

Taylor Series Expansion of Function

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

Question # 1

Apply the Maclaurin series expansion to prove that:

$$(i) \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$(ii) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$(iii) \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

$$(iv) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$(v) e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$$

Solution

$$(i) \text{ Let } f(x) = \ln(1+x)$$

$$\Rightarrow f(0) = \ln(1+0) = 0$$

$$f'(x) = \frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$$

$$\Rightarrow f'(0) = \frac{1}{1+0} = 1$$

$$f''(x) = \frac{d}{dx} (1+x)^{-1} = -(1+x)^{-2}$$

$$\Rightarrow f''(0) = -(1+0)^{-2} = -1$$

$$f'''(x) = \frac{d}{dx} [-(1+x)^{-2}] = +2(1+x)^{-3}$$

$$\Rightarrow f'''(0) = 2(1+0)^{-3} = 2$$

$$f^{(iv)}(x) = \frac{d}{dx} 2(1+x)^{-3} = -6(1+x)^{-4}$$

$$\Rightarrow f^{(iv)}(0) = -6(1+0)^{-4} = -6$$

By Maclaurin series

$$\begin{aligned}
 f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\
 \Rightarrow \ln(1+x) &= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots \\
 &= x - \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1}(2) - \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1}(6) + \dots \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots
 \end{aligned}$$

(ii) Let $f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$

$$f'(x) = \frac{d}{dx} \cos x = -\sin x \quad \Rightarrow \quad f'(0) = -\sin(0) = 0$$

$$f''(x) = \frac{d}{dx}(-\sin x) = -\cos x \quad \Rightarrow \quad f''(0) = -\cos(0) = -1$$

$$f'''(x) = \frac{d}{dx}(-\cos x) = +\sin x \quad \Rightarrow \quad f'''(0) = \sin(0) = 0$$

$$f^{(iv)}(x) = \frac{d}{dx} \sin x = \cos x \quad \Rightarrow \quad f^{(iv)}(0) = \cos(0) = 1$$

$$f^{(v)}(x) = \frac{d}{dx} \cos x = -\sin x \quad \Rightarrow \quad f^{(v)}(0) = -\sin(0) = 0$$

$$f^{(vi)}(0) = \frac{d}{dx}(-\sin x) = -\cos x \quad \Rightarrow \quad f^{(vi)}(0) = -\cos(0) = -1$$

Now by Maclaurin series

$$\begin{aligned}
 f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\
 \Rightarrow \cos x &= 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \dots \\
 &= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \dots \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots
 \end{aligned}$$

(iii) Let $f(x) = \sqrt{1+x}$

$$= (1+x)^{\frac{1}{2}} \quad \Rightarrow \quad f(0) = (1+0)^{\frac{1}{2}} = 1$$

$$f'(x) = \frac{d}{dx} (1+x)^{\frac{1}{2}}$$

$$= \frac{1}{2} (1+x)^{-\frac{1}{2}} (1) = \frac{1}{2} (1+x)^{-\frac{1}{2}}$$

$$\begin{aligned} \Rightarrow f'(0) &= \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2} \\ f''(x) &= \frac{d}{dx} \left[\frac{1}{2}(1+x)^{-\frac{1}{2}} \right] = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \\ \Rightarrow f''(0) &= -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4} \\ f'''(x) &= -\frac{1}{4} \frac{d}{dx} \left[(1+x)^{-\frac{3}{2}} \right] \\ &= -\frac{1}{4} \left[-\frac{3}{2}(1+x)^{-\frac{5}{2}} \right] = \frac{3}{8}(1+x)^{-\frac{5}{2}} \\ \Rightarrow f'''(0) &= \frac{3}{8}(1+0)^{-\frac{5}{2}} = \frac{3}{8} \end{aligned}$$

Now by Maclaurin series

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\ \Rightarrow \sqrt{1+x} &= 1 + x \cdot \frac{1}{2} + \frac{x^2}{2!} \cdot \left(-\frac{1}{4} \right) + \frac{x^3}{3!} \cdot \frac{3}{8} + \dots \\ &= 1 + x \cdot \frac{1}{2} + \frac{x^2}{2} \cdot \left(-\frac{1}{4} \right) + \frac{x^3}{6} \cdot \frac{3}{8} + \dots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots \end{aligned}$$

(iv) Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^x) = e^x \Rightarrow f'(0) = e^0 = 1 \\ f''(x) &= \frac{d}{dx}(e^x) = e^x \Rightarrow f''(0) = e^0 = 1 \\ f'''(x) &= \frac{d}{dx}(e^x) = e^x \Rightarrow f'''(0) = e^0 = 1 \end{aligned}$$

By Maclaurin series

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\ \Rightarrow e^x &= 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

(v) Let $f(x) = e^{2x} \Rightarrow f(0) = e^{2(0)} = e^0 = 1$

$$f'(x) = \frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\Rightarrow f'(0) = 2e^{2(0)} = 2(1) = 2$$

$$f''(x) = 2 \frac{d}{dx}(e^{2x}) = 2(2e^{2x}) = 4e^{2x}$$

$$\Rightarrow f''(0) = 4e^{2(0)} = 4(1) = 4$$

$$f'''(x) = 4 \frac{d}{dx}(e^{2x}) = 4(2e^{2x}) = 8e^{2x}$$

$$\Rightarrow f'''(0) = 8e^{2(0)} = 8$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\begin{aligned} \Rightarrow e^{2x} &= 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \dots \\ &= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots \end{aligned}$$

Question # 2

Show that

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2} \cos x + \frac{h^3}{3} \sin x + \dots$$

and evaluate $\cos 61^\circ$.

Solution Let $f(x) = \cos x$

$$f'(x) = \frac{d}{dx} \cos x = -\sin x$$

$$f''(x) = -\frac{d}{dx} \sin x = -\cos x$$

$$f'''(x) = -\frac{d}{dx} \cos x = -(-\sin x) = \sin x$$

By Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\Rightarrow \cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}(\sin x) + \dots$$

$$\Rightarrow \cos(x+h) = \cos x - h \sin x - \frac{h^2}{2} \cos x + \frac{h^3}{3} \sin x + \dots$$

Put $x = 60^\circ$ and $h = 1^\circ = \frac{\pi}{180} = 0.01745$ rad

$$\cos(60+1) = \cos 60 - (0.01745) \sin 60 - \frac{(0.01745)^2}{2} \cos 60 + \frac{(0.01745)^3}{3} \sin 60 + \dots$$

$$\begin{aligned}\Rightarrow \cos 61^\circ &= 0.5 - (0.01745)(0.866) - \frac{(0.000305)}{2}(0.5) + \frac{(0.00000531)}{6}(0.866) + \dots \\ &= 0.5 - 0.0151117 - 0.000076125 + 0.000000072 + \dots \\ &= 0.484812247 \approx 0.4848\end{aligned}$$

Question # 3

Show that

$$2^{x+h} = 2^x \left[1 + (\ln 2)h + (\ln 2)^2 \frac{h^2}{2!} + (\ln 2)^3 \frac{h^3}{3!} + \dots \right]$$

Solution

Let $f(x) = 2^x$

$$\begin{aligned}f'(x) &= \frac{d}{dx} 2^x \quad \because \frac{d}{dx} a^x = a^x \ln a \\ &= 2^x \ln 2\end{aligned}$$

$$f''(x) = \ln 2 \cdot \frac{d}{dx} 2^x = \ln 2 (2^x \ln 2) = (\ln 2)^2 2^x$$

$$f'''(x) = (\ln 2)^2 \frac{d}{dx} 2^x = (\ln 2)^2 2^x \cdot \ln 2 = (\ln 2)^3 2^x$$

Now by Taylor series

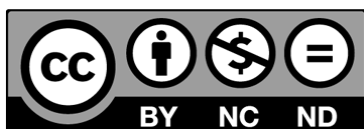
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\begin{aligned}\Rightarrow 2^{x+h} &= 2^x + h \cdot 2^x \cdot \ln 2 + \frac{h^2}{2!} \cdot (\ln 2)^2 2^x + \frac{h^3}{3!} (\ln 2)^3 2^x + \dots \\ &= 2^x \left[1 + (\ln 2)h + (\ln 2)^2 \frac{h^2}{2!} + (\ln 2)^3 \frac{h^3}{3!} + \dots \right]\end{aligned}$$

Book:*Exercise 2.8, page 101**Text Book of Algebra and Trigonometry Class XII**Punjab Textbook Board, Lahore.*

Available online at <http://www.MathCity.org> in PDF Format
(Picture format to view online).

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