2.10 Derivative of General Exponential Function (Page 80)

A function define by

$$f(x)=a^x \text{ where } a>0,\ a\neq 1$$

is called general exponential function.

Suppose $y=a^x$

$$\Rightarrow \quad y+\delta y=a^{x+\delta x} \quad \Rightarrow \delta y=a^{x+\delta x}-y$$

$$\Rightarrow \quad \delta y=a^{x+\delta x}-a^x \qquad \text{Since } y=a^x$$

$$\Rightarrow \quad \delta y=a^x(a^{\delta x}-1)$$

Dividing by δx

$$\frac{\delta y}{\delta x}=\frac{a^x(a^{\delta x}-1)}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{a^x(a^{\delta x}-1)}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\lim_{\delta x\to 0}a^x\left(\frac{a^{\delta x}-1}{\delta x}\right) \quad \Rightarrow \quad \frac{dy}{dx}=a^x\lim_{\delta x\to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right)$$

$$\Rightarrow \quad \boxed{\frac{d}{dx}(a^x)=a^x.\ln a} \qquad \text{Since } \lim_{x\to 0}\frac{a^x-1}{x}=\ln a$$

Derivative of Natural Exponential Function

The exponential function $f(x)=e^x$, where $e=2.71828...$, is called Natural Exponential Function.

Suppose $y = e^x$

Do yourself … Just Change *a* by *e* in above article. You'll get

$$\boxed{\frac{d}{dx}e^x = e^x}$$

2.11 Derivative of General Logarithmic Function (page 81)

If $a>0,\ a\neq 1$ and $x=a^y$, then the function defined by $y=\log_a x \quad (x>0)$ is called General Logarithmic Function.

Suppose $y=\log_a x$

$$\Rightarrow \quad y+\delta y=\log_a(x+\delta x) \quad \Rightarrow \quad \delta y=\log_a(x+\delta x)-y$$

$$\Rightarrow \quad \delta y=\log_a(x+\delta x)-\log_a x$$

$$=\log_a\left(\frac{x+\delta x}{x}\right) \qquad \text{Since } \log_a m-\log_a n=\log_a\frac{m}{n}$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$=\lim_{\delta x\to 0}\frac{x}{x}\cdot\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right) \qquad \div\text{ing and }\times\text{ing by } x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{Since } m\log_a x=\log_a x^m$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\log_a\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\log_a e \qquad \text{Since } \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x)=\frac{1}{x}\frac{1}{\log_e a} \qquad \text{Since } \log_a b=\frac{1}{\log_b a}$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x)=\frac{1}{x\ln a} \qquad \text{Since } \log_e a=\ln a$$

Derivative of Natural Logarithmic Function

The logarithmic function $f(x)=\log_e x$ where $e=2.71828...$ is called Natural Logarithmic Function. And we write $\ln x$ instead of $\log_e x$ for our ease.

Suppose $y=\ln x$

$$\Rightarrow \quad y+\delta y=\ln(x+\delta x) \quad \Rightarrow \quad \delta y=\ln(x+\delta x)-y$$

$$\Rightarrow \quad \delta y=\ln(x+\delta x)-\ln x$$

$$\Rightarrow \quad \delta y=\ln\left(\frac{x+\delta x}{x}\right) \qquad \text{Since} \Rightarrow \quad \ln m-\ln n=\ln\frac{m}{n}$$

$$=\ln\left(1+\frac{\delta x}{x}\right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}=\ln\left(1+\frac{\delta x}{x}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{x}{x}\cdot\frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right) \qquad \div\text{ing and }\times\text{ing by } x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\frac{x}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\ln\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{Since } m\ln x=\ln x^m$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\ln\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}\ln e \qquad \text{Since } \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

$$\Rightarrow \quad \frac{d}{dx}(\ln x)=\frac{1}{x}\cdot 1 \qquad \text{Since } \ln e=\log_e e=1$$

$$\Rightarrow \quad \frac{d}{dx}(\ln x)=\frac{1}{x}$$

Exercise 2.6 (Questions)

Question # 1

Find $f'(x)$ if

(i) $f(x)=e^{\sqrt{x}-1}$ (ii) $f(x)=x^3e^{\frac{1}{x}},(x\neq 0)$ (iii) $f(x)=e^x(1+\ln x)$

(iv) $f(x)=\dfrac{e^x}{e^{-x}+1}$ (v) $f(x)=\ln\left(e^x+e^{-x}\right)$ (vi) $f(x)=\dfrac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

(vii) $f(x)=\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}$ (viii) $f(x)=\ln\sqrt{\left(e^{2x}+e^{-2x}\right)}$

Solution

(i) $f(x)=e^{\sqrt{x}-1}$

Diff. w.r.t x

$$\frac{d}{dx}f(x)=\frac{d}{dx}e^{\sqrt{x}-1}$$

$$\Rightarrow f'(x)=e^{\sqrt{x}-1}\frac{d}{dx}\left(\sqrt{x}-1\right)$$

$$=e^{\sqrt{x}-1}\left(\frac{1}{2}x^{-\frac{1}{2}}-0\right)=\frac{e^{\sqrt{x}-1}}{2\sqrt{x}} \quad \textit{Ans.}$$

(ii) $f(x)=x^3e^{\frac{1}{x}}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^3e^{\frac{1}{x}}$$

$$\Rightarrow f'(x) = x^3\frac{d}{dx}e^{\frac{1}{x}} + e^{\frac{1}{x}}\frac{d}{dx}x^3$$

$$= x^3e^{\frac{1}{x}}\frac{d}{dx}\left(\frac{1}{x}\right) + e^{\frac{1}{x}}\left(3x^2\right)$$

$$= x^3e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + e^{\frac{1}{x}}\left(3x^2\right) \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

$$= -xe^{\frac{1}{x}} + 3x^2e^{\frac{1}{x}} = xe^{\frac{1}{x}}(3x-1) \quad \textit{Ans.}$$

(iii) $f(x) = e^x(1+\ln x)$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}e^x(1+\ln x)$$

$$\Rightarrow f'(x) = e^x\frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}e^x$$

$$= e^x\left(0+\frac{1}{x}\right) + (1+\ln x)e^x$$

$$\Rightarrow f'(x) = e^x\left(\frac{1}{x}+1+\ln x\right) \quad \text{or} \quad f'(x) = e^x\left(\frac{1+x(1+\ln x)}{x}\right)$$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{e^x}{e^{-x}+1}\right)$$

$$\Rightarrow f'(x) = \frac{\left(e^{-x}+1\right)\frac{d}{dx}e^x - e^x\frac{d}{dx}\left(e^{-x}+1\right)}{\left(e^{-x}+1\right)^2}$$

$$= \frac{\left(e^{-x}+1\right)e^x - e^x\left(e^{-x}(-1)+0\right)}{\left(e^{-x}+1\right)^2} = \frac{e^x\left(e^{-x}+1+e^{-x}\right)}{\left(e^{-x}+1\right)^2}$$

$$\Rightarrow f'(x) = \frac{e^x\left(2e^{-x}+1\right)}{\left(e^{-x}+1\right)^2} \quad \textit{Ans.}$$

(v) $f(x) = \ln\left(e^x + e^{-x}\right)$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\ln\left(e^x + e^{-x}\right)$$

$$\Rightarrow f'(x) = \frac{1}{\left(e^x + e^{-x}\right)}\frac{d}{dx}\left(e^x + e^{-x}\right)$$

$$= \frac{1}{\left(e^x + e^{-x}\right)}\left(e^x + e^{-x}(-1)\right)$$

$$\Rightarrow f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{or} \quad f'(x) = \tanh x \qquad \because \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(vi) $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}\right)$$

$$= \frac{\left(e^{ax} + e^{-ax}\right)\frac{d}{dx}\left(e^{ax} - e^{-ax}\right) - \left(e^{ax} - e^{-ax}\right)\frac{d}{dx}\left(e^{ax} + e^{-ax}\right)}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{\left(e^{ax} + e^{-ax}\right)\left(e^{ax}(a) - e^{-ax}(-a)\right) - \left(e^{ax} - e^{-ax}\right)\left(e^{ax}(a) + e^{-ax}(-a)\right)}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left(e^{ax} + e^{-ax}\right)\left(e^{ax} + e^{-ax}\right) - a\left(e^{ax} - e^{-ax}\right)\left(e^{ax} - e^{-ax}\right)}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[\left(e^{ax} + e^{-ax}\right)^2 - \left(e^{ax} - e^{-ax}\right)^2\right]}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[(e^{2ax} + e^{-2ax} + 2e^{ax}e^{-ax}) - (e^{2ax} + e^{-2ax} - 2e^{ax}e^{-ax})\right]}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[e^{2ax} + e^{-2ax} + 2 - e^{2ax} - e^{-2ax} + 2\right]}{\left(e^{ax} + e^{-ax}\right)^2} \qquad \because e^{ax}e^{-ax} = e^0 = 1$$

$$\Rightarrow f'(x) = \frac{4a}{\left(e^{ax} + e^{-ax}\right)^2} \quad \textit{Ans.}$$

(vii) $f(x) = \sqrt{\ln\left(e^{2x}+e^{-2x}\right)}$

$\Rightarrow \dfrac{d}{dx}f(x) = \dfrac{d}{dx}\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{\frac{1}{2}}$

$\Rightarrow f'(x) = \dfrac{1}{2}\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{-\frac{1}{2}}\dfrac{d}{dx}\ln\left(e^{2x}+e^{-2x}\right)$

$= \dfrac{1}{2\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{\frac{1}{2}}}\cdot\dfrac{1}{\left(e^{2x}+e^{-2x}\right)}\dfrac{d}{dx}\left(e^{2x}+e^{-2x}\right)$

$= \dfrac{1}{2\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}\cdot\dfrac{1}{\left(e^{2x}+e^{-2x}\right)}\left(e^{2x}(2)+e^{-2x}(-2)\right)$

$= \dfrac{1}{2\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}\cdot\dfrac{2\left(e^{2x}-e^{-2x}\right)}{\left(e^{2x}+e^{-2x}\right)} = \dfrac{\left(e^{2x}-e^{-2x}\right)}{\left(e^{2x}+e^{-2x}\right)\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}$ *Ans.*

(viii) $f(x) = \ln\sqrt{\left(e^{2x}+e^{-2x}\right)}$

$= \ln\left(e^{2x}+e^{-2x}\right)^{\frac{1}{2}} \Rightarrow f(x)=\dfrac{1}{2}\ln\left(e^{2x}+e^{-2x}\right)$ $\because \ln x^m = m\ln x$

Now diff. w.r.t x

$\dfrac{d}{dx}f(x) = \dfrac{1}{2}\dfrac{d}{dx}\ln\left(e^{2x}+e^{-2x}\right)$

Now do yourself

Question # 2

Find $\dfrac{dy}{dx}$ if

(i) $y = x^2\ln\sqrt{x}$ (ii) $y = x\sqrt{\ln x}$ (iii) $y = \dfrac{x}{\ln x}$

(iv) $y = x^2\ln\dfrac{1}{x}$ (v) $y = \ln\sqrt{\dfrac{x^2-1}{x^2+1}}$ (vi) $y = \ln\left(x+\sqrt{x^2+1}\right)$

(vii) $y = \ln\left(9-x^2\right)$ (viii) $y = e^{-2x}\sin 2x$ (ix) $y = e^{-x}\left(x^3+2x^2+1\right)$

(x) $y = xe^{\sin x}$ (xi) $y = 5e^{3x-4}$ (xii) $y = (x+1)^x$

(xiii) $y = (\ln x)^{\ln x}$ (xiv) $y = \dfrac{\sqrt{x^2-1}\,(x+1)}{\left(x^3+1\right)^{3/2}}$

Solution

(i) $y = x^2\ln\sqrt{x}$

$$\Rightarrow\ y = x^2\ln(x)^{\frac{1}{2}} \qquad \Rightarrow\ y = \frac{1}{2}x^2\ln x \qquad \because\ \ln x^m = m\ln x$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}x^2\ln x$$

$$= \frac{1}{2}\left(x^2\frac{d}{dx}\ln x + \ln x\frac{d}{dx}x^2\right)$$

$$= \frac{1}{2}\left(x^2\cdot\frac{1}{x} + \ln x(2x)\right) = \frac{1}{2}x + x\ln x \text{ or } \frac{1}{2}x + 2x\ln\sqrt{x} \quad \textit{Ans.}$$

(ii) $y = x\sqrt{\ln x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x(\ln x)^{\frac{1}{2}}$$

$$= x\frac{d}{dx}(\ln x)^{\frac{1}{2}} + (\ln x)^{\frac{1}{2}}\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{2}(\ln x)^{-\frac{1}{2}}\frac{d}{dx}(\ln x) + (\ln x)^{\frac{1}{2}}(1) = \frac{x}{2(\ln x)^{\frac{1}{2}}}\left(\frac{1}{x}\right) + (\ln x)^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1+2\ln x}{2\sqrt{\ln x}} \quad \textit{Answer}$$

(iii) $y = \dfrac{x}{\ln x}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\ln x}\right)$$

$$= \frac{\ln x\frac{dx}{dx} - x\frac{d}{dx}\ln x}{(\ln x)^2} = \frac{\ln x(1) - x\cdot\frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \qquad \textit{Answer}$$

(iv) $y = x^2\ln\dfrac{1}{x}$

$$\Rightarrow\ y = x^2\ln x^{-1} \qquad \Rightarrow\ y = -x^2\ln x$$

Now do yourself.

(v) $y = \ln\sqrt{\dfrac{x^2-1}{x^2+1}}$

$$\Rightarrow\ y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}} \quad \Rightarrow\ y = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\left(\frac{x^2-1}{x^2+1}\right)}\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{x^2+1}{2\left(x^2-1\right)}\cdot\left(\frac{\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)-\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}\right)$$

$$= \frac{1}{2\left(x^2-1\right)}\cdot\left(\frac{\left(x^2+1\right)(2x)-\left(x^2-1\right)(2x)}{\left(x^2+1\right)}\right)$$

$$= \frac{1}{2\left(x^2-1\right)}\cdot\left(\frac{2x\left(x^2+1-x^2+1\right)}{\left(x^2+1\right)}\right) = \frac{1}{\left(x^2-1\right)}\cdot\left(\frac{x(2)}{\left(x^2+1\right)}\right) = \frac{2x}{\left(x^4-1\right)}$$ *Ans.*

(vi) $y = \ln\left(x+\sqrt{x^2+1}\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\frac{d}{dx}\left(x^2+1\right)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\left(x^2+1\right)^{\frac{1}{2}}}\cdot(2x)\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$ *Answer*

(vii) $y = \ln\left(9-x^2\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(9-x^2\right)$$

$$= \frac{1}{9-x^2}\cdot\frac{d}{dx}\left(9-x^2\right) \quad = \frac{1}{9-x^2}\cdot(-2x)$$

$$\Rightarrow \quad \frac{dy}{dx} \quad = \frac{-2x}{9-x^2}$$

(viii) $y = e^{-2x}\sin 2x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}e^{-2x}\sin 2x$$

$$= e^{-2x}\frac{d}{dx}\sin 2x + \sin 2x\frac{d}{dx}e^{-2x}$$

$$= e^{-2x}\cos 2x\,(2) + \sin 2x\; e^{-2x}(-2) \quad = \quad 2e^{-2x}\left(\cos 2x - \sin 2x\right) \quad \textit{Answer}$$

(ix) $y = e^{-x}\left(x^3+2x^2+1\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}e^{-x}\left(x^3+2x^2+1\right)$$

$$= e^{-x}\frac{d}{dx}\left(x^3+2x^2+1\right)+\left(x^3+2x^2+1\right)\frac{d}{dx}e^{-x}$$

$$= e^{-x}\left(3x^2+4x+0\right)+\left(x^3+2x^2+1\right)\cdot e^{-x}\,(-1)$$

$$= e^{-x}\left(3x^2+4x\right)-\left(x^3+2x^2+1\right)\cdot e^{-x} \quad = \quad e^{-x}\left(3x^2+4x-x^3-2x^2-1\right)$$

$$= e^{-x}\left(-x^3+x^2+4x-1\right) \qquad \textit{Answer}$$

(x) $y = xe^{\sin x}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}xe^{\sin x}$$

$$= x\frac{d}{dx}e^{\sin x} + e^{\sin x}\frac{d}{dx}x$$

$$= x\cdot e^{\sin x}\frac{d}{dx}\sin x + e^{\sin x}(1) \quad = \quad x\cdot e^{\sin x}\cos x + e^{\sin x}$$

$$= e^{\sin x}\left(x\cos x+1\right) \qquad \textit{Answer}$$

(xi) *Do yourself*

(xii) $y = (x+1)^x$

Taking log on both sides

$$\ln y = \ln(x+1)^x \quad \Rightarrow \ln y = x\ln(x+1)$$

Diff w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln(x+1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(x+1)+\ln(x+1)\frac{dx}{dx}$$

$$= x\cdot\frac{1}{x+1}\frac{d}{dx}(x+1)+\ln(x+1)(1)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{x}{x+1}(1)+\ln(x+1)\right)$$

$$= (x+1)^x\left(\frac{x}{x+1}+\ln(x+1)\right) \qquad \textit{Answer}$$

(xiii) $y = (\ln x)^{\ln x}$

Taking log on both sides

$$\ln y = \ln(\ln x)^{\ln x} \quad \Rightarrow \ln y = (\ln x)\cdot\ln(\ln x)$$

Diff w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x)\cdot\ln(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\ln x)\frac{d}{dx}\ln(\ln x)+\ln(\ln x)\frac{d}{dx}(\ln x)$$

$$= (\ln x)\cdot\frac{1}{\ln x}\frac{d}{dx}(\ln x)+\ln(\ln x)\cdot\frac{1}{x}$$

$$= \frac{1}{x}+\frac{\ln(\ln x)}{x} = \frac{1+\ln(\ln x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1+\ln(\ln x)}{x}\right) \qquad \Rightarrow \frac{dy}{dx} = (\ln x)^{\ln x}\left(\frac{1+\ln(\ln x)}{x}\right)$$

(xiv) $$y = \frac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}} \quad \Rightarrow y = \frac{((x+1)(x-1))^{\frac{1}{2}}(x+1)}{\left[(x+1)(x^2-x+1)\right]^{\frac{3}{2}}}$$

$$\Rightarrow y = \frac{(x+1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x+1)}{(x+1)^{\frac{3}{2}}(x^2-x+1)^{\frac{3}{2}}} \quad \Rightarrow y = \frac{(x+1)^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}}(x^2-x+1)^{\frac{3}{2}}}$$

$$\Rightarrow \quad y = \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}$$

Taking log on both sides

$$\ln y = \ln \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}$$

$$= \ln(x-1)^{\frac{1}{2}} - \ln\left(x^2-x+1\right)^{\frac{3}{2}}$$

$$\Rightarrow \ln y = \frac{1}{2}\ln(x-1) - \frac{3}{2}\ln\left(x^2-x+1\right)$$

Now diff. w.r.t x

$$\frac{d}{dx}\ln y = \frac{1}{2}\frac{d}{dx}\ln(x-1) - \frac{3}{2}\frac{d}{dx}\ln\left(x^2-x+1\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x-1}\frac{d}{dx}(x-1) - \frac{3}{2}\frac{1}{\left(x^2-x+1\right)}\frac{d}{dx}\left(x^2-x+1\right)$$

$$= \frac{1}{2(x-1)}(1) - \frac{3}{2\left(x^2-x+1\right)}(2x-1) \quad = \frac{1}{2(x-1)} - \frac{3(2x-1)}{2\left(x^2-x+1\right)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{x^2-x+1-3(2x-1)(x-1)}{2(x-1)\left(x^2-x+1\right)}\right]$$

$$= \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}} \cdot \left[\frac{x^2-x+1-3\left(2x^2-x-2x+1\right)}{2(x-1)\left(x^2-x+1\right)}\right]$$

$$= \left[\frac{x^2-x+1-6x^2+3x+6x-3}{2(x-1)^{1-\frac{1}{2}}\left(x^2-x+1\right)^{\frac{3}{2}+1}}\right] \quad = \frac{-5x^2+8x-2}{2(x-1)^{\frac{1}{2}}\left(x^2-x+1\right)^{\frac{5}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{5x^2-8x+2}{2\sqrt{x-1}\left(x^2-x+1\right)^{\frac{5}{2}}} \text{ Ans.}$$

(xv) $$y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x^2+x-2}}$$

$$\Rightarrow y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x^2+2x-x-2}} \quad \Rightarrow y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{x(x+2)-1(x+2)}}$$

$$\Rightarrow\ y = \frac{(x+2)^2\cdot\sqrt{x-1}}{\sqrt{(x+2)(x-1)}} \quad \Rightarrow\ y = (x+2)^{2-\frac{1}{2}} \Rightarrow\ y = (x+2)^{\frac{3}{2}}$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x+2)^{\frac{3}{2}}$$

Do yourself

2.1.3 Derivative of Hyperbolic Function (page 85)

The hyperbolic functions are define by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad ,\ x\in R\ ; \qquad \cosh x = \frac{e^x + e^{-x}}{2} \quad ,\ x\in R$$

$$\text{and} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad ,\ x\in R$$

The reciprocal of these functions are defined as;

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad ,\ x\in R-\{0\}\ ; \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad ,\ x\in R$$

$$\text{and} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad ,\ x\in R-\{0\}$$

and there derivatives are

(i) $\dfrac{d}{dx}(\sinh x) = \cosh x$ (ii) $\dfrac{d}{dx}(\cosh x) = \sinh x$

(iii) $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ (iv) $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

(v) $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x\tanh x$ (vi) $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x\coth x$

Proof:

(i) $$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{d}{dx}\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) = \frac{1}{2}\frac{d}{dx}\left(e^x - e^{-x}\right)$$

$$= \frac{1}{2}\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) = \frac{1}{2}\left(e^x - e^{-x}(-1)\right) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$= \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x$$

(ii) *Similar as above.*

(iii) *See the below (iv) proof.*

(iv) $$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$$

$$= \frac{\left(e^x - e^{-x}\right)\frac{d}{dx}\left(e^x + e^{-x}\right) - \left(e^x + e^{-x}\right)\frac{d}{dx}\left(e^x - e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{\left(e^x - e^{-x}\right)\left(e^x + e^{-x}(-1)\right) - \left(e^x + e^{-x}\right)\left(e^x - e^{-x}(-1)\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{\left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right) - \left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{\left(e^x - e^{-x}\right)^2 - \left(e^x + e^{-x}\right)^2}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{(e^{2x} + e^{-2x} - 2e^x e^{-x}) - (e^{2x} + e^{-2x} + 2e^x e^{-x})}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} - 2 - e^{2x} - e^{-2x} - 2}{\left(e^x - e^{-x}\right)^2} \qquad \because\ e^x e^{-x} = e^0 = 1$$

$$= \frac{-4}{\left(e^x - e^{-x}\right)^2} = -\left(\frac{2}{e^x - e^{-x}}\right)^2 = -\operatorname{csch}^2 x$$

(v) $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right) = \frac{d}{dx} 2\left(e^x + e^{-x}\right)^{-1} = 2\frac{d}{dx}\left(e^x + e^{-x}\right)^{-1}$$

$$= 2\left[(-1)\left(e^x + e^{-x}\right)^{-1-1}\frac{d}{dx}\left(e^x + e^{-x}\right)\right]$$

$$= -2\left(e^x + e^{-x}\right)^{-2}\left(e^x + e^{-x}(-1)\right) = \frac{-2}{\left(e^x + e^{-x}\right)^2}\left(e^x - e^{-x}\right)$$

$$= \frac{-2\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)} = -\frac{2}{\left(e^x + e^{-x}\right)}\frac{\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)}$$

$$= -\operatorname{sech} x \tanh x$$

(vi) *Do yourself as above (v).*

2.14 Derivative of Inverse Hyperbolic Function (page 86)

(i) $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$ (ii) $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$

(iii) $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$ (iv) $\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}$

(v) $\dfrac{d}{dx}\operatorname{sech}^{-1} x = \dfrac{-1}{x\sqrt{1-x^2}}$ (vi) $\dfrac{d}{dx}\operatorname{csch}^{-1} x = \dfrac{-1}{x\sqrt{1+x^2}}$

Proof:

(i) Let $y = \sinh^{-1} x \Rightarrow \sinh y = x$

differentiate w.r.t. x.

$$\frac{d}{dx}\sinh y = \frac{d}{dx}x \Rightarrow \cosh y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}} \quad \because \cosh^2 x - \sinh^2 x = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad \because \sinh y = x$$

(ii) Do yourself as above.

(iii) Do yourself as (iv) below *or* see book at page 88.

(iv) Let $y = \coth^{-1} x \Rightarrow \coth y = x$

differentiate w.r.t. x

$$\frac{d}{dx}\coth y = \frac{d}{dx}x \Rightarrow -\operatorname{csch}^2 y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{-\operatorname{csch}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{-(\coth^2 y - 1)} \quad \because \coth^2 y - 1 = \operatorname{csch}^2 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{-\coth^2 y + 1} = \frac{1}{1-\coth^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2} \quad \because \coth y = x$$

(v) Suppose $y = \operatorname{sech}^{-1} x \Rightarrow \operatorname{sech} y = x$

differentiate w.r.t. x

$$\frac{d}{dx}\operatorname{sech} y = \frac{d}{dx}x \Rightarrow -\operatorname{sech} y\tanh y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{-\operatorname{sech} y\tanh y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{sech} y\sqrt{1-\tanh^2 y}} \quad \because 1-\tanh^2 y = \operatorname{sech}^2 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} \quad \because \operatorname{sech} y = x$$

(vi) *Do yourself as above*

Question # 3

Find $\frac{dy}{dx}$ if

(i) $y = \cosh 2x$ (ii) $y = \sinh 3x$ (iii) $y = \tanh^{-1}(\sin x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $y = \sinh^{-1}\left(x^3\right)$ (v) $y = \left(\ln \tanh x\right)$ (vi) $y = \sinh^{-1}\left(\frac{x}{2}\right)$

Solution

(i) $y = \cosh 2x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cosh 2x \Rightarrow \frac{dy}{dx} = \sinh 2x \frac{d}{dx}(2x) \Rightarrow \frac{dy}{dx} = 2\sinh 2x$$

(ii) *Do yourself*

(iii) $y = \tanh^{-1}(\sin x) \Rightarrow \tanh y = \sin x$

Diff. w.r.t x

$$\frac{d}{dx}\tanh y = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{\operatorname{sech}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{1 - \tanh^2 y} \qquad \because \cosh^2\theta - \sinh^2\theta = 1$$

$$\therefore 1 - \tanh^2\theta = \operatorname{sech}^2\theta$$

$$= \frac{\cos x}{1 - \sin^2 x} \qquad \because \sin x = \tanh y$$

$$= \frac{\cos x}{\cos^2 x} \Rightarrow \frac{dy}{dx} = \sec x$$

(iv) $y = \sinh^{-1}\left(x^3\right) \Rightarrow \sinh y = x^3$

$$\Rightarrow \frac{d}{dx}\sinh y = \frac{d}{dx}x^3 \Rightarrow \cosh y \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{\cosh y}$$

$$= \frac{3x^2}{\sqrt{1 + \sinh^2 y}} \qquad \because \cosh^2 y - \sinh^2 y = 1$$

$$= \frac{3x^2}{\sqrt{1+\left(x^3\right)^2}} = \frac{3x^2}{\sqrt{1+x^6}}. \quad \textit{Answer}$$

(v) *Do yourself*

(vi) $y = \sinh^{-1}\left(\frac{x}{2}\right) \Rightarrow \sinh y = \frac{x}{2}$

Now diff w.r.t x

$$\frac{d}{dx}\sinh y = \frac{d}{dx}\left(\frac{x}{2}\right) \Rightarrow \cosh y\frac{dy}{dx} = \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\cosh y}$$

$$= \frac{1}{2\sqrt{1+\sinh^2 y}} \qquad \because \cosh^2 y - \sinh^2 y = 1 \quad \therefore \cosh^2 y = 1 + \sinh^2 y$$

$$= \frac{1}{2\sqrt{1+(x/2)^2}} = \frac{1}{2\sqrt{(4+x^2)/2}} = \frac{1}{\sqrt{4+x^2}} \quad \textit{Answer.}$$

Error Analyst		
Muzammil Ahsan	**2009-11**	Govt. Post Graduate Collage Jauharabad Distt. Khushab
Muhammad Usman Saleem	**2013-14**	DPS & IC Jauharabad
Asma Mussarat		Fazaia Degree College Risalpur
Irfan Mehmood		Fazaia Degree College Risalpur

Please report us error at www.mathcity.org/error

Book: ***Exercise 2.4***
Calculus and Analytic Geometry Mathematic 12
Punjab Textbook Board, Lahore.

Available online at http://www.MathCity.org *in PDF Format (Picture format to view online).*
Updated: September,12,2017.

These resources are shared under the licence Attribution-NonCommercial-NoDerivatives 4.0 International
https://creativecommons.org/licenses/by-nc-nd/4.0/
Under this licence if you remix, transform, or build upon the material, you may not distribute the modified material.