Linear Programming

Objective Function:

A function which is to be maximized or minimized is called an objective function.

Optical solution:

The feasible solution which maximizes or minimizes the objective function is called the optimal solution.

Theorem of linear programming

The theorem of linear programming states that the maximum and minimum values of the objective function occur at corner points of the feasible region.

Procedure for determining optimal solution.

(i) Graph the solution set of linear inequality constraints to determine feasible region.

(ii) Find the corner points of the feasible region.

(iii) Evaluate the objective function at each corner point to find the optimal solution.

Example: 1 Find the maximum and minimum values of the function defined as

$f(x,y) = 2x+3y$ subject to the constraints

$x-y \le 2$, $x+y \le 4$, $2x-y \le 6$, $x \ge 0$

Solution: Given that objective function is

$$f(x,y) = 2x+3y \rightarrow (I)$$

Given constraints are

$$x-y \le 2 \rightarrow (i)$$
$$x+y \le 4 \rightarrow (ii)$$
$$2x-y \le 6 \rightarrow (iii)$$
$$x \ge 0 \rightarrow (iv)$$

Associated equations of (i), (ii) and (iii) are

$$x-y = 2 \rightarrow (1)$$
$$x+y = 4 \rightarrow (2)$$
$$2x-y \le 6 \rightarrow (3)$$

For $y=0$, (1) $\Rightarrow x-0=2 \quad \therefore x=2$

for $x=0$, (1) $\Rightarrow 0-y=2 \quad \therefore y=-2$

∴ Line (1) cuts x-axis at (2, 0) and y-axis at (0, −2)

For $y = 0$, (2) ⇒ $x + 0 = 4$ ∴ $x = 4$

for $x = 0$, (2) ⇒ $0 + y = 4$ ∴ $y = 4$

∴ Line (2) cuts x-axis at (4, 0) and y-axis at (0, 4)

For $y = 0$, (3) ⇒ $2x - 0 = 6$ ∴ $x = 3$

for $x = 0$, (3) ⇒ $0 - y = 6$ ∴ $y = -6$

∴ Line (3) cuts x-axis at (3, 0) and y-axis at (0, −6).

We take origin (0, 0) as test point.

As (0, 0) satisfies inequalities (i), and (iii)

∴ The graphs of (i), and (iii) are the closed half planes on the origin side of (1), and (3). | As (0,0) does not satisfy (ii) ∴ Graph of (ii) is the closed half plane not on the origin side of (2).

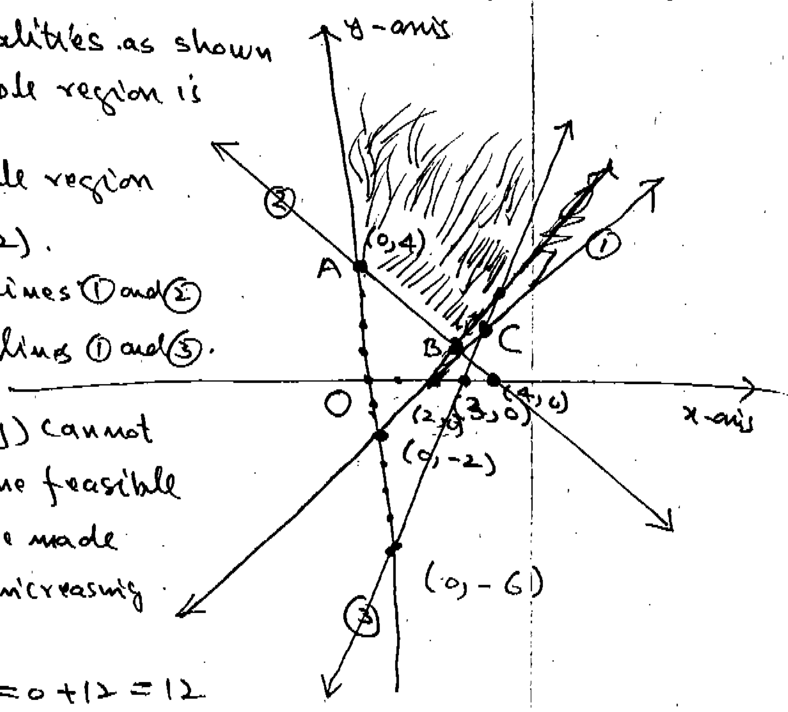
The graph of $x \geq 0$ is the closed right half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region of the system of linear inequalities as shown in the figure, and this feasible region is unbounded upwards.

The corner points of feasible region are A (0, 4), B(3, 1), C(4, 2).

B(3, 1) is point of intersection of lines (1) and (2)

C(4, 2) is point of intersection of lines (1) and (3).



Here objective function $f(x,y)$ cannot have maximum value in the feasible region because its value can be made larger than any number by increasing x and y.

Now $f(0,4) = 2(0) + 3(4) = 0 + 12 = 12$

$f(3,1) = 2(3) + 3(1) = 6 + 3 = 9$

$f(4,2) = 2(4) + 3(2) = 8 + 6 = 14$

Thus minimum value of $f(x,y)$ is at the corner point B(3, 1).

Note: If $f(x,y) = 2x + 2y$, then $f(0,4) = 2(0) + 2(4) = 0 + 8 = 8$

$f(3,1) = 2(3) + 2(1) = 6 + 2 = 8$

$f(4,2) = 2(4) + 2(2) = 8 + 4 = 12$

The minimum value of $f(x,y)$ is same at two corner points A(0, 4) and B(3, 1).

Further, we observe that the minimum value $2x + 2y$ at each point of the line segment AB is 8: because $f(x,y) = 2x + 2y$

$= 2x + 2(4 - x)$ ($\because x + y = 4$ ∴ $y = 4 - x$)

$= 2x + 8 - 2x = 8$.

Example: 2 Find the minimum and maximum values of f and ϕ defined by $f(x,y) = 4x+5y$, $\phi(x,y) = 4x+6y$ under the constraints

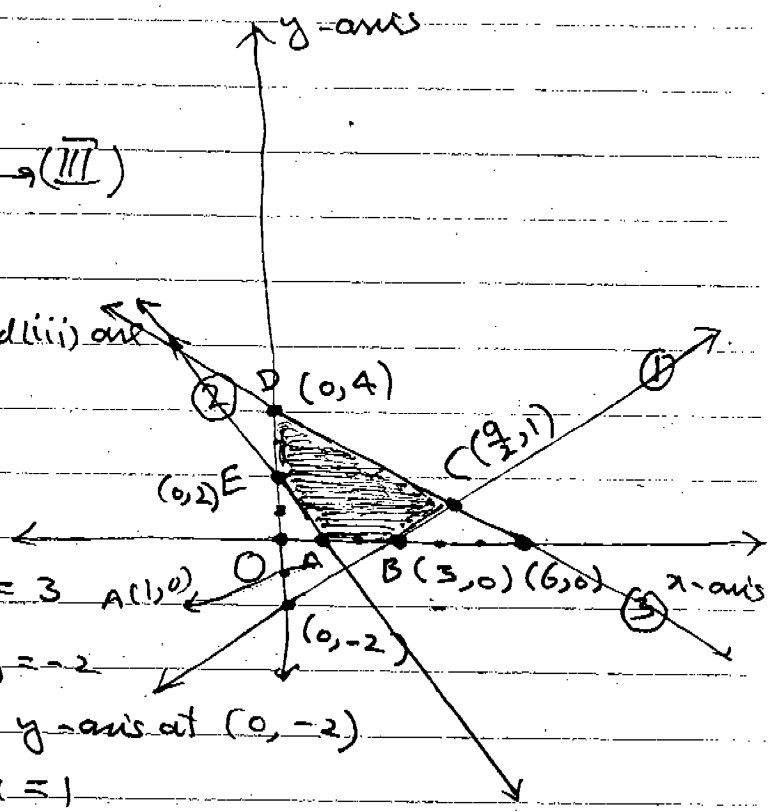
$$2x-3y \le 6,\ 2x+y \ge 2,\ 2x+3y \le 12,\ x \ge 0,\ y \ge 0.$$

Solution: Given that $f(x,y) = 4x+5y \rightarrow (\mathrm{I})$

$$\phi(x,y) = 4x+6y \rightarrow (\mathrm{II})$$

Given constraints are

$$\left.\begin{array}{l} 2x-3y \le 6 \rightarrow (i) \\ 2x+y \ge 2 \rightarrow (ii) \\ 2x+3y \le 12 \rightarrow (iii) \\ x \ge 0 \rightarrow (iv),\ y \ge 0 \rightarrow (v) \end{array}\right\} \rightarrow (\mathrm{III})$$



Associated equations of (i), (ii) and (iii) are

$$2x-3y = 6 \rightarrow (1)$$

$$2x+y = 2 \rightarrow (2)$$

$$2x+3y = 12 \rightarrow (3)$$

For $y=0$, (1) $\Rightarrow 2x-0 = 6 \quad \therefore x = 3$

for $x=0$, (1) $\Rightarrow 0-3y = 6 \quad \therefore y = -2$

$\therefore$ Line (1) cuts x-axis at $(3,0)$ and y-axis at $(0,-2)$

for $y=0$, (2) $\Rightarrow 2x+0 = 2 \quad \therefore x = 1$

for $x=0$, (2) $\Rightarrow 0+y = 2 \quad \therefore y = 2$

$\therefore$ Line (2) cuts x-axis at $(1,0)$ and y-axis at $(0,2)$

for $y=0$, (3) $\Rightarrow 2x+0 = 12 \quad \therefore x = 6$

for $x=0$, (3) $\Rightarrow 0+3y = 12 \quad \therefore y = 4$

$\therefore$ Line (3) cuts x-axis at $(6,0)$ and y-axis at $(0,4)$.

We take $(0,0)$ as test point.

As $(0,0)$ satisfies (i) and (iii)

$\therefore$ The graphs of (i) and (iii) are the closed half planes on the origin sides of (1) and (3).

As $(0,0)$ does not satisfy (2)

$\therefore$ The graph of (ii) is the closed half plane not on the origin side of (2)

The graph of $x \geq 0$ is the closed right half plane of the xy-plane.

and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of the graphs of (i), (ii), (iii), (iv) and (v) is the feasible region ABCDE of the given system of inequalities or constraints as shown in the figure by shading the region.

$A(1,0)$, $B(3,0)$, $C(\frac{9}{2},1)$, $D(0,4)$, $E(0,2)$ are corner points of feasible region.

$$f(x,y) = 4x+5y$$

$$\therefore f(1,0) = 4(1)+5(0) = 4+0 = 4$$

$$f(3,0) = 4(3)+5(0) = 12+0 = 12$$

$$f(\tfrac{9}{2},1) = 4(\tfrac{9}{2})+5(1) = 18+5 = 23$$

$$f(0,4) = 4(0)+5(4) = 0+20 = 20$$

$$f(0,2) = 4(0)+5(2) = 0+10 = 10$$

$\therefore$ Minimum value of $f(x,y)$ is 4 at the corner point $(1,0)$

Maximum value of $f(x,y)$ is 23 at the corner point $(\frac{9}{2},1)$.

Now $\phi(x,y) = 4x+6y$.

$$\therefore \phi(1,0) = 4(1)+6(0) = 4+0 = 4$$

$$\phi(3,0) = 4(3)+6(0) = 12+0 = 12$$

$$\phi(\tfrac{9}{2},1) = 4(\tfrac{9}{2})+6(1) = 18+6 = 24$$

$$\phi(0,4) = 4(0)+6(4) = 0+24 = 24$$

$$\phi(0,2) = 4(0)+6(2) = 0+12 = 12$$

$\therefore$ The minimum value of $\phi(x,y)$ is 4 at the corner point $(1,0)$.

The maximum value of $\phi(x,y)$ is 24 at the corner points $C(\frac{9}{2},1)$ and $D(0,4)$.

$\therefore \phi(x,y)$ has maximum value at all points of line segement CD.

Note: 1 Some times it may happen that each point of constraint line gives the optimal value of the objective function.

Note: 2 For different values of K, the equation $4x+5y=K$ represents lines parallel to the line $4x+5y=0$. For a certain admissible value of K, the intersection of $4x+5y=K$ with the feasible region gives feasible solutions for which the profit is K.

Linear Programming Problems

Example: 1 A farmer possesses 100 canals of land and want to grow corn and wheat. Cultivation of corn requires 3 hours per canal while cultivation of wheat requireds 2 hours per canal. Working hours cannot exceed 240. If he gets a profit of Rs. 20 per canal for corn and Rs. 15/- per canal for wheat, how many canals of each he should cultivate to maximize his profit.

Solution: Suppose that he cultivates x canals of corn and y canals of wheat.

Let $f(x,y)$ be the profit function.

$$\therefore f(x,y) = 20x+15y \longrightarrow (I)$$

constraints are

$$\left.\begin{array}{l} x+y \le 100 \rightarrow (i) \\ 3x+2y \le 240 \rightarrow (ii) \\ x \geq 0 \rightarrow (iii), \quad y \geq 0 \rightarrow (iv) \end{array}\right] \rightarrow (II)$$

Associated equations of (i) and (ii) are

$$x+y = 100 \longrightarrow (1)$$

$$3x+2y = 240 \longrightarrow (2)$$

For $y=0$, ① $\Rightarrow x+0=100 \quad \therefore x=100$

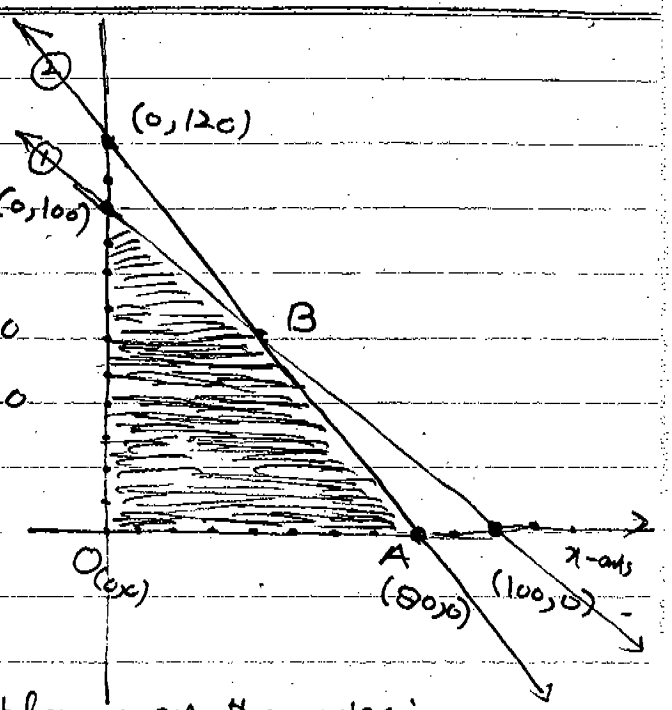
for $x=0$, ① $\Rightarrow 0+y=100 \quad \therefore y=100$

$\therefore$ The line ① cuts x-axis at $(100,0)$ and y-axis at $(0,100)$.

For $y=0$, ② $\Rightarrow 3x+0=240 \quad \therefore x=80$

for $x=0$, ② $\Rightarrow 0+2y=240 \quad \therefore y=120$

$\therefore$ Line ② cuts x-axis at $(80,0)$ and y-axis at $(0,120)$



We take $(0,0)$ as test point.

As $(0,0)$ satisfies (i) and (ii)

$\therefore$ Graphs of (i) and (ii) are closed half planes on the origin side of ① and ②.

The graph of $x \geq 0$ is the closed right half plane of xy-plane. and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region OABC as shown in the figure by shading the region.

$O(0,0)$, $A(80,0)$, $B(40,60)$, $C(0,100)$ are corner points of feasible region.

Now $f(x,y) = 20x+15y$

$\therefore f(0,0) = 20(0)+15(0) = 0+0 = 0$

$f(80,0) = 20(80)+15(0) = 1600+0 = 1600$

$f(40,60) = 20(40)+15(60) = 800+900 = 1700$

$f(0,100) = 20(0)+15(100) = 0+1500 = 1500$

$\therefore f(x,y)$ is maximum at the corner point $B(40,60)$ and maximum profit is Rs. 1700.

$\therefore$ The farmer get maximum profit if he cultivates 40 canals of corn and 60 canals of wheat.

Example: 2 A factory produces bicycles and motorcycles by using two machines A and B. Machine A has at most 120 hours available and machine B has a maximum of 144 hours available. Manufacturing a bicycle requires 5 hours in machine A and 4 hours in machine B while manufacturing of a motorcycle requires 4 hours in machine A and 8 hours in machine B. If he gets profit of Rs. 40 per bicycle and profit of Rs. 50 per motorcycle, how many bicycles and motorcycles should be manufactured to get maximum profit?

Solution. Let the number of bicycles to be manufactured be x and number of motorcycles to be manufactured be y

Let ~~The~~ $P(x,y)$ be the profit function

$$\therefore P(x,y) = 40x + 50y \longrightarrow (I)$$

and constraints are

$$\left.\begin{array}{l} 5x + 4y \leq 120 \rightarrow (i) \\ 4x + 8y \leq 144 \\ \text{or } 2x + 4y \leq 72 \rightarrow (ii) \\ x \geq 0 \rightarrow (iii), \quad y \geq 0 \rightarrow (iv) \end{array}\right] \rightarrow (II)$$

The associated equations of (i) and (ii) are

$$5x + 4y = 120 \rightarrow (1)$$

$$2x + 4y = 72 \rightarrow (2)$$

For $y = 0$, (1) $\Rightarrow 5x + 0 = 120 \quad \therefore x = 24$

for $x = 0$, (1) $\Rightarrow 0 + 4y = 120 \quad \therefore y = 30$

$\therefore$ Line (1) cuts x-axis at $(24, 0)$ and y-axis at $(0, 30)$

For $y = 0$, (2) $\Rightarrow 2x + 0 = 72 \quad \therefore x = 36$

for $x = 0$, (2) $\Rightarrow 0 + 4y = 72 \quad \therefore y = 18$

$\therefore$ Line (2) cuts x-axis at $(36, 0)$ and y-axis at $(0, 18)$.

We take (0,0) as test point.

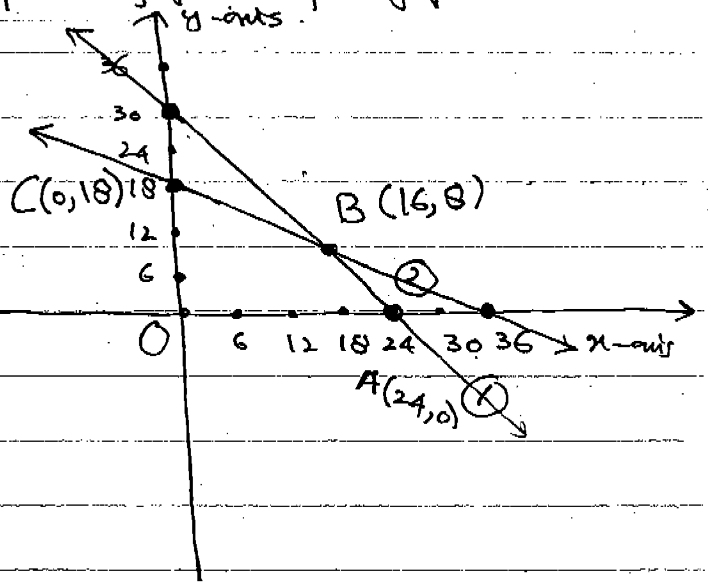
As (0,0) satisfies (i) and (ii)

∴ The graphs of (i) and (ii) are the closed half plane on the origin sides of (1) and (2)

The graph of $x \geq 0$ is the closed right half plane of xy-plane.

and graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii) (iii) and (iv) is the feasible region OABC of the system of linear inequalities as shown in the figure by shading the region.



B is the intersection of lines (1) and (2)

$$5x + 4y = 120 \rightarrow (1)$$
$$2x + 4y = 72 \rightarrow (2)$$

subtracting (2) from (1)

$$3x = 48 \quad \therefore x = 16$$

Putting value of x in (2) we get $2(16) + 4y = 72$ $\therefore 4y = 72 - 32 = 40$ $\therefore y = 10$

∴ Point B is (16,8)

O(0,0), A(24,0), B(16,8), C(0,18) are corner of feasible region.

Now $P(0,0) = 40(0) + 50(0) = 0 + 0 = 0$

$P(24,0) = 40(24) + 50(0) = 960 + 0 = 960$

$P(16,10) = 40(16) + 50(10) = 640 + 500 = 1140$

$P(0,18) = 40(0) + 50(18) = 0 + 900 = 900$

∴ P(x,y) is maximum at corner point B (~~16,8~~) (16,10)

and maximum profit is Rs. ~~1040~~ 1140.

Hence manufacturer should manufacture 16 bicycles and 10 motorcycles to get the maximum profit.

Exercise: 5.3

(1) objective function is

$f(x,y) = 2x + 5y \rightarrow (I)$

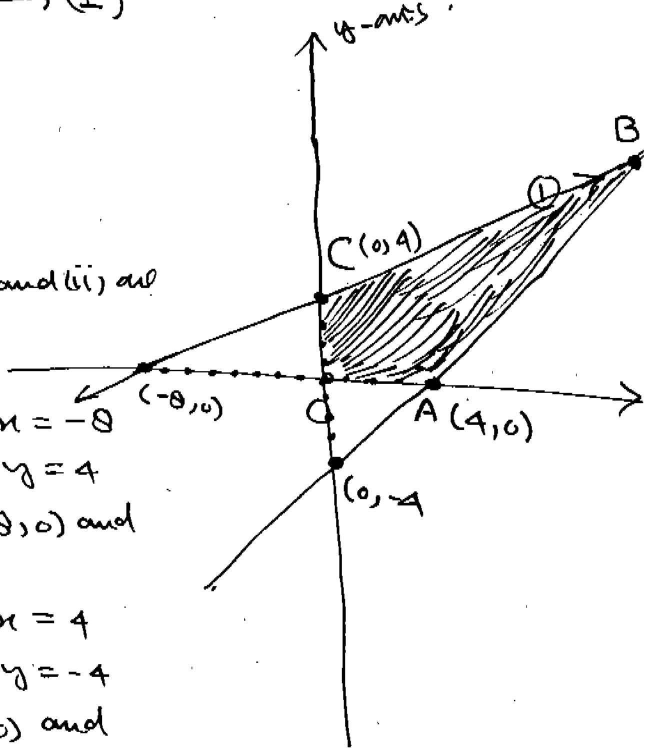
Given constraints are

$2y - x \le 8 \rightarrow (i)$

$x - y \le 4 \rightarrow (ii)$

$x \geq 0 \rightarrow (iii)$

$y \geq 0 \rightarrow (iv)$



Associated equations of (i) and (ii) are

$2y - x = 8 \rightarrow ①$

$x - y = 4 \rightarrow ②$

For $y=0$, ① $\Rightarrow 0 - x = 8 \quad \therefore x = -8$

for $x=0$, ① $\Rightarrow 2y - 0 = 8 \quad \therefore y = 4$

$\therefore$ Line ① cuts x-axis at $(-8,0)$ and y-axis at $(0,4)$.

For $y=0$, ② $\Rightarrow x - 0 = 4 \quad \therefore x = 4$

for $x=0$, ② $\Rightarrow 0 - y = 4 \quad \therefore y = -4$

$\therefore$ Line ② cuts x-axis at $(4,0)$ and y-axis at $(0,-4)$

We take origin $(0,0)$ as test point.

As $(0,0)$ satisfies both (i) and (ii)

$\therefore$ The graphs of (i) and (ii) are the closed half planes on the origin side of ① and ②.

The graph of $x \geq 0$ is the closed right half plane of xy-plane and graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region OABC as shown in the figure by shading the region.

The corner points of feasible region OABC are

$O(0,0)$, $A(4,0)$, $B(16,12)$, $C(0,4)$.

$f(0,0) = 2(0) + 5(0) = 0 + 0 = 0$

$f(4,0) = 2(4) + 5(0) = 8 + 0 = 8$

$f(16,12) = 2(16) + 5(12) = 32 + 60 = 92$

$f(0,4) = 2(0) + 5(4) = 0 + 20 = 20$

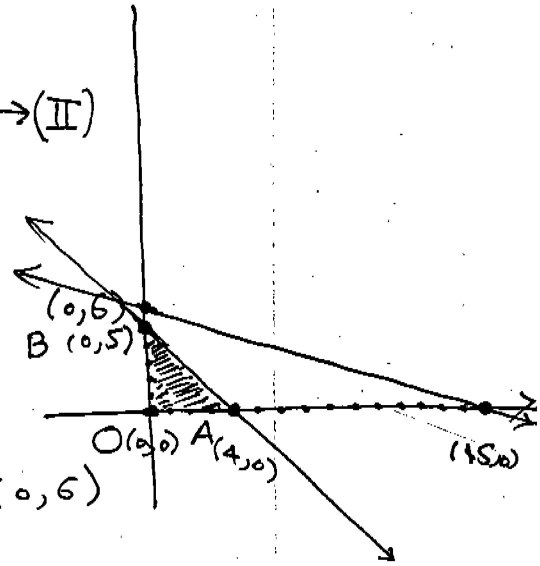
$\therefore$ Maximum value of $f(x,y)$ is at the corner point $B(16,12)$

and maximum value of $f(x,y) = 92$.

(2) Given that $f(x,y) = x + 3y \rightarrow (\text{I})$

Given constraints are

$$\left.\begin{array}{r} 2x+5y \le 30 \rightarrow (i) \\ 5x+4y \le 20 \rightarrow (ii) \\ x \ge 0 \rightarrow (iii),\ y \ge 0 \rightarrow (iv) \end{array}\right] \rightarrow (\text{II})$$



Associated equations of (i) and (ii) are

$2x+5y = 30 \longrightarrow (1)$

$5x+4y = 20 \longrightarrow (2)$

For $y=0$, $(1) \Rightarrow 2x+0 = 30 \quad \therefore x = 15$

for $x=0$, $(1) \Rightarrow 0+5y = 30 \quad \therefore y = 6$

$\therefore$ Line (1) cuts x-axis at $(15,0)$ and y-axis at $(0,6)$

for $y=0$, $(2) \Rightarrow 5x+0=20 \quad \therefore x = 4$

for $x=0$, $(2) \Rightarrow 0+4y=20 \quad \therefore y = 5$

$\therefore$ Line (2) cuts x-axis at $(4,0)$ and y-axis at $(0,5)$.

We take $(0,0)$ as the test point.

As $(0,0)$ satisfies (i) and (ii).

$\therefore$ Graphs of (i) and (ii) are the closed half planes on the origion side of (1) and (2).

The graph of $x \ge 0$ is the closed right half plane of xy-plane.

and graph of $y \ge 0$ is the closed upper half plane of xy-plane

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region OAB of the given system of inequalities, as shown in the figure by shading the region.

$O(0,0)$, $A(4,0)$, $B(0,5)$ are corner points of feasible region.

Now $f(0,0) = 0+3(0) = 0+0 = 0$

$f(4,0) = 4+3(0) = 4+0 = 4$

$f(0,5) = 0+3(5) = 0+15 = 15$.

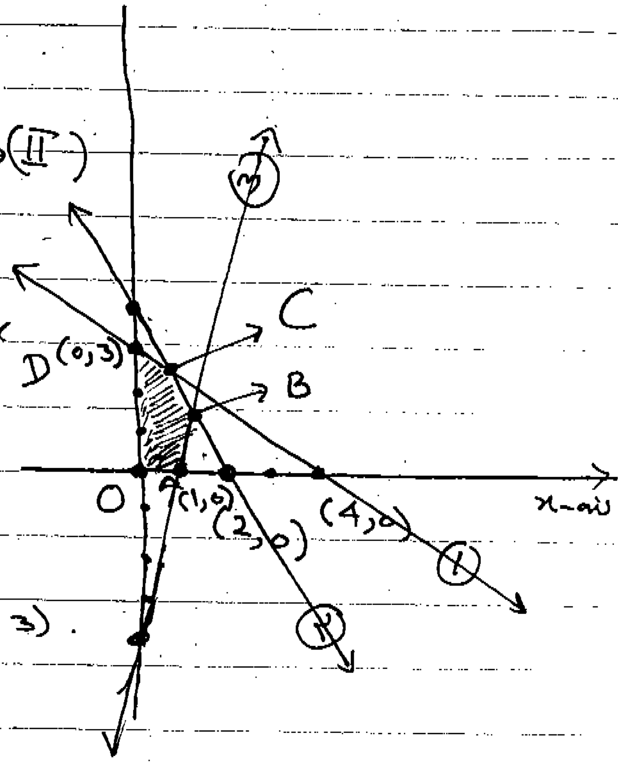
$\therefore f(x,y)$ has maximum value at corner point $(0,5)$

and maximum value of $f(x,y)$ is 15.

(3) Given that $z = 2x + 3y \rightarrow (I)$

Constraints are

$$\left.\begin{array}{l} 3x + 4y \le 12 \rightarrow (i) \\ \cancel{5x + 4y} \\ 2x + y \le 4 \rightarrow (ii) \\ 4x - y \le 4 \rightarrow (iii) \\ x \ge 0 \rightarrow (iv),\ y \ge 0 \rightarrow (v) \end{array}\right\} \rightarrow (II)$$



The associated equations of (i), (ii) and (iii) are

$3x + 4y = 12 \longrightarrow (1)$

$2x + y = 4 \longrightarrow (2)$

$4x - y = 4 \longrightarrow (3)$

For $y = 0$, (1) $\Rightarrow 3x + 0 = 12 \quad \therefore x = 4$

for $x = 0$, (1) $\Rightarrow 0 + 4y = 12 \quad \therefore y = 3$

$\therefore$ Line (1) cuts x-axis at $(4, 0)$ and y-axis at $(0, 3)$.

For $y = 0$, (2) $\Rightarrow 2x + 0 = 4 \quad \therefore x = 2$

for $x = 0$, (2) $\Rightarrow 0 + y = 4 \quad \therefore y = 4$

$\therefore$ Line (2) cuts x-axis at $(2, 0)$ and y-axis at $(0, 4)$

For $y = 0$, (3) $\Rightarrow 4x - 0 = 4 \quad \therefore x = 1$

for $x = 0$, (3) $\Rightarrow 0 - y = 4 \quad \therefore y = -4$

$\therefore$ Line (3) cuts x-axis at $(1, 0)$ and y-axis at $(0, -4)$

We take $(0, 0)$ as test point.

As $(0, 0)$ satisfies (i), (ii) and (iii).

$\therefore$ The graphs of (i), (ii), (iii) & are the closed half planes on the origin sides of (1), (2), (3).

The graph of $x \ge 0$ is the closed right half plane of xy-plane and the graph of $y \ge 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii), (iv) and (v) is the feasible region OABCD, of the given system of inequalities as shown in the figure by shading the region.

B is point of intersection of lines (2) and (3), C is the point of intersection of (1) and (2).

$2x+y=4 \rightarrow (2)$

$4x-y=4 \rightarrow (3)$

Adding (2) and (3), $6x=8 \quad \therefore x=\frac{8}{6}=\frac{4}{3}$

$\therefore (2) \Rightarrow y=4-2x=4-2\left(\frac{4}{3}\right)=4-\frac{8}{3}=\frac{12-8}{3}=\frac{4}{3}$.

$\therefore$ point B is $\left(\frac{4}{3}, \frac{4}{3}\right)$.

$3x+4y=12 \rightarrow (1)$

$2x+y=4 \rightarrow (2)$

$\cancel{(1) \Rightarrow 12x+16y=48}$

$(2) \Rightarrow 8x+4y=16 \rightarrow (2)'$

subtracting $(2)'$ from (1)

$-5x=-4 \quad \therefore x=\frac{4}{5}$

$\therefore (2) \Rightarrow y=4-2x=4-2\left(\frac{4}{5}\right)=\frac{20-8}{5}=\frac{12}{5}$.

$\therefore$ point C is $\left(\frac{4}{5}, \frac{12}{5}\right)$

$\therefore$ Corners of feasible region are $O(0,0)$, $A(1,0)$, $B\left(\frac{4}{3}, \frac{4}{3}\right)$

$C\left(\frac{4}{5}, \frac{12}{5}\right)$, $D(0,3)$

Now $z=2x+3y$

$\therefore z(0,0)=2(0)+3(0)=0+0=0$

$z(1,0)=2(1)+3(0)=2+0=2$

$z\left(\frac{4}{3}, \frac{4}{3}\right)=2\left(\frac{4}{3}\right)+3\left(\frac{4}{3}\right)=\frac{8}{3}+\frac{12}{3}=\frac{20}{3}$

$z\left(\frac{4}{5}, \frac{12}{5}\right)=2\left(\frac{4}{5}\right)+3\left(\frac{12}{5}\right)=\frac{8}{5}+\frac{36}{5}=\frac{44}{5}$

$z(0,3)=2(0)+3(3)=0+9=9$

$\therefore z$ is maximum at corner point $(0,3)$

and maximum value of z is 9.

4 Given that $z = 2x + y \longrightarrow$ (I)

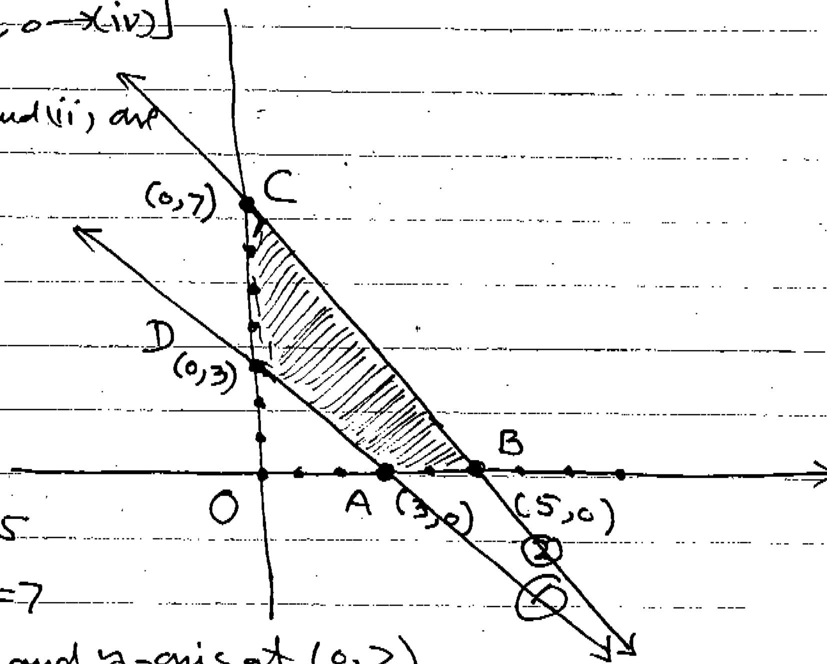
Given constraints are

$$\left.\begin{aligned} x + y &\geq 3 \rightarrow (i) \\ 7x + 5y &\leq 35 \rightarrow (ii) \\ x \geq 0 \rightarrow (iii)&,\ y \geq 0 \rightarrow (iv) \end{aligned}\right\} \rightarrow (II)$$

The associated equations of (i) and (ii) are

$x + y = 3 \rightarrow ①$

$7x + 5y = 35 \rightarrow ②$



For $y = 0$, ① $\Rightarrow x + 0 = 3 \therefore x = 3$

for $x = 0$, ① $\Rightarrow 0 + y = 3 \therefore y = 3$

$\therefore$ Line ① cuts x-axis at $(3,0)$ and y-axis at $(0,3)$.

for $y = 0$, ② $\Rightarrow 7x + 0 = 35 \therefore x = 5$

for $x = 0$, ② $\Rightarrow 0 + 5y = 35 \therefore y = 7$

$\therefore$ Line ② cuts x-axis at $(5,0)$ and y-axis at $(0,7)$

We take $(0,0)$ as test point

$(0,0)$ does not satisfy (i)

$\therefore$ The graph of (i) is the closed half plane not on the origin side of ①.

As $(0,0)$ satisfies (ii)

$\therefore$ The graph of (ii) is the closed half plane on the origin side of ①.

The graph of $x \geq 0$ is the closed right half plane of xy-plane and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region ABCD as shown in the figure by shading the region.

$A(3,0), B(5,0), C(0,7), D(0,3)$ are corners of feasible region.

$z = 2x + y$

$z\big|_A = 2(3) + 0 = 6 + 0 = 6$

$z\big|_B = 2(5) + 0 = 10 + 0 = 10$

$z\big|_C = 2(0) + 7 = 0 + 7 = 7$

$z\big|_D = 2(0) + 3 = 0 + 3 = 3$

$\therefore$ z is minimum at $D(0, 3)$ and minimum value of z is 3.

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(5) Given that $f(x, y) = 2x + 3y \longrightarrow (I)$

Given constraints are

$$\left.\begin{array}{r} 2x + y \leq 8 \rightarrow (i) \\ x + 2y \leq 14 \rightarrow (ii) \\ x \geq 0 \rightarrow (iii),\ y \geq 0 \rightarrow (iv) \end{array}\right] \longrightarrow (II)$$

Associated equations of (i) and (ii) are

$$2x + y = 8 \rightarrow (1)$$

$$x + 2y = 14 \rightarrow (2)$$

For $y = 0$, (1) $\Rightarrow 2x + 0 = 8 \quad \therefore x = 4$

for $x = 0$, (1) $\Rightarrow 0 + y = 8 \quad \therefore y = 8$

$\therefore$ The line (1) cuts x-axis at $(4, 0)$ and y-axis at $(0, 8)$.

For $y = 0$, (2) $\Rightarrow x + 0 = 14 \quad \therefore x = 14$

for $x = 0$, (2) $\Rightarrow 0 + 2y = 14 \quad \therefore y = 7$

$\therefore$ Line (2) cuts x-axis at $(14, 0)$ and y-axis at $(0, 7)$

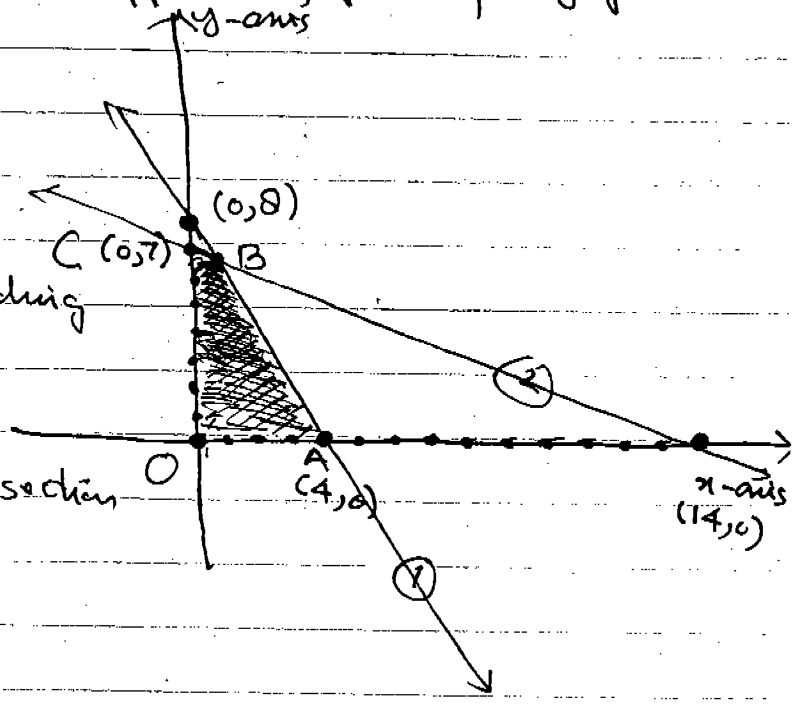
We take $(0, 0)$ as test point.

As $(0, 0)$ satisfies (i) and (ii)

∴ The graphs of (i) and (ii) are the closed half planes on the origin sides of ① and ②.

The graph of $x \geq 0$ is the closed right half plane of xy-plane and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region of the given system of linear inequalities as shown in the figure by shading the region.



Here OABC is the feasible region where B is point of intersection of line ① and ②

$① \Rightarrow 4x + 2y = 16 \rightarrow ①'$

$x + 2y = 14 \rightarrow ②$

subtracting ② from ①', we get $3x = 2 \quad \therefore x = \frac{2}{3}$

Putting value of x in ① $2\left(\frac{2}{3}\right) + y = 8$

$\therefore y = 8 - \frac{4}{3} = \frac{20}{3}$

$\therefore$ B is $\left(\frac{2}{3}, \frac{20}{3}\right)$

O(0,0), A(4,0), B$\left(\frac{2}{3}, \frac{20}{3}\right)$, C(0,7) are corners of feasible region.

$f(0,0) = 2(0) + 3(0) = 0 + 0 = 0$

$f(4,0) = 2(4) + 3(0) = 8 + 0 = 8$

$f\left(\frac{2}{3}, \frac{20}{3}\right) = 2\left(\frac{2}{3}\right) + 3\left(\frac{20}{3}\right) = \frac{4}{3} + \frac{60}{3} = \frac{64}{3}$

$f(0,7) = 2(0) + 3(7) = 21$

$\therefore f(x,y)$ is maximum at corner point $\left(\frac{2}{3}, \frac{20}{3}\right)$ and maximum value is $\frac{64}{3}$.

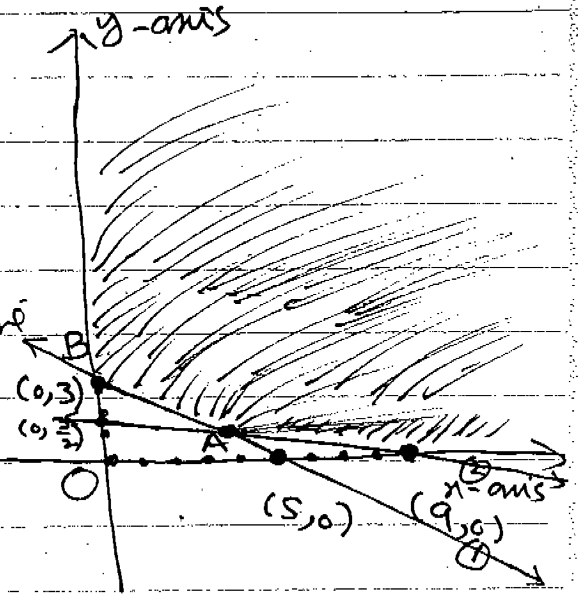
(6) Given that $z = 3x + y \longrightarrow (I)$

Given constraints are

$$3x + 5y \geq 15 \longrightarrow (i)$$

$$x + 6y \geq 9 \longrightarrow (ii)$$

$$x \geq 0 \longrightarrow (iii), \quad y \geq 0 \longrightarrow (iv)$$



The associated equations of (i) and (ii) are

$$3x + 5y = 15 \longrightarrow (1)$$

$$x + 6y = 9 \longrightarrow (2)$$

For $y = 0$, (1) $\Rightarrow 3x + 0 = 15 \quad \therefore x = 5$

for $x = 0$, (1) $\Rightarrow 0 + 5y = 15 \quad \therefore y = 3$

$\therefore$ Line (1) cuts x-axis at $(5, 0)$ and y-axis at $(0, 3)$.

For $y = 0$, (2) $\Rightarrow x + 0 = 9 \quad \therefore x = 9$

for $x = 0$, (2) $\Rightarrow 0 + 6y = 9 \quad \therefore y = \frac{9}{6} = \frac{3}{2}$

$\therefore$ Line (2) cuts x-axis at $(9, 0)$ and y-axis at $(0, \frac{3}{2})$.

We take $(0, 0)$ as the test point.

As $(0, 0)$ does not satisfy (i) and (ii)

$\therefore$ The graphs of (i) and (ii) are the closed half planes not on the origin sides of (1) and (2).

The graph of $x \geq 0$ is the closed right half plane of xy-plane.

and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region of the given system of inequalities as shown in the figure by shading the region.

Here feasible region is unbounded.

Its two corner points are A and B.

where A is point of intersection of lines (1) and (2).

$$3x + 5y = 15 \rightarrow (1)$$

$(2) \Rightarrow \quad 3x + 18y = 27 \rightarrow (2)'$

subtracting we have

$$-13y = -12 \quad \therefore y = \frac{12}{13}$$

Putting value of y in (2)

$$x + 6\left(\frac{12}{13}\right) = 9$$

$$\therefore x = 9 - \frac{72}{13} = \frac{117-72}{13} = \frac{45}{13}$$

$\therefore$ A is $\left(\frac{45}{13}, \frac{12}{13}\right)$

$\therefore$ $A\left(\frac{45}{13}, \frac{12}{13}\right)$ and $B(0,3)$ are corner points of feasible region

Now $Z\Big|_A = 3\left(\frac{45}{13}\right) + \frac{12}{13} = \frac{135}{13} + \frac{12}{13} = \frac{147}{13}$

$Z\Big|_B = 3(0) + 3 = 0 + 3 = 3$

$\therefore$ Z is minimum at corner point of $B(0,3)$.

and minimum value of Z is 3.

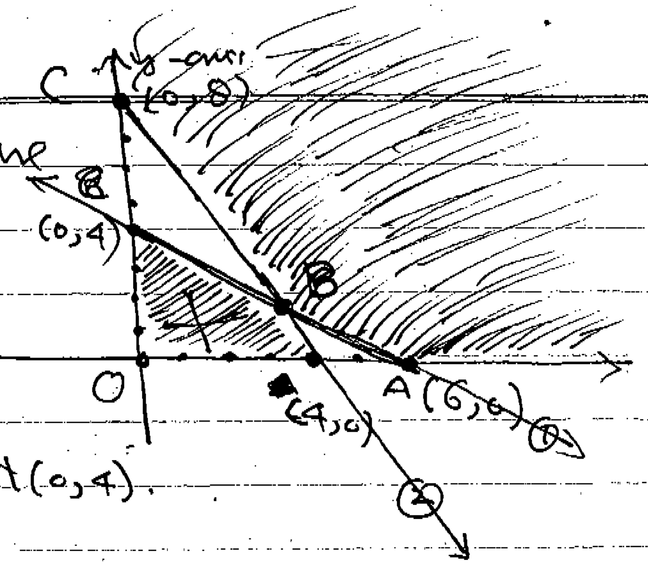
(7) suppose that x units of food X and y units of food Y should be fed to each animal each day.

Let Z be the cost of food for each animal each day.

$$\therefore Z = 25x + 30y \longrightarrow (\mathrm{I})$$

~~Given constraints are~~ According to given constraints we have

$$\left.\begin{array}{l} 2x + 3y \geq 12 \rightarrow (i) \\ 4x + 2y \geq 16 \\ \text{or } 2x + y \geq 8 \longrightarrow (ii) \\ x \geq 0 \rightarrow (iii),\ y \geq 0 \rightarrow (iv) \end{array}\right] \longrightarrow (\mathrm{II})$$



The associated equations of (i) and (ii) are

$2x + 3y = 12 \rightarrow (1)$

$2x + y = 8 \rightarrow (2)$

For $y = 0$, $(1) \Rightarrow 2x + 0 = 12 \therefore x = 6$

for $x = 0$, $(1) \Rightarrow 0 + 3y = 12 \therefore y = 4$

$\therefore$ Line (1) cuts x-axis at $(6,0)$ and y-axis at $(0,4)$.

For $y = 0$, $(2) \Rightarrow 2x + 0 = 8 \therefore x = 4$

for $x = 0$, $(2) \Rightarrow 0 + y = 8 \therefore y = 8$

$\therefore$ Line (2) cuts x-axis at $(4,0)$ and y-axis at $(0,8)$.

We take $(0,0)$ as the test point.

As $(0,0)$ does not satisfy (i) and (ii)

$\therefore$ The graphs of (i) and (ii) are the closed half planes not on the side of origin of lines (1) and (2).

The graph of $x \geq 0$ is the closed right half plane of xy-plane. and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region of the system of given inequalities as shown in the figure by shading the region.

Here feasible region is unbounded and ~~these~~ A, B, C are three corners of feasible region.

The corner B is intersection of lines (1) and (2)

subtracting (2) from (1), $2y = 4 \therefore y = 2$ $\quad \therefore (2) \Rightarrow 2x + 2 = 8$

$\therefore 2x = 8 - 2 = 6 \therefore x = 3$.

$\therefore$ B is $(3,2)$

$\therefore A(6,0), B(3,2)$ and $C(0,8)$ are three corners of feasible region.

$\therefore Z|_A = 25(6) + 0 = 150$

$Z|_B = 25(3) + 30(2) = 75 + 60 = 135$

$Z|_C = 25(0) + 30(8) = 0 + 240 = 240$

$\therefore$ Z is minimum at corner point $B(3,2)$

and minimum cost is Rs.135.

Hence 3 units of food X and 2 units of food Y should be fed to each animal at the smallest possible cost.

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(8) Suppose that dealer should purchase x fans and y ~~...~~ sewing machines.

Let $P(x,y)$ be the profit function.

$\therefore P(x,y) = 22x + 18y \longrightarrow$ (I)

According to given conditions we have

$x + y \le 20 \longrightarrow$ (i)

$360x + \cancel{240}y \le 5760$

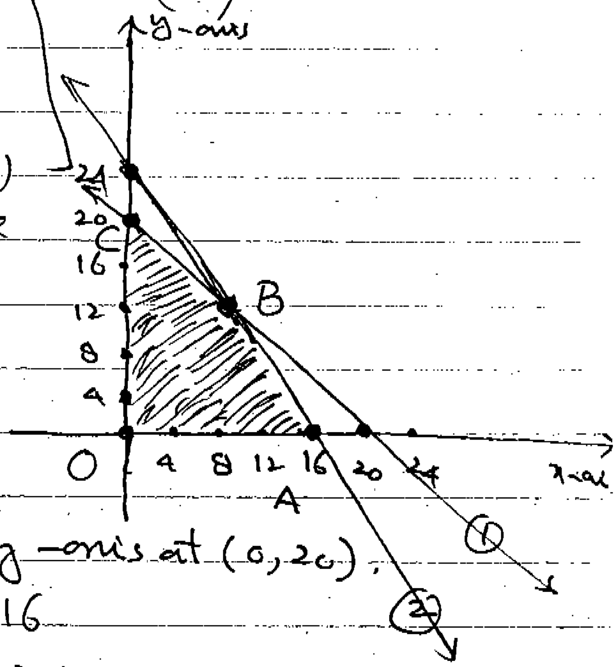
or $36x + 24y \le 576$

or $9x + 6y \le 144$

or $3x + 2y \le 48 \longrightarrow$ (ii)

$x \geq 0 \to$ (iii), $y \geq 0 \longrightarrow$ (iv)

$\longrightarrow$ (II)



The associated equations of (i) and (ii) are

$x + y = 20 \longrightarrow$ (1)

$3x + 2y = 48 \longrightarrow$ (2)

For $y = 0$, (1) $\Rightarrow x + 0 = 20 \quad \therefore x = 20$

for $x = 0$, (1) $\Rightarrow 0 + y = 20 \quad \therefore y = 20$

$\therefore$ Lines (1) cuts x-axis at $(20,0)$ and y-axis at $(0,20)$.

for $y = 0$, (2) $\Rightarrow 3x + 0 = 48 \quad \therefore x = 16$

for $x = 0$, (2) $\Rightarrow 0 + 2y = 48 \quad \therefore y = 24$

$\therefore$ Line (2) cuts x-axis at $(16,0)$ and y-axis at $(0,24)$.

We take $(0,0)$ as the test point.

As $(0,0)$ satisfies both (i) and (ii)

$\therefore$ The graphs of (i) and (ii) are the closed half planes on the origin sides of (1) and (2).

The graph of $x \geq 0$ is the closed right half plane of xy-plane and the graph of $y \geq 0$ is the closed upper half plane of xy-plane.

The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region OABC of the system of linear inequalities as shown in the figure by shading the region.

B is point of intersection of lines (1) and (2).

$$3x + 2y = 48 \rightarrow (2)$$

$$(1) \Rightarrow \quad 2x + 2y = 40 \rightarrow (1)'$$

subtracting (1)' from (2) we get

$x = 8$ $\quad \therefore (1) \Rightarrow 8 + y = 20 \quad \therefore y = 20 - 8 = 12$

$\therefore$ $O(0,0)$, $A(16,0)$, $B(8,12)$, $C(0,20)$ are corners of feasible region.

Now $P(0,0) = 22(0) + 18(0) = 0 + 0 = 0$

$P(16,0) = 22(16) + 18(0) = 352 + 0 = 352$

$P(8,12) = 22(8) + 18(12) = 176 + 216 = 392$

$P(0,20) = 22(0) + 18(20) = 360$

$\therefore P(x,y)$ is maximum at the corner point $B(8,12)$ and maximum profit is Rs. 392.

$\therefore$ The dealer should purchase 8 fans and 12 sewing machines to get the maximum profit.

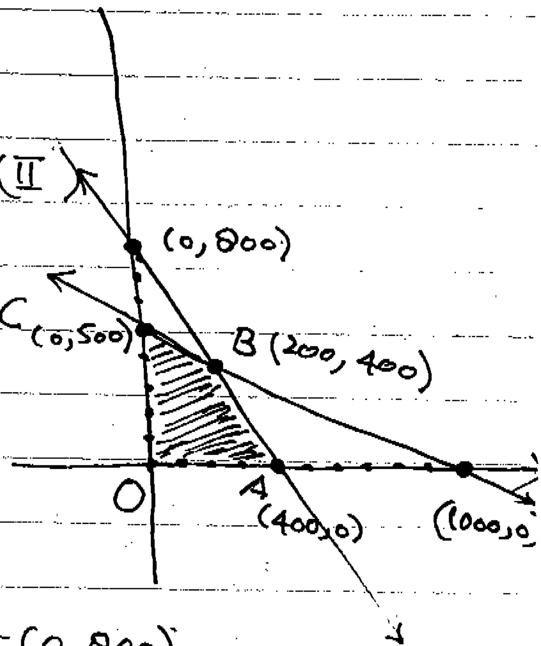
(9) Suppose that machine produces x units of product A and y units of product B.

Let $P(x,y)$ be the profit function.

$\therefore P(x,y) = 30x + 20y \rightarrow (I)$

According to given conditions

$$\left.\begin{array}{l} 2x + y \leq 800 \rightarrow (i) \\ x + 2y \leq 1000 \rightarrow (ii) \\ x \geq 0 \rightarrow (iii),\ y \geq 0 \rightarrow (iv) \end{array}\right\} \rightarrow (II)$$



Associated equations of (i) and (ii) are

$$2x + y = 800 \rightarrow (1)$$

$$x + 2y = 1000 \rightarrow (2)$$

For $y = 0$, (1) $\Rightarrow$ $2x + 0 = 800$ $\therefore x = 400$

for $x = 0$, (1) $\Rightarrow$ $0 + y = 800$ $\therefore y = 800$

$\therefore$ Line (1) cuts x-axis at $(400, 0)$ and y-axis at $(0, 800)$

For $y = 0$, (2) $\Rightarrow$ $x + 0 = 1000$ $\therefore x = 1000$

for $x = 0$, (2) $\Rightarrow$ $0 + 2y = 1000$ $\therefore y = 500$

$\therefore$ Line (2) cuts x-axis at $(1000, 0)$ and y-axis at $(0, 500)$

We take $(0,0)$ as test point.

As $(0,0)$ satisfies (i) and (ii)

$\therefore$ The graphs of (i) and (ii) are closed half planes on the origin side of (1) and (2).

The graph of $x \geq 0$ is the closed right half plane of xy-plane. and the graph of $y \geq 0$ is the closed upper half plane of xy-plane. The intersection of graphs of (i), (ii), (iii) and (iv) is the feasible region OABC of the system of linear inequalities as shown in the figure by shading the region.

The point B is the intersection of lines (1) and (2).

$$(2) \Rightarrow \quad 2x + 4y = 2000 \longrightarrow (2)'$$

$$2x + y = 800 \longrightarrow (1)$$

subtracting (1) from (2)', we get

$$3y = 1200 \quad \therefore y = 400$$

$$\therefore (1) \Rightarrow 2x + 400 = 800$$

$$\therefore 2x = 800 - 400$$

$$2x = 400 \quad \therefore x = 200$$

$\therefore$ Point B is $(200, 400)$.

$O(0,0)$, $A(400,0)$, $B(200,400)$, $C(0,500)$ are corners of feasible region.

Now $\quad P(0,0) = 30(0) + 20(0) = 0 + 0 = 0$

$$P(400,0) = 30(400) + 20(0) = 12000 + 0 = 12000$$

$$P(200,400) = 30(200) + 20(400) = 6000 + 8000 = 14000$$

$$P(0,500) = 30(0) + 20(500) = 0 + 10000 = 10000$$

$\therefore P(x,y)$ is maximum at the corner point $B(200,400)$ and maximum profit is Rs. 14000.

$\therefore$ The machine should produce 200 units of product A and 400 units of product B to get maximum profit.