$$\sim\begin{bmatrix} & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5\end{bmatrix} \text{ by } R_1 \leftrightarrow R_4$$

$$R\sim\begin{bmatrix}1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 6 & 30 & 54\end{bmatrix} \begin{matrix}\text{by } R_1+(-2)R_2 \to R_1' \\ R_3+(-1)R_2 \to R_3' \\ R_4+7R_2 \to R_4'\end{matrix}$$

$$R\sim\begin{bmatrix}1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 1 & 5 & 9\end{bmatrix} \text{ by } \frac{1}{6}R_4 \to R_4'$$

$$R\sim\begin{bmatrix}1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 6 & 11 & 16\end{bmatrix} \text{ by } R_3 \leftrightarrow R_4$$

$$R\sim\begin{bmatrix}1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & -19 & -38\end{bmatrix} \begin{matrix}\text{by } R_1+(1)R_3 \to R_1' \\ R_4+(-6)R_3 \to R_4'\end{matrix}$$

$$R\sim\begin{bmatrix}1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 & 2\end{bmatrix} \text{ by } -\frac{1}{19}R_4 \to R_4'$$

$$R\sim\begin{bmatrix}1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2\end{bmatrix} \begin{matrix}\text{by } R_1+4R_4 \to R_1' \\ R_2+(-3)R_4 \to R_2' \\ R_3+(-5)R_4 \to R_3\end{matrix}$$

Here number of non-zero rows = 4

Hence rank of the matrix = 4 Ans.

Ex 3.5

① i) $2x+2y+z=3$

$3x-2y-2z=1$

$5x+y-3z=2$

Here $|A| = \begin{vmatrix}2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3\end{vmatrix}$

$|A| = 2(6+2)-2(-9+10)+1(3+10)$

$|A| = 16-2+13 = 27 \neq 0$

Now by Cramer's Rule

$$x = \frac{\begin{vmatrix}3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3\end{vmatrix}}{27} = \frac{3(6+2)-2(-3+4)+1(1+4)}{27}$$

$$x = \frac{24-2+5}{27} = \frac{27}{27} = 1 \Rightarrow \boxed{x=1}$$

$$y = \frac{\begin{vmatrix}2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3\end{vmatrix}}{27} = \frac{2(-3+4)-3(-9+10)+1(6-5)}{27}$$

$$y = \frac{2-3+1}{27} = \frac{0}{27} = 0 \Rightarrow \boxed{y=0}$$

and

$$z = \frac{\begin{vmatrix}2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2\end{vmatrix}}{27} = \frac{2(-4-1)-2(6-5)+3(3+10)}{27}$$

$$z = \frac{-10-2+39}{27} = \frac{27}{27} = 1 \Rightarrow z = 1$$

Hence $x=-1$, $y=0$, $z=1$

(ii) $2x_1 - x_2 + x_3 = 2$

$4x_1 + 2x_2 + 3x_3 = 8$

$3x_1 - 4x_2 - x_3 = 3.$

Here $|A| = \begin{vmatrix}2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1\end{vmatrix} = 2(-2+12)+1(-4-9)+1(-16-6)$

$|A| = 20-13-22 = -15 \neq 0$

Now by Crammer's Rule.

$$x_1 = \frac{\begin{vmatrix}5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1\end{vmatrix}}{-15} = \frac{5(-2+12)+1(-8-9)+1(-32-6)}{-15}$$

$$x_1 = \frac{50-17-38}{-15} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \boxed{x_1 = \frac{1}{3}}$$

$$x_2 = \frac{\begin{vmatrix}2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1\end{vmatrix}}{-15} = \frac{2(-8-9)-5(-4-9)+1(12-24)}{-15}$$

$$x_2 = \frac{-34+65-12}{-15} = \frac{19}{-15} \Rightarrow \boxed{x_2 = -\frac{19}{15}}$$

and

$$x_3 = \frac{\begin{vmatrix}2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3\end{vmatrix}}{-15} = \frac{2(6+32)+1(12-24)+5(-16-6)}{-15}$$

$$x_3 = \frac{76-12-110}{-15} = \frac{-46}{-15} \Rightarrow x_3 = \frac{46}{15}$$

Hence $x_1 = \frac{1}{3}$, $x_2 = -\frac{19}{15}$, $x_3 = \frac{46}{15}$ Ans.

(iii) $2x_1 - x_2 + x_3 = 8$

$x_1 + 2x_2 + 2x_3 = 6$

$x_1 - 2x_2 - x_3 = 1$

Here

$|A| = \begin{vmatrix}2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1\end{vmatrix} = 2(-2+4)+1(-1-2)+1(-2-2)$

$|A| = 4-3-4 = -3 \neq 0$

Now by Cramer's Rule

$$x_1 = \frac{\begin{vmatrix}8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1\end{vmatrix}}{-3} = \frac{8(-2+4)+1(-6-2)+1(-12-2)}{-3}$$

$$x_1 = \frac{16-8-14}{-3} = \frac{-6}{-3} = 2 \Rightarrow \boxed{x_1 = 2}$$

$$x_2 = \frac{\begin{vmatrix}2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1\end{vmatrix}}{-3} = \frac{2(-6-2)-8(-1-2)+1(1-6)}{-3}$$

$$x_2 = \frac{-16+24-5}{-3} = \frac{3}{-3} = -1 \Rightarrow \boxed{x_2 = -1}$$

and

$$x_3 = \frac{\begin{vmatrix}2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1\end{vmatrix}}{-3} = \frac{2(2+12)+1(1-6)+8(-2-2)}{-3}$$

$$x_3 = \frac{28-5-32}{-3} = \frac{-9}{-3} = 3 \Rightarrow \boxed{x_3 = 3}$$

Hence $x_1 = 2$, $x_2 = -1$, $x_3 = 3$ Ans.

جو شخص اپنا دوست نہیں وہ کسی کا دوست نہیں ۔

Q2 (i)

$$\left.\begin{aligned} x-2y+z&=-1\\ 3x+y-2z&=4\\ y-z&=1 \end{aligned}\right\}$$

The matrix form of the given system is

$$\begin{bmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

or $AX=B$ ----- (i)

Where $A=\begin{bmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}-1\\4\\1\end{bmatrix}$

As $|A|=\begin{vmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{vmatrix}=1(-1+2)-3(2-1)+0$

$|A|=1-3=-2\neq 0$ so A^{-1} Exists.

Now matrix of cofactors of A

$$=\begin{bmatrix}\begin{vmatrix}1&-2\\1&-1\end{vmatrix} & -\begin{vmatrix}3&-2\\0&-1\end{vmatrix} & \begin{vmatrix}3&1\\0&1\end{vmatrix}\\ -\begin{vmatrix}-2&1\\1&-1\end{vmatrix} & \begin{vmatrix}1&1\\0&-1\end{vmatrix} & -\begin{vmatrix}1&-2\\0&1\end{vmatrix}\\ \begin{vmatrix}-2&1\\1&-2\end{vmatrix} & -\begin{vmatrix}1&1\\3&-2\end{vmatrix} & \begin{vmatrix}1&-2\\3&1\end{vmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}(-1+2)&-(-3-0)&(3-0)\\-(2-1)&(-1-0)&-(1-0)\\(4-1)&-(-2-3)&(1+6)\end{bmatrix}$$

$$=\begin{bmatrix}1&3&3\\-1&-1&-1\\3&5&7\end{bmatrix}$$

$\therefore$ $\text{adj}\,A=\begin{bmatrix}1&-1&3\\3&-1&5\\3&-1&7\end{bmatrix}$

and $A^{-1}=\dfrac{1}{|A|}\text{adj} = \dfrac{1}{-2}\begin{bmatrix}1&-1&3\\3&-1&5\\3&-1&7\end{bmatrix}$

Equation (i) can be written as

$$X=A^{-1}B$$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}1&-1&3\\3&-1&5\\3&-1&7\end{bmatrix}\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-1-4+3\\-3-4+5\\-3-4+7\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2\\-2\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Hence $x=1$, $y=1$, $z=0$ Ans.

(ii)

$$\left.\begin{aligned} 2x_1+x_2+3x_3&=3\\ x_1+x_2-2x_3&=0\\ -3x_1-x_2+2x_3&=-4 \end{aligned}\right\}$$

The matrix form the given system is

$$\begin{bmatrix}2&1&3\\1&1&-2\\-3&-1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3\\0\\-4\end{bmatrix}$$

or $AX=B$ ----- (i)

Where $A=\begin{bmatrix}2&1&3\\1&1&-2\\-3&-1&2\end{bmatrix}$, $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ & $B=\begin{bmatrix}3\\0\\-4\end{bmatrix}$

As $|A|=\begin{vmatrix}2&1&3\\1&1&-2\\-3&-1&2\end{vmatrix}=2(2-2)-1(2-6)+3(-1+3)$

$|A|=0+4+6=10\neq 0$ $\therefore$ A^{-1} Exists

and (i) can be written as

$$X=A^{-1}B \quad ---(ii)$$

Now we find adj A

$A_{11}=(-1)^{1+1}\begin{vmatrix}1&-2\\-1&2\end{vmatrix}=1(2-2)=0$

$A_{12}=(-1)^{1+2}\begin{vmatrix}1&-2\\-3&2\end{vmatrix}=-1(2-6)=4$

$A_{13}=(-1)^{1+3}\begin{vmatrix}1&1\\-3&-1\end{vmatrix}=1(-1+3)=2$

$A_{21}=(-1)^{2+1}\begin{vmatrix}1&3\\-1&2\end{vmatrix}=-1(+2+3)=-5$

$A_{22}=(-1)^{2+2}\begin{vmatrix}2&3\\-3&2\end{vmatrix}=1(4+9)=13$

$A_{23}=(-1)^{2+3}\begin{vmatrix}2&1\\-3&-1\end{vmatrix}=-1(-2+3)=-1$

$A_{31}=(-1)^{3+1}\begin{vmatrix}1&3\\1&-2\end{vmatrix}=1(-2-3)=-5$

$A_{32}=(-1)^{3+2}\begin{vmatrix}2&3\\1&-2\end{vmatrix}=-1(-4-3)=7$

$A_{33}=(-1)^{3+3}\begin{vmatrix}2&1\\1&1\end{vmatrix}=1(2-1)=1$

Thus Matrix of cofactors of A

$$=\begin{bmatrix}0&4&2\\-5&13&-1\\-5&7&1\end{bmatrix}$$

and so $\text{adj}\,A=\begin{bmatrix}0&-5&-5\\4&13&7\\2&-1&1\end{bmatrix}$

Now $A^{-1}=\dfrac{1}{|A|}\text{adj}\,A=\dfrac{1}{10}\begin{bmatrix}0&-5&-5\\4&13&7\\2&-1&1\end{bmatrix}$.

Thus $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\dfrac{1}{10}\begin{bmatrix}0&-5&-5\\4&13&7\\2&-1&1\end{bmatrix}\begin{bmatrix}3\\0\\-4\end{bmatrix}$

$$\left(=\frac{1}{10}\begin{bmatrix}0-0+20\\12+0-28\\6+0-4\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\-16\\2\end{bmatrix}\right)$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-8/5\\1/5\end{bmatrix}$$

Hence $x_1=2$, $x_2=-\dfrac{8}{5}$, $x_3=\dfrac{1}{5}$ Ans

(iii)

$$\left.\begin{aligned} x+y&=2\\ 2x-z&=1\\ 2y-3z&=-1 \end{aligned}\right\}$$

The matrix form of the given system is

$$\begin{bmatrix}1&1&0\\2&0&-1\\0&2&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

or $AX=B$ ——— (i)

where $A=\begin{bmatrix}1&1&0\\2&0&-1\\0&2&-3\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, and $B=\begin{bmatrix}2\\1\\-1\end{bmatrix}$.

$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 1(0+2) - 1(-6-0) + 0$

$|A| = 2 + 6 = 8 \neq 0 \quad \therefore A^{-1}$ Exists

Now we find adj A.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = 1(0+2) = 2$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = -1(-6-0) = 6$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = +1(4-0) = 4$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -1(-3-0) = 3$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = 1(-3-0) = -3$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -1(2-0) = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = 1(-1-0) = -1$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1(-1-0) = 1$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1(0-2) = -2$

So matrix of cofactors of A

$$= \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

and adj $A = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$

So $A^{-1} = \frac{1}{|A|}\,\text{adj}A = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$.

Equation (i) can be written as

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence $x = 1,\ y = 1,\ z = 1$ Ans.

3(i)
$$\left.\begin{aligned} x_1 - 2x_2 - 2x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}\right\}$$

The augmented matrix of the given system is $\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 2 & 3 & 1 & 1 \\ 5 & -4 & -3 & 1 \end{array}\right]$

$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 7 & 5 & 3 \\ 0 & 6 & 7 & 6 \end{array}\right]$ by $R_2 + (-2)R_1 \to R_2'$ & $R_3 + (-5)R_1 \to R_3'$

$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 6 & 7 & 6 \end{array}\right]$ by $R_2 + (-1)R_3 \to R_2'$

$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 19 & 24 \end{array}\right]$ by $R_3 + (-6)R_2 \to R_3'$

$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{24}{19} \end{array}\right]$ by $\frac{1}{19}R_3 \to R_3'$

The equivalent system in echelon form is

$$\left.\begin{aligned} x_1 - 2x_2 - 2x_3 &= -1 \\ x_2 - 2x_3 &= -3 \\ x_3 &= \frac{24}{19} \end{aligned}\right\}$$

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put $x_3 = \frac{24}{19}$ in the second eq.

$x_2 - 2\left(\frac{24}{19}\right) = -3 \Rightarrow x_2 = -3 + \frac{48}{19}$

$\Rightarrow x_2 = \frac{-57+48}{19} \Rightarrow x_2 = -\frac{9}{19}$

Now from first eq.

$x_1 - 2\left(-\frac{9}{19}\right) - 2\left(\frac{24}{19}\right) = -1$

$x_1 + \frac{18}{19} - \frac{48}{19} = -1$

$x_1 = \frac{48}{19} - \frac{18}{19} - 1 = \frac{48-18-19}{19}$

$x_1 = \frac{11}{19}$

Thus the solution is $x_1 = \frac{11}{19}$, $x_2 = -\frac{9}{19}$ & $x_3 = \frac{24}{19}$.

Now we reduce the matrix

$\left[\begin{array}{ccc:c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{24}{19} \end{array}\right]$ to reduced echelon form.

$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -6 & -7 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{24}{19} \end{array}\right]$ by $R_1 + 2R_2 \to R_1'$

$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{11}{19} \\ 0 & 1 & 0 & -9/19 \\ 0 & 0 & 1 & 24/19 \end{array}\right]$ by $R_1 + 6R_3 \to R_1'$

The equivalent system in the reduced echelon form is

$x_1 = \frac{11}{19}$, $x_2 = -\frac{9}{19}$, $x_3 = \frac{24}{19}$

which is the solution set of the given system.

(ii)
$$\left.\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \\ 2x + 3y - z &= 9 \end{aligned}\right\}$$

The augmented matrix of the given matrix is $\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 2 & 3 & -1 & 9 \end{array}\right]$

we reduce the above matrix by applying elementary row operations

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -5 \\ 0 & -1 & -3 & 5 \end{array}\right] \text{ by } R_2+(-2)R_1 \to R_2' \;\&\; R_3+(-2)R_1 \to R_3'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & -1 & -3 & 5 \\ 0 & -3 & 0 & -5 \end{array}\right] \text{ by } R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -3 & 0 & -5 \end{array}\right] \text{ by } (-1)R_2 \to R_2'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 9 & -20 \end{array}\right] \text{ by } R_1+(-2)R_2 \to R_1' \;\&\; R_3+3R_2 \to R_3'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right] \quad \tfrac{1}{9}R_3 \to R_3'$$

The equivalent system in echelon form is

$x - 5z = 12$ —— (1)

$y + 3z = -5$ —— (2)

$z = -\frac{20}{9}$ —— (3)

put $\boxed{z = -\frac{20}{9}}$ in (2) we get

$y + 3\left(-\frac{20}{9}\right) = -5 \Rightarrow y = -5 + \frac{20}{3}$

$y = \frac{-15+20}{3} = \frac{5}{3} \Rightarrow \boxed{y = \frac{5}{3}}$

Now from (1) $x - 5\left(-\frac{20}{9}\right) = 12$

$x = 12 - \frac{100}{9} = \frac{108-100}{9} = \frac{8}{9} \Rightarrow \boxed{x = \frac{8}{9}}$

Thus the solution is

$x = \frac{8}{9}, \; y = \frac{5}{3} \;\&\; z = -\frac{20}{9}.$

Now we reduce the matrix $\left[\begin{array}{ccc:c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right]$ in the reduced echelon form.

$$\left[\begin{array}{ccc:c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{8}{9} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right] \text{ by } R_1+5R_3 \to R_1', \; R_2+(-3)R_3 \to R_2'$$

The equivalent system in reduced (row) echelon form is

$x = \frac{8}{9}, \; y = \frac{5}{3}, \; z = -\frac{20}{9}$ Ans.

(iii) $x_1 + 4x_2 + 2x_3 = 2$

$2x_1 + x_2 - 2x_3 = 9$

$3x_1 + 2x_2 - 2x_3 = 12$

The augmented matrix of the given system is

$$\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 2 & 1 & -2 & 9 \\ 3 & 2 & -2 & 12 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & -7 & -6 & 5 \\ 0 & -10 & -8 & 6 \end{array}\right] \text{ by } R_2+(-2)R_1 \to R_2' \;\&\; R_3+(-3)R_1 \to R_3'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 1 & \frac{6}{7} & -\frac{5}{7} \\ 0 & -10 & -8 & 6 \end{array}\right] \text{ by } -\tfrac{1}{7}R_2 \to R_2'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -\frac{10}{7} & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & -\frac{5}{7} \\ 0 & 0 & \frac{4}{7} & -\frac{8}{7} \end{array}\right] \text{ by } R_1+(-4)R_2 \to R_1' \;\&\; R_3+(10)R_2 \to R_3'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -\frac{10}{7} & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & -\frac{5}{7} \\ 0 & 0 & 1 & -2 \end{array}\right] \text{ by } \tfrac{7}{4}R_3 \to R_3'$$

$x_1 - \frac{10}{7}x_3 = \frac{34}{7}$ —— (1)

$x_2 + \frac{6}{7}x_3 = -\frac{5}{7}$ —— (2)

$x_3 = -2$ —— (3)

Equivalent system

$\Rightarrow \boxed{x_3 = -2}$

From (2) $x_2 + \frac{6}{7}(-2) = -\frac{5}{7}$

$\Rightarrow x_2 = \frac{12}{7} - \frac{5}{7} = \frac{12-5}{7} = \frac{7}{7} = 1 \Rightarrow \boxed{x_2 = 1}$

From eq (1) $x_1 - \frac{10}{7}(-2) = \frac{34}{7}$

$\Rightarrow x_1 + \frac{20}{7} = \frac{34}{7} \Rightarrow x_1 = \frac{34}{7} - \frac{20}{7} = \frac{34-20}{7}$

$x_1 = \frac{14}{7} \Rightarrow \boxed{x_1 = 2}$

Thus the solution is

$x_1 = 2, \quad x_2 = 1, \quad x_3 = -2$

Now we reduce the matrix

$$\left[\begin{array}{ccc:c} 1 & 0 & -\frac{10}{7} & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & -\frac{5}{7} \\ 0 & 0 & 1 & -2 \end{array}\right] \text{ to reduced (row) echelon form}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \text{ by } R_1 + \tfrac{10}{7}R_3 \to R_1', \; R_2 + \left(-\tfrac{6}{7}\right)R_3 \to R_2'$$

The equivalent system in reduced (row) echelon form is

$x_1 = 2, \quad x_2 = 1, \quad x_3 = -2$ Ans.

4(i) $x + 2y - 2z = 0$ —— (1)

$2x + y + 5z = 0$ —— (2)

$5x + 4y + 8z = 0$ —— (3)

As $|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$

$|A| = 1(8-20) - 2(16-25) - 2(8-5)$

$= -12 + 18 - 6 = 0$

So the system of eqs. has a non-trivial solution.

Now $2(1) - (2) \Rightarrow$

$$\begin{array}{r} 2x + 4y - 4z = 0 \\ \underline{2x \pm y \pm 5z = 0} \\ 3y - 9z = 0 \end{array}$$

$1 - 3z$ — (4)

gain by $2(2) - (1) \Rightarrow$
$$\begin{aligned} 4x+2y+10z&=0\\ -x \mp 2y \pm 2z&=0 \\ \hline 3x+12z&=0\end{aligned}$$

$\Rightarrow 3x = -12z \Rightarrow x = -4z$ — (5)

using (4) & (5) in (3)

$LHS = 5(-4z) + 4(3z) + 8z$
$= -20z + 12z + 8z$
$= -20z + 20z$
$= 0$
$= RHS.$

$\therefore$ Eq (3) is satisfied by (4) & (5) which shows that eqs (1), (2) & (3) are satisfied by

$x = -4t,\ y = 3t\ \&\ z = t$

for any real value of t

Thus the given system has infinitely many solutions.

(ii) $x_1 + 4x_2 + 2x_3 = 0$ — (i)

$2x_1 + x_2 - 3x_3 = 0$ — (ii)

$3x_1 + 2x_2 - 4x_3 = 0$ — (iii)

The matrix form of the above system

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{pmatrix}$

As $|A| = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{vmatrix} = 1(-4+6) - 4(-8+9) + 2(4-3)$

$|A| = 2 - 4 + 2 = 4 - 4 = 0$

$\therefore$ The system has a non trivial sol.

Solving Eqs (i) and (ii)

By $2(i) - (ii) \Rightarrow$
$$\begin{aligned} 2x_1 + 8x_2 + 4x_3 &= 0 \\ 2x_1 \pm x_2 \mp 3x_3 &= 0 \\ \hline 7x_2 + 7x_3 &= 0 \end{aligned}$$

$\Rightarrow 7x_2 = -7x_3 \Rightarrow x_2 = -x_3$ — (iv)

and by $(i) - 4(ii) \Rightarrow$
$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ -8x_1 \pm 4x_2 \mp 12x_3 &= 0 \\ \hline -7x_1 + 14x_3 &= 0 \end{aligned}$$

$\Rightarrow -7x_1 = -14x_3 \Rightarrow x_1 = 2x_3$ — (v)

using (iv) & (v) in (iii)

$3(2x_3) + 2(-x_3) - 4x_3$
$= 6x_3 - 2x_3 - 4x_3$
$= 0$

which shows that eq (i), (ii) & (iii) are satisfied by $x_1 = 2t,\ x_2 = -t,\ x_3 = t$

for any real value of t.

Thus system is consistant & has infinitely many solutions.

(iii) $x_1 - 2x_2 - x_3 = 0$ — (i)

$x_1 + x_2 + 5x_3 = 0$ — (ii)

$2x_1 - x_2 + 4x_3 = 0$ — (iii)

Here As $|A| = \begin{vmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{vmatrix}$

$= 1(4+5) + 2(4-10) - 1(-1-2)$
$= 9 - 12 + 3 = 0$

$\therefore$ The system has a non-trivial solution

Solve eqs. (ii) & (iii)

Adding (ii) & (iii)
$$\begin{aligned} x_1 + x_2 + 5x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 0 \\ \hline 3x_1 + 9x_3 &= 0 \end{aligned}$$

$\Rightarrow 3x_1 = -9x_3 \Rightarrow x_1 = -3x_3$ — (iv)

Also $2(ii) - (iii) \Rightarrow$
$$\begin{aligned} 2x_1 + 2x_2 + 10x_3 &= 0 \\ -2x_1 \mp x_2 \pm 4x_3 &= 0 \\ \hline 3x_2 + 6x_3 &= 0 \end{aligned}$$

$\Rightarrow 3x_2 = -6x_3 \Rightarrow x_2 = -2x_3$ — (v)

using (iv) & (v) in (i)

$-3x_3 - 2(-2x_3) - x_3$
$= -3x_3 + 4x_3 - x_3 = 0$

which shows that the eqs (i), (ii) & (iii) are satisfied by

$x_1 = -3t,\ x_2 = -2t,\ x_3 = t$

for any real value of t

Thus the system consisting (i), (ii) & (iii) has infinitely many solutions

(5) (ii) $x + y + z = 0$ — (i)

$2x + y - \lambda z = 0$ — (ii)

$x + 2y - 2z = 0$ — (iii)

Given system has a non trivial sol. if $|A| = 0$

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$

$\Rightarrow 1(-2+2\lambda) - 1(-4+\lambda) + 1(4-1) = 0$

$\Rightarrow -2 + 2\lambda + 4 - \lambda + 3 = 0 \Rightarrow \lambda + 5 = 0$

$\Rightarrow \boxed{\lambda = -5}$

Now the given system becomes

2(b) $x+y+z=0$ —— (1)

$2x+y+5z=0$ —— (2)

$x+2y-2z=0$ —— (3)

we solve (1) & (2)

By subtracting (2) from (1) we get

$$\begin{array}{l} x+y+z=0 \\ \underline{2x+y+5z=0} \\ -x-4z=0 \end{array} \Rightarrow \boxed{x=-4z} \quad —(4)$$

Also by $2(1)-(2) \Rightarrow$

$$\begin{array}{r} 2x+2y+2z=0 \\ \underline{2x \pm y \pm 5z=0} \\ y-3z=0 \end{array}$$

$\boxed{y=3z}$ —(5)

using (4) & (5) in (3)

$-4z+3z+z=0$

which shows that eqs (i) (ii) & (iii) are satisfied by $x=-4t$, $y=3t$ and $z=t$ for any real value of t.

—x—x—x—x—x—

(ii) $x_1+4x_2+\lambda x_3=0$ —— (i)

$2x_1+x_2-3x_3=0$ —— (ii)

$3x_1+\lambda x_2-4x_3=0$ —— (iii)

The given system has a non-trivial solution if $|A|=0$ that is

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix}=0$$

$\Rightarrow 1(-4+3\lambda)-4(-8+9)+\lambda(2\lambda-3)=0$

$\Rightarrow -4+3\lambda+32-36+2\lambda^2-3\lambda=0$

$\Rightarrow 2\lambda^2-8=0 \Rightarrow 2\lambda^2=8 \Rightarrow \lambda^2=4$

$\lambda=\pm 2$

For $\lambda=2$, the given system becomes

$x_1+4x_2+2x_3=0$ —— (1)

$2x_1+x_2-3x_3=0$ —— (2)

$3x_1+2x_2-4x_3=0$ —— (3)

we solve (1) and (2)

By $(1)-4(2) \Rightarrow$

$$\begin{array}{r} x_1+4x_2+2x_3=0 \\ \underline{8x_1 \pm 4x_2 \mp 12x_3=0} \\ -7x_1+14x_3=0 \end{array}$$

$\Rightarrow 7x_1=14x_3$

$\Rightarrow \boxed{x_1=2x_3}$ —(4)

Now By $2(1)-(2) \Rightarrow$

$$\begin{array}{r} 2x_1+8x_2+4x_3=0 \\ \underline{2x_1 \pm x_2 \mp 3x_3=0} \\ 7x_2+7x_3=0 \end{array}$$

$\Rightarrow 7x_2=-7x_3 \Rightarrow \boxed{x_2=-x_3}$ —(5)

using (4) & (5) in (3)

$3(2x_3)+2(-x_3)-4x_3=6x_3-2x_3-4x_3=0$

which shows that eqs. (1), (2) and (3) are satisfied by $x_1=2t$, $x_2=-t$ & $x_3=t$ for any real value of t. Ans

Again for $\lambda=-2$

The given system becomes

$x_1+4x_2-2x_3=0$ —— (a)

$2x_1+x_2-3x_3=0$ —— (b)

$3x_1-2x_2-4x_3=0$ —— (c)

we solve equations (a) & (b)

By $(a)-4(b) \Rightarrow$

$$\begin{array}{r} x_1+4x_2-2x_3=0 \\ \underline{-8x_1 \pm 4x_2 \mp 12x_3=0} \\ -7x_1+10x_3=0 \end{array}$$

$\Rightarrow 10x_3=7x_1 \Rightarrow \boxed{x_3=\frac{7}{10}x_1}$ —(d)

By $3(a)-2(b) \Rightarrow$

$$\begin{array}{r} 3x_1+12x_2-6x_3=0 \\ \underline{-4x_1 \pm 2x_2 \mp 6x_3=0} \\ -x_1+10x_2=0 \end{array}$$

$\Rightarrow 10x_2=x_1 \Rightarrow \boxed{x_2=\frac{1}{10}x_1}$ —(f)

using (d) & (f) in (c)

$3x_1-\frac{2}{10}x_1-4(\frac{7}{10}x_1)$

$=\dfrac{30x_1-2x_1-28x_1}{10}=\dfrac{0}{10}=0$

which shows that equations (a), (b) & (c) are satisfied by

$x_1=t$, $x_2=\frac{1}{10}t$, & $x_3=\frac{7}{10}t$

for any real value t.

—x—x—x—x—

(b) $x_1+4x_2+\lambda x_3=2$ —— (1)

$2x_1+x_2-2x_3=11$ —— (2)

$3x_1+2x_2-2x_3=16$ —— (3)

The matrix of coefficients of the given system is

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{bmatrix}$$

$$R\begin{bmatrix} 1 & 4 & \lambda \\ 0 & -7 & -2-2\lambda \\ 0 & -10 & -2-3\lambda \end{bmatrix} \quad \text{by } R_2+(-2)R_1 \rightarrow R_2' \ \& \ R_3+(-3)R_1 \rightarrow R_3'$$

$$R\begin{bmatrix} 1 & 4 & \lambda \\ 0 & 1 & \frac{2+2\lambda}{7} \\ 0 & -10 & -2-3\lambda \end{bmatrix} \quad \text{by } -\tfrac{1}{7}R_2 \rightarrow R_2'$$

$$R\begin{bmatrix} 1 & 4 & \lambda \\ 0 & 1 & \frac{2}{7}+\frac{2}{7}\lambda \\ 0 & 0 & -2-3\lambda+\frac{20}{7}+\frac{20}{7}\lambda \end{bmatrix} \quad \text{by } R_3+10R_2 \rightarrow R_3'$$

$$\sim\begin{pmatrix} 1 & 4 & \lambda \\ 0 & 1 & \frac{2}{7}+\frac{2}{7}\lambda \\ 0 & 0 & \frac{6-\lambda}{7} \end{pmatrix}$$

Now the system does not possess a unique solution if

$$\frac{6-\lambda}{7}=0 \Rightarrow 6-\lambda=0 \Rightarrow \lambda=6$$

For $\lambda=6$ we have the augmented matrix of the given system

$$\begin{bmatrix} 1 & 4 & 6 & \vdots & 2 \\ 2 & 1 & -2 & \vdots & 11 \\ 3 & 2 & -2 & \vdots & 16 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 6 & \vdots & 2 \\ 0 & -7 & -14 & \vdots & 7 \\ 0 & -10 & -20 & \vdots & 10 \end{bmatrix} \begin{matrix} \text{by} \\ R_2+(-2)R_1\to R_2' \\ R_3+(-3)R_1\to R_3' \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 6 & \vdots & 2 \\ 0 & 1 & 2 & \vdots & -1 \\ 0 & 1 & 2 & \vdots & -1 \end{bmatrix} \begin{matrix} \text{by } -\frac{1}{7}R_2\to R_2' \\ -\frac{1}{10}R_3\to R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 6 & \vdots & 2 \\ 0 & 1 & 2 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \text{ by } R_3+(-1)R_2\to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -2 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{matrix} \text{by} \\ R_1+(-4)R_2\to R_1' \end{matrix}$$

Now the system is reduced to the equivalent system

$$x_1-2x_3=6 \Rightarrow x_1=2x_3+6$$

$$x_2+2x_3=-1 \qquad x_2=-2x_3-1$$

Thus we see that for $\lambda=6$ and after this the system is satisfied by $x_1=2t+6$,

$x_2=-2t-1$ and $x_3=t$

for any real value of t.

Hence for the value of λ which is $\lambda=6$

The solution of the system is

$x_1=2t+6,\ x_2=-2t-1,\ x_3=t$