Some Important Derivative Formulas

- $\frac{d}{dx}c = 0$ where c is constant
- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\csc x = -\csc x\cot x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\cot x = -\csc^2 x$
- $\frac{d}{dx}\sec x = \sec x\tan x$
- $\frac{d}{dx}Sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$
- $\frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}Cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}Cot^{-1}x = \frac{-1}{1+x^2}$
- $\frac{d}{dx}Csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

Question # 1

Difference the following trigonometric functions from the first principles.

(i) $\sin 2x$ (ii) $\tan 3x$ (iii) $\sin 2x + \cos 2x$ (iv) $\cos x^2$

(v) $\tan^2 x$ (vi) $\sqrt{\tan x}$ (vii) $\cos\sqrt{x}$

Solution

(i) Suppose $y = \sin 2x$

$$\Rightarrow y + \delta y = \sin 2(x + \delta x)$$

$$\Rightarrow \delta y = \sin 2(x + \delta x) - y$$

$$= \sin 2(x + \delta x) - \sin 2x$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{\sin(2x + 2\delta x) - \sin 2x}{\delta x}$$

$$= \frac{2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right)\sin\left(\frac{2x + 2\delta x - 2x}{2}\right)}{\delta x}$$

$$= \frac{2\cos(2x + \delta x)\sin(\delta x)}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{2\cos(2x + \delta x)\sin(\delta x)}{\delta x}$$

$$\frac{dy}{dx} = 2\lim_{\delta x \to 0}\cos(2x + \delta x)\cdot\frac{\sin(\delta x)}{\delta x}$$

$$= 2\lim_{\delta x \to 0}\cos(2x + \delta x)\cdot\lim_{\delta x \to 0}\frac{\sin(\delta x)}{\delta x}$$

$$= 2\cos(2x + 0)\cdot(1) \qquad \because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x}$$

(ii) Let $y = \tan 3x$

$$\Rightarrow y+\delta y = \tan 3(x+\delta x)$$

$$\Rightarrow \delta y = \tan(3x+3\delta x)-\tan 3x$$

$$= \frac{\sin(3x+3\delta x)}{\cos(3x+3\delta x)}-\frac{\sin 3x}{\cos 3x} = \frac{\sin(3x+3\delta x)\cos 3x-\cos(3x+3\delta x)\sin 3x}{\cos(3x+3\delta x)\cos 3x}$$

$$= \frac{\sin(3x+3\delta x-3x)}{\cos(3x+3\delta x)\cos 3x} = \frac{\sin(3\delta x)}{\cos(3x+3\delta x)\cos 3x}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\cdot\frac{\sin(3\delta x)}{\cos(3x+3\delta x)\cos 3x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin(3\delta x)}{\delta x\ \cos(3x+3\delta x)\cos 3x}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\sin(3\delta x)}{\delta x}\cdot\frac{1}{\cos(3x+3\delta x)\cos 3x}\cdot\frac{3}{3} \qquad \times\text{ing and }\div\text{ing 3 on R.H.S}$$

$$= 3\lim_{\delta x\to 0}\frac{\sin(3\delta x)}{3\delta x}\cdot\lim_{\delta x\to 0}\frac{1}{\cos(3x+3\delta x)\cos 3x}$$

$$= 3(1)\cdot\frac{1}{\cos(3x+3(0))\cos 3x}$$

$$= \frac{3}{\cos 3x\,\cos 3x} = \frac{3}{\cos^2 3x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 3\sec^2 3x}$$

(iii) Let $y = \sin 2x+\cos 2x$

$$\Rightarrow y+\delta y = \sin 2(x+\delta x)+\cos 2(x+\delta x)$$

$$\Rightarrow \delta y = \sin 2(x+\delta x)+\cos 2(x+\delta x)-y$$

$$= \sin 2(x+\delta x)+\cos 2(x+\delta x)-\sin 2x-\cos 2x$$

$$= \left[\sin(2x+2\delta x)-\sin 2x\right]+\left[\cos(2x+2\delta x)-\cos 2x\right]$$

$$= \left[2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$+\left[-2\sin\left(\frac{2x+2\delta x-2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$= 2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\left[2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left[2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)\right]$$

$$\frac{dy}{dx} = 2\lim_{\delta x\to 0}\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x} - 2\lim_{\delta x\to 0}\sin(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}$$

$$= 2\cos(2x+0)\cdot(1) - 2\sin(2x+0)\cdot(1) \qquad \text{Since } \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x - 2\sin 2x}$$

(iv) Let $y = \cos x^2$

$$\Rightarrow y + \delta y = \cos(x+\delta x)^2$$

$$\Rightarrow \delta y = \cos(x+\delta x)^2 - \cos x^2$$

$$= -2\sin\left(\frac{(x+\delta x)^2 + x^2}{2}\right)\sin\left(\frac{(x+\delta x)^2 - x^2}{2}\right)$$

$$= -2\sin\left(\frac{x^2 + 2x\delta x + \delta x^2 + x^2}{2}\right)\sin\left(\frac{x^2 + 2x\delta x + \delta x^2 - x^2}{2}\right)$$

$$= -2\sin\left(\frac{2x^2 + 2x\delta x + \delta x^2}{2}\right)\cdot\sin\left(\frac{2x\delta x + \delta x^2}{2}\right)$$

$$= -2\sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right)\cdot\sin\left(x + \frac{\delta x}{2}\right)\delta x$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{1}{\delta x}\cdot 2\sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right)\cdot\sin\left(x + \frac{\delta x}{2}\right)\delta x$$

$\times$ing and $\div$ing $\left(x + \frac{\delta x}{2}\right)$ on R.H.S

$$\Rightarrow \frac{\delta y}{\delta x} = -\left[\frac{2}{\delta x}\sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right)\cdot\sin\left(x + \frac{\delta x}{2}\right)\delta x\right]\cdot\frac{\left(x + \frac{\delta x}{2}\right)}{\left(x + \frac{\delta x}{2}\right)}$$

$$= -\left[2\sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right)\cdot\frac{\sin\left(x + \frac{\delta x}{2}\right)\delta x}{\left(x + \frac{\delta x}{2}\right)\delta x}\right]\cdot\left(x + \frac{\delta x}{2}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x\to 0}\left[2\sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right)\cdot\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\right]\cdot\left(x+\frac{\delta x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = -2\lim_{\delta x\to 0}\sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\cdot\lim_{\delta x\to 0}\left(x+\frac{\delta x}{2}\right)$$

$$= -2\sin\left(x^2+(0)+(0)\right)\cdot(1)\cdot\left(x+(0)\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -2x\sin x^2}$$

(v) Let $y = \tan^2 x$

$$\Rightarrow y+\delta y = \tan^2(x+\delta x)$$

$$\Rightarrow \delta y = \tan^2(x+\delta x) - \tan^2 x$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\tan(x+\delta x)-\tan x\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x)\cos x - \sin x\cos(x+\delta x)}{\cos(x+\delta x)\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x - x)}{\cos(x+\delta x)\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\tan(x+\delta x)+\tan x}{\cos(x+\delta x)\cos x}\right)\cdot\lim_{\delta x\to 0}\left(\frac{\sin\delta x}{\delta x}\right)$$

$$= \left(\frac{\tan(x+0)+\tan x}{\cos(x+0)\cos x}\right)\cdot(1) = \frac{\tan x+\tan x}{\cos x\cdot\cos x} = \frac{2\tan x}{\cos^2 x}$$

$$\Rightarrow \boxed{\dfrac{dy}{dx} = 2\tan x\ \sec^2 x}$$

(vi) Let $y = \sqrt{\tan x}$

$$\Rightarrow y+\delta y = \sqrt{\tan(x+\delta x)}$$

$$\Rightarrow \delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}$$

$$= \left(\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\right)\cdot\left(\frac{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\right)$$

$$= \frac{\tan(x+\delta x) - \tan x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$= \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\cdot\left(\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right)$$

Now do yourself as above.

(vii) Let $y = \cos\sqrt{x}$

$$\Rightarrow y+\delta y = \cos\sqrt{x+\delta x}$$

$$\Rightarrow \delta y = \cos\sqrt{x+\delta x} - \cos\sqrt{x}$$

$$= -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -2\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

As $\delta x = \left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)$, putting in above

$$\Rightarrow \frac{dy}{dx} = -2\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)}$$

$$= -\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)}\cdot\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

$$= -\frac{\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right)}{\left(\sqrt{x+0}+\sqrt{x}\right)}\cdot(1) \qquad \Rightarrow \boxed{\frac{dy}{dx} = -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}}$$

Question # 2

Differentiate the following w.r.t. the variable involved.

(i) $x^2\sec 4x$ (ii) $\tan^3\theta\ \sec^2\theta$ (iii) $\left(\sin 2\theta-\cos 3\theta\right)^2$ (iv) $\cos\sqrt{x}+\sqrt{\sin x}$

Solution

(i) Assume $y = x^2\sec 4x$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2\sec 4x$$

$$= x^2\frac{d}{dx}\sec 4x+\sec 4x\frac{d}{dx}x^2$$

$$= x^2\sec 4x\tan 4x\frac{d}{dx}(4x)+\sec 4x\ (2x)$$

$$= x^2\sec 4x\tan 4x(4)+2x\sec 4x$$

$$= 2x\sec 4x\left(2x\tan 4x+1\right)$$

(ii) Let $y = \tan^3\theta\ \sec^2\theta$

Diff. w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\tan^3\theta\ \sec^2\theta$$

$$= \tan^3\theta\frac{d}{d\theta}\sec^2\theta+\sec^2\theta\frac{d}{d\theta}\tan^3\theta$$

$$= \tan^3\theta\left(2\sec\theta\frac{d}{d\theta}\sec\theta\right)+\sec^2\theta\left(3\tan^2\theta\frac{d}{d\theta}\tan\theta\right)$$

$$= \tan^3\theta\left(2\sec\theta\cdot\sec\theta\tan\theta\right)+\sec^2\theta\left(3\tan^2\theta\cdot\sec^2\theta\right)$$

$$= \sec^2\theta\tan^2\theta\left(2\tan^2\theta+3\sec^2\theta\right)$$

(iii) Let $y = \left(\sin 2\theta-\cos 3\theta\right)^2$

Diff. w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\sin 2\theta-\cos 3\theta\right)^2$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\frac{d}{d\theta}\left(\sin 2\theta-\cos 3\theta\right)$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\left(\cos 2\theta\cdot\frac{d}{d\theta}(2\theta)+\sin 3\theta\cdot\frac{d}{d\theta}(3\theta)\right)$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\left(\cos 2\theta\cdot(2)+\sin 3\theta\cdot(3)\right)$$

$$= 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta)$$

(iv) Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

$$= \cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}\right)$$

$$= -\sin(x)^{\frac{1}{2}}\frac{d}{dx}x^{\frac{1}{2}} + \frac{1}{2}(\sin x)^{-\frac{1}{2}}\frac{d}{dx}(\sin x)$$

$$= -\sin(x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$$

$$= \frac{1}{2}\left(\frac{\cos x}{\sqrt{\sin x}} - \frac{\sin\sqrt{x}}{\sqrt{x}}\right)$$

Question # 3

Find $\frac{dy}{dx}$ if

(i) $y = x\cos y$ (ii) $x = y\cos y$

Solution

(i) Since $y = x\cos y$

$$\frac{dy}{dx} = \frac{d}{dx}x\cos y$$

$$= x\frac{d}{dx}\cos y + \cos y\frac{dx}{dx}$$

$$= x(-\sin y)\frac{dy}{dx} + \cos y(1)$$

$$\Rightarrow \frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y \quad \Rightarrow (1 + x\sin y)\frac{dy}{dx} = \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}$$

(ii) *Do yourself as above*

Question # 4

Find the derivative w.r.t. "x"

(i) $\cos\sqrt{\frac{1+x}{1+2x}}$ (ii) $\sin\sqrt{\frac{1+2x}{1+x}}$

Solution

(i) Since $y = \cos\sqrt{\frac{1+x}{1+2x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cos\sqrt{\frac{1+x}{1+2x}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\left(\sqrt{\frac{1+x}{1+2x}}\right) = -\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)^{\frac{1}{2}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1+2x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1+2x)}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{1+2x-2-2x}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{-1}{(1+2x)^2}\right)$$

$$= \frac{1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}(1+2x)^{2-\frac{1}{2}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}(1+2x)^{\frac{3}{2}}}\sin\sqrt{\frac{1+x}{1+2x}}}$$

(ii) *Do yourself as above.*

Question # 5

Differentiate

(i) $\sin x$ w.r.t. $\cot x$ (ii) $\sin^2 x$ w.r.t. $\cos^4 x$

Solution

(i) Let $y = \sin x$ and $u = \cot x$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin x$$

$$= \cos x$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\cot x$$

$$= -\csc^2 x$$

$$\Rightarrow \frac{dx}{du} = -\frac{1}{\csc^2 x}$$
$$= -\sin^2 x$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= (\cos x)(-\sin^2 x) = -\sin^2 x \cos x$$

(ii) Let $y = \sin^2 x$ and $u = \cos^4 x$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^2 x$$
$$= 2\sin x\frac{d}{dx}(\sin x) = 2\sin x \cos x$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\cos^4 x$$
$$= 4\cos^3 x\frac{d}{dx}(\cos x) = 4\cos^3 x(-\sin x)$$
$$= -4\sin x\cos^3 x$$
$$\Rightarrow \frac{dx}{du} = -\frac{1}{4\sin x\cos^3 x}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= (2\sin x \cos x)\left(-\frac{1}{4\sin x\cos^3 x}\right)$$
$$= -\frac{1}{2}\sec^2 x$$

Question # 6

If $\tan y(1+\tan x) = 1-\tan x$, show that $\dfrac{dy}{dx} = -1$

Solution

Since $\tan y(1+\tan x) = 1-\tan x$

$$\Rightarrow \tan y = \frac{1-\tan x}{1+\tan x}$$
$$= \frac{1-\tan x}{1+1\cdot\tan x} = \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x} = \tan\left(\frac{\pi}{4}-x\right)$$
$$\Rightarrow y = \frac{\pi}{4}-x$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right)$$

$$= 0 - 1 \Rightarrow \frac{dy}{dx} = -1$$

Question # 7

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$, Prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

Solution

Since $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$

Taking square on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}$$

$$= \tan x + \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$$

$$\Rightarrow y^2 = \tan x + y$$

Diff. w.r.t x

$$\frac{d}{dx}y^2 = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \sec^2 x$$

Question # 8

If $x = a\cos^3\theta, \quad y = b\sin^3\theta$, Show that $a\frac{dy}{dx} + b\tan\theta = 0$

Solution

$$x = a\cos^3\theta, \quad y = b\sin^3\theta$$

Diff. x w.r.t θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos^3\theta\right)$$

$$= a \cdot 3\cos^2\theta\frac{d}{d\theta}(\cos\theta) = 3a\cos^2\theta(-\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta \quad \Rightarrow \frac{d\theta}{dx} = \frac{-1}{3a\sin\theta\cos^2\theta}$$

Now diff. y w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\sin^3\theta\right)$$

$$= b \cdot 3\sin^2\theta\frac{d}{d\theta}(\sin\theta) = 3b\sin^2\theta\cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= 3b\sin^2\theta\cos\theta\cdot-\frac{1}{3a\sin\theta\cos^2\theta}$$

$$= -\frac{b}{a}\tan\theta$$

$$\Rightarrow a\frac{dy}{dx} = -b\tan\theta \qquad \Rightarrow a\frac{dy}{dx} + b\tan\theta = 0$$

Question # 9

Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$ and $y = a(\sin t - t\cos t)$

Solution

$x = a(\cos t + \sin t)$ and $y = a(\sin t - t\cos t)$

Do yourself

Derivative of inverse trigonometric formulas

(i) $\frac{d}{dx}Sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

See proof on book page 76

(ii) $\frac{d}{dx}Cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

Proof

Let $y = \cos^{-1}x$ where $x \in [0, \pi]$

$\Rightarrow \cos y = x$

Diff. w.r.t x

$$\frac{d}{dx}\cos y = \frac{dx}{dx} \qquad \Rightarrow -\sin y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$ Since $\sin y$ is positive for $x \in [0, \pi]$

$$= \frac{-1}{\sqrt{1-x^2}}$$

(iii) $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$

See proof on book at page 77

(iv) $\frac{d}{dx}Cot^{-1}x = \frac{-1}{1+x^2}$

Proof

Let $y = \cot^{-1}x$

$\Rightarrow \cot y = x$

Diff. w.r.t x

$$\frac{d}{dx}\cot y = \frac{d}{dx}x \quad \Rightarrow -\csc^2 y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc^2 y}$$

$$= \frac{-1}{1+\cot^2 y} \qquad \because 1+\cot^2 y = \csc^2 y$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$$

(v) $\frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

Proof

Let $y = \sec^{-1}x \quad \Rightarrow \sec y = x$

Diff. w.r.t x

$$\frac{d}{dx}\sec y = \frac{d}{dx}x \quad \Rightarrow \sec y\tan y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y\tan y}$$

$$= \frac{1}{\sec y\sqrt{\sec^2 y-1}} \qquad \because 1+\tan^2 y = \sec^2 y$$

$$\Rightarrow \frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \because \sec y = x$$

(vi) $\frac{d}{dx}Csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

See on book at page 77

Question # 10

Differentiate w.r.t. "x"

(i) $Cos^{-1}\frac{x}{a}$ (ii) $\cot^{-1}\frac{x}{a}$ (iii) $\frac{1}{a}Sin^{-1}\frac{a}{x}$

(iv) $Sin^{-1}\sqrt{1-x^2}$ (v) $Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ (vi) $Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution

(i) Let $y = Cos^{-1}\frac{x}{a}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Cos^{-1}\frac{x}{a}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\frac{d}{dx}x$$

$$= \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a}(1) = \frac{-a}{\sqrt{a^2-x^2}}\cdot\frac{1}{a} = \frac{-1}{\sqrt{a^2-x^2}} \quad Ans$$

(ii) Let $y = \cot^{-1}\frac{x}{a}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}\frac{x}{a}$$

$$= \frac{-1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}\frac{d}{dx}(x)$$

$$= \frac{-a^2}{a^2+x^2}\cdot\frac{1}{a}(1) = \frac{-a}{a^2+x^2}.$$

(iii) Let $y = \frac{1}{a}Sin^{-1}\frac{a}{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{a}\frac{d}{dx}Sin^{-1}\frac{a}{x}$$

$$= \frac{1}{a}\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\frac{d}{dx}\left(\frac{a}{x}\right) = \frac{1}{a}\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}\cdot a\frac{d}{dx}\left(x^{-1}\right)$$

$$= \frac{x}{\sqrt{x^2-a^2}}\left(-x^{-2}\right) = \frac{x}{\sqrt{x^2-a^2}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x\sqrt{x^2-a^2}} \quad Ans$$

(iv) Let $y = Sin^{-1}\sqrt{1-x^2}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Sin^{-1}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{\sqrt{1-1+x^2}}\cdot\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{1}{2}\frac{1}{\left(1-x^2\right)^{\frac{1}{2}}}(-2x) = -\frac{1}{x}\cdot\frac{x}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$$

(v) Let $y = Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\cdot\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{\left(x^2+1\right)^2-\left(x^2-1\right)^2}{(x^2-1)^2}}}\cdot\left(\frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\cdot\frac{\sqrt{\left(x^4+2x^2+1\right)-\left(x^4+2x^2+1\right)}}{(x^2-1)}}\cdot\left(\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{\left(x^2-1\right)^2}{\left(x^2+1\right)\cdot\sqrt{x^4+2x^2+1-x^4+2x^2-1}}\cdot\left(\frac{2x\left(x^2-1-x^2-1\right)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{1}{\left(x^2+1\right)\cdot\sqrt{4x^2}}\cdot\left(2x(-2)\right) = \frac{-4x}{\left(x^2+1\right)\cdot 2x} = \frac{-2}{\left(x^2+1\right)} \quad Ans$$

(vi) *Do yourself as above.*

(vii) *Do yourself as above.*

Question # 11

Show that $\frac{dy}{dx}=\frac{y}{x}$ if $\frac{y}{x}=\tan^{-1}\frac{x}{y}$

Solution

Since $\frac{y}{x} = Tan^{-1}\frac{x}{y} \quad \Rightarrow \quad y = x\,Tan^{-1}\frac{x}{y}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\,Tan^{-1}\frac{x}{y}\right)$$

$$= x\frac{d}{dx}\left(Tan^{-1}\frac{x}{y}\right)+Tan^{-1}\frac{x}{y}\cdot\frac{d}{dx}(x)$$

$$= x\left(\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)\right)+Tan^{-1}\frac{x}{y}\cdot(1)$$

$$= x\left(\frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{y(1)-x\frac{dy}{dx}}{y^2}\right)\right)+Tan^{-1}\frac{x}{y} \quad = \frac{x}{y^2+x^2}\left(y-x\frac{dy}{dx}\right)+\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{y^2+x^2}-\frac{x^2}{y^2+x^2}\cdot\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}+\frac{x^2}{y^2+x^2}\cdot\frac{dy}{dx} = \frac{xy}{y^2+x^2}+\frac{y}{x} \quad \Rightarrow \left(1+\frac{x^2}{y^2+x^2}\right)\cdot\frac{dy}{dx} = \frac{y}{x}\left(\frac{x^2}{y^2+x^2}+1\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \quad \textit{Proved}$$

Question # 12

If $y = \tan\left(pTan^{-1}x\right)$, show that $\left(1+x^2\right)y_1 - p\left(1+y^2\right) = 0$

Solution

Since $y = \tan\left(pTan^{-1}x\right) \Rightarrow Tan^{-1}y = pTan^{-1}x$

Differentiating w.r.t x

$$\frac{d}{dx}Tan^{-1}y = p\frac{d}{dx}Tan^{-1}x$$

$$\Rightarrow \frac{1}{1+y^2}\frac{dy}{dx} = p\cdot\frac{1}{1+x^2} \quad \Rightarrow \left(1+x^2\right)\frac{dy}{dx} = p\left(1+y^2\right)$$

$$\Rightarrow \left(1+x^2\right)y_1 - p\left(1+y^2\right) = 0 \qquad \text{Since } \frac{dy}{dx} = y_1$$

Error Analyst		
Muzammil Ahsan	**(2009-11)**	Govt. Post Graduate Collage Jauharabad Distt. Khushab
Nain Malik	**(2012-14)**	
Muhammad Usman Saleem	**(2013-15)**	DPS & IC Jauharabad Distt. Khushab
Faizan Ali	**(2017)**	Punjab Group Of Colleges, Sargodha

Book: **Exercise 2.5, page 79**
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