DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SARGODHA
ADMISSION TEST PH.D

Max Marks: 70 **Date:** 28-01-12 **Time:** 1.5 Hours

Instructions:

- *Attempt all questions.*
- *Read all questions very carefully before answering them.*
- *Write only solutions of the problems on answer sheet.*

Question 1(a). Suppose f is defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$, then prove that $g(x) \to 0$ as $x \to +\infty$.

(b). Let E be the set of all continuous real valued functions $u : [0,1] \to \mathbb{R}$ satisfying

$$|u(x) - u(y)| \leq |x - y|, \quad 0 \leq x, y \leq 1, \quad u(0) = 0.$$

Let $f : E \to \mathbb{R}$ be define by

$$f(u) = \int_0^1 \left(u^2(x) - u(x)\right) dx.$$

Show that the maximum value of f on E is $5/6$, that is, $|f(u)| \leq 5/6$ for $u \in E$.

Question 2(a). If $A_{ij}B_j$ is a vector where B_j is an arbitrary vector. Prove that A_{ij} is also a tensor of rank 2.

(b). Show that the general rigid body motion is a screw motion.

Question 3(a).

(i) Write CR-equation in rectangular coordinates.

(ii) Write CR-equation in polar coordinates.

(iii) When C is any positively oriented simple closed contour surrounding the origin then

$$\int_C \frac{dz}{z} = \ldots\ldots\ldots\ldots\ldots\ldots$$

(iv) Show that $u(x, y) = 2x(1 - y)$ is harmonic.

(v) $\text{Arg}(-z) = \ldots\ldots\ldots\ldots\ldots\ldots$

(b). Use residues to evaluate the improper integral

$$\int_0^\infty \frac{dx}{x^2 + 1}$$

Question 4(a). Determine whether the following statements true or false. (3)

(i) Every subset of a discrete metric space is open. (T/F)

(ii) $\text{Int}(A) \cup \text{Int}(B) = \text{Int}(A \cup B)$. (T/F)

(iii) A Cauchy sequence in a metric space converges in the space. (T/F)

(b). Fill in the blanks. (3)

(i) In a metric space (X, d) every .... sequence is Cauchy.

(ii) X is a Hausdorff space iff the diagonal $D = \{(x, x) | x \in X\}$ is .... in $X \times X$.

(iii) Cofinite topology of finite set is a .... topology.

(b). Let $X = \{a, b, c, d, e\}$ and $\tau = \{\varphi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be topology on X. Find interior of $A = \{b, c, d\} \subset X$.

Question 5(a). Fill in the blanks meaningfully: (3)

(i) A group homomorphism f is injective iff ....

(ii) The only non-trivial proper subgroup of $\mathbb{Z}_4$ under addition modulo is ....

(iii) The characteristic of ring of integers is ....

(iv) Index of the alternating group A_n in S_n is ....

(v) Centre of an abelian group is ....

(vi) If the characteristic polynomial of a matrix A is $x^2 - 5x + 6$, then its eigen values are ....

(b). Define Normalizer of a subgroup of a group G, show that $N_G(H)$ is a subgroup. (3)

(c). If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear transformation define by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, 2x_1 + x_3, x_1 - 2x_2 + 2x_3)$$

then find imT and kerT.

Question 6(a). Define eigenvalue problem and find the eigen solution of

$$\frac{d^2u}{dx^2} + \lambda u = 0 \quad \text{with} \quad , u(0) = 0, u(a) = 0$$

(b). Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Question 7(a). Solve the following system of equation using Gaussian elimination method.

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ -2x + 3y - z &= 1 \end{aligned}$$

Good Luck