Formula for the sum

i) $\sum_{k=1}^{n} k = 1+2+3+.........+n = \dfrac{n(n+1)}{2}$

ii) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+.........+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

iii) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+.........+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

iv) $\sum_{k=1}^{n} (1) = 1+1+1+.........+1$ (n times) $= n$

Sum the following series up to n terms.

Question # 1

$1\times1 + 2\times4 + 3\times7 + ...$

Solution

$1\times1 + 2\times4 + 3\times7 + ...$

If T_k denotes the kth term of the series then

$$T_k = k(3k-2) = 3k^2 - 2k$$

$$\begin{aligned}&1+(k-1)3\\ &=1+3k-3\\ &=3k-2\end{aligned}$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(3k^2-2k\right)$$

$$= 3\sum_{k=1}^{n}k^2 - 2\sum_{k=1}^{n}k = 3\left(\frac{n(n+1)(2n+1)}{6}\right) - 2\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)(2n+1)}{2} - n(n+1)$$

$$= \frac{n(n+1)}{2}\left(2n+1-2\right) \qquad = \frac{n(n+1)(2n-1)}{2}$$ *Answer*

Question # 2

$1\times3 + 3\times6 + 5\times9 + ...$

Solution

$1\times3 + 3\times6 + 5\times9 + ...$

If T_k denotes the kth term of the series then

$$T_k = (2k-1)(3k) = 6k^2 - 3k$$

$$\begin{aligned}&1+(k-1)2\\ &=1+2k-2\\ &=2k-1\end{aligned}$$

$$\begin{aligned}&3+(k-1)3\\ &=3+3k-3\\ &=3k\end{aligned}$$

Now do yourself as above

Question # 3

$1\times4+2\times7+3\times10+$.........................

Solution

Do yourself as Question # 1

Question # 4

$3\times5+5\times9+7\times13+$.........................

Solution

Do yourself as Question # 1

Question # 5

$1^2+3^2+5^2+$....................

Solution

$1^2+3^2+5^2+$....................

$$\begin{aligned}&1+(k-1)2\\&=1+2k-2\\&=2k-1\end{aligned}$$

If T_k denotes the kth term of the series then

$$T_k=(2k-1)^2=4k^2-4k+1$$

Let S_n denotes the sum of first n terms of the series then

$$S_n=\sum_{k=1}^{n}\left(4k^2-4k+1\right)$$

$$=4\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+\sum_{k=1}^{n}(1)=4\left(\frac{n(n+1)(2n+1)}{6}\right)-4\left(\frac{n(n+1)}{2}\right)+n$$

$$=\frac{2n(n+1)(2n+1)}{3}-2\big(n(n+1)\big)+n$$

$$=n\left(\frac{2(n+1)(2n+1)}{3}-2(n+1)+1\right)\quad=n\left(\frac{2(2n^2+2n+n+1)}{3}-2n-2+1\right)$$

$$=n\left(\frac{2(2n^2+3n+1)}{3}-2n-1\right)\quad=n\left(\frac{4n^2+6n+2-6n-3}{3}\right)$$

$$=n\left(\frac{4n^2-1}{3}\right)\quad=\frac{n}{3}\left(4n^2-1\right)\qquad \textit{Answer}$$

Question # 6

$2^2+5^2+8^2+$....................

Solution

$2^2+5^2+8^2+$....................

$$\begin{aligned}&2+(k-1)3\\&=2+3k-3\\&=3k-1\end{aligned}$$

Let S_n denotes the sum of first n terms of the series then

$$T_k=(3k-1)^2=9k^2-6k+1$$

Let S_n be the sum of first n term of the series then

Now do yourself as above

Question # 7

$2\times1^2+4\times2^2+6\times3^2+$...................

Solution

$2\times1^2+4\times2^2+6\times3^2+$...................

If T_k denotes the kth term of the series then

$$T_k = (2k)(k)^2 = 2k^3$$

$$\begin{aligned}&2+(k-1)2\\&=2+2k-2\\&=2k\end{aligned}$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}(2k^3) = 2\sum_{k=1}^{n}k^3$$

$$\begin{aligned}&1+(k-1)1\\&=1+k-1\\&=k\end{aligned}$$

$$= 2\left(\frac{n(n+1)}{2}\right)^2 = 2\,\frac{n^2(n+1)^2}{4} = \frac{n^2(n+1)^2}{2} \quad \textit{Answer}$$

Question # 8

$3\times 2^2 + 5\times 3^2 + 7\times 4^2 + \dots\dots\dots\dots$

Solution

$3\times 2^2 + 5\times 3^2 + 7\times 4^2 + \dots\dots\dots\dots$

$$\begin{aligned}&3+(k-1)2\\&=3+2k-2\\&=2k+1\end{aligned}$$

If T_k denotes the kth term of the series then

$$\begin{aligned}T_k &= (2k+1)(k+1)^2 = (2k+1)(k^2+2k+1)\\&= 2k^3 + 4k^2 + 2k + k^2 + 2k + 1\\&= 2k^3 + 5k^2 + 4k + 1\end{aligned}$$

$$\begin{aligned}&2+(k-1)1\\&=2+k-1\\&=k+1\end{aligned}$$

Let S_n denotes the sum of first n terms of the series then

$$\begin{aligned}S_n &= \sum_{k=1}^{n}(2k^3 + 5k^2 + 4k + 1)\\
&= 2\sum_{k=1}^{n}k^3 + 5\sum_{k=1}^{n}k^2 + 4\sum_{k=1}^{n}k + \sum_{k=1}^{n}(1)\\
&= 2\left(\frac{n(n+1)}{2}\right)^2 + 5\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\frac{n(n+1)}{2} + n\\
&= 2\,\frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + 2n(n+1) + n\\
&= n\left(\frac{n(n^2+2n+1)}{2} + \frac{5(2n^2+2n+n+1)}{6} + 2(n+1) + 1\right)\\
&= n\left(\frac{n^3+2n^2+n}{2} + \frac{5(2n^2+3n+1)}{6} + 2n + 2 + 1\right)\\
&= n\left(\frac{n^3+2n^2+n}{2} + \frac{10n^2+15n+5}{6} + 2n + 3\right)\\
&= n\left(\frac{3n^3+6n^2+3n+10n^2+15n+5+12n+18}{6}\right)\\
&= \frac{n}{6}\left(3n^3+16n^2+30n+23\right) \quad \textit{Answer}\end{aligned}$$

Question # 9

$2\times 4\times 7 + 3\times 6\times 10 + 4\times 8\times 13 + \dots\dots\dots\dots$

Solution

$2\times 4\times 7 + 3\times 6\times 10 + 4\times 8\times 13 + \dots\dots\dots\dots$

$$\begin{aligned}&2+(k-1)1\\&=2+k-1\\&=k+1\end{aligned}$$

If T_k denotes the kth term of the series then

$$T_k = 2(k+1)(k+1)(3k+4) = 2(k+1)^2(3k+4)$$
$$= 2(k^2+2k+1)(3k+4)$$
$$= 3k^3+6k^2+3k+4k^2+8k+4$$
$$= 3k^3+10k^2+11k+4$$

$$7+(k-1)3 \\ = 7+3k-3 \\ = 3k+4$$
$$4+(k-1)2 \\ = 4+2k-2 \\ = 2k+2 = 2(k+1)$$

Now do yourself.

Question # 10

$1\times4\times6+4\times7\times10+7\times10\times14+$..................

Solution

$1\times4\times6+4\times7\times10+7\times10\times14+$..................

If T_k denotes the kth term of the series then

$$T_k = (3k-2)(3k+1)\ 2(2k+1) = 2(3k-2)(3k+1)(2k+1)$$
$$= 2(3k-2)(6k^2+2k+3k+1) = 2(3k-2)(6k^2+5k+1)$$
$$= 2(18k^3+15k^2+3k-12k^2-10k-2)$$
$$= 2(18k^3+3k^2-7k-2)$$

$$1+(k-1)3 \\ = 1+3k-3 \\ = 3k-2$$
$$4+(k-1)3 \\ = 4+3k-3 \\ = 3k+1$$
$$6+(k-1)4 \\ = 6+4k-4 \\ = 4k+2 = 2(2k+1)$$

Now do yourself.

Question # 11

$1+(1+2)+(1+2+3)+$......................

Solution

$1+(1+2)+(1+2+3)+$......................

If T_k denotes the kth term of the series then

$$T_k = 1+2+3+\ldots\ldots\ldots+k = \frac{k(k+1)}{2} = \frac{1}{2}(k^2+k)$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{2}(k^2+k)\right) = \frac{1}{2}\sum_{k=1}^{n}k^2 + \frac{1}{2}\sum_{k=1}^{n}k$$
$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3}+1\right) = \frac{n(n+1)}{4}\left(\frac{2n+1+3}{3}\right)$$
$$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right) = \frac{n(n+1)(2n+4)}{12} \qquad \textit{Answer}$$

Question # 12

$1^2+(1^2+2^2)+(1^2+2^2+3^2)+$.................

Solution

$1^2+(1^2+2^2)+(1^2+2^2+3^2)+$.................

If T_k denotes the kth term of the series then

$$T_k = 1^2 + 2^2 + 3^2 + ...+ k^2 = \frac{k(k+1)(2k+1)}{6}$$
$$= \frac{1}{6}k\left(2k^2 + 2k + k + 1\right) = \frac{1}{6}k\left(2k^2 + 3k + 1\right)$$
$$= \frac{1}{6}\left(2k^3 + 3k^2 + k\right) = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right) = \frac{1}{3}\sum_{k=1}^{n}k^3 + \frac{1}{2}\sum_{k=1}^{n}k^2 + \frac{1}{6}\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\left(\frac{n^2(n+1)^2}{4}\right) + \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{6}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{n(n+1)}{12}\left(n(n+1) + (2n+1) + 1\right) = \frac{n(n+1)}{12}\left(n^2 + n + 2n + 1 + 2\right)$$
$$= \frac{n(n+1)(n^2+3n+3)}{12} \qquad \textit{Answer}$$

Question # 13

$$2 + (2+5) + (2+5+8) + ...$$

Solution

$$2 + (2+5) + (2+5+8) + ...$$

If T_k denotes the kth term of the series then

$\because\ S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$$T_k = 2 + 5 + 8 + ... + \text{ up to } k \text{ terms}$$
$$= \frac{k}{2}\left[2(2) + (k-1)(3)\right] = \frac{k}{2}\left[4 + 3k - 3\right] = \frac{k}{2}\left[3k+1\right] = \frac{3}{2}k^2 + \frac{1}{2}k$$

Now do yourself

Question # 14

Sum the series

(i) $1^2 - 2^2 + 3^2 - 4^2 + ... + (2n-1)^2 - (2n)^2$

(ii) $1^2 - 3^2 + 5^2 - 7^2 + ... + (4n-3)^2 - (4n-1)^2$

(iii) $\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + ...$

Solution

(i)

$$1^2 - 2^2 + 3^2 - 4^2 + ... + (2n-1)^2 - (2n)^2$$

If T_k denotes the kth term of the series then

$$T_k = (2k-1)^2 - (2k)^2 = 4k^2 - 4k + 1 - 4k^2 = -4k + 1$$

Now do yourself

(ii) $1^2 - 3^2 + 5^2 - 7^2 + ... + (4n-3)^2 - (4n-1)^2$

If T_k denotes the kth term of the series then

$$T_k = (4k-3)^2 - (4k-1)^2 = (16k^2 - 24k + 9) - (16k^2 - 8k + 1)$$

$$=16k^2-24k+9-16k^2+8k-1=-16k+8$$

Now do yourself

(iii) $\dfrac{1^2}{1}+\dfrac{1^2+2^2}{2}+\dfrac{1^2+2^2+3^2}{3}+\ldots\ldots\ldots$ to n terms

If T_k denotes the kth term of the series then

$$T_k=\frac{1^2+2^2+3^2+\ldots\ldots+k^2}{k}$$

$$=\frac{k(k+1)(2k+1)\big/6}{k}=\frac{k(k+1)(2k+1)}{6k}=\frac{(k+1)(2k+1)}{6}$$

$$=\frac{2k^2+2k+k+1}{6}=\frac{2k^2+3k+1}{6}=\frac{2}{6}k^2+\frac{3}{6}k+\frac{1}{6}$$

$$=\frac{1}{3}k^2+\frac{1}{2}k+\frac{1}{6}$$

Let S_n denotes the sum of first n terms of the series then

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{3}k^2+\frac{1}{2}k+\frac{1}{6}\right)=\frac{1}{3}\sum_{k=1}^{n}k^2+\frac{1}{2}\sum_{k=1}^{n}k+\frac{1}{6}\sum_{k=1}^{n}(1)$$

$$=\frac{1}{3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{1}{2}\left(\frac{n(n+1)}{2}\right)+\frac{n}{6}$$

$$=\frac{n(n+1)(2n+1)}{18}+\frac{n(n+1)}{4}+\frac{n}{3}=\frac{n}{2}\left(\frac{(n+1)(2n+1)}{9}+\frac{n+1}{2}+\frac{1}{3}\right)$$

$$=\frac{n}{2}\left(\frac{(2n^2+2n+n+1)}{9}+\frac{n+1}{2}+\frac{1}{3}\right)=\frac{n}{2}\left(\frac{2n^2+3n+1}{9}+\frac{n+1}{2}+\frac{1}{3}\right)$$

$$=\frac{n}{2}\left(\frac{4n^2+6n+2+9n+9+6}{18}\right)=\frac{n}{2}\left(\frac{4n^2+15n+17}{18}\right)$$

$$=\frac{n}{36}\left(4n^2+15n+17\right)\quad\text{Answer}$$

Question # 15

Find the sum of n term of the series whose nth term are given.

(i) $3n^2+n+1$ (ii) n^2+4n+1

Solution

(i)

Since $T_n=3n^2+n+1$

Therefore $T_k=3k^2+k+1$

Now do yourself

(ii)

Do yourself

Question # 16

Given nth term of the series, find the sum to $2n$ term.

(i) $3n^2+2n+1$ (ii) n^3+2n+3

Solution

(i)

Do yourself as below (Q # 16 (ii))

(ii) Since $T_n = n^3 + 2n + 3$

Therefore $T_k = k^3 + 2k + 3$

Let S_n denotes the sum of first n terms of the series then

$$S_n = \sum_{k=1}^{n}\left(k^3 + 2k + 3\right) = \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k + 3\sum_{k=1}^{n}(1)$$

$$= \frac{n^2(n+1)^2}{4} + 2\,\frac{n(n+1)}{2} + 3n = n\left(\frac{n(n+1)^2}{4} + n + 1 + 3\right)$$

$$= n\left(\frac{n(n^2+2n+1)}{4} + n + 4\right) = n\left(\frac{n^3+2n^2+n}{4} + n + 4\right)$$

$$= n\left(\frac{n^3+2n^2+n+4n+16}{4}\right)$$

$$= \frac{n}{4}\left(n^3 + 2n^2 + 5n + 16\right)$$

Now for sum of first $2n$ terms put $n = 2n$

$$S_{2n} = \frac{2n}{4}\left((2n)^3 + 2(2n)^2 + 5(2n) + 16\right)$$

$$= \frac{n}{2}\left(8n^3 + 8n^2 + 10n + 16\right) = \frac{2n}{2}\left(4n^3 + 4n^2 + 5n + 8\right)$$

$$= n\left(4n^3 + 4n^2 + 5n + 8\right) \quad \text{Answer}$$

Book: **Exercise 6.11 (Page 229)**
Text Book of Algebra and Trigonometry Class XI
Punjab Textbook Board, Lahore.