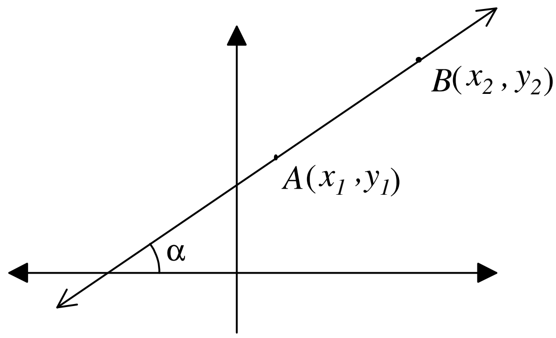
Inclination of a Line:

The angle α $(0^\circ \le \alpha < 180^\circ)$ measure anti-clockwise from positive $x-$axis to the straight line l is called *inclination* of a line l.



Slope or Gradient of Line

The slope m of the line l is defined by:

$$m = \tan\alpha$$

If $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two distinct points on the line l then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

See proof on book at page: 191

Note: l is horizontal, iff $m = 0$ $(\because \alpha = 0^\circ)$

l is vertical, iff $m = \infty$ i.e. m is not defined. $(\because \alpha = 90^\circ)$

If slope of AB = slope of BC, then the points A, B and C are collinear i.e. lie on the same line.

Theorem

The two lines l_1 and l_2 with respective slopes m_1 and m_2 are

(i) Parallel iff $m_1 = m_2$

(ii) Perpendicular iff $m_1 m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$

Question # 1

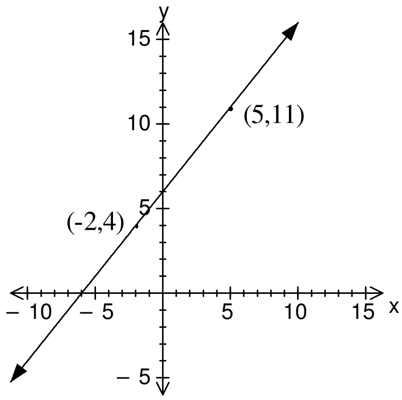
Find the slope and inclination of the line joining the points:

(i) $(-2,4)$; $(5,11)$ (ii) $(3,-2)$; $(2,7)$

(iii) $(4,6)$; $(4,8)$

Solution

(i) $(-2,4)$; $(5,11)$



Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{11-4}{5+2} = \dfrac{7}{7} = 1$

Since $\tan\alpha = m = 1$

$\Rightarrow \alpha = \tan^{-1}(1) = 45^\circ$

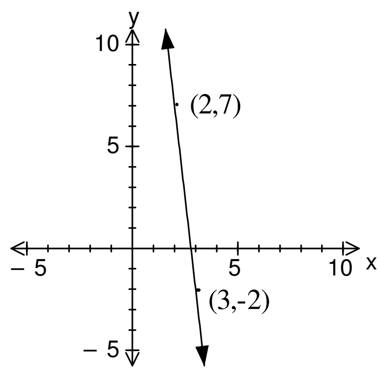
(ii) $(3,-2)$; $(2,7)$

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7+2}{2-3} = \dfrac{9}{-1} = -9$

Since $\tan\alpha = m = -9$

$\Rightarrow -\tan\alpha = 9 \Rightarrow \tan(180-\alpha) = 9$

$\Rightarrow 180-\alpha = \tan^{-1}(9)$



$\Rightarrow\ 180-\alpha = 83^\circ 40'$

$\Rightarrow\ \alpha = 180 - 83^\circ 40' = 96^\circ 20'$

(ii) $(4,6)$; $(4,8)$

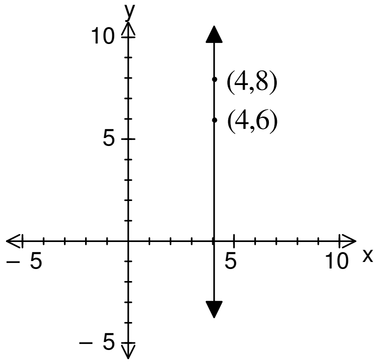
Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{8-6}{4-4} = \dfrac{2}{0} = \infty$

Since $\tan\alpha = m = \infty$

$\Rightarrow\ \alpha = \tan^{-1}(\infty)$

$= 90^\circ$



Question # 2

In the triangle $A(8,6)$, $B(-4,2)$ and $C(-2,-6)$, find the slope of

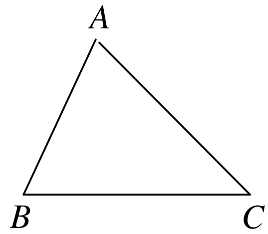
(i) each side of the triangle (ii) each median of the triangle

(iii) each altitude of the triangle

Solution

Since $A(8,6)$, $B(-4,2)$ and $C(-2,-6)$ are vertices of triangle therefore

(i) Slope of side $AB = \dfrac{2-6}{-4-8} = \dfrac{-4}{-12} = \dfrac{1}{3}$



Slope of side $BC = \dfrac{-6-2}{-2+4} = \dfrac{-8}{2} = -4$

Slope of side $CA = \dfrac{6+6}{8+2} = \dfrac{12}{10} = \dfrac{6}{5}$

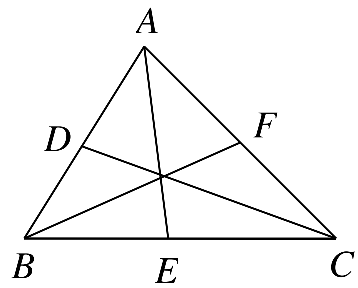
(ii) Let D, E and F are midpoints of sides AB, BC and CA respectively.

Then

Coordinate of $D = \left(\dfrac{8-4}{2}, \dfrac{6+2}{2}\right) = \left(\dfrac{4}{2}, \dfrac{8}{2}\right) = (2,4)$

Coordinate of $E = \left(\dfrac{-4-2}{2}, \dfrac{2-6}{2}\right) = \left(\dfrac{-6}{2}, \dfrac{-4}{2}\right) = (-3,-2)$

Coordinate of $F = \left(\dfrac{-2+8}{2}, \dfrac{-6+6}{2}\right) = \left(\dfrac{6}{2}, \dfrac{0}{2}\right) = (3,0)$



Hence Slope of median $AE = \dfrac{-2-6}{-3-8} = \dfrac{-8}{-11} = \dfrac{8}{11}$

Slope of median $BF = \dfrac{0-2}{3+4} = \dfrac{-2}{7}$

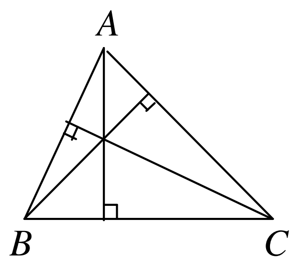
Slope of median $CD = \dfrac{4+6}{2+2} = \dfrac{10}{4} = \dfrac{5}{2}$

(iii) Since altitudes are perpendicular to the sides of a triangle therefore

Slope of altitude from vertex $A = \dfrac{-1}{\text{slope of side } BC} = \dfrac{-1}{-4} = \dfrac{1}{4}$

Slope of altitude from vertex $B = \dfrac{-1}{\text{slope of side } AC} = \dfrac{-1}{6/5} = -\dfrac{5}{6}$

Slope of altitude from vertex $C = \dfrac{-1}{\text{slope of side } AB} = \dfrac{-1}{1/3} = -3$



Question # 3

By means of slopes, show that the following points lie in the same line:

(a) $(-1,-3)$; $(1,5)$; $(2,9)$ (b) $(4,-5);(7,5);(10,15)$

(c) $(-4,6);(3,8);(10,10)$ (d) $(a,2b);(c,a+b);(2c-a,2a)$

Solution

(a) Let $A(-1,-3)$, $B(1,5)$ and $C(2,9)$ be given points

$$\text{Slope of } AB = \frac{5+3}{1+1} = \frac{8}{2} = 4$$

$$\text{Slope of } BC = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

Since slope of $AB =$ slope of BC

Therefore A, B and C lie on the same line.

(b) *Do yourself as above*

(c) *Do yourself as above*

(d) Let $A(a,2b), B(c,a+b)$ and $C(2c-a,2a)$ be given points.

$$\text{Slope of } AB = \frac{(a+b)-2b}{c-a} = \frac{a-b}{c-a}$$

$$\text{Slope of } BC = \frac{2a-(a+b)}{(2c-a)-c} = \frac{2a-a-b}{2c-a-c} = \frac{a-b}{c-a}$$

Since slope of $AB =$ slope of BC

Therefore A, B and C lie on the same line.

Question # 4

Find k so that the line joining $A(7,3)$; $B(k,-6)$ and the line joining $C(-4,5)$; $D(-6,4)$ are (i) parallel (ii) perpendicular.

Solution

Since $A(7,3)$, $B(k,-6)$, $C(-4,5)$ and $D(-6,4)$

$$\text{Therefore slope of } AB = m_1 = \frac{-6-3}{k-7} = \frac{-9}{k-7}$$

$$\text{Slope of } CD = m_2 = \frac{4-5}{-6+4} = \frac{-1}{-2} = \frac{1}{2}$$

(i) If AB and CD are parallel then $m_1 = m_2$

$$\Rightarrow \frac{-9}{k-7}=\frac{1}{2} \quad \Rightarrow -18=k-7$$
$$\Rightarrow k=-18+7 \quad \Rightarrow \boxed{k=-11}$$

(ii) If AB and CD are perpendicular then $m_1m_2=-1$

$$\Rightarrow \left(\frac{-9}{k-7}\right)\left(\frac{1}{2}\right)=-1 \quad \Rightarrow -9=-2(k-7)$$
$$\Rightarrow 9=2k-14 \quad \Rightarrow 2k=9+14=23$$
$$\Rightarrow \boxed{k=\frac{23}{2}}$$

Question # 5
Using slopes, show that the triangle with its vertices $A(6,1), B(2,7)$ and $C(-6,-7)$ is a right triangle.
Solution
Since $A(6,1), B(2,7)$ and $C(-6,-7)$ are vertices of triangle therefore

Slope of $\overline{AB}$ $=m_1=\frac{7-1}{2-6}=\frac{6}{-4}=-\frac{3}{2}$

Slope of $\overline{BC}$ $=m_2=\frac{-7-7}{-6-2}=\frac{-12}{-8}=\frac{7}{4}$

Slope of $\overline{CA}$ $=m_3=\frac{1+7}{6+6}=\frac{8}{12}=\frac{2}{3}$

Since $m_1m_3=\left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)=-1$

$\Rightarrow$ The triangle ABC is a right triangle with $m\angle A=90^\circ$

> **REMEMBER**
> The symbols
> (i) || stands for 'parallel"
> (ii) $\nparallel$ stands for "not parallel"
> (iii) $\perp$ stands for "perpendicular"

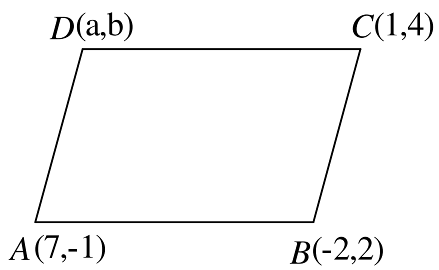
Question # 6
The three points $A(7,-1), B(-2,2)$ and $C(1,4)$ are consecutive vertices of a parallelogram. Find the fourth vertex.
Solution
Let $D(a,b)$ be a fourth vertex of the parallelogram.

Slope of $\overline{AB}=\frac{2+1}{-2-7}=\frac{3}{-9}=-\frac{1}{3}$

Slope of $\overline{BC}=\frac{4-2}{1+2}=\frac{2}{3}$

Slope of $\overline{CD}=\frac{b-4}{a-1}$

Slope of $\overline{DA}=\frac{-1-b}{7-a}$



Since $ABCD$ is a parallelogram therefore
Slope of $\overline{AB}=$ Slope of $\overline{CD}$

$\Rightarrow\ -\dfrac{1}{3}=\dfrac{b-4}{a-1} \qquad \Rightarrow\ -(a-1)=3(b-4)$

$\Rightarrow\ -a+1-3b+12=0 \qquad \Rightarrow\ -a-3b+13=0 \ldots$(i)

Also slope of $\overline{BC}=$ slope of $\overline{DA}$

$\Rightarrow\ \dfrac{2}{3}=\dfrac{-1-b}{7-a} \qquad \Rightarrow\ 2(7-a)=3(-1-b) \qquad \Rightarrow\ 14-2a=-3-3b$

$\Rightarrow\ 14-2a+3+3b=0 \quad \Rightarrow\ -2a+3b+17=0\ldots$ (ii)

Adding (i) and (ii)

$$\begin{array}{l} -a-3b+13=0 \\ \underline{-2a+3b+17=0} \\ -3a\qquad +30=0 \end{array} \quad \Rightarrow\ 3a=30 \quad \Rightarrow\ \boxed{a=10}$$

Putting value of a in (i)

$-10-3b+13=0 \quad \Rightarrow\ -3b+3=0 \quad \Rightarrow\ 3b=3 \Rightarrow\ \boxed{b=1}$

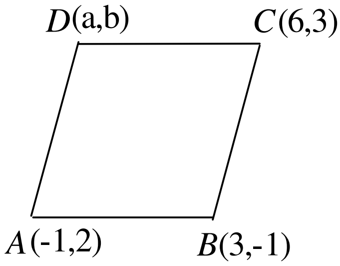
Hence $D(10,1)$ is the fourth vertex of parallelogram.

Question # 7

The points $A(-1,2), B(3,-1)$ and $C(6,3)$ are consecutive vertices of a rhombus. Find the fourth vertex and show that the diagonals of the rhombus are perpendicular to each other.

Solution

Let $D(a,b)$ be a fourth vertex of rhombus.



Slope of $\overline{AB}=\dfrac{-1-2}{3+1}=\dfrac{-3}{4}$

Slope of $\overline{BC}=\dfrac{3+1}{6-3}=\dfrac{4}{3}$

Slope of $\overline{CD}=\dfrac{b-3}{a-6}$

Slope of $\overline{DA}=\dfrac{2-b}{-1-a}$

Since $ABCD$ is a rhombus therefore

Slope of $\overline{AB}=$ Slope of $\overline{CD}$

$\Rightarrow\ -\dfrac{3}{4}=\dfrac{b-3}{a-6} \qquad \Rightarrow\ -3(a-6)=4(b-3)$

$\Rightarrow\ -3a+18=4b-12 \qquad \Rightarrow\ -3a+18-4b+12=0$

$\Rightarrow\ -3a-4b+30=0\ldots$ (i)

Also slope of $\overline{BC}=$ slope of $\overline{DA}$

$\Rightarrow\ \dfrac{4}{3}=\dfrac{2-b}{-1-a} \qquad \Rightarrow\ 4(-1-a)=3(2-b)$

$\Rightarrow\ -4-4a=6-3b \qquad \Rightarrow\ -4-4a-6+3b=0$

$\Rightarrow\ -4a+3b-10=0 \ldots$ (ii)

$\times$ing eq. (i) by 3 and (ii) by 4 and adding.

$$\begin{array}{l} -9a - 12b + 90 = 0 \\ \underline{-16a + 12b - 40 = 0} \\ -25a \qquad\quad + 50 = 0 \end{array} \quad \Rightarrow \quad 25a = 50 \quad \Rightarrow \quad \boxed{a = 2}$$

Putting value of a in (ii)

$$-4(2) + 3b - 10 = 0 \Rightarrow 3b - 18 = 0 \quad \Rightarrow \quad 3b = 18 \Rightarrow \boxed{b = 6}$$

Hence $D(2,6)$ is the fourth vertex of rhombus.

Now slope of diagonal $\overline{AC} = \dfrac{3-2}{6+1} = \dfrac{1}{7}$

Slope of diagonal $\overline{BD} = \dfrac{b-(-1)}{a-3} = \dfrac{6+1}{2-3} = \dfrac{7}{-1} = -7$

Since

(Slope of $\overline{AC}$)(Slope of $\overline{BD}$) = $\left(\dfrac{1}{7}\right)(-7) = -1$

$\Rightarrow$ Diagonals of a rhombus are $\perp$ to each other.

Question # 8

Two pairs of points are given. Find whether the two lines determined by these points are:

(i) Parallel (ii) perpendicular (iii) none

(a) $(1,-2),(2,4) and (4,1),(-8,2)$ (b) $(-3,4),(6,2) and (4,5),(-2,-7)$

Solution

(a) Slope of line joining $(1,-2)$ and $(2,4) = m_1 = \dfrac{4+2}{2-1} = \dfrac{6}{1} = 6$

Slope of line joining $(4,1)$ and $(-8,2) = m_2 = \dfrac{2-1}{-8-4} = \dfrac{1}{-12}$

Since $m_1 \neq m_2$

Also $m_1 m_2 = 6 \cdot \dfrac{1}{-12} = -\dfrac{1}{2} \neq -1$

$\Rightarrow$ lines are neither parallel nor perpendicular.

(b) *Do yourself as above.*

Equation of Straight Line:

(i) **Slope-intercept form**

Equation of straight line with slope m and $y-$intercept c is given by:

$$\boxed{y = mx + c}$$

See proof on book at page 194

(ii) **Point-slope form**

Let m be a slope of line and $A(x_1, y_1)$ be a point lies on a line then equation of line is given by:

$$\boxed{y - y_1 = m(x - x_1)}$$

See proof on book at page 195

(iii) **Symmetric form**

Let α be an inclination of line and $A(x_1, y_1)$ be a point lies on a line then equation of line is given by:

$$\boxed{\frac{y-y_1}{\cos\alpha}=\frac{x-x_1}{\sin\alpha}}$$

See proof on book at page 195

(iv) **Two-points form**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points lie on a line then it's equation is given by:

$$\boxed{y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)} \quad \text{or} \quad \boxed{y-y_2=\frac{y_2-y_1}{x_2-x_1}(x-x_2)} \quad \text{or} \quad \boxed{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0}$$

See proof on book at page 196

(v) **Two-intercept form**

When a line intersect $x-$axis at $x=a$ and $y-$axis at $y=b$

i.e. $x-$intercept $=a$ and $y-$intercept $=b$, then equation of line is given by:

$$\boxed{\frac{x}{a}+\frac{y}{b}=1}$$

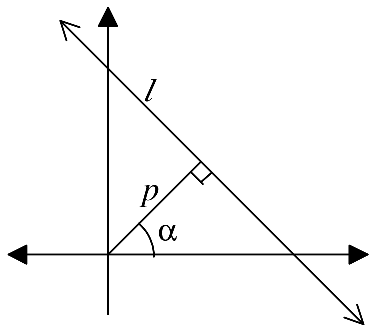
See proof on book at page 197

(vi) **Normal form**

Let p denoted length of perpendicular from the origin to the line and α is the angle of the perpendicular from +ive x-axis then equation of line is given by:

$$\boxed{x\cos\alpha+y\sin\alpha=p}$$

See proof on book at page 198



Question # 9

Find an equation of

(a) the horizontal line through $(7,-9)$

(b) the vertical line through $(-5,3)$

(c) the line bisecting the first and third quadrants.

(d) the line bisecting the second and fourth quadrants.

Solution

(a) Since slope of horizontal line $=m=0$

$$\& \ (x_1, y_1)=(7,-9)$$

therefore equation of line:

$$y-(-9)=0(x-7)$$

$$\Rightarrow \ y+9=0 \qquad \text{Answer}$$

(b) Since slope of vertical line $m=\infty=\dfrac{1}{0}$

$$\& \ (x_1, y_1)=(-5,3)$$

therefore required equation of line

$$y-3=\infty\left(x-(-5)\right)$$

$$\Rightarrow\ y-3=\frac{1}{0}(x+5) \qquad \Rightarrow\ 0(y-3)=1(x+5)$$

$$\Rightarrow\ x+5=0 \qquad \text{Answer}$$

(c) The line bisecting the first and third quadrant makes an angle of 45° with the $x-$axis therefore slope of line $=m=\tan 45^\circ=1$

Also it passes through origin $(0,0)$, so its equation

$$y-0=1(x-0) \quad \Rightarrow \quad y=x$$

$$\Rightarrow\ x-y=0 \qquad \text{Answer}$$

(d) The line bisecting the second and fourth quadrant makes an angle of 135° with $x-$axis therefore slope of line $=m=\tan 135^\circ=-1$

Also it passes through origin $(0,0)$, so its equation

$$y-0=-1(x-0) \quad \Rightarrow \quad y=-x$$

$$\Rightarrow\ x+y=0 \qquad \text{Answer}$$

Question # 10

Find an equation of the line

(a) through $A(-6,5)$ having slope 7

(b) through $(8,-3)$ having slope 0

(c) through $(-8,5)$ having slope undefined

(d) through $(-5,-3)$ and $(9,-1)$

(e) $y-intercept -7$ and $slope-5$

(f) $x-intercept:-3$ and $y-intercept:-4$

(g) $x-intercept:-9$ and $slope:-4$

Solution

(a) $\because\ (x_1,y_1)=(-6,5)$

and slope of line $=m=7$

so required equation

$$y-5=7\left(x-(-6)\right)$$

$$\Rightarrow\ y-5=7(x+6) \qquad \Rightarrow\ y-5=7x+42$$

$$\Rightarrow\ 7x+42-y+5=0 \qquad \Rightarrow\ 7x-y+47=0 \qquad \text{Answer}$$

(b) *Do yourself as above.*

(c) $\because\ (x_1,y_1)=(-8,5)$

and slope of line $=m=\infty$

So required equation

$$y-5=\infty\left(x-(-8)\right)$$

$$\Rightarrow\ y-5=\frac{1}{0}(x+8) \qquad \Rightarrow\ 0(y-5)=1(x+8)$$

$$\Rightarrow\ x+8=0 \qquad \text{Answer}$$

(d) The line through $(-5,-3)$ and $(9,-1)$ is

$$y-(-3)=\frac{-1-(-3)}{9-(-5)}\big(x-(-5)\big) \quad \Rightarrow \quad y+3=\frac{2}{14}(x+5)$$

$$\Rightarrow \quad y+3=\frac{1}{7}(x+5) \qquad \Rightarrow \quad 7y+21=x+5$$

$$\Rightarrow \quad x+5-7y-21=0 \qquad \Rightarrow \quad x-7y-16=0 \qquad \text{Answer}$$

(e) $\quad \because \; y-$intercept $=-7$

$\Rightarrow$ $(0,-7)$ lies on a required line

Also slope $=m=-5$

So required equation

$y-(-7)=-5(x-0)$

$\Rightarrow \; y+7=-5x \qquad \Rightarrow \; 5x+y+7=0 \qquad$ Answer

(f) $\quad \because \; x-$intercept $=-9$

$\Rightarrow$ $(-9,0)$ lies on a required line

Also slope $=m=4$

Therefore required line

$y-0=4(x+9)$

$\Rightarrow \; y=4x+9 \qquad \Rightarrow \; 4x-y+9=0 \qquad$ Answer

(g) $\quad x-$intercept $=a=-3$

$\quad y-$intercept $=b=4$

Using two-intercept form of equation line

$$\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow \quad \frac{x}{-3}+\frac{y}{4}=0$$

$\Rightarrow \; 4x-3y=-12 \qquad \times$ing by -12

$\Rightarrow \; 4x-3y+12=0 \qquad$ Answer

Question # 11

Find an equation of the perpendicular bisector of the segment joining the points $A(3,5)$ and $B(9,8)$

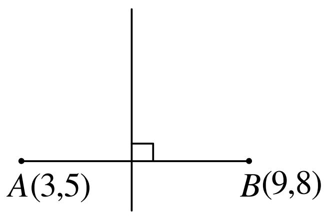
Solution

Given points $A(3,5)$ and $B(9,8)$

Midpoint of $\overline{AB}=\left(\frac{3+9}{2},\frac{5+8}{2}\right)=\left(\frac{12}{2},\frac{13}{2}\right)=\left(6,\frac{13}{2}\right)$

Slope of $\overline{AB}=m=\frac{8-5}{9-3}=\frac{3}{6}=\frac{1}{2}$

Slope of line $\perp$ to $\overline{AB}=-\frac{1}{m}=-\frac{1}{1/2}=--2$



Now equation of $\perp$ bisector having slope -2 through $\left(6,\frac{13}{2}\right)$

$$\Rightarrow\ y-\frac{13}{2}=-2(x-6)$$

$$\Rightarrow\ y-\frac{13}{2}=-2x+12 \qquad \Rightarrow\ y-\frac{13}{2}+2x-12=0$$

$$\Rightarrow\ 2x+y-\frac{37}{2}=0 \qquad \Rightarrow\ 4x+2y-37=0$$

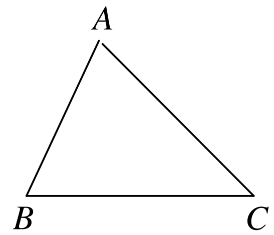
Question # 12

Find equations of the sides, altitudes and medians of the triangle whose vertices are $A(-3,2), B(5,4)$ and $C(3,-8)$.

Solution

Given vertices of triangle are $A(-3,2)$, $B(5,4)$ and $C(3,-8)$.

Equation of sides:



Slope of $\overline{AB}=m_1=\dfrac{4-2}{5-(-3)}=\dfrac{2}{8}=\dfrac{1}{4}$

Slope of $\overline{BC}=m_2=\dfrac{-8-4}{3-5}=\dfrac{-12}{-2}=6$

Slope of $\overline{CA}=m_3=\dfrac{2-(-8)}{-3-3}=\dfrac{10}{-6}=-\dfrac{5}{3}$

Now equation of side $\overline{AB}$ having slope $\dfrac{1}{4}$ passing through $A(-3,2)$

[You may take $B(5,4)$ instead of $A(-3,2)$]

$$y-2=\frac{1}{4}\big(x-(-3)\big) \quad\Rightarrow\ 4y-8=x+3$$

$$\Rightarrow\ x+3-4y+8=0 \quad\Rightarrow\ \boxed{x-4y+11=0}$$

Equation of side $\overline{BC}$ having slope 6 passing through $B(5,4)$.

$$y-4=6(x-5) \qquad\Rightarrow\ y-4=6x-30$$

$$\Rightarrow\ 6x-30-y+4=0 \quad\Rightarrow\ \boxed{6x-y-26=0}$$

Equation of side $\overline{CA}$ having slope $-\dfrac{5}{3}$ passing through $C(3,-8)$

$$y-(-8)=-\frac{5}{3}(x-3) \qquad\Rightarrow\ 3(y+8)=-5(x-3)$$

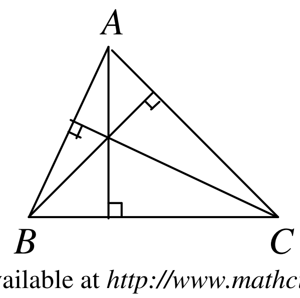
$$\Rightarrow\ 3y+24=-5x+15 \qquad\Rightarrow\ 5x-15+3y+24=0$$

$$\Rightarrow\ \boxed{5x+3y+9=0}$$

Equation of altitudes:

Since altitudes are perpendicular to the sides of triangle therefore

Slope of altitude on $\overline{AB}=-\dfrac{1}{m_1}=-\dfrac{1}{1/4}=-4$



Equation of altitude from $C(3,-8)$ having slope -4

$$y+8=-4(x-3) \qquad\Rightarrow\ y+8=-4x+12$$

$$\Rightarrow\ 4x-12+y+8=0 \quad \Rightarrow\ \boxed{4x+y-4=0}$$

Slope of altitude on $\overline{BC}=-\dfrac{1}{m_2}=-\dfrac{1}{6}$

Equation of altitude from $A(-3,2)$ having slope $-\dfrac{1}{6}$

$$y-2=-\frac{1}{6}(x+3) \quad \Rightarrow\ 6y-12=-x-3$$

$$\Rightarrow\ x+3+6y-12=0 \quad \Rightarrow\ \boxed{x+6y-9=0}$$

Slope of altitude on $\overline{CA}=-\dfrac{1}{m_3}=-\dfrac{1}{-5/3}=\dfrac{3}{5}$

Equation of altitude from $B(5,4)$ having slope $\dfrac{3}{5}$

$$y-4=\frac{3}{5}(x-5) \quad \Rightarrow\ 5y-20=3x-15$$

$$\Rightarrow\ 3x-15-5y+20=0 \quad \Rightarrow\ \boxed{3x-5y+5=0}$$

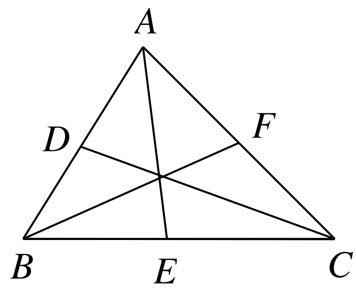
Equation of Medians:

Suppose D,E and F are midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively.

Then coordinate of $D=\left(\dfrac{-3+5}{2},\dfrac{2+4}{2}\right)=\left(\dfrac{2}{2},\dfrac{6}{2}\right)=(1,3)$

Coordinate of $E=\left(\dfrac{5+3}{2},\dfrac{4-8}{2}\right)=\left(\dfrac{8}{2},\dfrac{-4}{2}\right)=(4,-2)$

Coordinate of $F=\left(\dfrac{3-3}{2},\dfrac{-8+2}{2}\right)=\left(\dfrac{0}{2},\dfrac{-6}{2}\right)=(0,-3)$



Equation of median $\overline{AE}$ by two-point form

$$y-2=\frac{-2-2}{4-(-3)}\left(x-(-3)\right)$$

$$\Rightarrow\ y-2=\frac{-4}{7}(x+3) \quad \Rightarrow\ 7y-14=-4x-12$$

$$\Rightarrow\ 7y-14+4x+12=0 \quad \Rightarrow\ \boxed{4x+7y-2=0}$$

Equation of median $\overline{BF}$ by two-point form

$$y-4=\frac{-3-4}{0-5}(x-5)$$

$$\Rightarrow\ y-4=\frac{-7}{-5}(x-5) \quad \Rightarrow\ -5y+20=-7x+35$$

$$\Rightarrow\ -5y+20+7x-35=0 \quad \Rightarrow\ \boxed{7x-5y-15=0}$$

Equation of median $\overline{CD}$ by two-point form

$$y-(-8)=\frac{3-(-8)}{1-3}(x-3)$$

$$\Rightarrow\ y+8=\frac{11}{-2}(x-3) \qquad \Rightarrow\ -2y-16=11x-33$$

$$\Rightarrow\ 11x-33+2y+16=0 \qquad \Rightarrow\ \boxed{11x+2y-17=0}$$

Question # 13

Find an equation of the line through $(-4,-6)$ and perpendicular to the line having slope $\frac{-3}{2}$.

Solution

Here $(x_1, y_1)=(-4,-6)$

Slope of given line $=m=\frac{-3}{2}$

$\because$ required line is $\perp$ to given line

$\therefore$ slope of required line $=-\frac{1}{m}=-\frac{1}{-3/2}=\frac{2}{3}$

Now equation of line having slope $\frac{2}{3}$ passing through $(-4,-6)$

$$y-(-6)=\frac{2}{3}(x-(-4))$$

$$\Rightarrow\ 3(y+6)=2(x+4) \qquad \Rightarrow\ 3y+18=2x+8$$

$$\Rightarrow\ 2x+8-3y-18=0 \qquad \Rightarrow\ 2x-3y-10=0$$

Question # 14

Find an equation of the line through $(11,-5)$ and parallel to a line with slope -24.

Solution

Here $(x_1, y_1)=(11,-5)$

Slope of given line $=m=-24$

$\because$ required line is $\|$ to given line

$\therefore$ slope of required line $=m=-24$

Now equation of line having slope -24 passing through $(11,-5)$

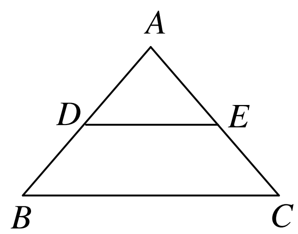
$$y-(-5)=-24(x-11)$$

$$\Rightarrow\ y+5=-24x+264 \qquad \Rightarrow\ 24x-264+y+5=0$$

$$\Rightarrow\ 24x+y-259=0$$

Question # 15

The points $A(-1,2), B(6,3)$ and $C(2,-4)$ are vertices of a triangle. Show that the line joining the midpoint D of AB and the midpoint E of AC is parallel to BC and $DE=\frac{1}{2}BC$.



Solution Given vertices $A(-1,2)$, $B(6,3)$ and $C(2,-4)$

Since D and E are midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively.

Therefore coordinate of $D=\left(\dfrac{-1+6}{2},\dfrac{2+3}{2}\right)=\left(\dfrac{5}{2},\dfrac{5}{2}\right)$

Coordinate of $E=\left(\dfrac{-1+2}{2},\dfrac{2-4}{2}\right)=\left(\dfrac{1}{2},\dfrac{-2}{2}\right)=\left(\dfrac{1}{2},-1\right)$

Now slope of $\overline{DE}=\dfrac{-1-{}^5\!/_2}{{}^1\!/_2-{}^5\!/_2}=\dfrac{{}^{-7}\!/_2}{{}^{-4}\!/_2}=\dfrac{7}{4}$

slope of $\overline{BC}=\dfrac{-4-3}{2-6}=\dfrac{-7}{-4}=\dfrac{7}{4}$

Since slope of $\overline{DE}=$ slope of $\overline{BC}$

Therefore $\overline{DE}$ is parallel to $\overline{BC}$.

Now $\quad \left|\overline{DE}\right|=\sqrt{\left(\dfrac{1}{2}-\dfrac{5}{2}\right)^2+\left(-1-\dfrac{5}{2}\right)^2}=\sqrt{\left(-\dfrac{4}{2}\right)^2+\left(-\dfrac{7}{2}\right)^2}$

$$=\sqrt{4+\frac{49}{4}}=\sqrt{\frac{65}{4}}=\frac{\sqrt{65}}{2}\ \ldots\ldots\ldots\ldots\text{(i)}$$

$$\left|\overline{BC}\right|=\sqrt{(2-6)^2+(-4-3)^2}=\sqrt{(-4)^2+(-7)^2}$$

$$=\sqrt{16+49}=\sqrt{65}\ \ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii)

$$\left|\overline{DE}\right|=\frac{1}{2}\left|\overline{BC}\right|$$

Question # 16

A milkman can sell 560 litres of milk at $Rs12.50$ per litre and 700 litres of milk at $Rs12.00$ per litre. Assuming the graph of the sale price and the milk sold to be a straight line, find the number of litres of milk that the milkman can sell at $Rs12.25$ per litre.

Solution

Let l denotes the number of litres of milk and p denotes the price of milk,

Then $\quad (l_1,p_1)=(560,12.50) \quad \& \quad (l_2,p_2)=(700,12.00)$

Since graph of sale price and milk sold is a straight line

Therefore, from two point form, it's equation

$$p-p_1=\frac{p_2-p_1}{l_2-l_1}(l-l_1)$$

$$\Rightarrow\ p-12.50=\frac{12.00-12.50}{700-560}(l-560)$$

$$\Rightarrow\ p-12.50=\frac{-0.50}{140}(l-560)$$

$$\Rightarrow\ 140p-1750=-0.50l+280$$

$$\Rightarrow\ 140p-1750+0.50l-280=0$$

$$\Rightarrow\ 0.50l+140p-2030=0$$

If $\ p=12.25$

ALTERNATIVE

You may use determinant form of two-point form to find an equation of line.

$$\begin{vmatrix} l & p & 1 \\ l_1 & p_1 & 1 \\ l_2 & p_2 & 1 \end{vmatrix}=0$$

$$\Rightarrow 0.50l + 140(12.25) - 2030 = 0$$
$$\Rightarrow 0.50l + 1715 - 2030 = 0 \qquad \Rightarrow 0.50l - 315 = 0$$
$$\Rightarrow 0.50l = 315 \qquad \Rightarrow l = \frac{315}{0.50} = 630$$

Hence milkman can sell 630 litres milk at Rs. 12.25 per litre.

Question # 17

The population of Pakistan to the nearest million was 60 million in 1961 and 95 million in 1981. Using t as the number of years after 1961, Find an equation of the line that gives the population in terms of t. Use this equation to find the population in

(a) 1947 (b) 1997

Solution

Let p denotes population of Pakistan in million and t denotes year after 1961,

Then $(p_1, t_1) = (60, 1961)$ and $(p_2, t_2) = (95, 1981)$

Equation of line by two point form:

$$t - t_1 = \frac{t_2 - t_1}{p_2 - p_1}(p - p_1)$$
$$\Rightarrow t - 1961 = \frac{1981 - 1961}{95 - 60}(p - 60)$$
$$\Rightarrow t - 1961 = \frac{20}{35}(p - 60) \qquad \Rightarrow t - 1961 = \frac{4}{7}(p - 60)$$
$$\Rightarrow 7t - 13727 = 4p - 240 \qquad \Rightarrow 7t - 13727 + 240 = 4p$$
$$\Rightarrow 4p = 7t - 13487 \quad \Rightarrow p = \frac{7}{4}t - \frac{13487}{4} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

This is the required equation which gives population in term of t.

(a) Put $t = 1947$ in eq. (i)

$$p = \frac{7}{4}(1947) - \frac{13487}{4} \quad = 3407.25 - 3371.75 \quad = 35.5$$

Hence population in 1947 is 35.5 millions.

(b) Put $t = 1997$ in eq. (i)

$$p = \frac{7}{4}(1997) - \frac{13487}{4} = 3494.75 - 3371.75 = 123$$

Hence population in 1997 is 123 millions.

Question # 18

A house was purchased for $Rs1$ million in 1980. It is worth $Rs4$ million in 1996. Assuming that the value increased by the same amount each year, find an equation that gives the value of the house after t years of the date of purchase. What was the value in 1990?

Solution

Let p denotes purchase price of house in millions and t denotes year then

$$(p_1, t_1) = (1, 1980) \quad \text{and} \quad (p_2, t_2) = (4, 1996)$$

Equation of line by two point form:

$$t-t_1=\frac{t_2-t_1}{p_2-p_1}(p-p_1)$$

$$\Rightarrow\ t-1980=\frac{1996-1980}{4-1}(p-1)$$

$$\Rightarrow\ t-1980=\frac{16}{3}(p-1)$$

$$\Rightarrow\ 3t-5940=16p-16$$

$$\Rightarrow\ 3t-5940+16=16p \quad \Rightarrow\ 16p=3t-5924$$

$$\Rightarrow\ p=\frac{3}{16}t-\frac{5924}{16} \quad \Rightarrow\ p=\frac{3}{16}t-\frac{1481}{4} \ \ldots\ldots\ldots\ldots \text{(i)}$$

ALTERNATIVE
You may use determinant form of two-point form to find an equation of line.

$$\begin{vmatrix} p & t & 1 \\ p_1 & t_1 & 1 \\ p_2 & t_2 & 1 \end{vmatrix}=0$$

This is the required equation which gives value of house in term of t.

Put $t=1990$ in eq. (i)

$$p=\frac{3}{16}(1990)-\frac{1481}{4} \quad =373.125-370.25 \quad =2.875$$

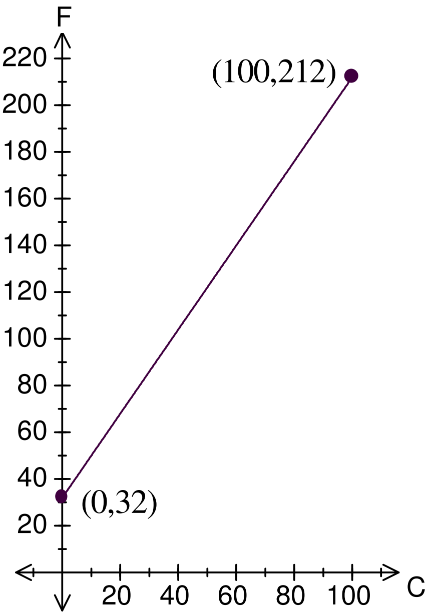
Hence value of house in 1990 is 2.875 millions.

Question # 19

Plot the Celsius (C) and Fahrenheit (F) temperature scales on the horizontal axis and the vertical axis respectively. Draw the line joining the freezing point and the boiling point of water. Find an equation giving F temperature in term of C.

Solution

Since freezing point of water $=0^\circ C$ = $32^\circ F$ and boiling point of water $=100^\circ C$ = $212^\circ F$ therefore we have points $(C_1,F_1)=(0,32)$ and $(C_2,F_2)=(100,212)$



Take scale 10ss = 20C and 10ss = 20F on x-axis and y-axis respectively to draw graph.

Equation of line by two point form

$$F-F_1=\frac{F_2-F_1}{C_2-C_1}(C-C_1)$$

$$\Rightarrow\ F-32=\frac{212-32}{100-0}(C-0)$$

$$\Rightarrow\ F-32=\frac{180}{100}C$$

$$\Rightarrow\ F=\frac{9}{5}C+32$$

Question # 20

The average entry test score of engineering candidates was in the year 1998 while the score was 564 in 2002. Assuming that the relationship between time and score is linear, find the average score for 2006.

Solution

Let s denotes entry test score and y denotes year.

Then we have $(s_1, y_1) = (592,1998)$ and $(s_2, y_2) = (564,2002)$

By two point form of equation of line

$$y - y_1 = \frac{y_2 - y_1}{s_2 - s_1}(s - s_1)$$

$$\Rightarrow \ y - 1998 = \frac{2002 - 1998}{564 - 592}(s - 592) \quad \Rightarrow \ y - 1998 = \frac{4}{-28}(s - 592)$$

$$\Rightarrow \ y - 1998 = -\frac{1}{7}(s - 592) \quad \Rightarrow \ 7y - 13986 = -s + 592$$

$$\Rightarrow \ 7y - 13986 + s - 592 = 0 \quad \Rightarrow \ s + 7y - 14578 = 0$$

Put $y = 2006$ in (i)

$$s + 7(2006) - 14578 = 0 \quad \Rightarrow \ s + 14042 - 14578 = 0$$

$$\Rightarrow \ s - 536 = 0 \quad \Rightarrow \ s = 536$$

Hence in 2006 the average score will be 536.

Question # 21

Convert each of the following equation into

(i) Slope intercept form (ii) Two-intercept form (iii) Normal form

(a) $2x - 4y + 11 = 0$ (b) $4x + 7y - 2 = 0$ (c) $15y - 8x + 3 = 0$

Also find the length of the perpendicular from $(0,0)$ to each line.

Solution

(a)

(i) **- Slope-intercept form**

$$\because \ 2x - 4y + 11 = 0$$

$$\Rightarrow \ 4y = 2x + 11 \quad \Rightarrow \ y = \frac{2x + 11}{4}$$

$$\Rightarrow \ y = \frac{1}{2}x + \frac{11}{4}$$

is the intercept form of equation of line with $m = \frac{1}{2}$ and $c = \frac{11}{4}$

(ii) **- Two-intercept form**

$$\because \ 2x - 4y + 11 = 0 \quad \Rightarrow \ 2x - 4y = -11$$

$$\Rightarrow \ \frac{2}{-11}x - \frac{4}{-11}y = 1 \quad \Rightarrow \ \frac{x}{-11/2} + \frac{y}{11/4} = 1$$

is the two-point form of equation of line with $a = -\frac{11}{2}$ and $b = \frac{11}{4}$.

(iii) **- Normal form**

$$\because \ 2x - 4y + 11 = 0 \quad \Rightarrow \ 2x - 4y = -11$$

Dividing above equation by $\sqrt{(2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

$$\frac{2x}{2\sqrt{5}} - \frac{4y}{2\sqrt{5}} = \frac{-11}{2\sqrt{5}} \quad \Rightarrow \ \frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}} = \frac{-11}{2\sqrt{5}}$$

$$\Rightarrow\ -\frac{x}{\sqrt{5}}+\frac{2y}{\sqrt{5}}=\frac{11}{2\sqrt{5}} \qquad \times\text{ing by } -1.$$

Suppose $\cos\alpha=-\frac{1}{\sqrt{5}}<0$ and $\sin\alpha=\frac{2}{\sqrt{5}}>0$

$\Rightarrow$ α lies in 2nd quadrant and $\alpha=\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)=116.57^\circ$

Hence the normal form is

$$x\cos(116.57^\circ)+y\sin(116.57^\circ)=\frac{11}{2\sqrt{5}}$$

And length of perpendicular from $(0,0)$ to line $=p=\frac{11}{2\sqrt{5}}$

(b)

(i) **- Slope-intercept form**

$$\because\ 4x+7y-2=0$$

$$\Rightarrow\ 7y=-4x+2 \qquad \Rightarrow\ y=\frac{-4x+2}{7}$$

$$\Rightarrow\ y=-\frac{4}{7}x+\frac{2}{7}$$

is the intercept form of equation of line with $m=-\frac{4}{7}$ and $c=\frac{2}{7}$

(ii) **- Two-intercept form**

$$\because\ 4x+7y-2=0 \qquad \Rightarrow\ 4x+7y=2$$

$$\Rightarrow\ 2x+\frac{7}{2}y=1 \qquad \div\text{ing by } 2$$

$$\Rightarrow\ \frac{x}{1/2}+\frac{y}{2/7}=1$$

is the two-point form of equation of line with $a=\frac{1}{2}$ and $b=\frac{2}{7}$.

(iii) **- Normal form**

$$\because\ 4x+7y-2=0$$

$$\Rightarrow\ 4x+7y=2$$

Dividing above equation by $\sqrt{(4)^2+(7)^2}=\sqrt{16+49}=\sqrt{65}$

$$\Rightarrow\ \frac{4}{\sqrt{65}}x+\frac{7}{\sqrt{65}}y=\frac{2}{\sqrt{65}} \quad .$$

Suppose $\cos\alpha=\frac{4}{\sqrt{65}}>0$ and $\sin\alpha=\frac{7}{\sqrt{65}}>0$

$\Rightarrow$ α lies in first quadrant and $\alpha=\cos^{-1}\left(\frac{4}{\sqrt{65}}\right)=60.26^\circ$

Hence the normal form is

$$x\cos\left(60.26^\circ\right)+y\sin\left(60.26^\circ\right)=\frac{2}{\sqrt{65}}$$

And length of perpendicular from $(0,0)$ to line $=p=\dfrac{2}{\sqrt{65}}$

(c)

(i) **- Slope-intercept form**

$$\because\ 15y-8x+3=0$$

$$\Rightarrow\ 15y=8x-3 \qquad \Rightarrow\ y=\frac{8x-3}{15}$$

$$\Rightarrow\ y=\frac{8}{15}x-\frac{3}{15} \qquad \Rightarrow\ y=\frac{8}{15}x-\frac{1}{5}$$

is the intercept form of equation of line with $m=\dfrac{8}{15}$ and $c=-\dfrac{1}{5}$

(ii) **- Two-intercept form**

$$\because\ 15y-8x+3=0 \qquad \Rightarrow\ -8x+15y=-3$$

$$\Rightarrow\ \frac{8x}{3}-5y=1 \qquad \Rightarrow\ \frac{x}{3/8}+\frac{y}{-1/5}=1$$

is the two-point form of equation of line with $a=\dfrac{3}{8}$ and $b=-\dfrac{1}{5}$.

(iii) **- Normal form**

$$\because\ 15y-8x+3=0$$

$$\Rightarrow\ 8x-15y=3$$

Dividing above equation by $\sqrt{(8)^2+(-15)^2}=\sqrt{64+225}=\sqrt{289}=17$

$$\Rightarrow\ \frac{8}{17}x-\frac{15}{17}y=\frac{3}{17}\quad .$$

Suppose $\cos\alpha=\dfrac{8}{17}>0$ and $\sin\alpha=-\dfrac{15}{17}<0$

$\Rightarrow$ α lies in 4th quadrant and $\alpha=\cos^{-1}\left(\dfrac{8}{17}\right)=298.07^\circ$

Hence the normal form is

$$x\cos\left(298.07^\circ\right)+y\sin\left(298.07^\circ\right)=\frac{3}{17}$$

And length of perpendicular from $(0,0)$ to line $=p=\dfrac{3}{17}$

> $\alpha=\cos^{-1}\left(\dfrac{8}{17}\right)$
> $=61.93^\circ$, 298.07°
> Taking value that lies in 4th quadrant.

General equation of the straight line

A general equation of straight line (General linear equation) in two variable x and y is given by:

$$ax+by+c=0$$

where a, b and c are constants and a and b are not simultaneously zero.

See proof on book at page: 199.

Note: Since $ax+by+c=0 \Rightarrow by=-ax-c \Rightarrow y=-\dfrac{a}{b}x-\dfrac{c}{b}$

Which is an intercept form of equation of line with slope $m=-\dfrac{a}{b}$ and $c=-\dfrac{c}{b}$

Question # 22

In each of the following check whether the two lines are

(i) parallel. (ii) perpendicular. (iii) neither parallel nor perpendicular.

(a) $2x+y-3=0;\ 4x+2y+5=0$

(b) $3y=2x+5;\ 3x+2y-8=0$

(c) $4y+2x-1=0;\ x-2y-7=0$

(d) $4x-y+2=0;\ 12x-3y+1=0$

(e) $12x+35y-7=0;\ 105x-36y+11=0$

Solution

(a) Let $l_1:\ 2x+y-3=0$

$l_2:\ 4x+2y+5=0$

Slope of $l_1=m_1=-\dfrac{2}{1}=-2$

Slope of $l_2=m_2=-\dfrac{4}{2}=-2$

Since $m_1=m_2$ therefore l_1 and l_2 are parallel.

(b) Let $l_1:\ 3y=2x+5 \Rightarrow 2x-3y+5=0$

$l_2:\ 3x+2y-8=0$

Slope of $l_1=m_1=-\dfrac{2}{-3}=\dfrac{2}{3}$

Slope of $l_2=m_2=-\dfrac{3}{2}=$

Since $m_1m_2=\left(\dfrac{2}{3}\right)\left(-\dfrac{3}{2}\right)=-1 \Rightarrow l_1$ and l_2 are perpendicular.

(c) Let $l_1:\ 4y+2x-1=0 \Rightarrow 2x+4y-1=0$

$l_2:\ x-2y-7=0$

Slope of $l_1=m_1=-\dfrac{2}{4}=-\dfrac{1}{2}$

Slope of $l_2=m_2=-\dfrac{1}{-2}=\dfrac{1}{2}$

Since $m_1\neq m_2$ and $m_1m_2=\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)=-\dfrac{1}{4}\neq -1$

$\Rightarrow l_1$ and l_2 are neither parallel nor perpendicular.

(d) *Do yourself as above.*

(e) *Do yourself as above.*

Question # 23

Find the distance between the given parallel lines. Sketch the line. Also find an equation of the parallel line lying midway between them.

(a) $3x-4y+3=0; 3x-4y+7=0$

(b) $12x+5y-6=0; 12x+5y+13=0$

(c) $x+2y-5=0; 2x+4y=1$

Solution

(a) $l_1:\ 3x-4y+3=0\ldots$ (i)

$l_2:\ 3x-4y+7=0\ldots$ (ii)

We first convert l_1 and l_2 in normal form

$$(i) \Rightarrow -3x+4y=3$$

Dividing by $\sqrt{(-3)^2+(4)^2}=\sqrt{25}=5$

$$-\frac{3}{5}x+\frac{4}{5}y=\frac{3}{5}$$

Let $\cos\alpha=-\frac{3}{5}<0$ and $\sin\alpha=\frac{4}{5}>0$

$\Rightarrow$ α lies in 2nd quadrant and $\alpha=\cos^{-1}\left(-\frac{3}{5}\right)=126.87^\circ$

$$\Rightarrow\ x\cos(126.87)+y\sin(126.87)=\frac{3}{5}$$

Hence distance of l_1 form origin $=\frac{3}{5}$

Now (ii) $\Rightarrow$ $-3x+4y=7$

Dividing by $\sqrt{(-3)^2+(4)^2}=\sqrt{25}=5$

$$-\frac{3}{5}x+\frac{4}{5}y=\frac{7}{5}$$

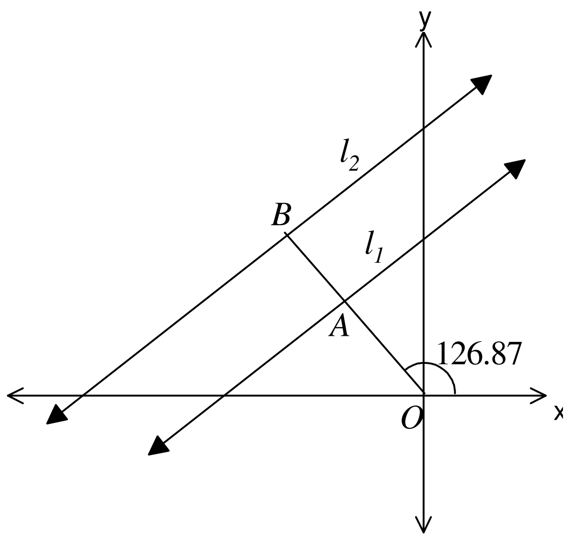
Let $\cos\alpha=-\frac{3}{5}<0$ and $\sin\alpha=\frac{4}{5}>0$

$\Rightarrow$ α lies in 1st quadrant

and $\alpha=\cos^{-1}\left(-\frac{3}{5}\right)=126.87^\circ$

$$\Rightarrow\ x\cos(126.87)+y\sin(126.87)=7/5$$

Hence distance of l_2 form origin $=7/5$



From graph we see that both lines lie on the same side of origin therefore

Distance between lines $=\left|\overline{AB}\right|=\left|\overline{OB}\right|-\left|\overline{OA}\right|$ $=\frac{7}{5}-\frac{3}{5}=\frac{4}{5}$

Let l_3 be a line parallel to l_1 and l_2, and lying midway between them. Then

Distance of l_3 from origin $=\left|\overline{OA}\right|+\dfrac{\left|\overrightarrow{AB}\right|}{2}$ $=\dfrac{3}{5}+\dfrac{4/5}{2}=\dfrac{3}{5}+\dfrac{4}{10}=1$

Hence equation of l_3

$$x\cos(126.87)+y\sin(126.87)=1$$

$$\Rightarrow\ x\left(-\frac{3}{5}\right)+y\left(\frac{4}{5}\right)=1 \quad \Rightarrow\ -3x+4y=5$$

$$\Rightarrow\ 3x-4y+5=0$$

(b) $l_1:\ 12x+5y-6=0...$ (i)

$l_2:\ 12x+5y+13=0...$ (ii)

We first convert l_1 and l_2 in normal form

(i) $\Rightarrow\ 12x+5y=6$

Dividing by $\sqrt{(12)^2+(5)^2}=\sqrt{169}=13$

$$\frac{12}{13}x+\frac{5}{13}y=\frac{6}{13}$$

Let $\cos\alpha=\dfrac{12}{13}>0$ and $\sin\alpha=\dfrac{5}{13}>0$

$\Rightarrow\ \alpha$ lies in 1st quadrant and $\alpha=\cos^{-1}\left(\dfrac{12}{13}\right)=22.62^\circ$

$$\Rightarrow\ x\cos(22.62)+y\sin(22.62)=\frac{6}{13}$$

Hence distance of l_1 form origin $=\dfrac{6}{13}$

Now (ii) $\Rightarrow\ -12x-5y=13$

Dividing by $\sqrt{(12)^2+(5)^2}=\sqrt{169}=13$

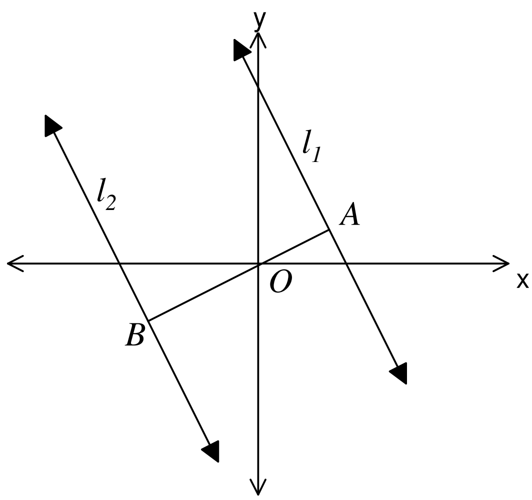
$$-\frac{12}{13}x-\frac{5}{13}y=1$$

Let $\cos\alpha=-\dfrac{12}{13}<0$ and $\sin\alpha=-\dfrac{5}{13}<0$

$\Rightarrow\ \alpha$ lies in 3rd quadrant

and $\alpha=\cos^{-1}\left(-\dfrac{12}{13}\right)=202.62^\circ$

$$\Rightarrow\ x\cos(202.62)+y\sin(202.62)=1$$



Hence distance of l_2 form origin $=1$

From graph we see that lines lie on the opposite side of origin therefore

Distance between lines $=\left|\overline{AB}\right|=\left|\overline{OA}\right|+\left|\overline{OB}\right|$ $=\dfrac{6}{13}+1=\dfrac{19}{13}$

Let l_3 be a line parallel to l_1 and l_2, and lying midway between them. Then

Distance of l_3 from origin $=\left|\overline{OB}\right|-\dfrac{\left|\overrightarrow{AB}\right|}{2}=1-\dfrac{19/13}{2}=1-\dfrac{19}{26}=\dfrac{7}{26}$

Hence equation of l_3

$$x\cos(202.62)+y\sin(202.62)=\frac{7}{26}$$

$$\Rightarrow\ x\left(-\frac{12}{13}\right)+y\left(-\frac{5}{13}\right)=\frac{7}{26}\qquad\Rightarrow\ -24x-10y=7$$

$$\Rightarrow\ 24x+10y+7=0$$

(c) *Do yourself as Question # 23 (a)*

Question # 24

Find an equation of the line through $(-4,7)$ and parallel to the line $2x-7y+4=0$

Solution

Let $l:\ 2x-7y+4=0$

Slope of $l=m=-\dfrac{2}{-7}=\dfrac{2}{7}$

Since required line is parallel to l

Therefore slope of required line $=m=\dfrac{2}{7}$

Now equation of line having slope $\dfrac{2}{7}$ passing through $(-4,7)$

$$y-7=\frac{2}{7}(x-(-4))$$

$$\Rightarrow\ 7(y-7)=2(x+4)$$

$$\Rightarrow\ 7y-49=2x+8\qquad\Rightarrow\ 2x+8-7y+49=0$$

$$\Rightarrow\ 2x-7y+57=0$$

REMEMBER

If $l:\ ax+by+c=0$

then slope of $l=-\dfrac{a}{b}$

Question # 25

Find an equation of the line through $(5,-8)$ and perpendicular to the join of $A(-15,-8),B(10,7)$

Solution

Given: $A(-15,-8)$, $B(10,7)$ and $(5,-8)$

Slope of $\overline{AB}\ =\ m\ =\ \dfrac{7-(-8)}{10-(-15)}$

$$=\frac{7+8}{10+15}=\frac{15}{25}=\frac{3}{5}$$

Since required line is perpendicular to $\overline{AB}$

Therefore slope of required line $=-\dfrac{1}{m}=-\dfrac{5}{3}$

Now equation of line having slope $-\dfrac{5}{3}$ through $(5,-8)$

$$y-(-8)=-\frac{5}{3}(x-5)$$

$$\Rightarrow\ 3y+24=-5x+25$$

$$\Rightarrow\ 5x+3y+24-25=0 \quad \Rightarrow\ 5x+3y-1=0$$

Question # 26

Find equations of two parallel lines perpendicular to $2x-y+3=0$ such that the product of the x-and y-intercepts of each is 3.

Solution

Let $l:\ 2x-y+3=0$

Slope of $l = m = -\dfrac{2}{-1} = 2$

Since required line is perpendicular to l

Therefore slope of required line $= -\dfrac{1}{m} = -\dfrac{1}{2}$

Let $y-$intercept of req. line $=c$

Then equation of req. line with slope $-\dfrac{1}{2}$ and $y-$intercept c

$$y=-\frac{1}{2}x+c \ \ldots\ldots\ldots\ldots \text{(i)}$$

$$\Rightarrow\ \frac{1}{2}x+y = c$$

$$\Rightarrow\ \frac{x}{2c}+\frac{y}{c} = 1$$

This is two intercept form of equation of line with

$x-$intercept $=2c$ and $y-$intercept $= c$

Since product of intercepts $=3$

$$\Rightarrow\ (c)(2c)=3 \quad \Rightarrow\ 2c^2=3 \quad \Rightarrow\ c^2=\frac{3}{2} \quad \Rightarrow\ c=\pm\sqrt{\frac{3}{2}}$$

Putting in (i)

$$\Rightarrow\ y = -\frac{1}{2}x\pm\sqrt{\frac{3}{2}}$$

$$\Rightarrow\ \frac{1}{2}x+y \mp \sqrt{\frac{3}{2}} = 0 \quad \Rightarrow\ \frac{1}{2}x+y \mp \sqrt{\frac{3\times 2}{2\times 2}} = 0$$

$$\Rightarrow\ \frac{1}{2}x+y \mp \frac{\sqrt{6}}{2} = 0$$

$\Rightarrow\ x+2y \mp \sqrt{6} = 0$ are the required equations.

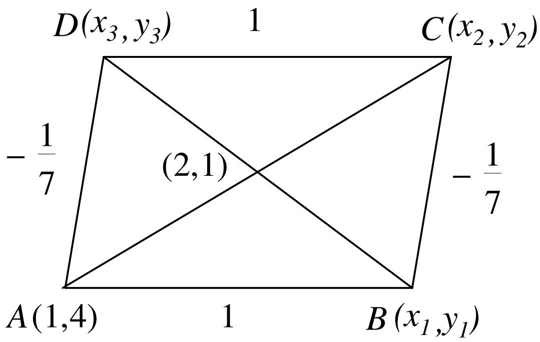
Question # 27
One vertex of a parallelogram is $(1,4)$, the diagonal intersect at $(2,1)$ and the sides have slopes 1 and $-\frac{1}{7}$. Find the other three vertices.

Solution

Let $A(1,4)$ be a given vertex and $B(x_1, y_1), C(x_2, y_2)$ and $D(x_3, y_3)$ are remaining vertices of parallelogram.

Since diagonals of parallelogram bisect at $(2,1)$ therefore

$$(2,1)=\left(\frac{1+x_2}{2},\frac{4+y_2}{2}\right)$$



$$\Rightarrow\ 2=\frac{1+x_2}{2} \quad \text{and} \quad 1=\frac{4+y_2}{2}$$
$$\Rightarrow\ 4=1+x_2 \quad , \quad 2=4+y_2$$
$$\Rightarrow\ x_2=4-1 \quad , \quad y_2=-4+2$$
$$\Rightarrow\ x_2=3 \quad , \quad y_2=-2$$

Hence $C(x_2, y_2) = C(3,-2)$

Now slope of $\overline{AB}=1$

$$\Rightarrow\ \frac{y_1-4}{x_1-1}=1 \quad \Rightarrow\ y_1-4=x_1-1$$
$$\Rightarrow\ x_1-y_1-1+4=0 \quad \Rightarrow\ x_1-y_1+3=0 \ \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also slope of $\overline{BC}=-\frac{1}{7}$

$$\Rightarrow\ \frac{y_2-y_1}{x_2-x_1}=-\frac{1}{7} \quad \Rightarrow\ \frac{-2-y_1}{3-x_1}=-\frac{1}{7}$$
$$\Rightarrow\ -14-7y_1=-3+x_1 \quad \Rightarrow\ -3-x_1+14+7y_1=0$$
$$\Rightarrow\ x_1+7y_1+11=0 \ \ldots\ldots\ldots\ldots \text{(ii)}$$

Subtracting (i) and (ii)

$$\begin{array}{r} x_1 - y_1 + 3 = 0 \\ \underset{-}{x_1} \underset{-}{+}7y_1 \underset{-}{+}11 = 0 \\ \hline -8y_1-8 = 0 \end{array}$$
$$\Rightarrow\ y_1+1=0 \quad \Rightarrow\ y_1=-1$$

Putting in (i)

$$x_1-(-1)+3 = 0 \quad \Rightarrow\ x_1+4=0 \quad \Rightarrow\ x_1=-4$$

$\Rightarrow\ B(x_2, y_2) = B(-4,-1)$

Now E is midpoint of BD

$$\Rightarrow\ (2,1) = \left(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2}\right)$$
$$= \left(\frac{-4+x_3}{2},\frac{-1+y_3}{2}\right)$$

$$\Rightarrow\ 2 = \frac{-4+x_3}{2} \quad , \quad 1 = \frac{-1+y_3}{2}$$
$$\Rightarrow\ 4 = -4+x_3 \quad , \quad 2 = -1+y_3$$
$$\Rightarrow\ x_3 = 8 \quad , \quad y_3 = 3$$
$$\Rightarrow\ D(x_3, y_3) = D(8,3)$$

Hence $(-4,-1)$, $(3,-2)$ and $D(8,3)$ are remaining vertex of $||_{gram}$.

Position of the point with respect to line (Page 204)

Consider $l:\ ax+by+c = 0$ with $b>0$

Then point $P(x_1, y_1)$ lies

i) above the line l if $ax_1+by_1+c > 0$

ii) below the line l if $ax_1+by_1+c < 0$

Corollary 1 (Page 205)

The point $P(x_1, y_1)$ lies above the line if ax_1+by_1+c and b have the same sign and the point $P(x_1, y_1)$ lies below the line if ax_1+by_1+c and b have opposite signs.

Question # 28

Find whether the given point lies above or below the given line

(a) $(5,8);2x-3y+6 = 0$

(b) $(-7,6);4x+3y-9 = 0$

Solution

(a) $2x-3y+6 = 0$

To make coefficient of y positive we multiply above eq. with -1.

$$-2x+3y-6 = 0$$

Putting $(5,8)$ on L.H.S of above

$$-2(5)+3(8)+6 = -10+24-6 = 8 > 0$$

Hence $(5,8)$ lies above the line.

(b) **Alternative Method**

$$4x+3y-9 = 0$$

Putting $(-7,6)$ in L.H.S of given eq.

$$4(-7)+3(6)-9 = -28+18-9 = -19 \quad \ldots\ldots\ldots \text{(i)}$$

Since coefficient of y and expression (i) have opposite signs therefore $(-7,6)$ lies below the line.

Question # 29

Check whether the given points are on the same or opposite sides of the given line.

(a) $(0,0)\,and\,(-4,7);6x-7y+70 = 0$

(b) $(2,3)\,and\,(-2,3);3x-5y+8 = 0$

Solution

(a) $6x-7y+70 = 0$

To make coefficient of y positive we multiply above eq. with -1.

$$-6x + 7y - 70 = 0 \ \ldots\ldots\ (i)$$

Putting $(0,0)$ on L.H.S of (i)

$$-6(0) + 7(0) - 70 = -70 < 0$$

$\Rightarrow$ $(0,0)$ lies below the line.

Putting $(-4,7)$ on L.H.S of (i)

$$-6(-4) + 7(7) - 70 = 24 + 49 - 70 = 3 > 0$$

$\Rightarrow$ $(-4,7)$ lies above the line.

Hence $(0,0)$ and $(-4,7)$ lies on the opposite side of line.

(b) $$3x - 5y + 8 = 0$$

To make coefficient of y positive we multiply above eq. with -1.

$$-3x + 5y - 8 = 0 \ \ldots\ldots\ (i)$$

Putting $(2,3)$ on L.H.S of (i)

$$-3(2) + 5(3) - 8 = -6 + 15 - 8 = 1 > 0$$

$\Rightarrow$ $(2,3)$ lies above the line.

Putting $(-2,3)$ on L.H.S of (i)

$$\begin{aligned} -3(-2) + 5(3) - 8 &= 6 + 15 - 8 \\ &= 13 > 0 \end{aligned}$$

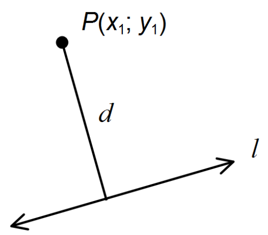
$\Rightarrow$ $(-2,3)$ lies above the line

Hence $(2,3)$ and $(-2,3)$ lies on the same side of line.

Perpendicular distance of P(x1,y1) from line (Page 212)

The distance d from the point $P(x_1, y_1)$ to the line l, where $l:\ ax + by + c = 0$,

is given by: $$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



Question # 30

Find the distance from the point $P(6,-1)$ to the line $6x - 4y + 9 = 0$.

Solution

$$l:\ 6x - 4y + 9 = 0$$

Let d denotes distance of $P(6,-1)$ from line l then

$$d = \frac{|6(6) - 4(-1) + 9|}{\sqrt{(6)^2 + (-4)^2}} = \frac{|36 + 4 + 9|}{\sqrt{36 + 16}} = \frac{|49|}{\sqrt{52}} = \frac{49}{2\sqrt{13}}$$

Area of Triangular Region

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of triangle then

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If A, B and C are collinear then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Question # 31
Find the area of the triangular region whose vertices are $A(5,3), B(-2,2), C(4,2)$.

Solution

Do yourself as below (Just find the area)

Question # 32
The coordinates of three points are $A(2,3), B(-1,1), C(4,-5)$. By computing the area bounded by ABC check whether the points are collinear.

Solution

Given: $A(2,3)$, $B(-1,1)$, $C(4,-5)$

$$\text{Area of } \Delta ABC = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 4 & -5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left(2(1+5)-3(-1-4)+1(5-4)\right)$$

$$= \frac{1}{2}(12+15+1) \quad = \quad \frac{1}{2}(28) \quad = \quad 14 \quad \text{sq. unit}$$

$\because$ Area of triangle $\neq 0$

$\Rightarrow$ A, B and C are not collinear.

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