Solutions of equations reducible to the quadratic equations

In this section we will discuss the equations which are not quadratic but can be reduced to quadratic equations.

Type I

Equations of the form $ax^{2n} + bx^n + c = 0$

Example 1: Solve the equation

$\boldsymbol{x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0}$ $\longrightarrow (\mathbf{1})$

Let $x^{\frac{1}{4}} = y$

$\Rightarrow\ x^{\frac{1}{2}} = y^2$

Therefore equation (1) becomes,

$y^2 - y - 6 = 0$

$y^2 - 3y + 2y - 6 = 0$

$y(y-3) + 2(y-3) = 0$

$(y-3)(y+2) = 0$

$y - 3 = 0\ or\ \ y + 2 = 0$

$y = 3\ or\ \ y = -2$

When $y = 3$

$x^{\frac{1}{4}} = 3$

$x = 3^4$

$x = 81$

When $y = -2$

$x^{\frac{1}{4}} = -2$

$x = (-2)^4$

$x = 16$

Hence solution set is $\{\mathbf{16}, \mathbf{81}\}$

Type II

Equations of the form $(x+a)(x+b)(x+c)(x+d) = k$, a scalar

Where $a + b = c + d$

Example 2: **Solve $\boldsymbol{(x-7)(x-3)(x+1)(x+5) - 1680 = 0}$**

Solution: $(x-7)(x-3)(x+1)(x+5) - 1680 = 0 \longrightarrow (1)$

Rearranging equation (1), we have

$$[(x-7)(x+5)][(x-3)(x+1)] - 1680 = 0$$

$$(x^2 - 2x - 35)(x^2 - 2x - 3) - 1680 = 0$$

Now we let $x^2 - 2x = y$

$$(y - 35)(y - 3) - 1680 = 0$$

$$y^2 - 38y + 105 - 1680 = 0$$

$$y^2 - 38y - 1575 = 0$$

$$y^2 - 38y - 1575 = 0$$

Using quadratic formula

$$y = \frac{38 \pm \sqrt{(38)^2 - 4(1)(-1575)}}{2}$$

$$y = \frac{38 \pm \sqrt{1444 + 6300}}{2}$$

$$y = \frac{38 \pm \sqrt{7744}}{2}$$

$$y = \frac{38 \pm 88}{2}$$

$$y = \frac{38 + 88}{2} \ or\ y = \frac{38 - 88}{2}$$

$$y = 63 \ or\ y = -25$$

When $y = 63$

$x^2 - 2x = 63$

$x^2 - 2x - 63 = 0$

$x^2 - 9x + 7x - 63 = 0$

$x(x - 9) + 7(x - 9) = 0$

$(x - 9)(x + 7) = 0$

$x - 9 = 0 \ or\ x + 7 = 0$

$x = 9 \ or\ x = -7$

When $y = -25$

$x^2 - 2x = -25$

$x^2 - 2x + 25 = 0$

Using quadratic formula

$$x = \frac{2 \pm \sqrt{4 - 4(1)(25)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 - 100}}{2}$$

$$x = \frac{2 \pm \sqrt{-96}}{2}$$

$$x = \frac{2 \pm 4\sqrt{6}}{2}$$

$x = 1 \pm 2\sqrt{6}$

Hence solution set is

$\{\mathbf{-7, 9,\ 1 + 2\sqrt{6}\, i, 1 - 2\sqrt{6}\, i}\}$

<u>Type III</u> Exponential Equations

Equations in which variable occurs in exponent, are called exponential equations.

Example 3: Solve the equation $2^{2x} - 3.2^{x+2} + 32 = 0$

Solution:

$2^{2x} - 3.2^{x+2} + 32 = 0$

$\Rightarrow\ 2^{2x} - 3.2^x.2^2 + 32 = 0$

$\Rightarrow\ 2^{2x} - 3.2^x.4 + 32 = 0$

$\Rightarrow\ 2^{2x} - 12.2^x + 32 = 0$

Now let $2^x = y$, we have

$\Rightarrow\ y^2 - 12y + 32 = 0$

$\Rightarrow\ y^2 - 4y - 8y + 32 = 0$

$\Rightarrow \; y(y-4)-8(y-4)=0$

$\Rightarrow \; (y-4)(y-8)=0$

$\Rightarrow \; y-4=0 \;\; or \;\; y-8=0$

$\Rightarrow \; y=4 \;\; or \;\; y=8$

When $y=4$

$\Rightarrow \; 2^x=4$

$\Rightarrow \; 2^x=2^2$

$\Rightarrow \; x=2$

When $y=8$

$\Rightarrow \; 2^x=8$

$\Rightarrow \; 2^x=2^3$

$\Rightarrow \; x=3$

Hence solution set is $\{2, 3\}$

Example 4: Solve the equation $4^{1+x}+4^{1-x}=10$

Solution:

$4^{1+x}+4^{1-x}=10$

$\Rightarrow \; 4^1.4^x+4^1.4^{-x}=10$

$\Rightarrow \; 4.4^x+\frac{4}{4^x}=10$

Multiplying by 4^x

$\Rightarrow \; 4.4^x.4^x+4=10.4^x$

$\Rightarrow \; 4.4^{2x}+4=10.4^x$

Now let $4^x=y$, we have

$\Rightarrow \; 4y^2+4=10y$

$\Rightarrow \; 4y^2-10y+4=0$

On dividing by 2, we get

$2y^2-5y+2=0$

$\Rightarrow \; 2y^2-4y-y+2=0$

$\Rightarrow \; 2y(y-2)-1(y-2)=0$

$\Rightarrow \; (y-2)(2y-1)=0$

$\Rightarrow \; y-2=0 \;\; or \;\; 2y-1=0$

$\Rightarrow \; y=2 \;\; or \;\; y=\frac{1}{2}$

When $y=2$

$\Rightarrow \; 4^x=2$

$\Rightarrow \; 2^{2x}=2^1$

$\Rightarrow \; 2x=1$

$\Rightarrow \; x=\frac{1}{2}$

When $y=\frac{1}{2}$

$\Rightarrow \; 4^x=\frac{1}{2}$

$\Rightarrow \; 2^{2x}=2^{-1}$

$\Rightarrow \; 2x=-1$

$\Rightarrow \; x=-\frac{1}{2}$

Hence solution set is $\left\{-\frac{1}{2}, \; \frac{1}{2}\right\}$

Type IV Reciprocal Equations

An equation which remain unchanged when x is replaced by $\frac{1}{x}$.

Example 5: Solve the equation $x^4-3x^3+4x^2-3x+1=0$

Solution:

$x^4-3x^3+4x^2-3x+1=0 \;\; \longrightarrow (1)$

Dividing by x^2

$\Rightarrow \; x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0$

Now re-arranging the terms

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)+4=0 \longrightarrow (2)$

Let $x+\frac{1}{x}=y$

$\Rightarrow \left(x+\frac{1}{x}\right)^2=y^2$

$\Rightarrow x^2+\frac{1}{x^2}+2=y^2$

$\Rightarrow x^2+\frac{1}{x^2}=y^2-2$

Using these values in equation (2)

$y^2-2-3y+4=0$

$\Rightarrow y^2-3y+2=0$

$\Rightarrow y^2-2y-y+2=0$

$\Rightarrow y^2-2y-y+2=0$

$\Rightarrow y(y-2)-1(y-2)=0$

$\Rightarrow (y-2)(y-1)=0$

$\Rightarrow y-2=0 \quad or \quad y-1=0$

$\Rightarrow y=2 \quad or \quad y=1$

When $y=2$

$\Rightarrow x+\frac{1}{x}=2$

On multiplying by x

$x^2+1=2x$

$\Rightarrow x^2-2x+1=0$

$\Rightarrow (x-1)^2=0$

$\Rightarrow x-1=0$

$\Rightarrow x=1$

When $y=1$

$\Rightarrow x+\frac{1}{x}=1$

On multiplying by x

$x^2+1=x$

$\Rightarrow x^2-x+1=0$

$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow x=\frac{1\pm\sqrt{1-4}}{2}$

$\Rightarrow x=\frac{1\pm\sqrt{-3}}{2}$

$\Rightarrow x=\frac{1\pm\sqrt{3}\,i}{2}$

Hence solution set is $\left\{\mathbf{1},\frac{\mathbf{1\pm\sqrt{3}\,i}}{\mathbf{2}}\right\}$

Exercise 4.2

Solve the following equations.

Q #1: $\boldsymbol{x^4-6x^2+8=0}$

Solution: $x^4-6x^2+8=0 \longrightarrow (1)$

$Let\ x^2=y \Rightarrow x^4=y^2$

Putting values in (1)

$y^2-6y+8=0$

$\Rightarrow y^2-4y-2y+8=0$

$\Rightarrow y(y-4)-2(y-4)=0$

$\Rightarrow (y-4)(y-2)=0$

$\Rightarrow y-4=0 \quad or \quad y-2=0$

$\Rightarrow y=4 \quad or \quad y=2$

When $y=4 \quad \Rightarrow x^2=4$

$\Rightarrow x=\pm 2$

When $y=2 \quad \Rightarrow x^2=2$

$\Rightarrow x=\pm\sqrt{2}$

Hence solution set = $\{\pm\mathbf{2},\ \pm\sqrt{\mathbf{2}}\}$

Q #2: $\boldsymbol{x^{-2}-10=3x^{-1}}$

Solution: $x^{-2}-10=3x^{-1} \longrightarrow (1)$

$Let\ x^{-1} = y \implies x^{-2} = y^2$

Putting values in (1)

$y^2 - 10 = 3y$

$y^2 - 3y - 10 = 0$

$y^2 - 5y + 2y - 10 = 0$

$y(y-5) + 2(y-5) = 0$

$(y-5)(y+2) = 0$

$y - 5 = 0 \ \ or\ \ y + 2 = 0$

$y = 5\ \ or\ \ \ y = -2$

When $y = 5 \quad \implies x^{-1} = 5$

$\implies \frac{1}{x} = 5 \qquad \implies x = \frac{1}{5}$

When $y = -2 \quad \implies x^{-1} = -2$

$\implies \frac{1}{x} = -2 \qquad \implies x = -\frac{1}{2}$

Hence solution set = $\left\{-\frac{1}{2}, \frac{1}{5}\right\}$

Q #3: $\boldsymbol{x^6 - 9x^3 + 8 = 0}$

Solution: $x^6 - 9x^3 + 8 = 0 \ \ \longrightarrow (1)$

$Let\ x^3 = y \implies x^6 = y^2$

Putting values in (1)

$y^2 - 9y + 8 = 0$

$\implies y^2 - 8y - y + 8 = 0$

$\implies y(y-8) - 1(y-8) = 0$

$\implies (y-8)(y-1) = 0$

$\implies y - 8 = 0\ \ or\ \ y - 1 = 0$

$y = 8\ \ or\ \ \ y = 1$

When $\ \ y = 8 \quad \implies x^3 = 8$

$\implies x^3 = 2^3$

$\implies x^3 - 2^3 = 0$

$\implies (x-2)(x^2 + 2x + 4) = 0$

$\implies x - 2 = 0\ or\ \ x^2 + 2x + 4 = 0$

$\implies x = 2$

$$or \quad x = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2}$$

$$x = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{(-2 \pm 2\sqrt{3}\,i\,)}{2}$$

$$x = -1 \pm \sqrt{3}\,i$$

When $y = 1 \ \ \implies x^3 = 1$

$\implies x^3 - 1 = 0$

$\implies (x-1)(x^2 + x + 1) = 0$

$\implies x - 1 = 0 \qquad or \qquad x^2 + x + 1 = 0$

$\implies x = 1$

$$or \quad x = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2}$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = \frac{-1 \pm \sqrt{3}\,i}{2}$$

Hence solution set is

$\left\{\boldsymbol{1,\ 2,\ -1 \pm \sqrt{3}\,i\,,\ \frac{-1 \pm \sqrt{3}\,i}{2}}\right\}$

Q # 4: $\boldsymbol{8x^6 - 19x^3 - 27 = 0}$

Solution: $8x^6 - 19x^3 - 27 = 0$

$Let\ x^3 = y \implies x^6 = y^2$

$8y^2 - 19y - 27 = 0$

$8y^2 + 8y - 27y - 27 = 0$

$8y(y+1) - 27(y+1) = 0$

$(y+1)(8y - 27) = 0$

$y + 1 = 0 \quad or \quad 8y - 27 = 0$

$y = -1 \quad or \quad y = \dfrac{27}{8}$

When $y = -1$

$\Rightarrow x^3 = -1$

$\Rightarrow x^3 + 1 = 0$

$\Rightarrow x^3 + 1^3 = 0$

$\Rightarrow (x+1)(x^2 - x + 1) = 0$

$\Rightarrow (x+1) = 0 \quad or \quad (x^2 - x + 1) = 0$

$\Rightarrow x = -1$

$$or \ \ x = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1 \pm \sqrt{3}\, i}{2}$$

When $y = \frac{27}{8} \Rightarrow x^3 = \frac{27}{8}$

$\Rightarrow x^3 - \dfrac{27}{8} = 0$

$\Rightarrow x^3 - \left(\dfrac{3}{2}\right)^3 = 0$

$\Rightarrow \left(x - \frac{3}{2}\right)\left(x^2 + \frac{3}{2}x + \frac{9}{4}\right) = 0$

$\Rightarrow x - \frac{3}{2} = 0 \quad or \quad x^2 + \frac{3}{2}x + \frac{9}{4} = 0$

$\Rightarrow x = \frac{3}{2} \qquad or \quad 4x^2 + 6x + 9 = 0$

$$or \quad x = \frac{-6 \pm \sqrt{36 - 4(4)(9)}}{8}$$

$$x = \frac{-6 \pm \sqrt{36 - 144}}{8}$$

$$x = \frac{-6 \pm \sqrt{-108}}{8}$$

$$x = \frac{-6 \pm 6\sqrt{-3}}{8}$$

$$x = \frac{6(-1 \pm \sqrt{3}\, i)}{8}$$

$$x = \frac{3(-1 \pm \sqrt{3}\, i)}{4}$$

Hence solution set

$= \left\{ \mathbf{-1},\ \frac{\mathbf{3}}{\mathbf{2}},\ \frac{\mathbf{1 \pm \sqrt{3}\, i}}{\mathbf{2}},\ \frac{\mathbf{3(-1 \pm \sqrt{3}\, i)}}{\mathbf{4}} \right\}$

Q #5: $\boldsymbol{x^{\frac{2}{5}} + 8 = 6x^{\frac{1}{5}}}$

Solution: $x^{\frac{2}{5}} + 8 = 6x^{\frac{1}{5}}$

$Let \quad x^{\frac{1}{5}} = y \Rightarrow x^{\frac{2}{5}} = y^2$

$y^2 + 8 = 6y$

$y^2 - 6y + 8 = 0$

$y^2 - 4y - 2y + 8 = 0$

$y(y-4) - 2(y-4) = 0$

$(y-4)(y-2) = 0$

$(y-4) = 0 \quad or \quad (y-2) = 0$

$y = 4 \quad or \quad y = 2$

When $y = 4 \quad \Rightarrow x^{\frac{1}{5}} = 4$

$\Rightarrow x = 4^5$

$\Rightarrow x = 1024$

When $y = 2 \quad \Rightarrow x^{\frac{1}{5}} = 2$

$\Rightarrow x = 2^5$

$\Rightarrow x = 32$

Hence solution set is $\{\mathbf{32}, \mathbf{1024}\}$

Q# 6:

$\boldsymbol{(x+1)(x+2)(x+3)(x+4) = 24}$

Solution:

$(x+1)(x+2)(x+3)(x+4) = 24 \longrightarrow (1)$

Re arranging equation (1), we have

$(x+1)(x+4)(x+2)(x+3) = 24$

$(x^2 + 5x + 4)(x^2 + 5x + 6) = 24$

Let $x^2 + 5x = y$

$\Rightarrow (y+4)(y+6) = 24$

$\Rightarrow y^2+10y+24=24$

$\Rightarrow y^2+10y+24-24=0$

$\Rightarrow y^2+10y=0$

$\Rightarrow y(y+10)=0$

$\Rightarrow y=0 \quad or \quad y+10=0$

$or \quad y=-10$

When $y=0 \Rightarrow x^2+5x=0$

$\Rightarrow x(x+5)=0$

$\Rightarrow x=0 \quad or \quad x+5=0$

$\Rightarrow x=0 \quad or \quad x=-5$

When $y=-10 \Rightarrow x^2+5x=-10$

$\Rightarrow x^2+5x+10=0$

$$\Rightarrow x=\frac{-5\pm\sqrt{25-4(1)(10)}}{2}$$

$$\Rightarrow x=\frac{-5\pm\sqrt{25-40}}{2}$$

$$\Rightarrow x=\frac{-5\pm\sqrt{-15}}{2}$$

$$\Rightarrow x=\frac{-5\pm\sqrt{15}\,i}{2}$$

Hence solution set = $\left\{\mathbf{-5,\ 0,\ \frac{-5\pm\sqrt{15}\,i}{2}}\right\}$

Q #7: $\boldsymbol{(x-1)(x+5)(x+8)(x+2)-880=0}$

Solution:

$(x-1)(x+5)(x+8)(x+2)-880=0$

Re arranging

$(x-1)(x+8)(x+5)(x+2)=880$

$(x^2+7x-8)(x^2+7x+10)=880$

Let $x^2+7x=y$

$\Rightarrow (y-8)(y+10)=880$

$\Rightarrow y^2+2y-80=880$

$\Rightarrow y^2+2y-80-880=0$

$\Rightarrow y^2+2y-960=0$

$$\Rightarrow y=\frac{-2\pm\sqrt{4-4(1)(-960)}}{2}$$

$$\Rightarrow y=\frac{-2\pm\sqrt{4+3840}}{2}$$

$$\Rightarrow y=\frac{-2\pm\sqrt{3844}}{2}$$

$\Rightarrow y=\frac{-2\pm 62}{2}$

$\Rightarrow y=\frac{-2+62}{2} \quad or \quad y=\frac{-2-62}{2}$

$\Rightarrow y=30 \quad or \quad y=-32$

When $y=30 \Rightarrow x^2+7x=30$

$\Rightarrow x^2+7x-30=0$

$$\Rightarrow x=\frac{-7\pm\sqrt{49-4(1)(-30)}}{2}$$

$$\Rightarrow x=\frac{-7\pm\sqrt{49+120}}{2}$$

$$\Rightarrow x=\frac{-7\pm\sqrt{169}}{2}$$

$$\Rightarrow x=\frac{-7\pm 13}{2}$$

$\Rightarrow x=3 \quad or \quad x=-10$

When $y=-32 \Rightarrow x^2+7x=-32$

$\Rightarrow x^2+7x+32=0$

$$\Rightarrow x=\frac{-7\pm\sqrt{49-4(1)(32)}}{2}$$

$$\Rightarrow x=\frac{-7\pm\sqrt{49-128}}{2}$$

$$\Rightarrow x=\frac{-7\pm\sqrt{-79}}{2}$$

$\Rightarrow x = \dfrac{-7 \pm \sqrt{79}\, i}{2}$

Hence solution set = $\left\{ \mathbf{-10},\ \mathbf{3},\ \frac{\mathbf{-7 \pm \sqrt{79}\, i}}{\mathbf{2}} \right\}$

Q #8:

$$\boldsymbol{(x-5)(x-7)(x+6)(x+4)-504=0}$$

Solution:

$(x-5)(x-7)(x+6)(x+4)-504=0 \longrightarrow (1)$

Re arranging equation (1), we have

$(x-5)(x+4)(x-7)(x+6)=504$

$(x^2-x-20)(x^2-x-42)=504$

Let $x^2-x=y$

$\Rightarrow (y-20)(y-42)=504$

$\Rightarrow y^2-62y+840=504$

$\Rightarrow y^2-62y+840-504=0$

$\Rightarrow y^2-62y+336=0$

$\Rightarrow y = \dfrac{62 \pm \sqrt{3844-4(1)(336)}}{2}$

$\Rightarrow y = \dfrac{62 \pm \sqrt{2500}}{2}$

$\Rightarrow y = \frac{62 \pm 50}{2}$

$\Rightarrow y = \frac{62+50}{2} \quad or \quad y = \frac{62-50}{2}$

$\Rightarrow y = 56 \quad or \quad y = 6$

When $y = 56 \ \Rightarrow x^2 - x = 56$

$\Rightarrow x^2-x-56=0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1-4(1)(-56)}}{2}$

$\Rightarrow x = \dfrac{1 \pm \sqrt{225}}{2}$

$\Rightarrow x = \dfrac{1 \pm 15}{2}$

$\Rightarrow x = \dfrac{1+15}{2} \quad or \quad \dfrac{1-15}{2}$

$\Rightarrow x = 8 \quad or \quad x = -7$

When $y = 6 \ \Rightarrow x^2 - x = 6$

$\Rightarrow x^2-x-6=0$

$\Rightarrow x^2-3x+2x-6=0$

$\Rightarrow x(x-3)+2(x-3)=0$

$\Rightarrow (x-3)(x+2)=0$

$\Rightarrow (x-3)=0 \quad or \quad (x+2)=0$

$\Rightarrow x = 3 \quad or \quad x = -2$

Hence solution set = $\{\mathbf{-7}, \mathbf{-2},\ \mathbf{3},\ \mathbf{8}\ \}$

Q #9:

$$\boldsymbol{(x-1)(x-2)(x-8)(x+5)+360=0}$$

Solution:

$(x-1)(x-2)(x-8)(x+5)+360=0 \ \longrightarrow (1)$

Re arranging equation (1), we have

$(x-1)(x-2)(x-8)(x+5)=-360$

$(x^2-3x+2)(x^2-3x-40)=-360$

Let $x^2-3x=y$

$\Rightarrow (y+2)(y-40)=-360$

$\Rightarrow y^2-38y-80=-360$

$\Rightarrow y^2-38y-80+360=0$

$\Rightarrow y^2-38y+280=0$

$\Rightarrow y = \dfrac{38 \pm \sqrt{1444-4(1)(280)}}{2}$

$\Rightarrow y = \dfrac{38 \pm \sqrt{324}}{2}$

$\Rightarrow y = \frac{38 \pm 18}{2}$

$\Rightarrow y = \frac{38+18}{2} \quad or \quad y = \frac{38-18}{2}$

$\Rightarrow y = 28 \qquad or \quad y = 10$

When $y = 56 \Rightarrow x^2 - 3x = 28$

$\Rightarrow x^2 - 3x - 28 = 0$

$\Rightarrow x^2 - 7x + 4x - 28 = 0$

$\Rightarrow x(x-7) + 4(x-7) = 0$

$\Rightarrow (x-7)(x+4) = 0$

$\Rightarrow (x-7) = 0 \qquad or \qquad (x+4) = 0$

$\Rightarrow x = 7 \qquad or \qquad x = -4$

When $y = 10 \Rightarrow x^2 - 3x = 10$

$\Rightarrow x^2 - 3x - 10 = 0$

$\Rightarrow x^2 - 5x + 2x - 10 = 0$

$\Rightarrow x(x-5) + 2(x-5) = 0$

$\Rightarrow (x-5)(x+2) = 0$

$\Rightarrow (x-5) = 0 \qquad or \qquad (x+2) = 0$

$\Rightarrow x = 5 \qquad or \qquad x = -2$

Hence solution set = $\{\mathbf{-4}, \mathbf{-2}, \mathbf{5}, \mathbf{7}\}$

Q #10:

$$(\boldsymbol{x}+\mathbf{1})(\mathbf{2}\boldsymbol{x}+\mathbf{3})(\mathbf{2}\boldsymbol{x}+\mathbf{5})(\boldsymbol{x}+\mathbf{3}) = \mathbf{945}$$

Solution:

$(x+1)(2x+3)(2x+5)(x+3) = 945 \longrightarrow (1)$

Re arranging equation (1), we have

$(x+1)\, 2\left(x+\frac{3}{2}\right)\, 2\left(x+\frac{5}{2}\right)(x+3) = 945$

$4(x+1)(x+3)\left(x+\frac{3}{2}\right)\left(x+\frac{5}{2}\right) = 945$

$4(x^2+4x+3)\left(x^2+4x+\frac{15}{4}\right) = 945$

Let $x^2 + 4x = y$

$\Rightarrow 4(y+3)\left(y+\frac{15}{4}\right) = 945$

$\Rightarrow (y+3)(4y+15) = 945$

$\Rightarrow 4y^2 + 27y + 45 = 945$

$\Rightarrow 4y^2 + 27y + 45 - 945 = 0$

$\Rightarrow 4y^2 + 27y - 900 = 0$

$\Rightarrow y = \dfrac{-27 \pm \sqrt{729 - 4(4)(-900)}}{8}$

$\Rightarrow y = \dfrac{-27 \pm \sqrt{729 + 14400}}{8}$

$\Rightarrow y = \frac{-27 \pm 123}{8}$

$\Rightarrow y = \dfrac{-27+123}{8} \qquad or \quad y = \dfrac{-27-123}{8}$

$\Rightarrow y = 12 \qquad or \quad y = -\dfrac{75}{4}$

When $y = 48 \Rightarrow x^2 + 4x = 12$

$\Rightarrow x^2 + 4x - 12 = 0$

$\Rightarrow x^2 + 6x - 2x - 12 = 0$

$\Rightarrow x^2 + 6x - 2x - 12 = 0$

$\Rightarrow x(x+6) - 2(x+6) = 0$

$\Rightarrow (x+6)(x-2) = 0$

$\Rightarrow (x+6) = 0 \qquad or \qquad (x-2) = 0$

$\Rightarrow \quad x = -6 \qquad or \qquad x = 2$

When $y = -\frac{75}{4} \Rightarrow x^2 + 4x = -\frac{75}{4}$

$\Rightarrow 4x^2 + 16x = -75$

$\Rightarrow 4x^2 + 16x + 75 = 0$

$\Rightarrow x = \dfrac{-16 \pm \sqrt{256 - 4(4)(75)}}{8}$

$\Rightarrow x = \dfrac{-16 \pm \sqrt{256 - 1200}}{8}$

$\Rightarrow x = \frac{-16 \pm \sqrt{-944}}{8}$

$\Rightarrow x = \dfrac{-16 \pm 4\sqrt{-59}}{8}$

$\Rightarrow x = \dfrac{-4 \pm \sqrt{-59}}{2}$

Hence solution set = $\left\{ \mathbf{-6,\ 2,\ \frac{-4\pm\sqrt{-59}}{2}} \right\}$

Q #11:

$$(\mathbf{2x-7})(\mathbf{x^2-9})(\mathbf{2x+5})-\mathbf{91}=\mathbf{0}$$

Solution:

$(2x-7)(x^2-9)(2x+5)-91=0 \longrightarrow (1)$

$(2x-7)\ (x+3)(x-3)(2x+5)=91$

$2\left(x-\frac{7}{2}\right)\ (x+3)(x-3)\ 2\left(x+\frac{5}{2}\right)=91$

$4\left(x-\frac{7}{2}\right)\ (x+3)(x-3)\ \left(x+\frac{5}{2}\right)=91$

$4\left(x^2-\frac{1}{2}x-\frac{21}{2}\right)\left(x^2-\frac{1}{2}x-\frac{15}{2}\right)=91$

$2\left(x^2-\frac{1}{2}x-\frac{21}{2}\right)\ 2\left(x^2-\frac{1}{2}x-\frac{15}{2}\right)=91$

$(2x^2-x-21)\ (2x^2-x-15)=91$

Let $2x^2-x=y$

$\Rightarrow (y-21)(y-15)=91$

$\Rightarrow y^2-36y+315=91$

$\Rightarrow y^2-36y+315-91=0$

$\Rightarrow y^2-36y+224=0$

$\Rightarrow y = \dfrac{36 \pm \sqrt{1296-4(1)(224)}}{2}$

$\Rightarrow y = \dfrac{36 \pm \sqrt{1296-896}}{2}$

$\Rightarrow y = \frac{36\pm\sqrt{400}}{2}$

$\Rightarrow y = \frac{36+20}{2} \quad or \quad y = \frac{36-20}{2}$

$\Rightarrow y = 28 \quad or \quad y = 8$

When $y=28 \Rightarrow 2x^2-x=28$

$\Rightarrow 2x^2-x-28=0$

$\Rightarrow 2x^2-8x+7x-28=0$

$\Rightarrow 2x(x-4)+7(x-4)=0$

$\Rightarrow (x-4)(2x+7)=0$

$\Rightarrow (x-4)=0 \quad or \quad (2x+7)=0$

$\Rightarrow \quad x=4 \quad or \quad x=-\dfrac{7}{2}$

When $y=10 \Rightarrow 2x^2-x=8$

$\Rightarrow 2x^2-x-8=0$

$\Rightarrow y = \dfrac{1 \pm \sqrt{1-4(2)(-8)}}{4}$

$\Rightarrow y = \dfrac{1 \pm \sqrt{65}}{4}$

Hence solution set = $\left\{ \mathbf{-\frac{7}{2},\ 4,\ \frac{1\pm\sqrt{65}}{4}} \right\}$

Q #12:

$$(\mathbf{x^2+6x+8})(\mathbf{x^2+14x+48})=\mathbf{105}$$

Solution:

$(x^2+6x+8)(x^2+14x+48)=105 \longrightarrow (1)$

$(x^2+4x+2x+8)\ (x^2+8x+6x+48)=105$

$(x(x+4)+2(x+4))\ (x(x+8)+6(x+8))=105$

$(x+4)(x+2)(x+8)(x+6)=105 \longrightarrow (2)$

Re arranging equation (2), we have

$(x+4)(x+6)(x+2)(x+8)=105$

$(x^2+10x+24)\ (x^2+10x+16)=105$

Let $x^2+10x=y$

$\Rightarrow (y+24)(y+16)=105$

$\Rightarrow y^2+40y+384=105$

$\Rightarrow y^2+40y+384-105=0$

$\Rightarrow y^2 + 40y + 279 = 0$

$\Rightarrow y = \dfrac{-40 \pm \sqrt{1600 - 4(1)(279)}}{2}$

$\Rightarrow y = \dfrac{-40 \pm \sqrt{1600 - 1116}}{2}$

$\Rightarrow y = \frac{-40 \pm \sqrt{484}}{2}$

$\Rightarrow y = \frac{-40+22}{2}$ $\quad or \quad y = \frac{-40-22}{2}$

$\Rightarrow y = -9 \quad or \quad y = -31$

When $y = -9 \Rightarrow x^2 + 10x = -9$

$\Rightarrow x^2 + 10x + 9 = 0$

$\Rightarrow x^2 + 9x + x + 9 = 0$

$\Rightarrow x(x + 9) + 1(x + 9) = 0$

$\Rightarrow (x + 9)(x + 1) = 0$

$\Rightarrow (x + 9) = 0 \quad or \quad (x + 1) = 0$

$\Rightarrow \quad x = -9 \quad or \quad x = -1$

When $y = -31 \Rightarrow x^2 + 10x = -31$

$\Rightarrow x^2 + 10x + 31 = 0$

$\Rightarrow y = \dfrac{-10 \pm \sqrt{100 - 4(1)(31)}}{2}$

$\Rightarrow y = \dfrac{-10 \pm \sqrt{100 - 124}}{2}$

$\Rightarrow y = \dfrac{-10 \pm \sqrt{-24}}{2}$

$\Rightarrow y = -5 \pm \sqrt{-6}$

Hence solution set = $\{\mathbf{-1, -9, \ -5 \pm \sqrt{-6}}\ \}$

Q #13:

$\boldsymbol{(x^2 + 6x - 27)(x^2 - 2x - 35) = 385}$

Solution:

$(x^2 + 6x - 27)(x^2 - 2x - 35) = 385 \longrightarrow (1)$

$(x^2 - 3x + 9x - 27)\ (x^2 - 7x + 5x - 35) = 385$

$(x(x - 3) + 9(x - 3))\ (x(x - 7) + 5(x - 7)) = 385$

$(x - 3)(x + 9)(x - 7)(x + 5) = 385 \longrightarrow (2)$

Re arranging equation (2), we have

$(x - 3)(x + 5)(x + 9)(x - 7) = 385$

$(x^2 + 2x - 15)\ (x^2 + 2x - 63) = 385$

Let $x^2 + 2x = y$

$\Rightarrow (y - 15)(y - 63) = 385$

$\Rightarrow y^2 - 78y + 945 = 385$

$\Rightarrow y^2 - 78y + 945 - 385 = 0$

$\Rightarrow y^2 - 78y + 560 = 0$

$\Rightarrow y^2 - 8y - 70y + 560 = 0$

$\Rightarrow y(y - 8) - 70(y - 8) = 0$

$\Rightarrow (y - 8)(y - 70) = 0$

$\Rightarrow y - 8 = 0 \quad or \quad y - 70 = 0$

$\Rightarrow y = 8 \quad or \quad y = 70$

When $y = 8 \Rightarrow x^2 + 2x = 8$

$\Rightarrow x^2 + 2x - 8 = 0$

$\Rightarrow x^2 + 4x - 2x - 8 = 0$

$\Rightarrow x(x + 4) - 2(x + 4) = 0$

$\Rightarrow (x + 4)(x - 2) = 0$

$\Rightarrow (x + 4) = 0 \quad or \quad (x - 2) = 0$

$\Rightarrow \quad x = -4 \quad or \quad x = 2$

When $y = 70 \Rightarrow x^2 + 2x = 70$

$\Rightarrow x^2 + 2x - 70 = 0$

$\Rightarrow y = \dfrac{-2 \pm \sqrt{4 - 4(1)(-70)}}{2}$

$\Rightarrow y = \dfrac{-2 \pm \sqrt{4 + 280}}{2}$

$\Rightarrow y = \frac{-2 \pm \sqrt{284}}{2}$

$\Rightarrow y = \frac{-2 \pm 2\sqrt{71}}{2}$

$\Rightarrow y = -1 \pm \sqrt{71}$

Hence solution set = $\{-\mathbf{4}, \mathbf{2},\ -1 \pm \sqrt{71}\ \}$

Q #14: $\mathbf{4.2^{2x+1} - 9.2^{x} + 1 = 0}$

Solution: $4.2^{2x+1} - 9.2^{x} + 1 = 0 \longrightarrow (1)$

$4.2^{2x}.2 - 9.2^{x} + 1 = 0$

$8.2^{2x} - 9.2^{x} + 1 = 0 \quad \longrightarrow (2)$

$Let\ \ 2^{x} = y \Longrightarrow\ 2^{2x} = y^{2}$

Then equation (2) becomes

$8y^{2} - 9y + 1 = 0$

$\Longrightarrow \quad 8y^{2} - 8y - y + 1 = 0$

$\Longrightarrow \quad 8y(y-1) - 1(y-1) = 0$

$\Longrightarrow \quad (y-1)(\,8y-1) = 0$

$\Longrightarrow \quad y - 1 = 0 \qquad or \qquad 8y - 1 = 0$

$\Longrightarrow \quad y = 1 \qquad or \qquad y = \frac{1}{8}$

When $y = 1 \qquad \Longrightarrow \ 2^{x} = 1$

$\Longrightarrow \quad 2^{x} = 2^{0}$

$\Longrightarrow \quad x = 0$

When $y = \frac{1}{8} \qquad \Longrightarrow \ 2^{x} = \frac{1}{8}$

$\Longrightarrow \quad 2^{x} = \frac{1}{2^{3}}$

$\Longrightarrow \quad 2^{x} = 2^{-3}$

$\Longrightarrow \quad x = -3$

Hence solution set = $\{-\mathbf{3}, \mathbf{0}\ \}$

Q #15: $\mathbf{2^{x} + 2^{-x+6} - 20 = 0}$

Solution: $2^{x} + 2^{-x+6} - 20 = 0 \ \longrightarrow (1)$

$2^{x} + 2^{-x}.2^{6} - 20 = 0$

$2^{x} + 2^{6}.2^{-x} - 20 = 0$

$2^{x} + 64.\frac{1}{2^{x}} - 20 = 0$

$2^{2x} + 64 - 20.2^{x} = 0 \ \longrightarrow (2)$

$Let\ \ 2^{x} = y \Longrightarrow\ 2^{2x} = y^{2}$

Then equation (2) becomes

$y^{2} + 64 - 20y = 0$

$\Longrightarrow \quad y^{2} - 20y + 64 = 0$

$\Longrightarrow \quad y^{2} - 16y - 4y + 64 = 0$

$\Longrightarrow \quad y(y-16) - 4(y-16) = 0$

$\Longrightarrow \quad (y-16)(\,y-4) = 0$

$\Longrightarrow \quad y - 16 = 0 \qquad or \qquad y - 4 = 0$

$\Longrightarrow \quad y = 1\,6 \qquad or \qquad y = 4$

When $y = 16 \qquad \Longrightarrow 2^{x} = 16$

$\Longrightarrow 2^{x} = 2^{4}$

$\Longrightarrow \ x = 4$

When $y = 4 \qquad \Longrightarrow \quad 2^{x} = 4$

$\Longrightarrow 2^{x} = 2^{2}$

$\Longrightarrow \ x = 2$

Hence solution set = $\{\mathbf{2}, \mathbf{4}\ \}$

Q #16: $\mathbf{4^{x} - 3.2^{x+3} + 128 = 0}$

Solution: $4^{x} - 3.2^{x+3} + 128 = 0 \longrightarrow (1)$

$2^{2x} - 3.2^{x}.2^{3} + 128 = 0$

$2^{2x} - 3.2^{3}.2^{x} + 128 = 0$

$2^{2x} - 24.2^{x} + 128 = 0 \ \longrightarrow (2)$

$Let\ \ 2^{x} = y \Longrightarrow\ 2^{2x} = y^{2}$

Then equation (2) becomes

$$y^2 - 24y + 128 = 0$$

$\Rightarrow \quad y^2 - 16y - 8y + 128 = 0$

$\Rightarrow \quad y(y-16) - 8(y-16) = 0$

$\Rightarrow \quad (y-16)(y-8) = 0$

$\Rightarrow \quad y - 16 = 0 \quad or \quad y - 8 = 0$

$\Rightarrow \quad y = 16 \quad or \quad y = 8$

When $y = 16 \Rightarrow 2^x = 16$

$\Rightarrow \quad 2^x = 2^4$

$\Rightarrow \quad x = 4$

When $y = 8 \Rightarrow 2^x = 8$

$\Rightarrow \quad 2^x = 2^3$

$\Rightarrow \quad x = 3$

Hence solution set = $\{\mathbf{3, 4}\}$

Q #17: $\mathbf{3^{2x-1} - 12.3^x + 81 = 0}$

Solution: $3^{2x-1} - 12.3^x + 81 = 0 \longrightarrow (1)$

$3^{2x}.3^{-1} - 12.3^x + 81 = 0$

$\frac{3^{2x}}{3} - 12.3^x + 81 = 0$

$3^{2x} - 36.3^x + 243 = 0 \longrightarrow (2)$

$Let\ 3^x = y \Rightarrow 3^{2x} = y^2$

Then equation (2) becomes

$$y^2 - 36y + 243 = 0$$

$\Rightarrow \quad y^2 - 27y - 9y + 243 = 0$

$\Rightarrow \quad y(y-27) - 9(y-27) = 0$

$\Rightarrow \quad (y-27)(y-9) = 0$

$\Rightarrow \quad y - 27 = 0 \quad or \quad y - 9 = 0$

$\Rightarrow \quad y = 27 \quad or \quad y = 9$

When $y = 27 \Rightarrow 3^x = 27$

$\Rightarrow \quad 3^x = 3^3$

$\Rightarrow \quad x = 3$

When $y = 9 \Rightarrow 3^x = 9$

$\Rightarrow \quad 3^x = 3^2$

$\Rightarrow \quad x = 2$

Hence solution set = $\{\mathbf{2, 3}\}$

Q #18: $\left(\boldsymbol{x + \frac{1}{x}}\right)^{\mathbf{2}} - \mathbf{3}\left(\boldsymbol{x + \frac{1}{x}}\right) - \mathbf{4} = \mathbf{0}$

Solution:

$$\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) - 4 = 0 \longrightarrow (1)$$

$Let\ \left(x + \frac{1}{x}\right) = y \quad \Rightarrow \left(x + \frac{1}{x}\right)^2 = y^2$

Using these values in (1), we have

$$y^2 - 3y - 4 = 0$$

$\Rightarrow \quad y^2 - 4y + y - 4 = 0$

$\Rightarrow \quad y(y-4) + 1(y-4) = 0$

$\Rightarrow \quad (y-4)(y+1) = 0$

$\Rightarrow \quad y - 4 = 0 \quad or \quad y + 1 = 0$

$\Rightarrow \quad y = 4 \quad or \quad y = -1$

When $y = 4 \Rightarrow x + \frac{1}{x} = 4$

$\Rightarrow \quad x^2 + 1 = 4x$

$\Rightarrow \quad x^2 - 4x + 1 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2}$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4}}{2}$

$\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$

$\Rightarrow x = \dfrac{4 \pm 2\sqrt{3}}{2}$

$\Rightarrow x = 2 \pm \sqrt{3}$

When $y = -1 \Rightarrow \quad x + \frac{1}{x} = -1$

$\Rightarrow \quad x^2 + 1 = -x$

$\Rightarrow \quad x^2 + x + 1 = 0$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{1 - 4(1)(1)}}{2}$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{3}\, i}{2}$

Hence solution set = $\left\{2 \pm \sqrt{3}, \frac{-1 \pm \sqrt{3}\, i}{2}\right\}$

Q #19: $\boldsymbol{x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0}$

Solution: $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = \mathbf{0} \longrightarrow (\mathbf{1})$

Re-arranging (1), we have

$\left(x^2 + \dfrac{1}{x^2}\right) + \left(x + \dfrac{1}{x}\right) - 4 = \mathbf{0} \longrightarrow (\mathbf{2})$

$Let \quad \left(x + \dfrac{1}{x}\right) = y \qquad \Rightarrow \quad \left(x + \dfrac{1}{x}\right)^2 = y^2$

Using these values in (2), we have

$x^2 + \frac{1}{x^2} + 2 = y^2$

$\Rightarrow \quad x^2 + \dfrac{1}{x^2} = y^2 - 2$

$\Rightarrow \quad y^2 - 2 + y - 4 = 0$

$\Rightarrow \quad y^2 + y - 6 = 0$

$\Rightarrow \quad y^2 - 2y + 3y - 6 = 0$

$\Rightarrow \quad y(y-2) + 3(y-2) = 0$

$\Rightarrow \quad (y-2)(y+3) = 0$

$\Rightarrow \quad y - 2 = 0 \qquad or \qquad y + 3 = 0$

$\Rightarrow \quad y = 2 \qquad or \qquad y = -3$

When $y = 2 \quad \Rightarrow \quad x + \frac{1}{x} = 2$

$\Rightarrow \quad x^2 + 1 = 2x$

$\Rightarrow \quad x^2 - 2x + 1 = 0$

$\Rightarrow \quad x^2 - x - x + 1 = 0$

$\Rightarrow x(x-1) - 1(x-1) = 0$

$\Rightarrow (x-1)(x-1) = 0$

$\Rightarrow (x-1) = 0 \qquad or \qquad (x-1) = 0$

$\Rightarrow \quad x = 1 \quad or \qquad x = 1$

When $y = 3 \quad \Rightarrow \quad x + \frac{1}{x} = -3$

$\Rightarrow \quad x^2 + 1 = -3x$

$\Rightarrow \quad x^2 + 3x + 1 = 0$

$\Rightarrow x = \dfrac{-3 \pm \sqrt{9 - 4(1)(1)}}{2}$

$\Rightarrow x = \dfrac{-3 \pm \sqrt{5}}{2}$

Hence solution set = $\left\{1, \frac{-3 \pm \sqrt{5}}{2}\right\}$

Q #20: $\boldsymbol{\left(x - \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 0}$

Solution: $\left(x - \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 0 \longrightarrow (1)$

$Let \quad \left(x + \dfrac{1}{x}\right) = y \quad \Rightarrow \quad \left(x + \dfrac{1}{x}\right)^2 = y^2$

$\Rightarrow \quad x^2 + \dfrac{1}{x^2} + 2 = y^2$

$\Rightarrow \quad x^2 + \dfrac{1}{x^2} = y^2 - 2$

$\Rightarrow \quad x^2 + \dfrac{1}{x^2} - 2 = y^2 - 2 - 2$

$\Rightarrow \quad \left(x-\frac{1}{x}\right)^2 = y^2-4$

Using these values in (1), we have

$y^2-4+3y=0$

$\Rightarrow \quad y^2+3y-4=0$

$\Rightarrow \quad y^2+4y-y-4=0$

$\Rightarrow \quad y(y+4)-1(y+4)=0$

$\Rightarrow \quad (y+4)(y-1)=0$

$\Rightarrow \quad y+4=0 \quad or \quad y-1=0$

$\Rightarrow \quad y=-4 \quad or \quad y=1$

When $y=-4$

$\Rightarrow \quad x+\frac{1}{x}=-4$

$\Rightarrow \quad x^2+1=-4x$

$\Rightarrow \quad x^2+4x+1=0$

$\Rightarrow x=\frac{-4\pm\sqrt{16-4(1)(1)}}{2}$

$\Rightarrow x=\frac{-4\pm\sqrt{12}}{2}$

$\Rightarrow x=\frac{-4\pm 2\sqrt{3}}{2}$

$\Rightarrow x=-2\pm\sqrt{3}$

When $y=1$

$\Rightarrow \quad x+\frac{1}{x}=1$

$\Rightarrow \quad x^2+1=x$

$\Rightarrow \quad x^2-x+1=0$

$\Rightarrow x=\frac{1\pm\sqrt{1-4(1)(1)}}{2}$

$\Rightarrow x=\frac{1\pm\sqrt{-3}}{2}$

$\Rightarrow x=\frac{1\pm\sqrt{3}\,i}{2}$

Hence solution set = $\left\{-2\pm\sqrt{3},\ \frac{1\pm\sqrt{3}\,i}{2}\right\}$

Q #21: $\boldsymbol{2x^4-3x^3-x^2-3x+2=0}$

Solution:

$2x^4-3x^3-x^2-3x+2=0 \longrightarrow (1)$

Dividing equation (1) by x^2

$2x^2-3x-1-\frac{3}{x}+\frac{2}{x^2}=0$

$2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-1=0 \longrightarrow (2)$

$Let \quad \left(x+\frac{1}{x}\right)=y \quad \Rightarrow \left(x+\frac{1}{x}\right)^2 = y^2$

$\Rightarrow \quad x^2+\frac{1}{x^2}+2 = y^2$

$\Rightarrow \quad x^2+\frac{1}{x^2} = y^2-2$

Using these values in (2), we have

$2(y^2-2)-3y-1=0$

$\Rightarrow \quad 2y^2-4-3y-1=0$

$\Rightarrow \quad 2y^2-3y-5=0$

$\Rightarrow \quad 2y^2-5y+2y-5=0$

$\Rightarrow \quad y(2y-5)+1(2y-5)=0$

$\Rightarrow \quad (2y-5)(y+1)=0$

$\Rightarrow \quad 2y-5=0 \quad or \quad y+1=0$

$\Rightarrow \quad y=\frac{5}{2} \quad or \quad y=-1$

When $y=\frac{5}{2} \quad \Rightarrow \quad x+\frac{1}{x}=\frac{5}{2}$

$\Rightarrow \quad x^2+1=\frac{5}{2}x$

$\Rightarrow \quad 2x^2+2=5x$

$\Rightarrow \quad 2x^2-5x+2=0$

$\Rightarrow\ 2x^2-4x-x+2=0$

$\Rightarrow\ 2x(x-2)-1(x-2)=0$

$\Rightarrow\ (x-2)(2x-1)=0$

$\Rightarrow\ x-2=0 \quad or \quad 2x-1=0$

$\Rightarrow\ x=2 \quad or \quad x=\frac{1}{2}$

When $y=-1 \Rightarrow\ x+\frac{1}{x}=-1$

$\Rightarrow\ x^2+1=-x$

$\Rightarrow\ x^2+x+1=0$

$$\Rightarrow x=\frac{-1\pm\sqrt{1-4(1)(1)}}{2}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{-3}}{2}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{3}\,i}{2}$$

Hence solution set = $\left\{\frac{1}{2},\ \mathbf{2}\ ,\frac{-1\pm\sqrt{3}\,i}{2}\ \right\}$

Q #22: $\mathbf{2x^4+3x^3-4x^2-3x+2=0}$

Solution: $2x^4+3x^3-4x^2-3x+2=0$

Dividing by x^2

$$2x^2+3x-4-\frac{3}{x}+\frac{2}{x^2}=0$$

$$2\left(x^2+\frac{1}{x^2}\right)+3\left(x-\frac{1}{x}\right)-4=0$$

$Let\ \left(x-\frac{1}{x}\right)=y \quad \Rightarrow\ \left(x-\frac{1}{x}\right)^2=\ y^2$

$\Rightarrow\ x^2+\frac{1}{x^2}-2=\ y^2$

$\Rightarrow\ x^2+\frac{1}{x^2}=\ y^2+2$

$\Rightarrow\ 2(\ y^2+2)+3y-4=0$

$\Rightarrow\ 2y^2+4+3y-4=0$

$\Rightarrow\ 2y^2+3y=0$

$\Rightarrow\ y(2y+3)=0$

$\Rightarrow\ y=0 \quad or \quad 2y+3=0$

$\Rightarrow\ y=0 \quad or \quad y=-\frac{3}{2}$

When $y=0 \quad \Rightarrow\ x-\frac{1}{x}=0$

$\Rightarrow\ x^2-1=0$

$\Rightarrow\ x^2=1$

$\Rightarrow\ x=\pm1$

When $y=-\frac{3}{2} \Rightarrow\ x-\frac{1}{x}=-\frac{3}{2}$

$\Rightarrow\ 2x^2-2=-3x$

$\Rightarrow\ 2x^2+3x-2=0$

$\Rightarrow\ 2x^2+4x-x-2=$

$\Rightarrow\ 2x(x+2)-1(x+2)=0$

$\Rightarrow\ (x+2)(2x-1)=0$

$\Rightarrow\ x+2=0 \quad or \quad 2x-1=0$

$\Rightarrow\ x=-2 \quad or \quad x=\frac{1}{2}$

Hence solution set = $\left\{\frac{1}{2},-\mathbf{2}\ ,\pm1\ \right\}$

Q #23: $\mathbf{6x^4-35x^3+62x^2-35x+6=0}$

Solution: $6x^4-35x^3+62x^2-35x+6=0$

Dividing by x^2

$$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0$$

$$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$$

$Let\ \left(x+\frac{1}{x}\right)=y \quad \Rightarrow\ \left(x+\frac{1}{x}\right)^2=\ y^2$

$\Rightarrow\ x^2+\frac{1}{x^2}+2=\ y^2$

$\Rightarrow\ x^2+\frac{1}{x^2}=\ y^2-2$

$\Rightarrow \quad 6(y^2-2)-35y+62=0$

$\Rightarrow \quad 6y^2-12-35y+62=0$

$\Rightarrow \quad 6y^2-35y+50=0$

$\Rightarrow \quad y=\dfrac{35\pm\sqrt{1225-4(6)(50)}}{12}$

$\Rightarrow \quad y=\dfrac{35\pm\sqrt{1225-1200}}{12}$

$\Rightarrow \quad y=\dfrac{35\pm\sqrt{25}}{12}$

$\Rightarrow \quad y=\dfrac{35\pm 5}{12}$

$\Rightarrow \quad y=\dfrac{35+5}{12} \quad or \quad y=\dfrac{35-5}{12}$

$\Rightarrow \quad y=\dfrac{10}{3} \quad or \quad y=\dfrac{5}{2}$

When $y=\frac{5}{2}$ $\quad \Rightarrow \quad x+\frac{1}{x}=\frac{5}{2}$

$\Rightarrow \quad x^2+1=\frac{5}{2}x$

$\Rightarrow \quad 2x^2+2=5x$

$\Rightarrow \quad 2x^2-5x+2=0$

$\Rightarrow \quad 2x^2-4x-x+2=0$

$\Rightarrow \quad 2x(x-2)-1(x-2)=0$

$\Rightarrow \quad (x-2)(2x-1)=0$

$\Rightarrow \quad x-2=0 \quad or \quad 2x-1$

$\Rightarrow \quad x=2 \quad or \quad x=\dfrac{1}{2}$

When $y=\frac{10}{3}$ $\quad \Rightarrow \quad x+\frac{1}{x}=\frac{10}{3}$

$\Rightarrow \quad 3x^2+3=10x$

$\Rightarrow \quad 3x^2-10x+3=0$

$\Rightarrow \quad 3x^2-9x-x+3=$

$\Rightarrow \quad 3x(x-3)-1(x-3)=0$

$\Rightarrow \quad (x-3)(3x-1)=0$

$\Rightarrow \quad x-3=0 \quad or \quad 3x-1=0$

$\Rightarrow \quad x=3 \quad or \quad x=\dfrac{1}{3}$

Hence solution set = $\left\{\frac{1}{2},\ \mathbf{2}, 3,\ \frac{1}{3}\right\}$

Q #24: $\boldsymbol{x^4-6x^2+10-\frac{6}{x^2}+\frac{1}{x^4}=0}$

Solution:

$$x^4-6x^2+10-\frac{6}{x^2}+\frac{1}{x^4}=0 \longrightarrow (1)$$

Re-arranging (1), we have

$$\left(x^4+\frac{1}{x^4}\right)-6\left(x^2+\frac{1}{x^2}\right)+10=0 \longrightarrow (2)$$

$Let \quad \left(x^2+\frac{1}{x^2}\right)=y \Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=y^2$

$\Rightarrow \quad x^4+\dfrac{1}{x^4}+2=y^2$

$\Rightarrow \quad x^4+\dfrac{1}{x^4}=y^2-2$

Using these values in (2), we have

$(y^2-2)-6y+10=0$

$\Rightarrow \quad y^2-2-6y+10=0$

$\Rightarrow \quad y^2-6y+8=0$

$\Rightarrow \quad y^2-4y-2y+8=0$

$\Rightarrow \quad y(y-4)-2(y-4)=0$

$\Rightarrow \quad (y-4)(y-2)=0$

$\Rightarrow \quad y-4=0 \quad or \quad y-2=0$

$\Rightarrow \quad y=4 \quad or \quad y=2$

When $y=4$ $\quad \Rightarrow \quad x^2+\frac{1}{x^2}=4$

$\Rightarrow \quad x^4+1=4x^2$

$\Rightarrow \quad x^4-4x^2+1=0$

$Let\ x^2=z \quad \Rightarrow \quad x^4=z^2$

$\Rightarrow \quad z^2 - 4z + 1 = 0$

$\Rightarrow z = \dfrac{4 \pm \sqrt{16-4}}{2}$

$\Rightarrow z = \dfrac{4 \pm \sqrt{12}}{2}$

$\Rightarrow z = \dfrac{4 \pm 2\sqrt{3}}{2}$

$\Rightarrow z = 2 \pm \sqrt{3}$

$\Rightarrow x^2 = 2 \pm \sqrt{3}$

$\Rightarrow x = \pm\sqrt{2 \pm \sqrt{3}}$

When $y = 2 \quad \Rightarrow \quad x^2 + \frac{1}{x^2} = 2$

$\Rightarrow \quad x^4 + 1 = 2x^2$

$\Rightarrow \quad x^4 - 2x^2 + 1 = 0$

$Let\ \ x^2 = z \quad \Rightarrow \quad x^4 = z^2$

$\Rightarrow \quad z^2 - 2z + 1 = 0$

$\Rightarrow \quad (z-1)^2 = 0$

$\Rightarrow \quad z - 1 = 0$

$\Rightarrow \quad z = 1$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

Hence solution set $=\left\{\pm 1,\ \pm\sqrt{2 \pm \sqrt{3}}\right\}$

"I could never have gone far in any science because on the path of every science the lion Mathematics lies in wait for you."

C.S. Lewis