

Question # 1Find y_2 if

$$(i) \ y = 2x^5 - 3x^4 + 4x^3 + x - 2 \quad (ii) \ y = (2x+5)^{\frac{3}{2}} \quad (iii) \ y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Solution

$$(i) \quad y = 2x^5 - 3x^4 + 4x^3 + x - 2$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(2x^5 - 3x^4 + 4x^3 + x - 2)$$

$$\begin{aligned} \Rightarrow y_1 &= 2(5x^4) - 3(4x^3) + 4(3x^2) + 1 - 0 \\ &= 10x^4 - 12x^3 + 12x^2 + 1 \end{aligned}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}(10x^4 - 12x^3 + 12x^2 + 1)$$

$$\begin{aligned} \Rightarrow y_2 &= 10(4x^3) - 12(3x^2) + 12(2x) + 0 \\ &= 40x^3 - 36x^2 + 24x \quad \text{Ans.} \end{aligned}$$

$$(ii) \quad y = (2x+5)^{\frac{3}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(2x+5)^{\frac{3}{2}}$$

$$\begin{aligned} \Rightarrow y_1 &= \frac{3}{2}(2x+5)^{\frac{3}{2}-1} \frac{d}{dx}(2x+5) \\ &= \frac{3}{2}(2x+5)^{\frac{1}{2}} (2) = 3(2x+5)^{\frac{1}{2}} \end{aligned}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3 \frac{d}{dx}(2x+5)^{\frac{1}{2}}$$

$$\Rightarrow y_2 = 3 \cdot \frac{1}{2}(2x+5)^{-\frac{1}{2}} (2) \quad \Rightarrow y_2 = \frac{3}{\sqrt{2x+5}}$$

$$(iii) \quad y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$\Rightarrow y = (x)^{\frac{1}{2}} + (x)^{-\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left[(x)^{\frac{1}{2}} + (x)^{-\frac{1}{2}} \right] \Rightarrow y_1 = \frac{1}{2}(x)^{-\frac{1}{2}} - \frac{1}{2}(x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\begin{aligned} \frac{dy_1}{dx} &= \frac{1}{2} \frac{d}{dx} \left[(x)^{-\frac{1}{2}} - (x)^{-\frac{3}{2}} \right] \\ \Rightarrow y_2 &= \frac{1}{2} \left[-\frac{1}{2}(x)^{-\frac{3}{2}} + \frac{3}{2}(x)^{-\frac{5}{2}} \right] \\ &= \frac{1}{4} \left[-\frac{1}{x^{\frac{3}{2}}} + \frac{3}{x^{\frac{5}{2}}} \right] = \frac{1}{4} \left[\frac{-x+3}{x^{\frac{5}{2}}} \right] \quad \text{or} \quad y_2 = \frac{3-x}{4x^{\frac{5}{2}}} \end{aligned}$$

Question # 2

Find y_2 if

(i) $y = x^2 e^{-x}$

(ii) $y = \ln \left(\frac{2x+3}{3x+2} \right)$

Solution

(i) $y = x^2 e^{-x}$

Diff. w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} x^2 e^{-x} \\ \Rightarrow y_1 &= x^2 \frac{d}{dx} e^{-x} + e^{-x} \frac{d}{dx} x^2 \\ &= x^2 e^{-x} (-1) + e^{-x} (2x) \\ &= e^{-x} (-x^2 + 2x) \end{aligned}$$

Again diff. w.r.t x

$$\begin{aligned} \frac{dy_1}{dx} &= \frac{d}{dx} e^{-x} (-x^2 + 2x) \\ y_2 &= e^{-x} \frac{d}{dx} (-x^2 + 2x) + (-x^2 + 2x) \frac{d}{dx} e^{-x} \\ &= e^{-x} (-2x + 2) + (-x^2 + 2x) e^{-x} (-1) \\ &= e^{-x} (-2x + 2 + x^2 - 2x) \\ &= e^{-x} (x^2 - 4x + 2) \end{aligned}$$

(ii) $y = \ln \left(\frac{2x+3}{3x+2} \right)$

$$\Rightarrow y = \ln(2x+3) - \ln(3x+2)$$

Diff. w.r.t x

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{d}{dx} \ln(2x+3) - \frac{d}{dx} \ln(3x+2) \\ \Rightarrow y_1 &= \frac{1}{2x+3} (2) - \frac{1}{3x+2} (3) \end{aligned}$$

$$= 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 2 \frac{d}{dx}(2x+3)^{-1} - 3 \frac{d}{dx}(3x+2)^{-1}$$

$$\Rightarrow y_2 = 2 \left[-(2x+3)^{-2} (2) \right] - 3 \left[-(3x+2)^{-2} (3) \right]$$

$$\Rightarrow y_2 = -\frac{4}{(2x+3)^2} + \frac{9}{(3x+2)^2} \quad \text{Ans.}$$

$$\text{OR } y_2 = \frac{-4(3x+2)^2 + 9(3x+2)^2}{(2x+3)^2 (3x+2)^2}$$

$$= \frac{-4(9x^2 + 12x + 4) + 9(4x^2 + 12x + 9)}{(2x+3)^2 (3x+2)^2}$$

$$= \frac{-36x^2 - 48x - 16 + 36x^2 + 108x + 81}{(2x+3)^2 (3x+2)^2} = \frac{60x + 65}{(2x+3)^2 (3x+2)^2} \quad \text{Ans.}$$

$$(iii) \quad y = \sqrt{\frac{1-x}{1+x}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}}$$

By solving, you will get (differentiate here)

$$\Rightarrow y_1 = \frac{-1}{(1-x)^{\frac{1}{2}} (1+x)^{\frac{3}{2}}} = -(1-x)^{-\frac{1}{2}} (1+x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx} \left[(1-x)^{-\frac{1}{2}} (1+x)^{-\frac{3}{2}} \right]$$

$$\Rightarrow y_2 = -(1-x)^{-\frac{1}{2}} \frac{d}{dx} (1+x)^{-\frac{3}{2}} - (1+x)^{-\frac{3}{2}} \frac{d}{dx} (1-x)^{-\frac{1}{2}}$$

$$= -(1-x)^{-\frac{1}{2}} \left(-\frac{3}{2} (1+x)^{-\frac{5}{2}} (1) \right) - (1+x)^{-\frac{3}{2}} \left(-\frac{1}{2} (1-x)^{-\frac{3}{2}} (-1) \right)$$

$$= \frac{3}{2(1-x)^{\frac{1}{2}} (1+x)^{\frac{5}{2}}} - \frac{1}{2(1+x)^{\frac{3}{2}} (1-x)^{\frac{3}{2}}}$$

$$= \frac{3(1-x) - (1+x)}{2(1-x)^{\frac{3}{2}} (1+x)^{\frac{5}{2}}} = \frac{3-3x-1-x}{2(1-x)^{\frac{3}{2}} (1+x)^{\frac{5}{2}}}$$

$$= \frac{2-4x}{2(1-x)^{\frac{3}{2}} (1+x)^{\frac{5}{2}}} = \frac{1-2x}{(1-x)^{\frac{3}{2}} (1+x)^{\frac{5}{2}}} \quad \text{Ans.}$$

Question # 3Find y_2 if

(i) $x^2 + y^2 = a^2$

(ii) $x^3 - y^3 = a^3$

(iii) $x = a \cos \theta, y = a \sin \theta$

(iv) $x = at^2, y = bt^4$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution

(i) $x^2 + y^2 = a^2$

Diff. w.r.t x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}a^2 \Rightarrow 2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2y y_1 = -2x \Rightarrow y_1 = -\frac{x}{y}$$

Again diff. w.r.t x

$$\Rightarrow \frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x}{y}\right) \Rightarrow y_2 = -\left(\frac{y \frac{dx}{dx} - x \frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow y_2 = -\left(\frac{y(1) - x\left(-\frac{x}{y}\right)}{y^2}\right) \quad \because \frac{dy}{dx} = -\frac{x}{y}$$

$$= -\left(\frac{y + \frac{x^2}{y}}{y^2}\right) = -\left(\frac{y^2 + x^2}{y^2}\right)$$

$$= -\left(\frac{x^2 + y^2}{y^3}\right) \text{ Ans.}$$

$$\text{OR } y_2 = -\frac{a^2}{y^3} \quad \because x^2 + y^2 = a^2$$

(ii) $x^3 - y^3 = a^3$

Diff. w.r.t x

$$\frac{d}{dx}(x^3 - y^3) = \frac{d}{dx}a^3$$

$$3x^2 - 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow -3y^2 y_1 = -3x^2 \Rightarrow y_1 = \frac{x^2}{y^2}$$

Again diff. w.r.t x

$$\begin{aligned} \Rightarrow \frac{dy_1}{dx} &= \frac{d}{dx} \left(\frac{x^2}{y^2} \right) \\ \Rightarrow y_2 &= \frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{(y^2)^2} \\ &= \frac{y^2(2x) - x^2 \left(2y \frac{dy}{dx} \right)}{y^4} \\ &= \frac{2xy^2 - 2x^2 y \left(\frac{x^2}{y^2} \right)}{y^4} \quad \because \frac{dy}{dx} = \frac{x^2}{y^2} \\ &= \frac{2xy^2 - 2x^4}{y^4} = \frac{2xy^3 - 2x^4}{y^4} \\ &= \frac{-2x(x^3 - y^3)}{y^5} \quad \text{Ans.} \end{aligned}$$

OR $y_2 = \frac{-2x(a^3)}{y^5} \quad \because x^3 - y^3 = a^3$

$$\Rightarrow y_2 = -\frac{2a^3 x}{y^5}$$

(iii) $x = a \cos \theta$, $y = a \sin \theta$

Diff. x w.r.t θ

$$\begin{aligned} \frac{dx}{d\theta} &= a \frac{d}{d\theta} \cos \theta \\ &= -a \sin \theta \end{aligned}$$

$$\Rightarrow \frac{d\theta}{dx} = -\frac{1}{a \sin \theta}$$

Now by chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \\ &= a \cos \theta \cdot \frac{-1}{a \sin \theta} \quad \Rightarrow y_1 = -\cot \theta \end{aligned}$$

Now diff. y_1 w.r.t θ

$$\begin{aligned} \frac{dy_1}{dx} &= -\frac{d}{dx} \cot \theta \\ \Rightarrow y_2 &= +\operatorname{cosec}^2 \theta \frac{d\theta}{dx} \end{aligned}$$

$$= \operatorname{cosec}^2 \theta \cdot \left(-\frac{1}{a \sin \theta} \right)$$

$$\Rightarrow y_2 = \frac{-1}{a \sin^3 \theta}$$

(iv) $x = at^2$, $y = bt^4$

Diff. x w.r.t t

$$\frac{dx}{dt} = a \frac{d}{dt} t^2$$

$$= 2at$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{2at}$$

Diff. y w.r.t t

$$\frac{dy}{dt} = b \frac{d}{dt} (t^4)$$

$$= 4bt^3$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 4bt^3 \cdot \frac{1}{2at} \Rightarrow y_1 = \frac{2b}{a} t^2$$

Now diff. y_1 w.r.t x

$$\frac{dy_1}{dx} = \frac{2b}{a} \frac{d}{dx} (t^2) = \frac{2b}{a} \frac{d}{dt} (t^2) \cdot \frac{dt}{dx}$$

$$\Rightarrow y_2 = \frac{2b}{a} (2t) \cdot \frac{1}{2at} \Rightarrow y_2 = \frac{2b}{a^2}$$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow \frac{d}{dx} (x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx} (0)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} + 2g(1) + 2f \frac{dy}{dx} + 0 = 0$$

$$\Rightarrow (2y + 2f) \frac{dy}{dx} + (2x + 2g) = 0$$

$$\Rightarrow (2y + 2f) \frac{dy}{dx} = -(2x + 2g)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2x + 2g)}{(2y + 2f)} \Rightarrow y_1 = -\frac{x + g}{y + f}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx} \left(\frac{x + g}{y + f} \right)$$

$$\Rightarrow y_2 = -\left(\frac{(y + f) \frac{d}{dx} (x + g) - (x + g) \frac{d}{dx} (y + f)}{(y + f)^2} \right)$$

$$= -\frac{(y+f)(1)-(x+g)\frac{dy}{dx}}{(y+f)^2} = -\frac{(y+f)-(x+g)\left(-\frac{x+g}{y+f}\right)}{(y+f)^2}$$

$$= -\frac{\frac{(y+f)^2+(x+g)^2}{y+f}}{(y+f)^2} = -\frac{(y+f)^2+(x+g)^2}{(y+f)^3} \quad \text{Ans.}$$

OR

$$y_2 = -\frac{y^2+2yf+f^2+x^2+2xg+g^2}{(y+f)^3}$$

$$= -\frac{(x^2+y^2+2gx+2fy+c)-c+f^2+g^2}{(y+f)^3}$$

$$= -\frac{0-c+f^2+g^2}{(y+f)^3} \quad \because x^2+y^2+2gx+2fy+c=0$$

$$\Rightarrow y_2 = \frac{c-f^2-g^2}{(y+f)^3} \quad \text{Ans.}$$

Question # 4Find y_4 if

(i) $y = \sin 3x$

(ii) $y = \cos^3 x$

(iii) $y = \ln(x^2 - 9)$

Solution

(i) $y = \sin 3x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$\Rightarrow y_1 = \cos 3x (3) \quad \Rightarrow y_1 = 3\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}\cos 3x \quad \Rightarrow y_2 = 3(-\sin 3x (3)) \quad \Rightarrow y_2 = -9\sin 3x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -9\frac{d}{dx}\sin 3x$$

$$\Rightarrow y_3 = -9\cos 3x (3) \quad \Rightarrow y_3 = -27\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = -27\frac{d}{dx}\cos 3x \quad \Rightarrow y_4 = -27(-\sin 3x (3))$$

$$\Rightarrow \boxed{y_4 = 81\sin 3x}$$

(ii) $y = \cos^3 x$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^3 x)$$

$$\Rightarrow y_1 = 3(\cos^2 x) \frac{d}{dx} \cos x$$

$$\Rightarrow y_1 = 3(\cos^2 x)(-\sin x)$$

$$\Rightarrow y_1 = 3(1 - \sin^2 x)(-\sin x) \quad \Rightarrow y_1 = -3\sin x + 3\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -3 \frac{d}{dx} \sin x + 3 \frac{d}{dx} \sin^3 x$$

$$\Rightarrow y_2 = -3\cos x + 9\sin^2 x \frac{d}{dx} \sin x$$

$$\begin{aligned} \Rightarrow y_2 &= -3\cos x + 9(1 - \cos^2 x)\cos x \\ &= -3\cos x + 9\cos x - 9\cos^3 x = 6\cos x - 9\cos^3 x \end{aligned}$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = 6 \frac{d}{dx} \cos x - 9 \frac{d}{dx} \cos^3 x$$

$$\begin{aligned} \Rightarrow y_3 &= 6(-\sin x) - 9(-3\sin x + 3\sin^3 x) \quad \because \frac{d}{dx}(\cos^3 x) = -3\sin x + 3\sin^3 x \\ &= -6\sin x + 27\sin x - 27\sin^3 x = 21\sin x - 27\sin^3 x \end{aligned}$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 21 \frac{d}{dx} \sin x - 27 \frac{d}{dx} \sin^3 x$$

$$\begin{aligned} \Rightarrow y_4 &= 21(\cos x) - 27(3\sin^2 x) \frac{d}{dx} \sin x \\ &= 21\cos x - 81\sin^2 x(\cos x) = 21\cos x - 81(1 - \cos^2 x)(\cos x) \\ &= 21\cos x - 81\cos x + 81\cos^3 x = -60\cos x + 54\cos^3 x \end{aligned}$$

Alternative:

$$y = \cos^3 x$$

$$\text{Since } \cos 3x = 4\cos^3 x - 3\cos x$$

$$\Rightarrow \cos 3x - 3\cos x = 4\cos^3 x \quad \Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x - 3\cos x)$$

Therefore

$$y = \frac{1}{4}(\cos 3x - 3\cos x)$$

Now diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4} \left(\frac{d}{dx} \cos 3x - 3 \frac{d}{dx} \cos x \right)$$

Do yourself

$$(iii) \quad y = \ln(x^2 - 9) \\ = \ln[(x+3)(x-3)] = \ln(x+3) + \ln(x-3)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \ln(x+3) + \frac{d}{dx} \ln(x-3) \\ \Rightarrow y_1 = \frac{1}{x+3} + \frac{1}{x-3} \\ = (x+3)^{-1} + (x-3)^{-1}$$

Again diff w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx} (x+3)^{-1} + \frac{d}{dx} (x-3)^{-1} \\ \Rightarrow y_2 = -(x+3)^{-2} - (x-3)^{-2}$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -\frac{d}{dx} (x+3)^{-2} - \frac{d}{dx} (x-3)^{-2} \Rightarrow y_3 = 2(x+3)^{-3} + 2(x-3)^{-3}$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 2\frac{d}{dx} (x+3)^{-3} + 2\frac{d}{dx} (x-3)^{-3} \\ \Rightarrow y_4 = 2(-3(x+3)^{-4}) + 2(-3(x-3)^{-4}) \\ = \frac{-6}{(x+3)^4} + \frac{-6}{(x-3)^4} = -6 \left[\frac{1}{(x+3)^4} + \frac{1}{(x-3)^4} \right] \text{ Ans.}$$

Question # 5

If $x = \sin \theta$, $y = \sin m\theta$, Show that $(1-x^2)y_2 - xy_1 + m^2y = 0$

Solution $x = \sin \theta$ (i), $y = \sin m\theta$ (ii)

From (i) $\theta = \sin^{-1} x$, putting in (ii)

$$y = \sin(m \sin^{-1} x)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \sin m(\sin^{-1} x) \\ \Rightarrow y_1 = \cos(m \sin^{-1} x) \frac{d}{dx} m \sin^{-1} x \\ = \cos(m \sin^{-1} x) \cdot m \frac{1}{\sqrt{1-x^2}} \\ \Rightarrow y_1 \sqrt{1-x^2} = m \cos(m \sin^{-1} x)$$

Taking square on both sides.

$$y_1^2 (1-x^2) = m^2 \cos^2(m \sin^{-1} x) \\ \Rightarrow y_1^2 (1-x^2) = m^2 (1 - \sin^2(m \sin^{-1} x)) \quad \because \cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow y_1^2(1-x^2) = m^2(1-y^2) \quad \text{From (ii)}$$

Now again diff. w.r.t x

$$\begin{aligned} \frac{d}{dx} y_1^2(1-x^2) &= m^2 \frac{d}{dx}(1-y^2) \\ \Rightarrow y_1^2 \frac{d}{dx}(1-x^2) + (1-x^2) \frac{d}{dx} y_1^2 &= m^2 \left(0 - 2y \frac{dy}{dx} \right) \\ \Rightarrow y_1^2(-2x) + (1-x^2) 2y_1 \frac{dy_1}{dx} &= -2m^2 y \frac{dy}{dx} \\ \Rightarrow -2xy_1^2 + (1-x^2) 2y_1 y_2 &= -2m^2 y y_1 \\ \Rightarrow 2y_1(-xy_1 + (1-x^2)y_2) &= 2y_1(-m^2 y) \\ \Rightarrow -xy_1 + (1-x^2)y_2 &= -m^2 y \\ \Rightarrow (1-x^2)y_2 - xy_1 + m^2 y &= 0 \quad \text{Proved} \end{aligned}$$

Question # 6

If $y = e^x \sin x$, show that $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$

Solution $y = e^x \sin x$

Diff. w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^x \sin x \\ &= e^x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} e^x \\ &= e^x \cos x + \sin x e^x = e^x (\cos x + \sin x) \end{aligned}$$

Again diff. w.r.t x

$$\begin{aligned} \frac{d}{dx} \left(\frac{dy}{dx} \right) &= \frac{d}{dx} e^x (\cos x + \sin x) \\ \Rightarrow \frac{d^2 y}{dx^2} &= e^x \frac{d}{dx} (\cos x + \sin x) + (\cos x + \sin x) \frac{d}{dx} e^x \\ &= e^x (-\sin x + \cos x) + (\cos x + \sin x) e^x = e^x (-\sin x + \cos x + \cos x + \sin x) \\ &= e^x (2 \cos x) = 2e^x \cos x \end{aligned}$$

Now

$$\begin{aligned} \text{L.H.S} &= \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y \\ &= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x \\ &= 2e^x (\cos x - \cos x - \sin x + \sin x) \\ &= 0 \end{aligned}$$

$$\text{i.e. } \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0 \quad \text{Proved}$$

Question # 7

If $y = e^{ax} \sin bx$, show that $\frac{d^2 y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$

Solution $y = e^{ax} \sin bx$

Diff. w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} e^{ax} \sin bx \\ &= e^{ax} \frac{d}{dx} \sin bx + \sin bx \frac{d}{dx} e^{ax} = e^{ax} \cos bx (b) + \sin bx e^{ax} (a) \\ &= e^{ax} (b \cos bx + a \sin bx)\end{aligned}$$

Again diff. w.r.t x

$$\begin{aligned}\frac{d}{dx} \left(\frac{dy}{dx} \right) &= \frac{d}{dx} e^{ax} (b \cos bx + a \sin bx) \\ \Rightarrow \frac{d^2 y}{dx^2} &= e^{ax} \frac{d}{dx} (b \cos bx + a \sin bx) + (b \cos bx + a \sin bx) \frac{d}{dx} e^{ax} \\ &= e^{ax} (-b \sin bx (b) + a \cos bx (b)) + (b \cos bx + a \sin bx) e^{ax} (a) \\ &= e^{ax} (-b^2 \sin bx + ab \cos bx + ab \cos bx + a^2 \sin bx) \\ &= e^{ax} (2ab \cos bx + a^2 \sin bx - b^2 \sin bx) \\ &= e^{ax} (2ab \cos bx + 2a^2 \sin bx - a^2 \sin bx - b^2 \sin bx) \\ &= e^{ax} [2a(b \cos bx + a \sin bx) - (a^2 + b^2) \sin bx] \\ &= 2ae^{ax} (b \cos bx + a \sin bx) - (a^2 + b^2) e^{ax} \sin bx \\ \Rightarrow \frac{d^2 y}{dx^2} &= 2a \frac{dy}{dx} - (a^2 + b^2)y \quad \Rightarrow \frac{d^2 y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0\end{aligned}$$

Question # 8

If $y = (\cos^{-1} x)^2$, prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$

Solution $y = (\cos^{-1} x)^2$

Diff. w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (\cos^{-1} x)^2 \quad \Rightarrow y_1 = 2(\cos^{-1} x) \frac{d}{dx} \cos^{-1} x \\ \Rightarrow y_1 &= 2(\cos^{-1} x) \cdot \frac{-1}{\sqrt{1-x^2}} \quad \Rightarrow y_1 \sqrt{1-x^2} = -2(\cos^{-1} x)\end{aligned}$$

On squaring both sides

$$\begin{aligned}y_1^2 (1 - x^2) &= 4(\cos^{-1} x)^2 \\ \Rightarrow y_1^2 (1 - x^2) &= 4y \quad \because y = (\cos^{-1} x)^2\end{aligned}$$

Again diff. w.r.t x

$$\frac{d}{dx} y_1^2 (1 - x^2) = 4 \frac{dy}{dx}$$

$$\Rightarrow (1-x^2) \frac{d}{dx} y_1^2 + y_1^2 \frac{d}{dx} (1-x^2) = 4 y_1$$

$$\Rightarrow (1-x^2) \cdot 2 y_1 \frac{dy_1}{dx} + y_1^2 (-2x) = 4 y_1 \Rightarrow 2 y_1 [(1-x^2) y_2 - x y_1] = 4 y_1$$

$$\Rightarrow (1-x^2) y_2 - x y_1 - 2 = 0$$

Question # 9

If $y = a \cos(\ln x) + b \sin(\ln x)$, prove that $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$

Solution $y = a \cos(\ln x) + b \sin(\ln x)$

Diff. w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= a \frac{d}{dx} \cos(\ln x) + b \frac{d}{dx} \sin(\ln x) \\ &= a [-\sin(\ln x)] \frac{d}{dx} (\ln x) + b \cos(\ln x) \frac{d}{dx} (\ln x) \\ &= -a \sin(\ln x) \frac{1}{x} + b \cos(\ln x) \frac{1}{x} \end{aligned}$$

$$\Rightarrow x \frac{dy}{dx} = -a \sin(\ln x) + b \cos(\ln x)$$

Again diff. w.r.t x

$$\frac{d}{dx} \left[x \frac{dy}{dx} \right] = -a \frac{d}{dx} \sin(\ln x) + b \frac{d}{dx} \cos(\ln x)$$

$$\Rightarrow x \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \left(\frac{dx}{dx} \right) = -a \cos(\ln x) \frac{d}{dx} (\ln x) + b (-\sin(\ln x)) \frac{d}{dx} (\ln x)$$

$$\Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot (1) = -a \cos(\ln x) \cdot \frac{1}{x} - b \sin(\ln x) \cdot \frac{1}{x}$$

$$\Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x} (a \cos(\ln x) + b \sin(\ln x)) \Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -y$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0 \quad \text{Proved}$$

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Book:

Exercise 2.7, page 94

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