UNIT 4

QUADRATIC EQUATIONS

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Learning Objectives:

(i) Definition of Quadratic Equation
(ii) Solution of Quadratic Equation
- Factorization
- Completing Square
- Quadratic Formula

(iii) Solutions of Equations Reducible to Quadratic Equations
(iv) Cube Roots of Unity and Properties
(v) Fourth Roots of Unity and Properties
(vi) Factor and Remainder Theorems
(vii) Synthetic Division
(viii) Relation Between Roots and Coefficients of Quadratic Equations
(ix) Formation of Quadratic Equation
(x) Nature of Roots of Quadratic Equations
(xi) System of Simultaneous Equations
(xii) Word Problems Associated to Quadratic Equations

Quadratic Equation:

An equation of degree 2 is called quadratic equation.

Degree: Highest power involved in an equation is called its degree.

Methods of Solution:

There are three methods to solve a quadratic equation.

(i) Factorization
(ii) Completing squares
(iii) Using quadratic formula.

Note: $ax^2 + bx + c = 0$ is General Quadratic equation

Exercise 4.1

Solve the following by Factorization.

Q #1: $\mathbf{3x^2 + 4x + 1 = 0}$

Solution: $3x^2 + 4x + 1 = 0$

$3x^2 + 3x + x + 1 = 0$

$3x(x + 1) + 1(x + 1) = 0$

$(x + 1)(3x + 1) = 0$

$(x + 1) = 0 \quad or \quad (3x + 1) = 0$

$x = -1 \quad or \quad x = -\frac{1}{3}$

Hence Solution Set = $\left\{-1, -\frac{1}{3}\right\}$

Q # 2: $\boldsymbol{x^2 + 7x + 12 = 0}$

Solution: $x^2 + 7x + 12 = 0$

$x^2 + 3x + 4x + 12 = 0$

$x(x + 3) + 4(x + 3) = 0$

$(x + 3)(x + 4) = 0$

$(x + 3) = 0 \quad or \quad (x + 4) = 0$

$x = -3 \quad or \quad x = -4$

Hence Solution Set = {-3 , -4 }

Q # 3: $\boldsymbol{9x^2 - 12x - 5 = 0}$

Solution: $9x^2 - 12x - 5 = 0$

$9x^2 + 3x - 15x - 5 = 0$

$3x(3x + 1) - 5(3x + 1) = 0$

$(3x + 1)(3x - 5) = 0$

$(3x + 1) = 0 \quad or \quad (3x - 5) = 0$

$x = -\frac{1}{3} \quad or \quad x = \frac{5}{3}$

Hence Solution Set = {$-\frac{1}{3}$, $\frac{5}{3}$}

Q # 4: $\boldsymbol{x^2 - x = 2}$

Solution: $x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x - 2) + 1(x - 2) = 0$

$(x - 2)(x + 1) = 0$

$(x - 2) = 0 \quad or \quad (x + 1) = 0$

$x = 2 \quad or \quad x = -1$

Hence Solution Set = {-1 , 2}

Q # 5: $\boldsymbol{x(x + 7) = (2x - 1)(x + 4)}$

Solution: $x(x + 7) = (2x - 1)(x + 4)$

$x^2 + 7x = 2x^2 + 7x - 4$

$x^2 + 7x - 2x^2 - 7x + 4 = 0$

$-x^2 + 4 = 0$

$x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

$x - 2 = 0 \quad or \quad x + 2 = 0$

$x = 2 \quad or \quad x = -2$

Hence Solution Set = {-2 , 2}

Q # 6: $\boldsymbol{\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}}$

Solution: $\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}$

Multiplying by $2x(x + 1)$

$2x^2 + 2x^2 + 4x + 2 = 5x(x + 1)$

$4x^2 + 4x + 2 = 5x^2 + 5x$

$4x^2 + 4x + 2 - 5x^2 - 5x = 0$

$-x^2 - x + 2 = 0$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x(x + 2) - 1(x + 2) = 0$

$(x + 2)(x - 1) = 0$

$x + 2 = 0 \quad or \quad x - 1 = 0$

$x = -2 \quad or \quad x = 1$

Hence Solution Set = {-2 , 1}

Q # 7: $\boldsymbol{\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5}}$

Solution: $\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5}$

Multiplying by $(x + 1)(x + 2)(x + 5)$

$(x + 2)(x + 5) + 2(x + 1)(x + 5) = 7(x + 1)(x + 2)$

$x^2 + 7x + 10 + 2x^2 + 12x + 10 = 7x^2 + 21x + 14$

$3x^2 + 19x + 20 - 7x^2 - 21x - 14 = 0$

$-4x^2 - 2x + 6 = 0$

$-2(2x^2 + x - 3) = 0$

$2x^2 + x - 3 = 0$

$2x^2 + 3x - 2x - 3 = 0$

$x(2x + 3) - 1(2x + 3) = 0$

$(2x + 3)(x - 1) = 0$

$2x + 3 = 0 \quad or \quad x - 1 = 0$

$x = -\frac{3}{2} \quad or \quad x = 1$

Hence Solution Set = $\{-\frac{3}{2}, 1\}$

Q # 8: $\frac{a}{ax-1} + \frac{b}{bx-1} = a + b$

Solution: $\frac{a}{ax-1} + \frac{b}{bx-1} = a + b$

$\frac{a}{ax-1} - b + \frac{b}{bx-1} - a = 0$

$\frac{a - b(ax-1)}{ax-1} + \frac{b - a(bx-1)}{bx-1} = 0$

$\frac{a - abx + b}{ax-1} + \frac{b - abx + a}{bx-1} = 0$

$\frac{a - abx + b}{ax-1} + \frac{a - abx + b}{bx-1} = 0$

$(a - abx + b)\left\{\frac{1}{ax-1} + \frac{1}{bx-1}\right\} = 0$

$(a - abx + b) = 0 \quad or \quad \frac{1}{ax-1} + \frac{1}{bx-1} = 0$

$abx = a + b \quad or \quad \frac{bx-1+ax-1}{(ax-1)(bx-1)} = 0$

$x = \frac{a+b}{ab} \quad or \quad bx - 1 + ax - 1 = 0$

$or \quad x(a + b) = 2$

$or \quad x = \frac{2}{a+b}$

Hence Solution Set = $\left\{\frac{a+b}{ab}, \frac{2}{a+b}\right\}$

Solve the following equations by completing squares method

Q # 9: $x^2 - 2x - 899 = 0$

Solution: $x^2 - 2x - 899 = 0$

$x^2 - 2\left(\frac{2}{2}\right)x = 899$

$x^2 - 2(1)x = 899$

Adding $(1)^2$ on both sides

$x^2 - 2(1)x + (1)^2 = 899 + (1)^2$

$(x - 1)^2 = 900$

Taking square root on both sides

$x - 1 = \pm 30$

$x - 1 = 30 \quad or \quad x - 1 = -30$

$x = 31 \quad or \quad x = -29$

Hence Solution set = $\{-29, 31\}$

Q # 10: $x^2 + 4x - 1085 = 0$

Solution: $x^2 + 4x - 1085 = 0$

$x^2 + 2\left(\frac{4}{2}\right)x = 1085$

$x^2 + 2(2)x = 1085$

Adding $(2)^2$ on both sides

$x^2 + 2(2)x + (2)^2 = 1085 + (2)^2$

$(x + 2)^2 = 1089$

Taking square root on both sides

$x + 2 = \pm 33$

$x + 2 = 33 \quad or \quad x + 2 = -33$

$x = 31 \quad or \quad x = -35$

Hence Solution set = $\{-35, 31\}$

Q # 11: $\boldsymbol{x^2 + 6x - 567 = 0}$

Solution: $x^2 + 6x = 567$

$x^2 + 2\left(\frac{6}{2}\right)x = 567$

$x^2 + 2(3)x = 567$

Adding $(3)^2$ on both sides

$x^2 + 2(3)x + (3)^2 = 567 + (3)^2$

$(x + 3)^2 = 576$

Taking square root on both sides

$x + 3 = \pm 24$

$x + 3 = 24 \quad or \quad x + 3 = -24$

$x = 21 \quad or \quad x = -27$

Hence Solution set = $\{-27,\ 21\}$

Q # 12: $\boldsymbol{x^2 - 3x - 648 = 0}$

Solution: $x^2 - 3x - 648 = 0$

$x^2 - 3x = 648$

$x^2 - 2\left(\frac{3}{2}\right)x = 648$

Adding $\left(\frac{3}{2}\right)^2$ on both sides

$x^2 - 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2 = 648 + \left(\frac{3}{2}\right)^2$

$\left(x - \frac{3}{2}\right)^2 = 648 + \frac{9}{4}$

$\left(x - \frac{3}{2}\right)^2 = \frac{2601}{4}$

Taking square root on both sides

$x - \frac{3}{2} = \pm\frac{51}{2}$

$x - \frac{3}{2} = \frac{51}{2} \quad or \quad x - \frac{3}{2} = -\frac{51}{2}$

$x = \frac{3}{2} + \frac{51}{2} \quad or \quad x = \frac{3}{2} - \frac{51}{2}$

Hence Solution set = $\{-24,\ 27\}$

Q # 13: $\boldsymbol{x^2 - x - 1806 = 0}$

Solution: $x^2 - x - 1806 = 0$

$x^2 - x = 1806$

$x^2 - 2\left(\frac{1}{2}\right)x = 1806$

Adding $\left(\frac{1}{2}\right)^2$ on both sides

$x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2 = 1806 + \left(\frac{1}{2}\right)^2$

$\left(x - \frac{1}{2}\right)^2 = 1806 + \frac{1}{4}$

$\left(x - \frac{1}{2}\right)^2 = \frac{7225}{4}$

Taking square root on both sides

$x - \frac{1}{2} = \pm\frac{85}{2}$

$x - \frac{1}{2} = \frac{85}{2} \quad or \quad x - \frac{1}{2} = -\frac{85}{2}$

$x = \frac{1}{2} + \frac{85}{2} \quad or \quad x = \frac{1}{2} - \frac{85}{2}$

Hence Solution set = $\{-42,\ 43\}$

Q # 14: $\boldsymbol{2x^2 + 12x - 110 = 0}$

Solution: $2x^2 + 12x - 110 = 0$

$2(x^2 + 6x - 55) = 0$

$x^2 + 6x - 55 = 0$

$x^2 + 2\left(\frac{6}{2}\right)x = 55$

$x^2 + 2(3)x = 55$

Adding $(3)^2$ on both sides

$x^2 + 2(3)x + (3)^2 = 55 + (3)^2$

$(x + 3)^2 = 64$

Taking square root on both sides

$x+3=\pm 8$

$x+3=8 \quad or \quad x+3=-8$

$x=5 \quad or \quad x=-11$

Hence Solution set = $\{-11,\ 5\}$

Find solutions of the following equations by Quadratic Formula

Q #15: $\boldsymbol{5x^2-13x+6=0}$

Solution: $5x^2-13x+6=0$

$a=5, b=-13\ , c=6$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-13)\pm\sqrt{169-4(5)(6)}}{10}$$

$$x=\frac{13\pm\sqrt{169-120}}{10}$$

$$x=\frac{13\pm\sqrt{49}}{10}$$

$$x=\frac{13\pm 7}{10}$$

$$x=\frac{13+7}{10} \quad or \quad x=\frac{13-7}{10}$$

$x=2 \quad or \quad x=\frac{3}{5}$

Hence Solution set = $\left\{\mathbf{2},\ \frac{\mathbf{3}}{\mathbf{5}}\right\}$

Q #16: $\boldsymbol{4x^2+7x-1=0}$

Solution: $4x^2+7x-1=0$

$a=4, b=7\ , c=-1$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-7\pm\sqrt{49-4(4)(-1)}}{8}$$

$$x=\frac{-7\pm\sqrt{49+16}}{8}$$

$$x=\frac{-7\pm\sqrt{65}}{8}$$

Hence Solution set = $\left\{\frac{\mathbf{-7\pm\sqrt{65}}}{\mathbf{8}}\right\}$

Q #17: $\boldsymbol{15x^2+2ax-a^2=0}$

Solution: $15x^2+2ax-a^2=0$

$a=15, b=2a\ , c=-a^2$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-2a\pm\sqrt{4a^2-4(15)(-a^2)}}{30}$$

$$x=\frac{-2a\pm\sqrt{4a^2+60a^2}}{30}$$

$$x=\frac{-2a\pm\sqrt{64a^2}}{30}$$

$$x=\frac{-2a\pm 8a}{30}$$

$$x=\frac{-2a+8a}{30} \quad or \quad x=\frac{-2a-8a}{30}$$

$$x=\frac{6a}{30} \quad or \quad x=\frac{-10a}{30}$$

$$x=\frac{a}{5} \quad or \quad x=\frac{-a}{3}$$

Hence Solution set = $\left\{\frac{\boldsymbol{a}}{\mathbf{5}},\ \frac{\boldsymbol{-a}}{\mathbf{3}}\right\}$

Q #18: $\boldsymbol{16x^2+8x+1=0}$

Solution: $16x^2+8x+1=0$

$a=16, b=8\ , c=1$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-8 \pm \sqrt{64-4(16)(1)}}{32}$$

$$x = \frac{-8 \pm \sqrt{64 - 64}}{32}$$

$$x = \frac{-8 \pm \sqrt{0}}{32}$$

$$x = \frac{-8}{32}$$

$$x = \frac{-1}{4}$$

Hence Solution set = $\left\{\frac{-1}{4}\right\}$

Q #19: $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

Solution: $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

$$x^2-(a+b)x+ab+x^2-(b+c)x+bc+x^2-(c+a)x+ca=0$$

$$3x^2-(a+b+b+c+a)x+ab+bc+ca=0$$

$$3x^2-2(a+b+c)x+ab+bc+ca=0$$

$$A=3, B=-2(a+b+c), C=ab+bc+ca$$

Using quadratic formula

$$x=\frac{2(a+b+c)\pm\sqrt{4(a+b+c)^2-4(3)(ab+bc+ca)}}{6}$$

$$x=\frac{2(a+b+c)\pm 2\sqrt{(a+b+c)^2-3(ab+bc+ca)}}{6}$$

$$x=\frac{2\left((a+b+c)\pm\sqrt{(a+b+c)^2-3(ab+bc+ca)}\right)}{6}$$

$$x=\frac{(a+b+c)\pm\sqrt{a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca)}}{3}$$

$$x=\frac{(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca)}}{3}$$

Hence Solution set = $\left\{\frac{(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca)}}{3}\right\}$

Q #20: $(a+b)x^2+(a+2b+c)x+b+c=0$

Solution: $(a+b)x^2+(a+2b+c)x+b+c=0$

$$A=a+b, B=a+2b+c, C=b+c$$

Using quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{(a+2b+c)^2 - 4(a+b)(b+c)}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{a^2+4b^2+c^2+4ab+4bc+2ca-4(a+b)(b+c)}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{a^2+4b^2+c^2+4ab+4bc+2ca-4ab-4ca-4b^2-4bc}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{a^2+c^2-2ca}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{(c-a)^2}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm (c-a)}{2(a+b)}$$

$$x = \frac{-(a+2b+c)+(c-a)}{2(a+b)} \quad or \quad x = \frac{-(a+2b+c)-(c-a)}{2(a+b)}$$

$$x = \frac{-a-2b-c+c-a}{2(a+b)} \quad or \quad x = \frac{-a-2b-c-c+a}{2(a+b)}$$

$$x = \frac{-2(a+b)}{2(a+b)} \quad or \quad x = \frac{-2(b+c)}{2(a+b)}$$

$$x = -1 \quad or \quad x = -\frac{(b+c)}{(a+b)}$$

Hence Solution set = $\left\{-1, -\frac{(b+c)}{(a+b)}\right\}$

If I were again beginning my studies, I would follow the advice of Plato and start with mathematics.

Galileo Galilei