

EXERCISE 3.5

1. Solve the following systems of linear equations by Cramer's Rule:

Solution (i) $2x + 2y + z = 3$, $3x - 2y - 2z = 1$, $5x + y - 3z = 2$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix} = 2(6+2) - 2(-9+10) + 1(3+10) \\ = 16 - 2 + 13 = 27 \neq 0 \\ \therefore \text{Solution is possible.}$$

$$\therefore x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{27} = \frac{3(6+2) - 2(-3+4) + 1(1+4)}{27} = \frac{24-2+5}{27} = \frac{27}{27} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{27} = \frac{2(-3+4) - 3(-9+10) + 1(6-5)}{27} = \frac{2-3+1}{27} = \frac{0}{27} = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{27} = \frac{2(-4-1) - 2(6-5) + 3(3+10)}{27} = \frac{-10-2+39}{27} = \frac{27}{27} = 1$$

1(ii). $2x_1 - x_2 + x_3 = 5$, $4x_1 + 2x_2 + 3x_3 = 8$, $3x_1 - 4x_2 - x_3 = 3$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix} = 2(-2+12) + 1(-4-9) + 1(-16-6) \\ = 20 - 13 - 22 = -15 \neq 0 \\ \therefore \text{Solution is possible.}$$

$$\therefore x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{-15} = \frac{5(-2+12) + 1(-8-9) + 1(-32-6)}{-15} = \frac{50-17-38}{-15} = \frac{1}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{-15} = \frac{2(-8-9) - 5(-4-9) + 1(12-24)}{-15} = \frac{-34+65-12}{-15} = -\frac{19}{15}$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{-15} = \frac{2(6+32) + 1(12-24) + 5(-16-6)}{-15} = \frac{76-12-110}{-15} = \frac{46}{15}$$

$$1(\text{iii}) \quad 2x_1 - x_2 + x_3 = 8, x_1 + 2x_2 + 2x_3 = 6, x_1 - 2x_2 - x_3 = 1$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 2(-2+4) + 1(-1-2) + 1(-2-2) \\ = 4 - 3 - 4 = -3 \neq 0 \\ \therefore \text{Solution is possible.}$$

$$\therefore x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{-3} = \frac{8(-2+4) + 1(-6-2) + 1(-12-2)}{-3} = \frac{16-8-14}{-3} = 2$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{-3} = \frac{2(-6-2) - 8(-1-2) + 1(1-6)}{-3} = \frac{-16+24-5}{-3} = -1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{-3} = \frac{2(2+12) + 1(1-6) + 8(-2-2)}{-3} = \frac{28-5-32}{-3} = 3$$

2. Use matrices to solve the following systems:

$$\text{Solution (i) } x - 2y + z = -1, 3x + y - 2z = 4, y - z = 1$$

$$\text{Here } |A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 1(-1+2) - 3(2-1) \\ = 1 - 3 = -2 \neq 0 \text{ (Exp. by } C_1) \\ \therefore \text{Solution is possible.}$$

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} (-1+2) & -(-3+0) & (3-0) \\ -(2-1) & (-1-0) & -(1+0) \\ (4-1) & -(-2-3) & (1+6) \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -1 & -1 & -1 \\ 3 & 5 & 7 \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-2} \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow x=1, y=1, z=0$$

Solution (ii) $2x_1 + x_2 + 3x_3 = 3$, $x_1 + x_2 - 2x_3 = 0$, $-3x_1 - x_2 + 2x_3 = -4$

Here $|A| = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix} = 2(2-2) - 1(2-6) - 3(-1+3) = 0 + 4 - 6 = -2 \neq 0$
 \therefore Solution is possible.

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & 4 & 2 \\ 1 & -5 & -1 \\ 1 & 1 & 1 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-2} \begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow = \frac{1}{-2} \begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 0+0-4 \\ 12-0-4 \\ 6-0-4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 0+0-4 \\ 12-0-4 \\ 6-0-4 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -4 \\ 16 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$$

$$\Rightarrow x_1 = 2, x_2 = -4, x_3 = -1.$$

Solution (iii) $x + y = 2, 2x - z = 1, 2y - 3z = -1$

$$\text{Here } |A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 1(0+2) - 1(-6-0) + 0 = 2+6 = 8 \neq 0$$

\therefore Solution is possible.

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} (0+2) & -(-6+0) & (4-0) \\ -(-3-0) & (-3-0) & (-2-0) \\ (-1-0) & -(-1+0) & (0-2) \end{bmatrix}^t = \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{8} \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = 1, y = 1, z = 1.$$

8. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

Solution. (i). $x_1 - 2x_2 - 2x_3 = -1$, $2x_1 + 3x_2 + x_3 = 1$, $5x_1 - 4x_2 - 3x_3 = 1$

(a) : Using Echelon Form :

Here A_b denotes the augmented matrix.

$$A_b = \begin{bmatrix} 1 & -2 & -2 & -1 \\ 2 & 3 & 1 & 1 \\ 5 & -4 & -3 & 1 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 7 & 5 & 3 \\ 0 & 6 & 7 & 6 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 6 & 7 & 6 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - R_3 \end{array} \xrightarrow{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 19 & 24 \end{bmatrix} \begin{array}{l} \text{by} \\ R_3 - 6R_2 \end{array}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} \begin{array}{l} \text{by} \\ \frac{1}{19} R_3 \end{array} \dots \text{ (I)}$$

Matrix in equation (I) is in Echelon Form.

$$\text{Now, } R_1 \Rightarrow x_1 - 2x_2 - 2x_3 = -1 \quad \dots(1),$$

$$R_2 \Rightarrow 0 + x_2 - 2x_3 = -3 \quad \dots(2),$$

$$R_3 \Rightarrow 0 + 0 + x_3 = \frac{24}{19} \quad \dots(3),$$

$$\text{From backward substitution : (2)} \Rightarrow x_2 = 2x_3 - 3 = \frac{2(24)}{19} - 3 = \frac{48 - 57}{19} = -\frac{9}{19}$$

$$\begin{aligned} \text{and (1)} \Rightarrow x_1 &= 2x_2 + 2x_3 - 1 = 2\left(-\frac{9}{19}\right) + 2\left(\frac{24}{19}\right) - 1 = -\frac{18}{19} + \frac{48}{19} - 1 \\ &\Rightarrow x_1 = \frac{-18 + 48 - 19}{19} = \frac{11}{19} \end{aligned}$$

(b) : Using Reduced Echelon Form :

Matrix in (I) can further be changed to *Reduced Echelon Form* :

$$\begin{aligned} \tilde{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} &\tilde{R} \begin{bmatrix} 1 & 0 & -6 & -7 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} \quad (\text{by } R_1 + 2R_2) \\ \tilde{R} \begin{bmatrix} 1 & 0 & 0 & 11/19 \\ 0 & 1 & 0 & -9/19 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} &\begin{matrix} \text{by} \\ R_1 + 6R_3 \\ R_2 + 2R_3 \end{matrix} \quad \dots \quad \text{(II)} \end{aligned}$$

From equation (II) :

$$\text{Now, } R_1 \Rightarrow x_1 + 0 + 0 = \frac{11}{19} \Rightarrow x_1 = \frac{11}{19}$$

$$R_2 \Rightarrow 0 + x_2 + 0 = -\frac{9}{19} \Rightarrow x_2 = -\frac{9}{19}$$

$$R_3 \Rightarrow 0 + 0 + x_3 = \frac{24}{19} \Rightarrow x_3 = \frac{24}{19}$$

Solution. (ii). $x + 2y + z = 2$, $2x + y + 2z = -1$, $2x + 3y - z = 9$

(a): Using Echelon Form :

$$\begin{aligned} A_b &= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 2 & 3 & -1 & 9 \end{bmatrix} \tilde{R} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -5 \\ 0 & -1 & -3 & 5 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \\ \tilde{R} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & -1 & -3 & 5 \end{bmatrix} &\begin{matrix} \text{by} \\ R_2 - 4R_3 \rightarrow R_2 \end{matrix} \end{aligned}$$

$$\begin{aligned} \tilde{R} & \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 9 & -20 \end{bmatrix} \quad \text{by} \\ & \quad R_3 + R_2 \rightarrow R_3 \\ \\ \tilde{R} & \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ \frac{1}{9}R_3 \end{array} \quad \dots \text{ (I)} \end{aligned}$$

Matrix in equation (I) is in Echelon Form.

$$\text{Now, } R_1 \Rightarrow x + 2y + z = 2 \quad \dots(1),$$

$$R_2 \Rightarrow 0 + y + 12z = -25 \quad \dots(2),$$

$$R_3 \Rightarrow 0 + 0 + z = -\frac{20}{9} \quad \dots(3),$$

Solving by backward substitution, using $z = -20/9$, we get

$$(2) \Rightarrow y + 12z = -25 \Rightarrow y = -25 - 12z = -25 - 12\left(-\frac{20}{9}\right)$$

$$y = -25 + \frac{80}{3} = \frac{-75 + 80}{3} = \frac{5}{3}$$

$$(1) \Rightarrow x + 2y + z = 2 \Rightarrow x = 2 - 2y - z = 2 - 2 \times \frac{5}{3} - \left(-\frac{20}{9}\right)$$

$$x = 2 - \frac{10}{3} + \frac{20}{9} = \frac{18 - 30 + 20}{9} = \frac{8}{9}$$

$$\text{Hence } x = \frac{8}{9}, y = \frac{5}{3}, z = -\frac{20}{9}.$$

(b): Using Reduced Echelon Form:

Equation (I) can further be reduced to Reduced echelon form:

$$\begin{aligned} & \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \quad \tilde{R} \begin{bmatrix} 1 & 0 & -23 & 52 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_1 - 2R_2 \end{array} \\ \\ \tilde{R} & \begin{bmatrix} 1 & 0 & 0 & 8/9 \\ 0 & 1 & 0 & -15/9 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_1 + 23R_3 \\ R_2 - 12R_3 \end{array} \quad \dots \text{ (II)} \end{aligned}$$

Matrix in Equation (II) is in reduced echelon form giving directly

$$x = \frac{8}{9}, y = \frac{15}{9} = \frac{5}{3}, z = -\frac{20}{9}.$$

Solution. (iii). $x_1 + 4x_2 + 2x_3 = 2$, $2x_1 + x_2 - 2x_3 = 9$, $3x_1 + 2x_2 - 2x_3 = 12$

(a): Using Echelon Form:

$$Ab = \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 1 & -2 & 9 \\ 3 & 2 & -2 & 12 \end{bmatrix}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & -7 & -6 & 5 \\ 0 & -10 & -8 & 6 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & -10 & -8 & 6 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ -\frac{1}{7}R_2 \end{array}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 4/7 & -8/7 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_3 + 10R_2 \dots \end{array} \quad \text{(I)}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ \frac{7}{4}R_3 \dots \end{array} \quad \text{(I)}$$

Matrix in equation (I) is in Echelon Form.

$$R_1 \Rightarrow x_1 + 4x_2 + 2x_3 = 2 \quad \dots(1),$$

$$R_2 \Rightarrow 0 + x_2 + \frac{6}{7}x_3 = -\frac{5}{7} \quad \dots(2),$$

$$R_3 \Rightarrow 0 + 0 + x_3 = -2 \quad \dots(3),$$

Solving by backward substitution, we get

$$(3) \Rightarrow x_3 = -2$$

$$(2) \Rightarrow x_2 + \frac{6}{7}x_3 = -\frac{5}{7} \Rightarrow x_2 = -\frac{5}{7} - \frac{6}{7}x_3 = -\frac{5}{7} - \frac{6}{7}(-2)$$

$$\Rightarrow x_2 = -\frac{5}{7} + \frac{12}{7} = \frac{7}{7} = 1$$

$$(1) \Rightarrow x_1 + 4x_2 + 2x_3 = 2 \Rightarrow x_1 = 2 - 4x_2 - 2x_3$$

$$\Rightarrow x_1 = 2 - 4(1) - 2(-2) = 2 - 4 + 4 = 2$$

Hence $x_1 = 2$, $x_2 = 1$, $x_3 = -2$.

(b): Using Reduced Echelon Form :

Equation (I) can further be reduced to Reduced echelon form :

$$\begin{aligned} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} & \xrightarrow{R} \begin{bmatrix} 1 & 0 & -10/7 & 34/7 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_1 - 4R_2 \\ R_3 + 10R_2 \end{array} \\ \\ \xrightarrow{R} \begin{bmatrix} 1 & 0 & 0 & 14/7 \\ 0 & 1 & 0 & 7/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} & \begin{array}{l} \text{by} \\ R_1 + \frac{10}{7}R_3 \\ R_2 - \frac{6}{7}R_3 \end{array} \quad \dots \text{ (II)} \end{aligned}$$

Matrix in equation (II) is in reduced echelon form giving directly

$$x_1 = \frac{14}{7} = 2, \quad x_2 = \frac{7}{7} = 1, \quad x_3 = -2.$$

4. Solve the following systems of homogeneous linear equations :

Note. In case of homogeneous system, it has non-zero solution only if the determinant of coefficients is zero, i.e., matrix of coefficients is singular, i.e., its rank is less than the number of variables. That is why we use the matrix of coefficients only for echelon form.

(i) $x + 2y - 2z = 0$, $2x + y + 5z = 0$, $5x + 4y + 8z = 0$

$$\begin{aligned} \text{Solution (i). } A &= \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & -6 & 18 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array} \\ \\ & \xrightarrow{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \quad \left(\begin{array}{l} \text{by} \\ -\frac{1}{3}R_2, \frac{1}{6}R_3 \end{array} \right) \\ \\ & \xrightarrow{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{by} \\ R_3 - R_2 \end{array} \quad \dots \text{ (I)} \end{aligned}$$

Thus rank of $A = 2 < 3$ (No. of variables). Therefore, (I) gives

$$R_1 \Rightarrow x + 2y - 2z = 0 \quad \dots \text{ (i),}$$

$$R_2 \Rightarrow y - 3z = 0 \quad \dots \text{ (ii),}$$

$$R_3 \Rightarrow 0z = 0 \dots \text{(iii)}$$

$$\text{(iii)} \Rightarrow z = t, \text{ then } \text{(ii)} \Rightarrow y = 3z = 3t.$$

$$\text{and } \text{(i)} \text{ gives } x + 6t - 2t = 0 \Rightarrow x = -4t.$$

Thus, S.S. = $\{ t, 3t, 4t \}$ where t is any real number.

REMARKS.-1. If $t = 1$, then solution is $\{ 1, 3, -4 \}$, if $t = 2$, then S.S. is $\{ 2, 6, -8 \}$, i.e., we can find infinity of solutions by giving different values to t .

In fact, when the number of equations is less than the number of variables, we have to give some arbitrary value of a variable and get the value of other/s in terms of that value.

2. If we are given a set of three homogeneous equations they will have $\{0,0,0\}$ as obvious solution. This solution is called *Trivial Solution*. And a solution of the system other than zero solution is called *Non-TRivial Solution*.

3. In such cases, we will use parametric values to free variables. Free variables are those variables which are in the last alphabatics, e.g., out of x, y, z , the free variable is z , out of xy , the free variable is y and out of x_1, x_2, x_3 the free variable is x_3 and parameters are ; t, s, u, v, w etc.

$$\text{Solution (ii). } x_1 + 4x_2 + 2x_3 = 0, 2x_1 + x_2 - 3x_3 = 0, 3x_1 + 2x_2 - 4x_3 = 0$$

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & -10 & -10 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \left(\begin{array}{l} \text{by} \\ -\frac{1}{7}R_2, -\frac{1}{10}R_3 \end{array} \right)$$

$$\xrightarrow{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{by} \\ R_3 - R_2 \end{array} \dots \text{(I)}$$

Thus rank of $A = 2 < 3$ (No. of variables). Therefore, (I) gives

$$R_1 \Rightarrow x_1 + 4x_2 + 2x_3 = 0 \dots \text{(i)},$$

$$R_2 \Rightarrow x_2 + x_3 = 0 \quad \dots \text{(ii)},$$

$$R_3 \Rightarrow 0x_3 = 0 \quad \dots \text{(iii)}$$

$$\text{(iii)} \Rightarrow x_3 = t, \text{ then}$$

$$\text{(ii)} \Rightarrow x_2 = -x_3 = -t$$

and (i) gives

$$x_1 - 4t + 2t = 0 \Rightarrow x_1 = 2t.$$

Thus, S.S. = $\{ 2t, -t, t \}$ where t is any real number.

Solution (iii). $x_1 - 2x_2 - x_3 = 0$, $x_1 + x_2 + 5x_3 = 0$, $2x_1 - x_2 + 4x_3 = 0$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \left(\begin{array}{l} \text{by} \\ \frac{1}{3}R_2, \frac{1}{3}R_3 \end{array} \right)$$

$$\xrightarrow{R} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{by} \\ R_3 - R_2 \end{array} \dots \text{(I)}$$

Thus rank of $A = 2 < 3$ (No. of variables). Therefore, (I) gives

$$x_1 - 2x_2 - x_3 = 0 \dots \text{(i)}, \quad x_2 + 2x_3 = 0 \dots \text{(ii)}, \quad 0x_3 = 0 \dots \text{(iii)}$$

$$\text{(iii)} \Rightarrow x_3 = t, \text{ then (ii) gives } x_2 = -2x_3 = -2t \text{ and (i) gives}$$

$$x_1 + 4t - t = 0 \Rightarrow x_1 = -3t.$$

Thus, S.S. = $\{ -3t, -2t, t \}$ where t is any real number.

5. Find the value of λ for which the following systems have a non-trivial solution. Also solve the system for the value of λ .

Solution. (i) $x + y + z = 0$, $2x + y - \lambda z = 0$, $x + 2y - 2z = 0$

The system has non-trivial solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$$1(-2 + 2\lambda) - 1(-4 + \lambda) + 1(4 - 1) = 0$$

$$\Rightarrow -2 + 2\lambda + 4 - \lambda + 3 = 0$$

$$\lambda + 5 = 0 \Rightarrow \lambda = -5$$

For the non-trivial solution (using $\lambda = -5$), we have

$$A_{\mathbf{b}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 2 & -2 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \text{ (by } -R_2 \text{)}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{by} \\ R_3 - R_2 \end{array} \dots \text{(I)}$$

$$\text{(I)} \Rightarrow x + y + z = 0, y - 3z = 0, 0z = 0 \Rightarrow z = t.$$

$$y = 3z = 3t, \text{ and } x + 3t + t = 0, \text{ i.e., } x = -4t$$

Thus, S.S. = $\{-4t, 3t, t\}$ where t is any real number.

$$\text{Solution. (ii). } x_1 + 4x_2 + \lambda x_3 = 0, 2x_1 + x_2 - 3x_3 = 0, 3x_1 + \lambda x_2 - 4x_3 = 0$$

The system has non-trivial solution if

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$$

$$1(-4 + 3\lambda) - 4(-8 + 9) + \lambda(2\lambda - 3) = 0$$

$$\text{or } -4 + 3\lambda - 4 + 2\lambda^2 - 3\lambda = 0$$

$$\text{or } 2\lambda^2 - 8 = 0 \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

When $\lambda = 2$:

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

$$\xrightarrow{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & -10 & -10 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & -10 & -10 \end{bmatrix} \text{ (by } -R_2 \text{)} \xrightarrow{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ by, } R_3 - R_2 \dots \text{(I)}$$

$$R_1 \Rightarrow x_1 + 4x_2 + 2x_3 = 0 \quad \dots \quad (i),$$

$$R_2 \Rightarrow x_3 = 0 \quad \dots \quad (ii),$$

Let $x_2 = t$, then

$$(i) \Rightarrow x_1 + 4t + 0 = 0 \Rightarrow x_1 = -4t$$

Thus, S.S. = $\{-4t, t, 0\}$ where t is any real number.

When $\lambda = -2$:

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 1 & -3 \\ 3 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & -14 & 2 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & -7 & 1 \end{bmatrix} \begin{array}{l} \text{(by } -R_2) \\ R_3 - R_2 \end{array} \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ by, } R_3 - R_2 \dots (I)$$

$$R_1 \Rightarrow x_1 + 4x_2 - 2x_3 = 0 \quad \dots \quad (i),$$

$$R_2 \Rightarrow -7x_2 + x_3 = 0 \quad \dots \quad (ii),$$

Let $x_2 = t$, then

$$(ii) \Rightarrow -7x_2 + x_3 = 0 \Rightarrow x_3 = \frac{t}{7}$$

$$(i) \Rightarrow x_1 + 4 \cdot \frac{t}{7} - 2 \cdot \frac{t}{7} = 0 \Rightarrow x_1 = -2t \cdot \frac{4}{7} = -\frac{10t}{7}$$

Thus, S.S. = $\left\{ \left(\frac{10t}{7}, \frac{t}{7}, t \right) \right\}$ where t is any real number.

$$= \left\{ (10t, t, 7t) \right\} \text{ Multiplying by } 7.$$

For $\lambda = -2$, (A) reads:

$$x_1 + 4x_2 - 2x_3 = 0, x_2 + \frac{3-4}{7}x_3 = 0, 0x_3 = 0 \Rightarrow x_3 = k.$$

$$x_2 = -\frac{1}{7}x_3 = -\frac{1}{7}k, \text{ and } x_1 - \frac{4}{7}k - 2k = 0, \text{ i.e., } x_1 = -\frac{18}{7}k$$

6. Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ .

$$x_1 + 4x_2 + \lambda x_3 = 2, 2x_1 + x_2 - 2x_3 = 11, 3x_1 + 2x_2 - 2x_3 = 16.$$

Solution.

The system does not possess a unique solution if

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

$$1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$$

$$2 - 8 + \lambda = 0 \Rightarrow \lambda = 6$$

For the solution put $\lambda = 6$ in the system, then solution is given by

$$A = \begin{bmatrix} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{bmatrix}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{bmatrix} \begin{array}{l} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{array}{l} \left(\text{by } -\frac{1}{7}R_2 \right) \\ \left(-\frac{1}{10}R_3 \right) \end{array}$$

$$\tilde{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{by} \\ R_3 - R_2 \dots \text{(I)} \\ \rightarrow R_3 \end{array}$$

$$R_1 \Rightarrow x_1 + 4x_2 + 6x_3 = 2, \dots \text{(i)},$$

$$R_2 \Rightarrow x_2 + 2x_3 = -1 \dots \text{(ii)},$$

Let $x_3 = t$

$$\text{(ii)} \Rightarrow x_2 + 2t = -1 \Rightarrow x_2 = -1 - 2t$$

$$\text{(i)} \Rightarrow x_1 + 4(-1 - 2t) + 6t = 2,$$

$$\Rightarrow x_1 - 4 - 8t + 6t = 2,$$

$$\Rightarrow x_1 = 6 + 2t,$$

Thus, S.S. = $\{ 6 + 2t, -1 - 2t, t \}$ where t is any real number.

