EXERCISE 3.5

1 Solve by Cramer's Rule

i)
$$2x + 2y + z = 3$$
$$3x - 2y - 2z = 1$$
$$5x + y - 3z = 2$$

Sol. $|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$

$= 2(6+2) - 2(-9+10) + 1(3+10)$

$= 16 - 2 + 13 = 27$

$|A_1| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}$

$= 3(6+2) - 2(-3+4) + 1(1+4)$

$= 24 - 2 + 5 = 27$

$x = \frac{|A_1|}{|A|} = \frac{27}{27} = 1$

$|A_2| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$

$= 2(-3+4) - 3(-9+10) + 1(6-5)$

$= 2 - 3 + 1 = 0$

$y = \frac{|A_2|}{|A|} = \frac{0}{27} = 0$

$|A_3| = \begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}$

$= 2(-4-1) - 2(6-5) + 3(3+10)$

$= -10 - 2 + 39 = 27$

$z = \frac{|A_3|}{|A|} = \frac{27}{27} = 1$

$x = 1 \quad y = 0 \quad z = 1$

ii)
$$2x_1 - x_2 + x_3 = 5$$
$$4x_1 + 2x_2 + 3x_3 = 8$$
$$3x_1 - 4x_2 - x_3 = 3$$

Sol. $|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$

$= 2(-2+12) + 1(-4-9) + 1(-16-6)$

$= 20 - 13 - 22 = -15$

$|A_1| = \begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$

$= 5(-2+12) + 1(-8-9) + 1(-32-6)$

$= 50 - 17 - 38 = -5$

$x_1 = \frac{|A_1|}{|A|} = \frac{-5}{-15} = \frac{1}{3}$

$|A_2| = \begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}$

$= 2(-8-9) - 5(-4-9) + 1(12-24)$

$= -34 + 65 - 12 = 19$

$x_2 = \frac{|A_2|}{|A|} = \frac{19}{-15}$

$|A_3| = \begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}$

$= 2(6+32) + 1(12-24) + 5(-16-6)$

$= 76 - 12 - 110 = -46$

$x_3 = \frac{|A_3|}{|A|} = \frac{-46}{-15} = \frac{46}{15}$

$x_1 = \frac{1}{3} \quad x_2 = -\frac{19}{15} \quad x_3 = \frac{46}{15}$

iii)
$$2x_1 - x_2 + x_3 = 8$$
$$x_1 + 2x_2 + 2x_3 = 6$$
$$x_1 - 2x_2 - x_3 = 1$$

Sol. $|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$

$|A| = 2(-2+4) + 1(-1-2) + 1(-2-2)$

$= 4 - 3 - 4 = -3$

$|A_1| = \begin{matrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{matrix}$

$= 8(-2+4) + 1(-6-2) + 1(-12-2)$

$= 16 - 8 - 14 = -6$

$x_1 = \frac{|A_1|}{|A|} = \frac{-6}{-3} = 2$

$|A_2| = \begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}$

$= 2(-6-2) - 8(-1-2) + 1(1-6)$

$= -16 + 24 - 5 = 3$

$x_2 = \dfrac{|A_2|}{|A|} = \dfrac{3}{-3} = -1$

$|A_3| = \begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}$

$= 2(2+12) + 1(1-6) + 8(-2-2)$

$= 28 - 5 - 32 = -9$

$x_3 = \dfrac{|A_3|}{|A|} = \dfrac{-9}{-3} = 3$

$x_1 = 2 \quad x_2 = -1 \quad x_3 = 3$

2. Use matrices to solve

i) $x - 2y + z = -1$
$3x + y - 2z = 4$
$y - z = 1$

The matrix form of the system

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $B = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

Then system becomes $AX = B$

$\Rightarrow \quad X = A^{-1}B$ ——— (1)

$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$

$= 1(-1+2) + 2(-3+0) + 1(3-0)$

$= 1 - 6 + 3 = -2 \neq 0$

Cofactors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = (-1)^2(-1+2) = 1$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = (-1)^3(-3+0) = 3$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = (-1)^4(3-0) = 3$

$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = (-1)^3(2-1) = -1$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = (-1)^5(1+0) = -1$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-1)^4(4-1) = 3$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = (-1)^5(-2-3) = 5$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = (-1)^6(1+6) = 7$

Cofactors of $A = \begin{pmatrix} 1 & 3 & 3 \\ -1 & -1 & -1 \\ 3 & 5 & 7 \end{pmatrix}$

Adj $A =$ (Cofactors of $A)^t = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$

$X = A^{-1}B = \dfrac{1}{|A|}(\text{Adj } A)\, B$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$$= \frac{1}{-2}\begin{pmatrix} -1-4+3 \\ -3-4+5 \\ -3-4+7 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$x = 1, \quad y = 1, \quad z = 0$

ii) $2x_1 + x_2 + 3x_3 = -3$
$x_1 + x_2 - 2x_3 = 0$
$-3x_1 - x_2 + 2x_3 = -4$

The matrix form of the system

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$$

$A \quad X = B$

$\Rightarrow \quad X = A^{-1}B$ -- (1)

where $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $B = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix}$

$= 2(2-2) - 1(2-6) + 3(-1+3)$

$= 0 + 4 + 6 = 10$

$$\text{Cofactors of } A = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & -(2-6) & -1+3 \\ -(2+3) & 4+9 & -(-2+3) \\ -2-3 & -(-4-3) & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 2 \\ -5 & 13 & -1 \\ -5 & 7 & 1 \end{bmatrix}$$

$$\text{Adj } A = (\text{Cofactors of } A)^t = \begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$$

$$(1) \Rightarrow X = A^{-1}B = \frac{1}{|A|}(\text{Adj } A)(B)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ -4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0-0+20 \\ -12+0-28 \\ -6-0-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$$

$x_1 = 2 \quad x_2 = -4 \quad x_3 = -1$

iii)
$$x + y = 2$$
$$2x - z = 1$$
$$2y - 3z = -1$$

Sol The matrix form of the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$AX = B$$
$$\Rightarrow X = A^{-1}B \text{ ----(1)}$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix}$$
$$= 1(0+2) - 1(-6+0) + 0(4-0)$$
$$= 2 + 6 + 0 = 8 \neq 0$$

$$\text{Cofactors of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = (-1)^2(0+2) = 2$$
$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = (-1)^3(-6+0) = 6$$
$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (-1)^4(4-0) = 4$$
$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = (-1)^3(-3-0) = +3$$
$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = (-1)^4(-3-0) = -3$$
$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = (-1)^5(2-0) = -2$$
$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (-1)^4(-1-0) = -1$$
$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (-1)^5(-1-0) = 1$$
$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = (-1)^6(0-2) = -2$$

$$\text{Cofactors of } A = \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

$$\text{Adj } A = \text{Cofactors of } A^t = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{|A|}(\text{Adj } A)(B)$$
$$= \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x = 1 \quad y = 1 \quad z = 1$

3. Solve the system by their augmented matrices to echelon and Reduced echelon forms

i)
$$x_1 - 2x_2 - 2x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 1$$
$$5x_1 - 4x_2 - 3x_3 = 1$$

Sol. The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 2 & 3 & 1 & 1 \\ 5 & -4 & -3 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 7 & 5 & 3 \\ 0 & 6 & 7 & 6 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-5R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 6 & 7 & 6 \end{array}\right] R_2-R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 19 & 24 \end{array}\right] R_3-6R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{array}\right] \frac{1}{19}R_3$$

The equivalent system in echelon form is

$$x_1 - 2x_2 - 2x_3 = -1 \quad ---(1)$$
$$x_2 - 2x_3 = -3 \quad ---(2)$$
$$x_3 = \frac{24}{19} \quad ---(3)$$

$(2) \Rightarrow x_2 = 2x_3 - 3 = 2\left(\frac{24}{19}\right) - 3$

$= \frac{48-57}{19} = -\frac{9}{19}$

$(1) \Rightarrow x_1 = 2x_2 + 2x_3 - 1$

$x_1 = 2\left(-\frac{9}{19}\right) + 2\left(\frac{24}{19}\right) - 1$

$= \frac{-18+48-19}{19} = \frac{11}{19}$

$x_1 = \frac{11}{19}, \ x_2 = -\frac{9}{19}, \ x_3 = \frac{24}{19}$

Now in Reduced echelon form

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -6 & -7 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{array}\right] R_1+2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 11/19 \\ 0 & 1 & 0 & -9/19 \\ 0 & 0 & 1 & 24/19 \end{array}\right] \begin{array}{l} R_1+6R_3 \\ R_2+2R_3 \end{array}$$

The system in reduced echelon form is

$x_1 = \frac{11}{19} \quad x_2 = -\frac{9}{19} \quad x_3 = \frac{24}{19}$

ii)
$$x + 2y + z = 2$$
$$2x + y + 2z = -1$$
$$2x + 3y - z = 9$$

Sol. The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 2 & 3 & -1 & 9 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -5 \\ 0 & -1 & -3 & 5 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-2R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 5/3 \\ 0 & -1 & -3 & 5 \end{array}\right] -\frac{1}{3}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & -3 & 20/3 \end{array}\right] R_3+R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -20/9 \end{array}\right] -\frac{1}{3}R_3$$

The system in echelon form is

$$x + 2y + z = 2 \quad ---(1)$$
$$y = \frac{5}{3} \qquad z = -\frac{20}{9}$$

$(1) \Rightarrow x = 2 - 2y - z$

$x = 2 - 2\left(\frac{5}{3}\right) - \left(-\frac{20}{9}\right)$

$= \frac{18-30+20}{9} = \frac{8}{9}$

$x = \frac{8}{9} \quad y = \frac{5}{3} \quad z = \frac{-20}{9}$

Now in reduced echelon form

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 1 & -4/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -20/9 \end{array}\right] R_1-2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 8/9 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -20/9 \end{array}\right] R_1-R_3$$

$x = \frac{8}{9} \quad y = \frac{5}{3} \quad z = \frac{-20}{9}$

iii)
$$x_1 + 4x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 2x_3 = 9$$
$$3x_1 + 2x_2 - 2x_3 = 12$$

Sol. The augmented Matrix is

$$\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 2 & 1 & -2 & 9 \\ 3 & 2 & -2 & 12 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & -7 & -6 & 5 \\ 0 & -10 & -8 & 6 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & +1 & 2 & -3 \\ 0 & -10 & -8 & 6 \end{array}\right] -3R_2+2R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 12 & -24 \end{array}\right] R_3+10R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \frac{1}{12}R_3$$

The system is in echelon form

$$x_1+4x_2+2x_3 = 2 \quad ---(1)$$
$$x_2+2x_3 = -3 \quad ---(2)$$
$$x_3 = -2 \quad ---(3)$$

$(2) \Rightarrow \quad x_2 = -2x_3 - 3$

$x_2 = -2(-2)-3 = 4-3 = 1$

$(1) \Rightarrow \quad x_1 = 2 - 4x_2 - 2x_3$

$x_1 = 2-4(1)-2(-2)$
$= 2-4+4 = 2$

$x_1 = 2 \quad x_2 = 1 \quad x_3 = -2$

Now in Reduced echelon form

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & -6 & 14 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] R_1-4R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R_1+6R_3 \\ R_2-2R_3 \end{array}$$

$x_1 = 2 \quad x_2 = 1 \quad x_3 = -2$

4. Solve the homogeneous linear equations

i)
$$x+2y-2z = 0 \quad \ldots(1)$$
$$2x+y+5z = 0 \quad ---(2)$$
$$5x+4y+8z = 0 \quad ---(3)$$

Sol. let the matrix of coefficients be A

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$|A| = 1(8-20) - 2(16-25) - 2(8-5)$
$= -12+18-6 = 0$

Thus system has infinite solutions.

$(1) \Rightarrow \quad x+2y-2z = 0$

$(2)\times 2 \Rightarrow \quad \underline{\pm 4x \pm 2y \pm 10z = 0}$

$\qquad -3x \quad -12z = 0$

$\Rightarrow \quad x = -4z$

$(2) \Rightarrow \quad y = -2x - 5z$
$= -2(-4z) - 5z = 3z$

putting values of x, y in (3)

L.H.S $= 5(-4z)+4(3z)+8z$
$= -20z+12z+8z = 0$
$=$ R.H.S Satisfied

Let $z = t$ where $t \in R$

Then $x = -4t \quad y = 3t$

The system has infinitely many solutions.

ii)
$$x_1+4x_2+2x_3 = 0 \quad ---(1)$$
$$2x_1+x_2-3x_3 = 0 \quad ---(2)$$
$$3x_1+2x_2-4x_3 = 0 \quad ---(3)$$

Sol let the matrix of the coefficients be A

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{vmatrix}$$

$= 1(-4+6) - 4(-8+9) + 2(4-3)$
$= 2-4+2 = 0$

Thus the system has infinite solutions

$(1) \Rightarrow \quad x_1+4x_2+2x_3 = 0$

$(2)\times 4 \Rightarrow \quad \underline{\pm 8x_1 \pm 4x_2 \mp 12x_3 = 0}$

$\qquad -7x_1 \quad +14x_3 = 0$

$\Rightarrow \quad x_1 = 2x_3$

(2) $\Rightarrow \quad x_2 = -2x_1 + 3x_3$

$x_2 = -2(2x_3) + 3x_3 = -x_3$

putting values of x_1, x_2 in (iii) L.H.S

$= 3(2x_3) + 2(-x_3) - 4x_3$

$= 6x_3 - 2x_3 - 4x_3 = 0 =$ R.H.S

Let $x_3 = t \quad t \in R$

$x_1 = 2t \quad x_2 = -t \quad x_3 = t$

The system has infinitely many solutions.

(iii)

$$x_1 - 2x_2 - x_3 = 0 \text{ ---(1)}$$
$$x_1 + x_2 + 5x_3 = 0 \text{ ---(2)}$$
$$2x_1 - x_2 + 4x_3 = 0 \text{ ---(3)}$$

Sol. Let the matrix of the coefficent be A

$$|A| = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

$= 1(4+5) + 2(4-10) - 1(-1-2)$

$= 9 - 12 + 3 = 0$

Thus the system has non trivial solution

(1) $\Rightarrow \quad x_1 - 2x_2 - x_3 = 0$

(2)×2 $\Rightarrow \quad \underline{2x_1 + 2x_2 + 10x_3 = 0}$

$3x_1 + 9x_3 = 0$

$\Rightarrow \quad x_1 = -3x_3$

(2) $\Rightarrow \quad x_2 = -x_1 - 5x_3$

$= 3x_3 - 5x_3 = -2x_3$

putting values of x_1, x_2 in (3) L.H.S

$= 2(-3x_3) - (-2x_3) + 4x_3$

$= -6x_3 + 2x_3 + 4x_3$

$= 0 =$ R.H.S

let $x_3 = t \quad t \in R$

$x_1 = -3t \quad x_2 = -2t \quad x_3 = t$

Thus the system has infinitely many solutions

5. Find the value of λ for which system has non trivial sol. Also solve the system for value of λ.

i)

$$x + y + z = 0 \text{ ---(1)}$$
$$2x + y - \lambda z = 0 \text{ ---(2)}$$
$$x + 2y - 2z = 0 \text{ ---(3)}$$

The matrix of coefficient of the system is A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix}$$

$= 1(-2+2\lambda) - 1(-4+\lambda) + 1(4-1)$

$= -2 + 2\lambda + 4 - \lambda + 3$

$= \lambda + 5$

System has non trivial sol if $|A| = 0 \Rightarrow \lambda + 5 = 0 \Rightarrow \lambda = -5$

The system becomes

$$x + y + z = 0 \text{ ----(4)}$$
$$2x + y + 5z = 0 \text{ ----(5)}$$
$$x + 2y - 2z = 0 \text{ ----(6)}$$

(4) $\Rightarrow \quad x + y + z = 0$

(5) $\Rightarrow \quad \underline{2x \pm y \pm 5z = 0}$

$-x - 4z = 0$

$\Rightarrow \quad x = -4z$

(4) $\Rightarrow \quad y = -x - z = 4z - z = 3z$

putting the values of x, y in (6)

L.H.S

$= -4z + 2(3z) - 2z$

$= -6z + 6z = 0$ R.H.S

Let $z = t \quad t \in R$

$x = -4t \quad y = 3t \quad z = t$

Thus the system has an infinitely many Solutions

ii $\quad x_1 + 4x_2 + \lambda x_3 = 0 \dots (1)$

$2x_1 + x_2 - 3x_3 = 0 \dots (2)$

$3x_1 + \lambda x_2 - 4x_3 = 0 \dots (3)$

The matrix of coefficient of the system is A

$$A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix}$$

$|A| = 1(-4+3\lambda) - 4(-8+9) + \lambda(2\lambda-3)$

$= -4 + 3\lambda - 4 + 2\lambda^2 - 3\lambda$

$= 2\lambda^2 - 8$

For non trivial sol $|A| = 0$

$\Rightarrow 2\lambda^2 - 8 = 0 \Rightarrow 2\lambda^2 = 8 \Rightarrow \lambda = \pm 2$

For $\lambda = 2$

The system becomes

$x_1 + 4x_2 + 2x_3 = 0 \text{ ---} (4)$

$2x_1 + x_2 - 3x_3 = 0 \text{ ---} (5)$

$3x_1 + 2x_2 - 4x_3 = 0 \text{ ---} (6)$

Same as Q 4(ii)

For $\lambda = -2$

The System becomes

$x_1 + 4x_2 - 2x_3 = 0 \dots (4)$

$2x_1 + x_2 - 3x_3 = 0 \dots (5)$

$3x_1 - 2x_2 - 4x_3 = 0 \text{ ---} (6)$

$(4) \Rightarrow \quad x_1 + 4x_2 - 2x_3 = 0$

$(5)\times 4 \Rightarrow \underline{8x_1 \pm 4x_2 \mp 12x_3 = 0}$

$-7x_1 + 10x_3 = 0 \Rightarrow x_1 = \frac{10}{7}x_3$

$(5) \Rightarrow \quad x_2 = -2x_1 + 3x_3$

$= -\frac{20}{7}x_3 + 3x_3 = \frac{1}{7}x_3$

putting values of x_1, x_2 in (6)

L.H.S $= 3\left(\frac{10}{7}x_3\right) - 2\left(\frac{1}{7}x_3\right) - 4x_3$

$= (30x_3 - 2x_3 - 28x_3)/7 = 0 =$ R.H.S

Let $x_3 = t \quad t \in R$

$x_1 = \frac{10}{7}t \quad x_2 = \frac{1}{7}t \quad x_3 = t$

Thus the System has an infinitely many Solutions

6 $\quad x_1 + 4x_2 + \lambda x_3 = 2$

$2x_1 + x_2 - 2x_3 = 11$

$3x_1 + 2x_2 - 2x_3 = 16$

Sol. The matrix of coefficient of the system is A

Then $|A| = \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix}$

$= 1(-2+4) - 4(-4+6) + \lambda(4-3)$

$= 2 - 8 + \lambda = \lambda - 6$

As system doesnot possess a unique solution if $|A| = 0$

$\Rightarrow \lambda - 6 = 0 \Rightarrow \lambda = 6$

The system becomes

$x_1 + 4x_2 + 6x_3 = 2$

$2x_1 + x_2 - 2x_3 = 11$

$3x_1 + 2x_2 - 2x_3 = 16$

The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{array}\right]$$

$$\underset{\sim}{R} \begin{array}{cccc} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{array} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\underset{\sim}{R} \begin{array}{cccc} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & -10 & -20 & 10 \end{array} \quad -\frac{1}{7}R_2$$

$$\underset{\sim}{R} \begin{array}{cccc} 1 & 0 & -2 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{l} R_1 - 4R_2 \\ R_3 + 10R_2 \end{array}$$

The equivalent system is

$x_1 - 2x_3 = 6 \quad , \quad x_2 + 2x_3 = -1$

Let $x_3 = t$ arbitrary

$x_1 = 2t + 6 \quad x_2 = -2t - 1$

Thus the system of eq has infinitely many Solutions.