)$ be centre and r be radius of circle

$\because$ A & B lies on circle

$\therefore$ $\left|\overline{CA}\right| = \left|\overline{CB}\right| = r$

$\Rightarrow \sqrt{(h-3)^2+(k+1)^2} = \sqrt{(h-0)^2+(k-1)^2} = r$ .... (i)

$\Rightarrow (h-3)^2+(k+1)^2 = h^2+(k-1)^2$ on squaring

$\Rightarrow h^2-6h+9+k^2+2k+1 = h^2+k^2-2k+1$

$\Rightarrow h^2-6h+9+k^2+2k+1-h^2-k^2+2k-1 = 0$

$\Rightarrow -6h+4k+9 = 0$ .... (ii)

Now since $C(h,k)$ lies on given equation l

$\therefore$ $4h-3k-3 = 0$ .... (iii)

$\times$ing equation (ii) by 3 & (iii) by 4 then adding

$$\begin{aligned} -18h + 12k + 27 &= 0 \\ 16h - 12k - 12 &= 0 \\ \hline -2h \qquad\quad + 15 &= 0 \end{aligned}$$

$\Rightarrow 2h = 15 \quad \Rightarrow \boxed{h = \frac{15}{2}}$

Putting in (iii)

$4\left(\frac{15}{2}\right)-3k-3 = 0 \quad \Rightarrow 30-3k-3 = 0$

$\Rightarrow -3k+27 = 0 \Rightarrow 3k = 27 \quad \Rightarrow \boxed{k = 9}$

Now from eq. (i)

$r = \sqrt{h^2+(k-1)^2}$

$= \sqrt{\left(\frac{15}{2}\right)^2+(9-1)^2} = \sqrt{\frac{225}{4}+64} = \sqrt{\frac{448}{4}}$

Now equation of circle with centre at $C(h,k)=\left(\frac{15}{2},9\right)$ and radius $\sqrt{\frac{481}{4}}$

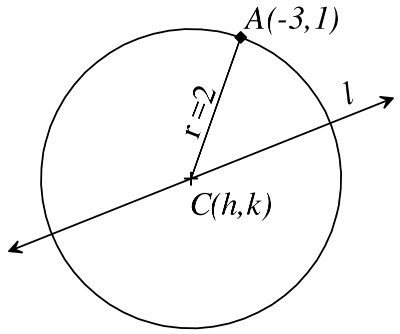
$$\left(x-\frac{15}{2}\right)^2+(y-9)^2 = \left(\sqrt{\frac{481}{4}}\right)^2$$

$$\Rightarrow\ x^2-15x+\frac{225}{4}+y^2-18y+81-\frac{481}{4} = 0$$

$$\Rightarrow\ x^2+y^2-15x-18y+17 = 0$$

(b) Given: $A(-3,1)$ lies on circle , radius $= r = 2$

$l:\ 2x-3y+3 = 0$



Let $C(h,k)$ be centre of circle.

Since $A(-3,1)$ lies on circle

$$\therefore\ r = \left|\overline{AC}\right|$$

$$\Rightarrow\ 2 = \sqrt{(h+3)^2+(k-1)^2}$$

$$\Rightarrow\ 4 = (h+3)^2+(k-1)^2$$

$$\Rightarrow\ 4 = h^2+6h+9+k^2-2k+1 \Rightarrow\ h^2+6h+9+k^2-2k+1-4 = 0$$

$$\Rightarrow\ h^2+6h+9+k^2-2k-3 = 0 \ \ldots\ldots\ldots\ \text{(i)}$$

Since centre $C(h,k)$ lies on l

$$\therefore\ 2h-3k+3 = 0$$

$$\Rightarrow\ 2h = 3k-3 \quad \Rightarrow\ h = \frac{3k-3}{2} \ \ldots\ldots\ldots\ \text{(ii)}$$

Putting value of h in (i)

$$\left(\frac{3k-3}{2}\right)^2+k^2+6\left(\frac{3k-3}{2}\right)-2k+6 = 0$$

$$\Rightarrow\ \frac{9k^2-18k+9}{4}+k^2+9k-9-2k+6 = 0$$

$$\Rightarrow\ 9k^2-18k+9+4k^2+36k-36-8k+24 = 0 \qquad \times\text{ing by } 4$$

$$\Rightarrow\ 13k^2+10k-3 = 0 \quad \Rightarrow\ 13k^2+13k-3k-3 = 0$$

$$\Rightarrow\ 13k(k+1)-3(k+1) = 0 \Rightarrow\ (k+1)(13k-3) = 0$$

$$\Rightarrow\ k = -1 \quad \text{or} \quad k = \frac{3}{13},$$

Putting value of k in (ii)

$$h = \frac{3(-1)-3}{2} = \frac{-6}{2} = -3$$

$\Rightarrow\ (-3,-1)$ is centre of circle

$$h = \frac{3\left(\frac{3}{13}\right)-3}{2} = \frac{\frac{9}{13}-3}{2} = \frac{-\frac{30}{13}}{2} = \frac{-15}{13}$$

$\Rightarrow\ \left(-\frac{15}{13},\frac{3}{13}\right)$ is centre of circle.

Now equation of circle with centre at $(-3,1)$ and radius 2

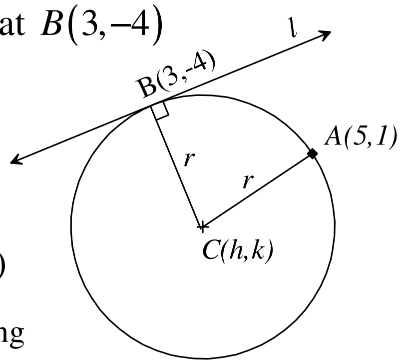
$$(x+3)^2+(y+1)^2 = (2)^2 \quad \Rightarrow \quad (x+3)^2+(y+1)^2 = 4$$

Now equation of circle with centre at $\left(-\frac{15}{13},\frac{3}{13}\right)$ and radius 2

$$\left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2 = (2)^2$$

$$\Rightarrow \left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2 = 4$$

(c) Given: $A(5,1)$ and l: $2x-y-10 = 0$ is tangent at $B(3,-4)$



Let $C(h,k)$ be centre and r be radius of circle.

$\because$ $A(5,1)$ and $B(3,-4)$ lies on circle

$\therefore$ $|AC| = |BC| = r$

$\Rightarrow \sqrt{(h-5)^2+(k-1)^2} = \sqrt{(h-3)^2+(k+4)^2} = r$ ..... (i)

$\Rightarrow (h-5)^2+(k-1)^2 = (h-3)^2+(k+4)^2$ On squaring

$\Rightarrow h^2-10h+25+k^2-2k+1 = h^2-6h+9+k^2+8k+16$

$\Rightarrow h^2-10h+25+k^2-2k+1-h^2+6h-9-k^2-8k-16 = 0$

$\Rightarrow -4h-10k+1 = 0$ ……….. (ii)

Now slope of tangent $l = m_1 = -\frac{a}{b} = -\frac{2}{-1} = 2$

And slope of radial segment $\overline{CB} = m_2 = \frac{k+4}{h-3}$

Since radial segment is perpendicular to tangent therefore

$$m_1m_2 = -1$$

$\Rightarrow 2\left(\frac{k+4}{h-3}\right) = -1 \quad \Rightarrow \quad 2k+8 = -h+3$

$\Rightarrow h-3+2k+8 = 0$

$\Rightarrow h+2k+5 = 0$ ……….. (iii)

Multiplying eq. (iii) by 4 and adding in (ii)

$$\begin{array}{rrrl} 4h & +\ 8k & +\ 20 & = 0 \\ -4h & -10k & +1 & = 0 \\ \hline & -\ 2k & +21 & = 0 \end{array}$$

$\Rightarrow 2k = 21 \quad \Rightarrow \quad \boxed{k = \frac{21}{2}}$

Putting value of k in (iii)

$h+2\left(\frac{21}{2}\right)+5 = 0 \quad \Rightarrow \quad h+21+5 = 0$

$$\Rightarrow \quad h+26 = 0 \quad \Rightarrow \quad \boxed{h = -26}$$

Now from eq. (i)

$$r = \sqrt{(h-3)^2+(k+4)^2}$$

$$= \sqrt{(-26-3)^2+\left(\frac{21}{2}+4\right)^2} \qquad = \sqrt{(-29)^2+\left(\frac{29}{2}\right)^2}$$

$$= \sqrt{841+\frac{841}{4}} \quad = \sqrt{\frac{4205}{4}}$$

Now equation of circle with centre at $\left(-26,\frac{21}{2}\right)$ and radius $\sqrt{\frac{4205}{4}}$

$$(x+26)^2+\left(y-\frac{21}{2}\right)^2 = \left(\sqrt{\frac{4205}{4}}\right)^2$$

$$\Rightarrow \quad x^2+52x+676+y^2-21y+\frac{441}{4}-\frac{4205}{4} = 0$$

$$\Rightarrow \quad x^2+y^2+52x-21y-265 = 0$$

(d) Given; $A(1,4)$, $B(-1,8)$

$l: \; x+3y-3 = 0$

Let $C(h,k)$ be centre and r be radius of circle then

$$\left|\overline{AC}\right| = \left|\overline{BC}\right| = r$$

$$\Rightarrow \quad \sqrt{(h-1)^2+(k-4)^2} = \sqrt{(h+1)^2+(k-8)^2} = r \quad \text{.........} \text{ (i)}$$

Also l is tangent to circle

$\therefore$ radius of circle $=$ $\perp$ar distance of $C(h,k)$ form l

$$\Rightarrow \quad r = \frac{|h+3k-3|}{\sqrt{(1)^2+(3)^2}}$$

$$\Rightarrow \quad r = \frac{|h+3k-3|}{\sqrt{10}} \quad \text{..........} \text{ (ii)}$$

Now from (i)

$$\sqrt{(h-1)^2+(k-4)^2} = \sqrt{(h+1)^2+(k-8)^2}$$

On squaring

$$(h-1)^2+(k-4)^2 = (h+1)^2+(k-8)^2$$

$$\Rightarrow \quad h^2-2h+1+k^2-8k+16 = h^2+2h+1+k^2-16k+64$$

$$\Rightarrow \quad h^2-2h+1+k^2-8k+16-h^2-2h-1-k^2+16k-64 = 0$$

$$\Rightarrow \quad -4h+8k-48 = 0$$

$$\Rightarrow \quad h-2k+12 = 0 \quad \text{............} \text{ (iii)}$$

Now from (i) & (ii)

$$\sqrt{(h-1)^2+(k-4)^2} = \frac{|h+3k-3|}{\sqrt{10}}$$

On squaring

$$(h-1)^2+(k-4)^2 = \frac{|h+3k-3|^2}{10}$$

$$\Rightarrow\ 10\left[(h-1)^2+(k-4)^2\right] = h^2+9k^2+9+6hk-18k-6h$$

$$\Rightarrow\ 10\left[h^2-2h+1+k^2-8k+16\right] = h^2+9k^2+9+6hk-18k-6h$$

$$\Rightarrow\ 10h^2-20h+10+10k^2-80k+160-h^2-9k^2-9-6hk+18k+6h = 0$$

$$\Rightarrow\ 9h^2+k^2-14h-62k-6hk+161 = 0 \ \ldots\ldots\ldots\ (iv)$$

From (iii)

$$h = 2k-12 \ \ldots\ldots\ (v)$$

Putting in (iv)

$$9(2k-12)^2+k^2-14(2k-12)-62k-6(2k-12)k+161 = 0$$

$$\Rightarrow\ 9(4k^2-48k+144)+k^2-28k+168-62k-12k^2+72k+161 = 0$$

$$\Rightarrow\ 36k^2-432k+1296+k^2-28k+168-62k-12k^2+72k+161 = 0$$

$$\Rightarrow\ 25k^2-450k+1625 = 0$$

$$\Rightarrow\ k^2-18k+65 = 0 \qquad \div\text{ing by } 25$$

$$\Rightarrow\ k^2-13k-5k+65 = 0 \qquad \Rightarrow\ k(k-13)-5(k-13) = 0$$

$$\Rightarrow\ (k-13)(k-5) = 0$$

$$\Rightarrow\ k=13 \quad \text{or} \quad k = 5$$

Putting in eq. (v)

$$h = 2(13)-12$$
$$= 26-12 = 14$$

Now from (i)

$$r = \sqrt{(h-1)^2+(k-4)^2}$$

$$\Rightarrow\ r = \sqrt{(14-1)^2+(13-4)^2}$$

$$= \sqrt{(13)^2+(9)^2} = \sqrt{169+81}$$

$$= \sqrt{250}$$

Now eq. of circle with centre $(14,13)$ and radius $\sqrt{170}$

$$(x-14)^2+(y-13)^2 = \left(\sqrt{250}\right)^2$$

$$\Rightarrow\ (x-14)^2+(y-13)^2 = 250$$

Putting in (v)

$$h = 2(5)-12$$
$$= 10-12 = -2$$

Now from (i)

$$r = \sqrt{(h-1)^2+(k-4)^2}$$

$$= \sqrt{(2-1)^2+(5-4)^2}$$

$$= \sqrt{1+1} = \sqrt{2}$$

Now eq. of circle with centre $(2,5)$ and radius $\sqrt{2}$

$$(x-2)^2+(y-5)^2 = \left(\sqrt{2}\right)^2$$

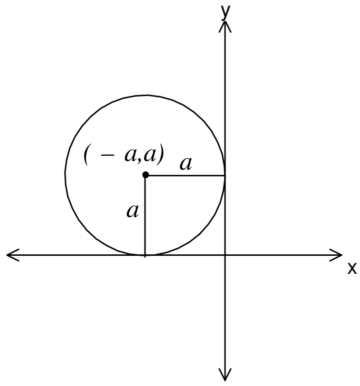
$$\Rightarrow\ (x-2)^2+(y-5)^2 = 2$$

Question # 5
Find an equation of a circle of radius a and lying in the second quadrant such that it is tangent to both the axes.

Solution

Radius of circle $= r = a$

$\because$ circle lies in second quadrant and touching both the axis therefore centre of circle is $(-a,a)$



So equation of circle

$$(x-(-a))^2+(y-a)^2 = (a)^2$$

$$\Rightarrow\ x^2-2ax+a^2+y^2-2ay+a^2-a^2 = 0$$

$$\Rightarrow\ x^2+y^2-2ax-2ay+a^2 = 0$$

Question # 6
Show that the lines $3x-2y=0$ and $2x+3y-13=0$ are tangents to the circle $x^2+y^2+6x-4y=0$

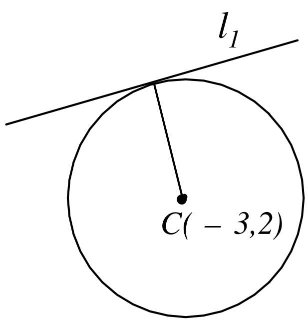
Solution

Suppose

$l_1:\ 3x-2y = 0$

$l_2:\ 2x+3y-13 = 0$

$S:\ x^2+y^2+6x-4y = 0$



From S

$2g = 6\ ,\ 2f = -4\ ,\ c = 0$

$\Rightarrow\ g = 3\ ,\ f = -2\ ,$

Centre $C(-g,-f) = C(-3,2)$

Radius $= r = \sqrt{g^2+f^2-c}$

$= \sqrt{(3)^2+(-2)^2-0}$

$= \sqrt{9+4=0} = \sqrt{13}$

Now to check l_1 is tangent to circle, we find

$$\perp\text{ar distance of } l_1 \text{ from centre } = \frac{|3(-3)-2(2)+0|}{\sqrt{(3)^2+(-2)^2}}$$

$$= \frac{|-9-4|}{\sqrt{9+4}} = \frac{|-13|}{\sqrt{13}} = \frac{13}{\sqrt{13}}$$

$$= \sqrt{13} = \text{radius of circle}$$

$\Rightarrow\ l_1$ is tangent to given circle.

Now to check l_2 is tangent to circle, let

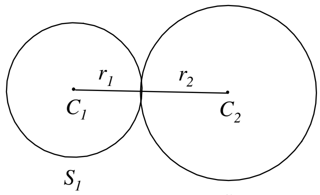
$$\perp\text{ar distance of } l_2 \text{ from centre } = \frac{|2(-3)+3(2)-13|}{\sqrt{(2)^2+(3)^2}}$$

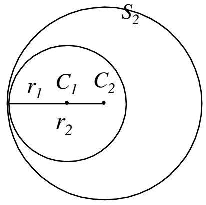
$$= \frac{|-6+6-13|}{\sqrt{4+9}} = \frac{|-13|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}} = \sqrt{13} = \text{Radius of circle}$$

$\Rightarrow$ l_2 is also tangent to given circle.

Circles touching each other externally or internally





Let C_1 be centre and r_1 be radius of circle S_1 and C_2 be centre and r_2 be radius of circle S_2.

Then they touch each other externally if

$$\left|\overline{C_1C_2}\right| = r_1 + r_2$$

And they touch each other internally if

$$\left|\overline{C_1C_2}\right| = |r_2 - r_1|$$

Question # 7

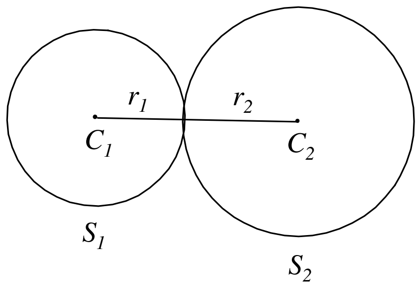
Show that the circles

$x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally.

Solution

Let $S_1 :\ x^2 + y^2 + 2x - 2y - 7 = 0$

$S_2 :\ x^2 + y^2 - 6x + 4y + 9 = 0$



For S_1:

$$2g = 2 \quad , \quad 2f = -2 \quad , \quad c = -7$$

$$\Rightarrow \ g = 1 \quad , \quad f = -1 \ ,$$

Let C_1 be centre and r_1 be radius of circle S_1, then

$$C_1(-g,-f) = C_1(-1,1)$$

$$\text{Radius} = r_1 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(1)^2 + (-1)^2 - (-7)} = \sqrt{1+1+7} = \sqrt{9} = 3$$

For S_2:

$$2g = -6 \quad , \quad 2f = 4 \quad , \quad c = 9$$

$$\Rightarrow \ g = -3 \quad , \quad f = 2$$

Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g,-f) = C_2(3,-2)$$

$$\text{Radius} = r_2 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(-3)^2 + (2)^2 - 9} = \sqrt{9+4-9} = \sqrt{4} = 2$$

Now circles touch each other externally if

$$|C_1C_2| = r_1 + r_2$$

$$\Rightarrow \sqrt{(3+1)^2 + (-2-1)^2} = 3 + 2$$

$$\Rightarrow \sqrt{16+9} = 5 \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$$

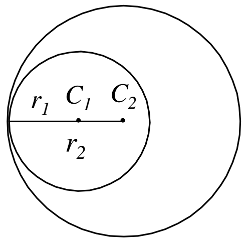
Hence both circles touch each other externally.

Question # 8

Show that the circles

$x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touches internally.

Solution



Suppose $S_1: x^2 + y^2 + 2x - 8 = 0$

$S_2: x^2 + y^2 - 6x + 6y - 46 = 0$

For S_1: $2g = 2$, $2f = 0$, $c = -8$

$\Rightarrow g = 1$, $f = 0$

Let C_1 be centre and r_1 be radius of circle S_1 then

$$C_1(-g,-f) = C(-1,0)$$

$$\text{Radius} = r_1 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(1)^2 + (0)^2 + 8} = \sqrt{9} = 3$$

For S_2: $2g = -6$, $2f = 6$, $c = -46$

$\Rightarrow g = -3$, $f = 3$

Let C_2 be centre and r_2 be radius of circle S_2 then

$$C_2(-g,-f) = C_2(3,-3)$$

$$\text{Radius} = r_2 = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(3)^2 + (-3)^2 - (-46)} = \sqrt{9+9+46} = \sqrt{64} = 8$$

Now circles touch each other internally if

$$\left|\overline{C_1C_2}\right| = |r_2 - r_1| \Rightarrow \sqrt{(3+1)^2 + (-3-0)^2} = |8-3|$$

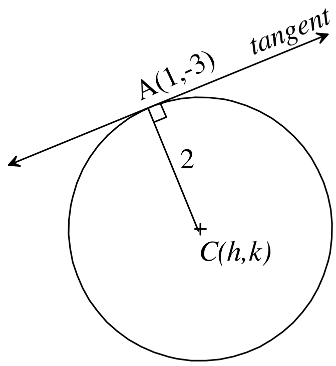
$$\Rightarrow \sqrt{16+9} = |5| \Rightarrow \sqrt{25} = 5 \Rightarrow 5 = 5$$

Hence circles are touching each other internally.

Question # 9

Find an equation of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at $A(1,-3)$.

Solution



Given: Radius $r = 2$,

Tangent: $x - y - 4 = 0$ at $A(1,-3)$

Suppose $C(h,k)$ be the centre then

$$|AC| = 2$$

$$\Rightarrow \sqrt{(h-1)^2 + (k+3)^2} = 2$$

On squaring

$$(h-1)^2 + (k+3)^2 = 4$$

$$\Rightarrow h^2 - 2h + 1 + k^2 + 6k + 9 - 4 = 0 \Rightarrow h^2 + k^2 - 2h + 6k + 6 = 0 \quad \text{........ (i)}$$

Now slope of radial line $AC = \dfrac{k+3}{h-1}$

Slope of line tangent $= -\dfrac{1}{-1} = 1$

Since radial line is $\perp$ar to tangent, therefore

(Slope of radial line) (Slope of tangent) $= -1$

$$\Rightarrow \left(\frac{k+3}{h-1}\right)(1) = -1$$

$$\Rightarrow\ k+3 = -(h-1) \quad \Rightarrow\ k = -h+1-3 \quad \Rightarrow\ k = -h-2 \ \ldots\ldots\text{(ii)}$$

Putting in (i) $h^2+(-h-2)^2-2h+6(-h-2)+6 = 0$

$$\Rightarrow\ h^2+h^2+4h+4-2h-6h-12+6 = 0 \Rightarrow\ 2h^2-4h-2 = 0 \quad \Rightarrow\ h^2-2h-1 = 0$$

$$\Rightarrow\ h = \frac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$

$$= \frac{2\pm\sqrt{4+4}}{2} = \frac{2\pm\sqrt{8}}{2}$$

$$= \frac{2\pm2\sqrt{2}}{2} = 1\pm\sqrt{2}$$

Putting $h=1+\sqrt{2}$ in (ii)

$k = -1-\sqrt{2}-2$

$\Rightarrow\ k = -3-\sqrt{2}$

Now equation of circle with centre $\left(1+\sqrt{2},-3-\sqrt{2}\right)$ and radius 2.

$$\left(x-(1+\sqrt{2})\right)^2-\left(y-(-3-\sqrt{2})\right)^2 = (2)^2$$

$$\Rightarrow\ \left(x-1-\sqrt{2}\right)^2-\left(y+3+\sqrt{2})\right)^2 = 4$$

Putting $h=1-\sqrt{2}$ in (ii)

$k = -1+\sqrt{2}-2 \Rightarrow\ k = -3+\sqrt{2}$

Now equation of circle with centre $\left(1-\sqrt{2},-3+\sqrt{2}\right)$ and radius 2.

$$\left(x-(1-\sqrt{2})\right)^2-\left(y-(-3+\sqrt{2})\right)^2 = (2)^2$$

$$\Rightarrow\ \left(x-1+\sqrt{2}\right)^2-\left(y+3-\sqrt{2})\right)^2 = 4$$

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Book: **Exercise 6.1 (Page 255)**
Calculus and Analytic Geometry Mathematic 12
Punjab Textbook Board, Lahore.

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