Homogenous 2nd Degree Equation
Every homogenous second degree equation
$$ax^2 + 2hxy + by^2 = 0$$
represents straight lines through the origin.

Consider the equations are $y = m_1 x$ and $y = m_2 x$
$\Rightarrow \; m_1 x - y = 0$ and $m_2 x - y = 0$

Taking product
$$(m_1 x - y)(m_2 x - y) = 0$$
$$\Rightarrow \; m_1 m_2 x^2 - m_1 xy - m_2 xy + y^2 = 0$$
$$\Rightarrow \; m_1 m_2 x^2 - (m_1 + m_2) xy + y^2 = 0 \; \ldots\ldots\ldots \text{(i)}$$

Also we have
$$ax^2 + 2hxy + by^2 = 0$$
$$\Rightarrow \; \frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0 \qquad \div\text{ing by } b$$
$$\Rightarrow \; \frac{a}{b}x^2 - \left(-\frac{2h}{b}\right)xy + y^2 = 0$$

Comparing it with (i), we have
$$\boxed{m_1 m_2 = \frac{a}{b}} \quad \text{and} \quad \boxed{m_1 + m_2 = -\frac{2h}{b}}$$

Let θ be the angles between the lines then
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$
$$= \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1 m_2} = \frac{\sqrt{m_1^2 + m_2^2 - 2m_1 m_2}}{1 + m_1 m_1}$$
$$= \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 - 4m_1 m_2}}{1 + m_1 m_1} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_1}$$
$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{b + a}{b}}$$
$$\Rightarrow \; \tan\theta = \frac{\sqrt{4(h^2 - ab)}}{b + a} \quad \Rightarrow \quad \boxed{\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}}$$

Find the lines represented by each of the following and also find measure of the angle between them (problem 1-6)

Question # 1
$10x^2 - 23xy - 5y^2 = 0$
Solution

$$10x^2 - 23xy - 5y^2 = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$
$$\Rightarrow \quad 10x^2 - 25xy + 2xy - 5y^2 = 0$$
$$\Rightarrow \quad 5x(2x-5y) + y(2x-5y) = 0 \quad \Rightarrow \quad (2x-5y)(5x+y) = 0$$
$$\Rightarrow \quad 2x - 5y = 0 \quad \text{and} \quad 5x + y = 0$$

are the required lines.

Comparing eq. (i) with
$$ax^2 + 2hxy + by^2 = 0$$
So $a = 10$, $2h = -23 \Rightarrow h = -\dfrac{23}{2}$, $b = -5$

Let θ be angle between lines then
$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$
$$= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - (10)(-5)}}{10-5} = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$$
$$= \frac{2\sqrt{729/4}}{5} = \frac{2\left(27/2\right)}{5} = \frac{27}{5}$$
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{27}{5}\right) = 79^\circ 31'$$

Hence acute angle between the lines $= 79^\circ 31'$

Question # 2
$3x^2 + 7xy + 2y^2 = 0$
Solution *Do yourself as above*

Question # 3
$9x^2 + 24xy + 16y^2 = 0$
Solution *Do yourself as above*

Question # 4
$2x^2 + 3xy - 5y^2 = 0$

Solution $2x^2 + 3xy - 5y^2 = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$
$$\Rightarrow \quad 2x^2 + 5xy - 2xy - 5y^2 = 0$$
$$\Rightarrow \quad x(2x+5y) - y(2x+5y) = 0$$
$$\Rightarrow \quad (2x+5y)(x-y) = 0$$
$$\Rightarrow \quad 2x + 5y = 0 \quad \text{and} \quad x - y = 0$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \quad a = 2 \quad , \quad 2h = 3 \Rightarrow h = \frac{3}{2} \quad , \quad b = -5$$

Let θ be angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2-5} = \frac{2\sqrt{\frac{9}{4}+10}}{-3}$$

$$= -\frac{2\sqrt{49/4}}{3} = -\frac{2\left(7/2\right)}{3} = -\frac{7}{3}$$

$$\Rightarrow \quad -\tan\theta = \frac{7}{3}$$

$$\Rightarrow \quad \tan(180-\theta) = \frac{7}{3} \qquad \because \tan(180-\theta) = -\tan\theta$$

$$\Rightarrow \quad 180-\theta = \tan^{-1}\left(\frac{7}{3}\right) \qquad \Rightarrow \quad 180-\theta = 66^\circ 48'$$

$$\Rightarrow \quad \theta = 180 - 66^\circ 48' = 113^\circ 12'$$

Hence acute angle between the lines $= 180 - 113^\circ 12' = 66^\circ 48'$

Question # 5

$$6x^2 - 19xy + 15y^2 = 0$$

Solution

Do yourself as above

Question # 6

$$x^2 - 2xy\sec\alpha + y^2 = 0$$

Solution

$$x^2 - 2xy\sec\alpha + y^2 = 0 \quad \ldots\ldots\ldots \text{(i)}$$

$\div$ing by y^2

$$\frac{x^2}{y^2} - \frac{2xy\sec\alpha}{y^2} + \frac{y^2}{y^2} = 0$$

$$\Rightarrow \quad \left(\frac{x}{y}\right)^2 - 2\sec\alpha\left(\frac{x}{y}\right) + 1 = 0$$

This is quadric equation in $\frac{x}{y}$ with $a = 1$, $b = -2\sec\alpha$, $c = 1$

$$\frac{x}{y} = \frac{2\sec\alpha \pm \sqrt{(-2\sec\alpha)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\sec\alpha \pm \sqrt{4\sec^2\alpha - 4}}{2(1)} \quad = \frac{2\sec\alpha \pm \sqrt{4\left(\sec^2\alpha - 1\right)}}{2}$$

$$= \frac{2\sec\alpha \pm \sqrt{4\tan^2\alpha}}{2} \qquad \because\ 1+\tan^2\alpha = \sec^2\alpha$$

$$= \frac{2\sec\alpha \pm 2\tan\alpha}{2}$$

$$\Rightarrow \frac{x}{y} = \sec\alpha \pm \tan\alpha$$

$$= \frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha} = \frac{1 \pm \sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{x}{y} = \frac{1+\sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{x}{y} = \frac{1-\sin\alpha}{\cos\alpha}$$

$$\Rightarrow x\cos\alpha = \left(1+\sin\alpha\right)y \quad \text{and} \quad x\cos\alpha = \left(1-\sin\alpha\right)y$$

$$\Rightarrow x\cos\alpha - \left(1+\sin\alpha\right)y = 0 \quad \text{and} \quad x\cos\alpha - \left(1-\sin\alpha\right)y = 0$$

These are required equations of lines.
Now comparing (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$a = 1 \ , \quad 2h = -2\sec\alpha \ \Rightarrow\ h = -\sec\alpha \quad , \quad b = 1\text{b}$$

If θ is angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1+1} = \frac{2\sqrt{\sec^2\alpha - 1}}{2} = \sqrt{\tan^2\alpha}$$

$$\Rightarrow \tan\theta = \tan\alpha \qquad \Rightarrow \theta = \alpha.$$

Note:
If one wish to get the answer similar to given at the end of textbook, then follow the solution as follows after getting:

$$x\cos\alpha - \left(1+\sin\alpha\right)y = 0 \quad \text{and} \quad x\cos\alpha - \left(1-\sin\alpha\right)y = 0$$

Multiplying equation at left with $\frac{1-\sin\alpha}{\cos\alpha}$ and equation at right with $\frac{1+\sin\alpha}{\cos\alpha}$ to get

$$x\left(1-\sin\alpha\right) - \frac{\left(1-\sin^2\alpha\right)}{\cos\alpha}y = 0 \quad \text{and} \quad x\left(1+\sin\alpha\right) - \frac{\left(1-\sin^2\alpha\right)}{\cos\alpha}y = 0$$

$$\Rightarrow x\left(1-\sin\alpha\right) - y\cos\alpha = 0 \quad \text{and} \quad x\left(1+\sin\alpha\right) - y\cos\alpha = 0$$

Question # 7
Find a joint equation of the lines through the origin and perpendicular to the lines:
$x^2 - 2xy\tan\alpha - y^2 = 0$

Solution Given: $x^2 - 2xy\tan\alpha - y^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b} = -\frac{-2\tan\alpha}{-1}$$
$$\Rightarrow \quad m_1 + m_2 = -2\tan\alpha$$

$a = 1$,
$2h = -2\tan\alpha$
$\Rightarrow \; h = -\tan\alpha$
$b = -1$

$$\& \qquad m_1 m_2 = \frac{a}{b} = \frac{1}{-1} \quad \Rightarrow \quad m_1 m_2 = -1$$

Now slopes of lines $\perp$ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \qquad \text{(Passing through origin)}$$
$$\Rightarrow \quad m_1 y = -x \quad \& \quad m_2 y = -x$$
$$\Rightarrow \quad x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$
$$\Rightarrow \quad x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$
$$\Rightarrow \quad x^2 + (-2\tan\alpha)xy + (-1)y^2 = 0$$
$$\Rightarrow \quad x^2 - 2xy\tan\alpha - y^2 = 0$$

Question # 8

Find a joint equation of the lines through the origin and perpendicular to the lines:
$ax^2 + 2hxy + by^2 = 0$

Solution

Given: $\quad ax^2 + 2hxy + by^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b} \quad \& \quad m_1 m_2 = \frac{a}{b}$$

Now slopes of lines $\perp$ar to given lines are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \qquad \text{(Passing through origin)}$$
$$\Rightarrow \quad m_1 y = -x \quad \& \quad m_2 y = -x$$
$$\Rightarrow \quad x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$
$$\Rightarrow \quad x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$
$$\Rightarrow \quad x^2 + \left(-\frac{2h}{b}\right)xy + \left(\frac{a}{b}\right)y^2 = 0$$
$$\Rightarrow \quad bx^2 - 2hxy + ay^2 = 0$$

Question # 9

Find the area of the region bounded by:

$$10x^2 - xy - 21y^2 = 0 \quad \text{and} \quad x + y + 1 = 0$$

Solution

$$10x^2 - xy - 21y^2 = 0 \quad , \quad x + y + 1 = 0$$

$$\Rightarrow 10x^2 - 15xy + 14xy - 21y^2 = 0$$

$$\Rightarrow 5x(2x - 3y) + 7y(2x - 3y) = 0$$

$$\Rightarrow (2x - 3y)(5x + 7y) = 0$$

$$\Rightarrow 2x - 3y = 0 \quad \text{or} \quad 5x + 7y = 0$$

So we have equation of lines

$l_1: \ 2x - 3y = 0$ ………. (i)

$l_2: \ 5x + 7y = 0$ ……… (ii)

$l_3: \ x + y + 1 = 0$ ……… (iii)

Now do yourself as Q # 14 (Ex. 4.4)

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